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Classical and Bayesian Estimation of Weibull Parameters Under Fuzzy Data Using Alpha Cuts

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ABSTRACT

This paper studies classical and Bayesian estimation of Weibull distribution parameters when lifetime observations are imprecise and represented as triangular fuzzy numbers. The fuzzy observations are handled using the alpha-cut approach, which converts each fuzzy observation into lower and upper endpoint samples at different membership levels. The scale parameter, shape parameter, and reliability function are estimated using maximum likelihood estimation, Lindley's approximation, and the Metropolis-Hastings algorithm. A simulation study is conducted for different sample sizes and alpha-cut levels, and the estimators are compared using mean squared error and bias. The numerical results show that estimation accuracy improves as the sample size increases, and the Bayesian estimators give competitive performance compared with maximum likelihood estimation. The study also shows that the endpoint ordering of the estimated Weibull shape parameter may not always follow the endpoint ordering of the fuzzy data, because the shape parameter depends on the relative spread and aging pattern of the sample. Therefore, the estimated shape values are interpreted using their minimum and maximum values when reporting fuzzy parameter intervals. A real-data application based on turbofan engine failure times demonstrates the practical usefulness of the proposed method and shows that the fuzzy reliability estimates reduce to the ordinary crisp-data estimates when the alpha level is one.

Keywords: Weibull distribution, fuzzy lifetime data, alpha-cut, Bayesian estimation, Metropolis-Hastings algorithm.

Introduction

Lifetime data analysis is an important area of reliability engineering, survival analysis, industrial quality control, and risk assessment. In many practical applications, the exact failure time of a unit may not be observed precisely. The recorded lifetime may be affected by measurement error, inspection delay, incomplete monitoring, expert judgment, or linguistic uncertainty. When such observations are treated as exact crisp values, part of the uncertainty present in the data may be ignored. Fuzzy set theory, introduced by Zadeh¹, provides a useful mathematical framework for representing and analyzing imprecise information.

The Weibull distribution is one of the most widely used lifetime models because of its flexibility in describing different failure-rate behaviors. Depending on the value of the shape parameter, the Weibull model can represent decreasing, constant, or increasing hazard rates. This makes it suitable for modeling early failures, random failures, and wear-out failures in engineering systems. The distribution was introduced by Weibull² and has since been extended in many directions to improve its flexibility for reliability and survival data. Several generalized and modified Weibull models have been proposed for handling complex data structures and different hazard shapes³⁻⁶. However, most standard estimation procedures for the Weibull model assume that the observed lifetime data are exact.

In real reliability studies, failure times are often imprecise rather than exact. Fuzzy numbers are useful in such situations because they allow each observation to be represented by a range of plausible values together with a membership structure. The alpha-cut representation, discussed by Dubois and Prade⁷ and Klir and Yuan⁸, converts a fuzzy number into an interval at a specified membership level. This makes it possible to apply statistical estimation procedures to the lower and upper endpoint samples obtained at different alpha levels. As the alpha level increases, the fuzzy interval becomes narrower, and at alpha equal to one, the fuzzy observation reduces to its core value.

Parameter estimation is a central problem in lifetime data analysis. In the classical framework, maximum likelihood estimation is widely used because of its desirable large-sample properties. Likelihood-based methods for lifetime and reliability models have been discussed extensively in the literature⁹⁻¹². Recent work has also considered improved and robust likelihood-based estimation procedures for Weibull-type models¹³⁻¹⁵. These studies show the usefulness of maximum likelihood estimation in crisp data settings. However, when the observations are fuzzy, the likelihood is affected by the lower and

upper endpoint samples obtained from alpha cuts, and the usual crisp-data estimation procedure must be adapted.

Bayesian methods provide an alternative framework for estimation under uncertainty. General Bayesian inference has been developed in standard references such as^{16,17}, and Bayesian reliability analysis has been discussed by¹⁸. When posterior distributions do not have closed-form expressions, approximation and simulation methods are commonly used. Lindley's approximation gives a convenient analytical approximation to Bayes estimates, while the Metropolis–Hastings algorithm^{19,20} provides a simulation-based approach for sampling from complex posterior distributions. Detailed treatments of Markov chain Monte Carlo methods are available in^{21–24}.

Several recent studies have considered estimation problems for lifetime models under fuzzy data. For example, Bayesian and classical estimation procedures have been applied to fuzzy lifetime models using alpha-cut, EM-type, and simulation-based methods^{25–27}. These works show that fuzzy-data methods can provide more realistic inference when observations are imprecise. Nevertheless, there remains a need for a unified comparison of classical and Bayesian estimation methods for the Weibull distribution under alpha-cut based fuzzy observations. In particular, the combined use of maximum likelihood estimation, Lindley's approximation, and the Metropolis–Hastings algorithm for fuzzy Weibull lifetime data has not been sufficiently explored.

The present study addresses this gap by developing a fuzzy alpha-cut based estimation framework for the Weibull distribution. The scale parameter, shape parameter, and reliability function are estimated using maximum likelihood estimation, Lindley's approximation, and the Metropolis–Hastings algorithm. The performance of the estimators is examined through a simulation study using mean squared error and bias, and the practical usefulness of the method is illustrated using turbofan engine failure-time data.

An important contribution of this study is the interpretation of fuzzy interval estimation. In ordinary crisp data analysis, a confidence interval or HPD interval has one lower endpoint and one upper endpoint. In the fuzzy-data setting, the lower alpha-cut sample produces its own interval and the upper alpha-cut sample produces another interval. Thus, the resulting inference captures both statistical uncertainty and data imprecision. This provides a wider and more informative uncertainty band, which can be useful for reliability decision-making, maintenance planning, and risk assessment.

The study also highlights an important feature of the Weibull shape parameter. In some cases, the estimate of the shape parameter obtained from the lower endpoint sample may be larger than the estimate obtained from the upper endpoint sample. This behaviour is not contradictory because the shape parameter depends on the relative spread and aging pattern of the data, not only on the magnitude of the observations. Therefore, when an ordered fuzzy estimate is required, the two endpoint estimates of the shape parameter are interpreted using their minimum and maximum values. The main objectives of this paper are as follows:

- To formulate the Weibull likelihood under fuzzy alpha-cut observations.
- To estimate the Weibull parameters using maximum likelihood estimation, Lindley's approximation, and the Metropolis–Hastings algorithm.
- To compare the estimators using MSE and bias.
- To study the behavior of fuzzy confidence and HPD intervals.
- To demonstrate the proposed method using real reliability data.

Preliminaries

In this section, some basic definitions are recalled. The Weibull distribution is first described because it forms the underlying lifetime model. The concepts of fuzzy sets and fuzzy numbers are then introduced to represent imprecise observations. In particular, triangular fuzzy numbers are considered due to their simple structure and practical suitability for modeling uncertain lifetime data. The alpha-cut technique is finally presented as a bridge between fuzzy observations and interval-valued data, allowing classical and Bayesian estimation methods to be applied at different levels of uncertainty.

Weibull Distribution

The Weibull distribution is widely used in reliability engineering and survival analysis because it can be used to model many different hazard functions, the probability density function (PDF) for a fuzzy failure times T is defined as:

$$f^*(t^*; \lambda, k) = \begin{cases} \frac{k}{\lambda} \left(\frac{t^*}{\lambda} \right)^{k-1} e^{-(t^*/\lambda)^k}, & t^* \geq 0, \\ 0, & t^* < 0, \end{cases} \quad (1)$$

Where $\lambda > 0$ is the scale parameter, which determines how spread out the distribution is, and $k > 0$ is the shape parameter, which governs the hazard function's behavior.

The reliability function, indicating the probability of an event occurring after time t , is:

$$R^*(t^*; \lambda, k) = P(T > t^*) = e^{-(t^*/\lambda)^k}. \quad (2)$$

The hazard function, $h^*(t^*) = \frac{f^*(t^*)}{R^*(t^*)}$, reflects the instantaneous event rate at time t^* . It varies depending on k :

- If $k = 1$: Constant hazard rate (exponential distribution).
- If $k < 1$: Decreasing hazard rate (e.g., "infant mortality").
- If $k > 1$: Increasing hazard rate (e.g., "wear-out failures").

Fuzzy Sets

Fuzzy sets¹ provide a structure for addressing uncertainty and imprecision, characterized by gradual rather than binary membership in a set. The membership function $\mu_{\tilde{A}}(x)$ for each element x is in the range $[0, 1]$:

$$\mu_{\tilde{A}}(x) \in [0, 1], \quad \forall x \in \mathbb{R}.$$

where $\mu_{\tilde{A}}(x)$ reflects the degree of truth for x belonging to the fuzzy set $\tilde{A}$.

Fuzzy Number

A fuzzy number $\tilde{A}$ is a special class of fuzzy sets defined on the real line $\mathbb{R}$, described by a membership function $\mu_{\tilde{A}}(x) : \mathbb{R} \rightarrow [0, 1]$. A fuzzy number satisfies the following conditions^{1,7}:

- *Normality*: There exists at least one $x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$,
- *Convexity*: For any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\},$$

- *Upper semi-continuity* of the membership function,
- *Bounded support*.

The triangular fuzzy number (TFN) is one of the most common types because it is easy to use. A TFN is represented as $\tilde{A} = (a, b, c)$, with the condition that $a < b < c$. The following is how its membership function is defined:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ \frac{c-x}{c-b}, & b \leq x \leq c, \\ 0, & x > c. \end{cases} \quad (3)$$

α -cuts

The idea of α -cuts is a useful way to show fuzzy numbers as clear intervals. The α -cut for a fuzzy number $\tilde{A}$ is defined as^{7,8}:

$$A_\alpha = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geq \alpha\}, \quad \alpha \in [0, 1]. \quad (4)$$

For any $\alpha \in [0, 1]$, the α -cut of a fuzzy number is a closed interval given by

$$A_\alpha = [\underline{A}(\alpha), \bar{A}(\alpha)], \quad \alpha \in [0, 1]. \quad (5)$$

where $\underline{A}(\alpha)$ and $\bar{A}(\alpha)$ represent the lower and upper bounds of the interval, respectively. In the case of a triangular fuzzy number $\tilde{A} = (a, b, c)$, the α -cut can be expressed as

$$A_\alpha = [a + \alpha(b - a), c - \alpha(c - b)], \quad \alpha \in [0, 1]. \quad (6)$$

The α -cut representation plays a crucial role in transforming fuzzy parameters into interval-valued quantities. This representation facilitates the extension of classical parametric estimation methods by allowing the analysis to be carried out at different levels of uncertainty.

In reliability engineering analysis, the fuzzy concept deals with uncertainties such as unclear failure times or missing data, which can happen when people don't remember things correctly or when follow-ups aren't regular. This concept gives a robust framework for modeling real-world datasets when the data is imprecise.

Methodology

In this section, we present the methodology adopted in this study. The estimation procedures based on Maximum Likelihood and Bayesian approaches are described in the following subsections. These methods are applied to estimate the model parameters under the specified framework.

Maximum Likelihood Estimation

MLE is a common method to estimate the parameters of a model's distribution which are assumed to be independent and identically distributed. The parameters are estimated by maximizing the likelihood function.

Let $t_1^*, t_2^*, \dots, t_n^*$ be fuzzy observations of size n obtained from the fuzzy density function $f^*(t^*; \lambda, k)$. The corresponding fuzzy likelihood function is defined as:

$$L_\alpha^*(\lambda, k; \mathbf{t}^*) = \prod_{i=1}^n f^*(t_i^*; \lambda, k), \quad \alpha \in [0, 1] \quad (7)$$

Substituting the weibull fuzzy probability density function into Eq. (7), we obtain

$$L_\alpha^*(\lambda, k; \mathbf{t}^*) = \prod_{i=1}^n \left[\frac{k}{\lambda} \left(\frac{t_i^*}{\lambda} \right)^{k-1} \exp \left\{ - \left(\frac{t_i^*}{\lambda} \right)^k \right\} \right], \quad \alpha \in [0, 1] \quad (8)$$

having lower and upper bounds of likelihood functions are $\underline{L}_\alpha(\lambda, k; \bar{\mathbf{t}})$ and $\bar{L}_\alpha(\lambda, k; \bar{\mathbf{t}})$ respectively.

Where

$$\underline{L}_\alpha(\lambda, k; \mathbf{t}) = \prod_{i=1}^n \underline{f}(t_i; \lambda, k) = \prod_{i=1}^n \left[\frac{k}{\lambda} \left(\frac{t_i}{\lambda} \right)^{k-1} \exp \left\{ - \left(\frac{t_i}{\lambda} \right)^k \right\} \right], \quad \alpha \in [0, 1] \quad (9)$$

and

$$\bar{L}_\alpha(\lambda, k; \mathbf{t}) = \prod_{i=1}^n \bar{f}(\bar{t}_i; \lambda, k) = \prod_{i=1}^n \left[\frac{k}{\lambda} \left(\frac{\bar{t}_i}{\lambda} \right)^{k-1} \exp \left\{ - \left(\frac{\bar{t}_i}{\lambda} \right)^k \right\} \right], \quad \alpha \in [0, 1] \quad (10)$$

For estimating the parameters λ, k that maximizes the fuzzy log likelihood function where,

$$\frac{\partial \log L_\alpha^*}{\partial \lambda} = 0, \quad \frac{\partial \log L_\alpha^*}{\partial k} = 0 \quad (11)$$

taking logarithms of Eq. (8), differentiating with respect to λ and k then equating to zero, the equations become,

$$\ln L_\alpha^*(\lambda, k; \mathbf{t}^*) = n \ln k - nk \ln \lambda + (k-1) \sum_{i=1}^n \ln t_i^* - \sum_{i=1}^n \left(\frac{t_i^*}{\lambda} \right)^k, \quad \alpha \in [0, 1] \quad (12)$$

$$\frac{\partial \ln L_{\alpha}^*}{\partial \lambda} = -\frac{nk}{\lambda} + \frac{k}{\lambda^{k+1}} \sum_{i=1}^n (t_i^*)^k = 0, \quad \alpha \in [0, 1] \quad (13)$$

$$\frac{\partial \ln L_{\alpha}^*}{\partial k} = \frac{n}{k} - n \ln \lambda + \sum_{i=1}^n \ln t_i^* - \sum_{i=1}^n \left(\frac{t_i^*}{\lambda} \right)^k \ln \left(\frac{t_i^*}{\lambda} \right) = 0, \quad \alpha \in [0, 1] \quad (14)$$

By eliminating λ from Eqs. (13) and (14), we obtain

$$\hat{\lambda} = \left(\frac{1}{n} \sum_{i=1}^n (t_i^*)^k \right)^{1/k} \quad (15)$$

$$\frac{1}{k} - \frac{\sum_{i=1}^n (t_i^*)^k \ln t_i^*}{\sum_{i=1}^n (t_i^*)^k} + \frac{\sum_{i=1}^n \ln t_i^*}{n} = 0 \quad (16)$$

The eqn. (16) has not yielded a closed form of analytical solution, so the Newton-Raphson method is required to estimate the parameter k must be solved numerically. The Newton-Raphson iterative method is defined as:

$$k_{i+1} = k_i - \frac{f(k_i)}{f'(k_i)} \quad (17)$$

where $f'(k_i)$ is the second derivative of the log-likelihood function with respect to k . Starting with an initial value $k_{(0)}$, the iteration is continued until convergence is achieved. The convergence criterion is given as ,

$$|k_{(i+1)} - k_{(i)}| < \varepsilon, \quad (18)$$

where ε is a small positive tolerance.

The above Newton–Raphson procedure is applied separately to the lower and upper α -cut samples to obtain the corresponding parameter estimates at each α level. That is, for each $\alpha \in [0, 1]$, the estimates

$$(\hat{\lambda}_{\alpha}, \hat{k}_{\alpha}) \quad \text{and} \quad (\bar{\hat{\lambda}}_{\alpha}, \bar{\hat{k}}_{\alpha})$$

are obtained based on the lower and upper bounds of the fuzzy observations, respectively.

Confidence Intervals

To account for uncertainty in the fuzzy observations, confidence intervals are constructed separately for the lower and upper α -cut samples.

Using the asymptotic normality of the maximum likelihood estimators, the approximate 95% confidence intervals for the parameters λ and k are obtained based on the observed Fisher information matrix.

For the lower α -cut sample, the confidence intervals are given by

$$\hat{\theta}_{\alpha} \pm z_{\alpha/2} \text{SE}(\hat{\theta}_{\alpha}),$$

and for the upper α -cut sample, they are given by

$$\bar{\hat{\theta}}_{\alpha} \pm z_{\alpha/2} \text{SE}(\bar{\hat{\theta}}_{\alpha}),$$

where $\hat{\theta} \in \{\lambda, k\}$, $\text{SE}(\cdot)$ denotes the standard error obtained from the inverse of the Fisher information matrix, and $z_{\alpha/2}$ is the standard normal critical value.

Thus, the interval estimates are obtained separately for both lower and upper bounds at each α -level.

Bayesian Estimation

In this section, the Bayesian estimation is used for estimating the unknown parameters λ and k , which are treated as random variables. The parameters λ and k are assumed to be independent and follow Gamma prior distributions given by

$$\lambda \sim \text{Gamma}(a_1, b_1), \quad k \sim \text{Gamma}(a_2, b_2),$$

$$\pi(\lambda) \propto \lambda^{a_1-1} e^{-b_1\lambda}, \quad \pi(k) \propto k^{a_2-1} e^{-b_2k}$$

with joint prior density

$$\pi(\lambda, k) \propto \lambda^{a_1-1} k^{a_2-1} \exp(-b_1\lambda - b_2k) \quad (19)$$

The posterior distribution for the parameters defined as:

$$\pi_{\alpha}^*(\lambda, k | \mathbf{t}^*) \propto \left(\frac{k}{\lambda}\right)^n \prod_{i=1}^n (t_i^*)^{k-1} \exp\left\{-\sum_{i=1}^n \left(\frac{t_i^*}{\lambda}\right)^k\right\} \cdot \lambda^{a_1-1} k^{a_2-1} \exp(-b_1\lambda - b_2k), \quad \alpha \in [0, 1] \quad (20)$$

Thus, for each α -cut level lower and upper bounds of posterior distributions are $\pi_{\alpha}(\lambda, k | \underline{t})$ and $\pi_{\alpha}(\lambda, k | \bar{t})$
Where

$$\pi_{\alpha}(\lambda, k | \underline{t}) \propto \left(\frac{k}{\lambda}\right)^n \prod_{i=1}^n (\underline{t}_i)^{k-1} \exp\left\{-\sum_{i=1}^n \left(\frac{\underline{t}_i}{\lambda}\right)^k\right\} \cdot \lambda^{a_1-1} k^{a_2-1} \exp(-b_1\lambda - b_2k), \quad \alpha \in [0, 1] \quad (21)$$

and

$$\pi_{\alpha}(\lambda, k | \bar{t}) \propto \left(\frac{k}{\lambda}\right)^n \prod_{i=1}^n (\bar{t}_i)^{k-1} \exp\left\{-\sum_{i=1}^n \left(\frac{\bar{t}_i}{\lambda}\right)^k\right\} \cdot \lambda^{a_1-1} k^{a_2-1} \exp(-b_1\lambda - b_2k), \quad \alpha \in [0, 1] \quad (22)$$

Since the posterior expectations cannot be obtained in closed form, Lindley's approximation is used to derive the Bayes estimates of λ and k .

Bayesian Estimation by Lindley Approximation

Lindley's approximation is numerical technique for computing Bayes estimators. Let $\hat{\lambda}$ and $\hat{k}$ denote the maximum likelihood estimates. Then the Bayes estimators under squared error loss are approximated by

$$\hat{\lambda}_B \approx \hat{\lambda} + \rho_1 s_{12} + \rho_2 s_{22} + \frac{1}{2} [L_{112}(s_{11}s_{22} + 2s_{12}^2) + 3L_{122}s_{12}s_{22} + L_{111}s_{11}s_{12} + L_{222}s_{22}^2], \quad (23)$$

and

$$\hat{k}_B \approx \hat{k} + \rho_1 s_{11} + \rho_2 s_{12} + \frac{1}{2} [L_{111}s_{11}^2 + 3L_{112}s_{11}s_{12} + L_{122}(s_{11}s_{22} + 2s_{12}^2) + L_{222}s_{12}s_{22}], \quad (24)$$

where

$$\rho_1 = \frac{a_1 - 1}{\hat{k}} - b_1, \quad \rho_2 = \frac{a_2 - 1}{\hat{\lambda}} - b_2, \quad (25)$$

and s_{ij} are the elements in $-L_{ij}^{-1}$. These approximations are computed separately for the lower and upper α -cut samples to obtain

$$(\hat{\lambda}_{B,\alpha}, \hat{k}_{B,\alpha}) \quad \text{and} \quad (\bar{\lambda}_{B,\alpha}, \bar{k}_{B,\alpha}).$$

Posterior Sampling and HPD Intervals

Since the posterior distribution does not admit a closed-form expression, samples are generated using the Metropolis–Hastings algorithm. A random walk proposal distribution is employed to generate candidate values for the parameters.

For each α -cut level, the algorithm is applied separately to the lower and upper samples. Starting from initial values $(\lambda^{(0)}, k^{(0)})$, candidate values (λ', k') at the m -th iteration are generated as

$$\lambda' \sim N(\lambda^{(m)}, \sigma_\lambda^2), \quad k' \sim N(k^{(m)}, \sigma_k^2),$$

where $N(\cdot)$ denotes the normal distribution.

The acceptance probability is given by

$$\delta = \min \left\{ 1, \frac{\pi(\lambda', k' | t)}{\pi(\lambda^{(m)}, k^{(m)} | t)} \right\}$$

The proposed values are accepted with probability δ , otherwise the previous values are retained. This process is repeated for a sufficiently large number of iterations to obtain samples from the posterior distribution.

After discarding an initial burn-in period, the remaining samples

$$\{\lambda^{(m)}, k^{(m)}\}_{m=1}^M$$

are used for inference.

The Bayes estimates of the parameters are obtained as the posterior means given by

$$\hat{\lambda}_{B,\alpha} = \frac{1}{M} \sum_{m=1}^M \lambda^{(m)}, \quad \hat{k}_{B,\alpha} = \frac{1}{M} \sum_{m=1}^M k^{(m)} \quad (26)$$

To quantify the uncertainty, highest posterior density (HPD) intervals are constructed from the posterior samples. The $100(1 - \gamma)\%$ HPD interval for a parameter $\theta \in \{\lambda, k\}$ is defined as the shortest interval $[l, u]$ such that

$$P(l \leq \theta \leq u | t) = 1 - \gamma.$$

Accordingly, for each α -cut level, the HPD intervals are obtained separately for the lower and upper samples as

$$[\underline{\theta}_{L,\alpha}, \underline{\theta}_{U,\alpha}] \quad \text{and} \quad [\bar{\theta}_{L,\alpha}, \bar{\theta}_{U,\alpha}],$$

where $\underline{\theta}$ and $\bar{\theta}$ denote the estimates based on the lower and upper bounds of the fuzzy observations, respectively.

As the value of α increases, the lower and upper HPD intervals tend to converge, reflecting reduced uncertainty in the parameter estimates.

Numerical Study

This section presents the numerical study for the Weibull distribution under fuzzy lifetime observations. The proposed estimation procedures are examined through a simulation study and a real data application. The Weibull distribution is considered with scale parameter λ and shape parameter k . For fuzzy observations, triangular fuzzy numbers are used and the corresponding α -cuts are taken at $\alpha = 0.2, 0.4, 0.6, 0.8$, and 1.0 . At each α -cut level, the lower and upper endpoints are analyzed separately. The maximum likelihood estimator (MLE), Lindley's approximation, and the Metropolis–Hastings (MH) algorithm are compared.

Simulation Study

In the simulation study, the true parameter values are taken as $\lambda = 3$ and $k = 2$. The sample sizes are $n = 30, 50$, and 100 . The purpose of using these three sample sizes is to study how the accuracy of the estimators changes when the amount of information increases. The comparison of the methods is mainly based on MSE and bias, while the parameter-estimate, confidence-interval, and HPD-interval tables are used to explain the behavior of the estimates and their uncertainty.

Figure 1 shows the convergence of the Newton–Raphson method for estimating the Weibull shape parameter k at different sample sizes. The plotted iterations indicate that the Newton–Raphson sequence stabilizes after a small number of iterations. This supports the numerical reliability of the MLE computation. Figures 2–5 give the MH diagnostic plots. The trace plots in Figure 2 show that the simulated chains fluctuate around stable regions for both λ and k . The density plots in Figure 3 summarize the posterior concentration of the parameters, and the running mean and autocorrelation plots for both λ and k

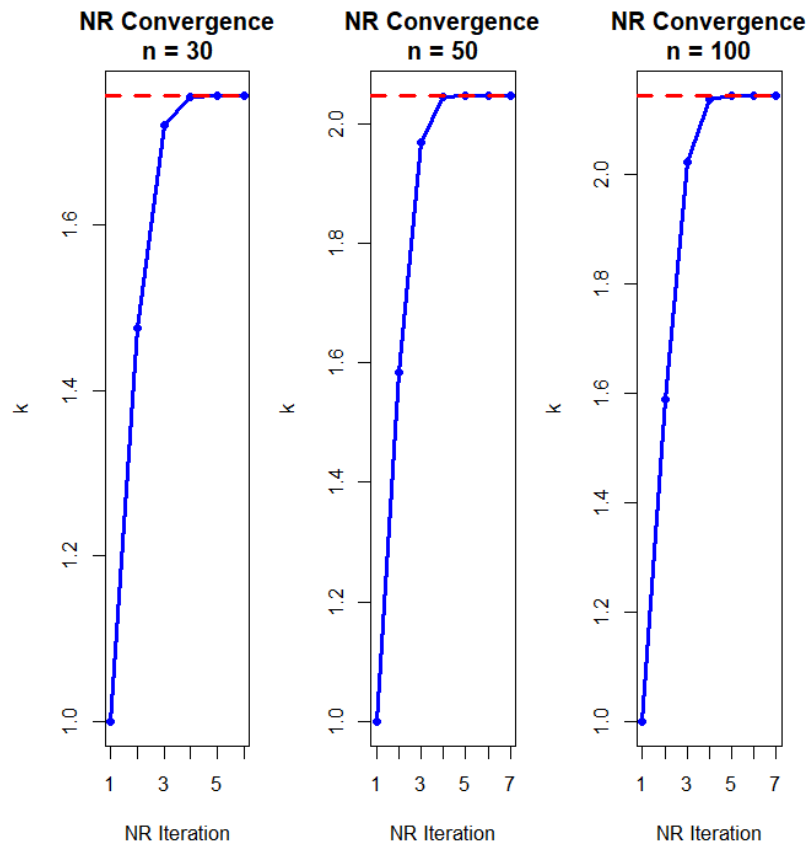


Figure 1. Convergence plot for κ by Newton-Raphson method for different sample sizes

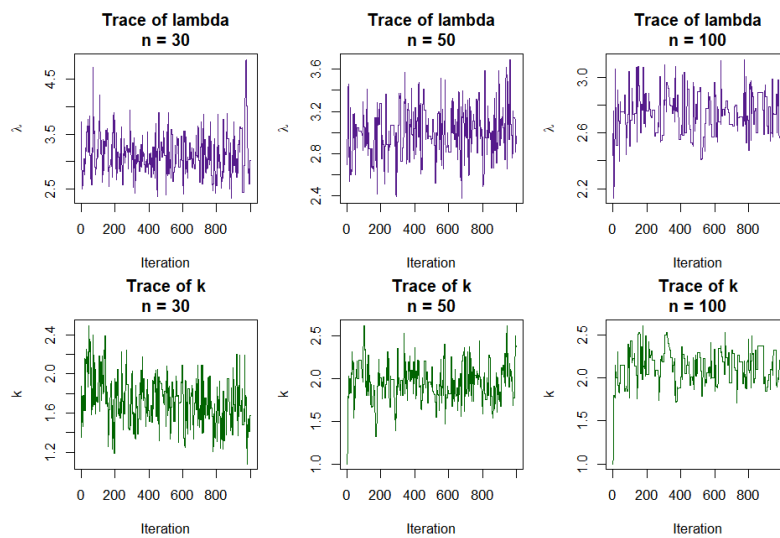


Figure 2. Trace plots for λ and κ for different sample sizes

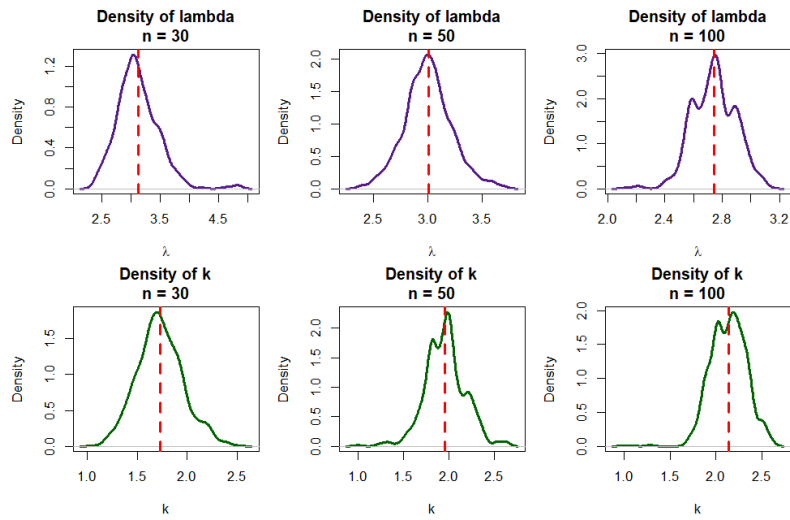


Figure 3. Density plots for λ and κ for different sample sizes

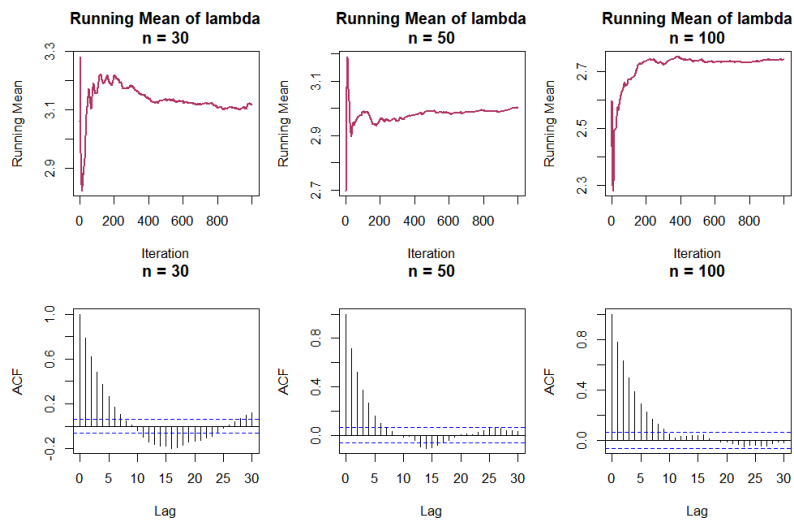


Figure 4. Running mean and Auto-correlation plots of λ for different sample sizes

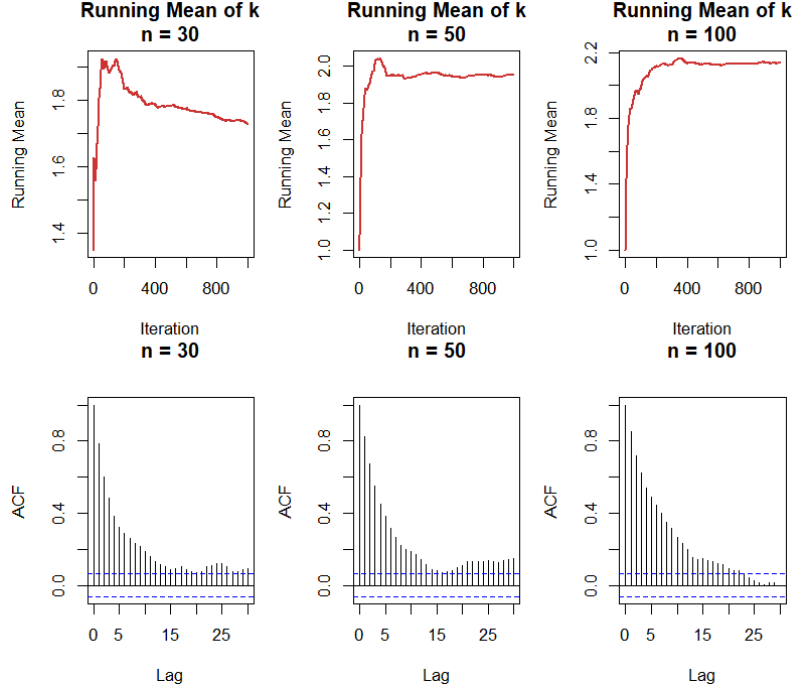


Figure 5. Running mean and Auto-correlation plots of κ for different sample sizes

in Figures 4 and 5 show the stability of the posterior simulation. These figures indicate that the MH algorithm gives stable Bayesian estimates for the fuzzy Weibull model.

The parameter estimates for different sample sizes and α -levels reports the estimated values of λ and k for MLE, Lindley, and MH in Table 1. The parameter estimates show the expected behaviour of fuzzy estimation. For the scale parameter λ , the lower endpoint estimates are generally smaller than the true value, while the upper endpoint estimates are generally larger than the true value for smaller α -levels. Hence, the true value $\lambda = 3$ lies between the lower and upper endpoint estimates in most cases. This confirms that the fuzzy endpoint estimates capture the uncertainty generated by the imprecise observations. As α increases, the lower and upper endpoint samples move closer to the core observations. Consequently, the lower and upper estimates also become closer. At $\alpha = 1$, there is no fuzziness left in the α -cut because both endpoints coincide with the core value; hence the lower and upper estimates are identical.

For the shape parameter k , the true value $k = 2$ is also well captured by the endpoint-based estimates in the simulation results. However, an important feature is observed: the lower endpoint estimate of k can be larger than the upper endpoint estimate. This is not a contradiction. The Weibull shape parameter is not a direct measure of the magnitude of lifetimes. It depends on the relative spread, log-spacing, and aging pattern of the sample. Therefore, the ordering of the original fuzzy lifetime endpoints need not be preserved after estimating k . For example, a lower endpoint sample may have smaller relative dispersion and may therefore produce a larger estimated shape parameter. For this reason, the two endpoint estimates of k should be interpreted as the possible range of the shape estimate. When an ordered fuzzy estimate is required, the smaller of the two values should be reported as the minimum estimate and the larger value as the maximum estimate.

The interval-estimation results require special interpretation under fuzzy data. In ordinary crisp data analysis, a confidence or HPD interval has two endpoints: a lower and an upper confidence limits. In the fuzzy-data setting, there are two endpoints at each α -level. Therefore, the lower and upper endpoints produces their own intervals. This means that fuzzy interval estimation contains two levels of uncertainty. The first level is statistical uncertainty, represented by the two endpoints of each CI or HPD interval. The second level is fuzzy-data uncertainty, represented by the lower and upper α -cut samples. Thus, for each parameter, the fuzzy interval output is richer than the ordinary crisp interval because it describes not only sampling uncertainty but also the imprecision in the observed lifetimes.

Tables 2 and 3 demonstrates this point clearly. For each α -level, the lower endpoint gives one interval and the upper endpoint gives another interval. Together, these intervals form a band of plausible parameter values. This band is especially useful in reliability decision making because it gives conservative and optimistic parameter ranges rather than a single crisp interval. When uncertainty is large, corresponding to smaller α -values, the two endpoint-based intervals are farther apart. As α increases, the fuzzy uncertainty decreases and the lower and upper endpoint intervals move closer. At $\alpha = 1$, the fuzzy

n	α	Method	$\hat{\lambda}$		$\hat{k}$	
			Lower	Upper	Lower	Upper
30	0.2	MLE	2.75311	3.22278	2.12048	2.10904
30	0.2	Lindley	2.69372	3.13941	1.99943	1.98488
30	0.2	MH	2.75511	3.20050	2.04471	2.02867
30	0.4	MLE	2.71192	3.06296	1.99212	2.00606
30	0.4	Lindley	2.65133	2.98488	1.88370	1.89366
30	0.4	MH	2.70421	3.04759	1.92889	1.93765
30	0.6	MLE	2.83974	3.07382	2.14541	2.13951
30	0.6	Lindley	2.77626	2.99862	2.02202	2.01447
30	0.6	MH	2.82604	3.05643	2.05322	2.05315
30	0.8	MLE	3.03396	3.15541	2.08868	2.08942
30	0.8	Lindley	2.95963	3.07453	1.96771	1.96725
30	0.8	MH	3.01595	3.13302	2.00420	1.99875
30	1.0	MLE	3.04949	3.04949	2.15245	2.15245
30	1.0	Lindley	2.97529	2.97529	2.02573	2.02573
30	1.0	MH	3.03370	3.03370	2.06375	2.06375
50	0.2	MLE	2.75311	3.23146	2.04991	2.04442
50	0.2	Lindley	2.71591	3.17925	1.98136	1.97366
50	0.2	MH	2.74672	3.21532	2.00948	1.99750
50	0.4	MLE	2.75078	3.09333	2.02461	2.02333
50	0.4	Lindley	2.71336	3.04522	1.95743	1.95437
50	0.4	MH	2.75350	3.08030	1.98201	1.97661
50	0.6	MLE	2.87780	3.12074	2.00650	2.00827
50	0.6	Lindley	2.83627	3.07149	1.93951	1.93995
50	0.6	MH	2.87467	3.10724	1.96620	1.95594
50	0.8	MLE	2.94612	3.06704	2.00137	1.99868
50	0.8	Lindley	2.90281	3.01977	1.93433	1.93116
50	0.8	MH	2.94288	3.05859	1.96023	1.95291
50	1.0	MLE	3.00975	3.00975	2.02543	2.02543
50	1.0	Lindley	2.96471	2.96471	1.95700	1.95700
50	1.0	MH	3.00094	3.00094	1.98053	1.98053
100	0.2	MLE	2.77859	3.25969	2.00196	2.00718
100	0.2	Lindley	2.75952	3.23304	1.96906	1.97293
100	0.2	MH	2.77422	3.25177	1.97644	1.98134
100	0.4	MLE	2.81328	3.17243	2.03934	2.03978
100	0.4	Lindley	2.79349	3.14699	2.00490	2.00438
100	0.4	MH	2.80194	3.16009	2.00584	2.00933
100	0.6	MLE	2.85475	3.09511	2.04595	2.04167
100	0.6	Lindley	2.83476	3.07137	2.01152	2.00672
100	0.6	MH	2.84817	3.09077	2.01405	2.01505
100	0.8	MLE	2.96455	3.08717	2.06490	2.06175
100	0.8	Lindley	2.94313	3.06382	2.02973	2.02634
100	0.8	MH	2.95772	3.07521	2.03716	2.02072
100	1.0	MLE	3.01444	3.01444	2.01749	2.01749
100	1.0	Lindley	2.99182	2.99182	1.98341	1.98341
100	1.0	MH	3.00781	3.00781	1.98714	1.98714

Table 1. Parameter estimates for different sample sizes at various α -levels

intervals reduce to the ordinary crisp-data interval because the lower and upper endpoint samples are identical.

n	α	$\hat{\lambda}$		$\hat{k}$	
		Lower-bound Interval	Upper-bound Interval	Lower-bound Interval	Upper-bound Interval
30	0.2	[2.25394, 2.63580]	[3.25228, 3.80976]	[1.51817, 1.51176]	[2.72279, 2.70632]
30	0.4	[2.18867, 2.47616]	[3.23517, 3.64976]	[1.42644, 1.43458]	[2.55780, 2.57754]
30	0.6	[2.33402, 2.52466]	[3.34546, 3.62299]	[1.53621, 1.53275]	[2.75461, 2.74628]
30	0.8	[2.47744, 2.57636]	[3.59048, 3.73446]	[1.49730, 1.49799]	[2.68005, 2.68086]
30	1.0	[2.50946, 2.50946]	[3.58951, 3.58951]	[1.55161, 1.55161]	[2.75329, 2.75329]
50	0.2	[2.35831, 2.76655]	[3.14792, 3.69638]	[1.60594, 1.60138]	[2.49388, 2.48745]
50	0.4	[2.35126, 2.64409]	[3.15029, 3.54256]	[1.58614, 1.58626]	[2.46309, 2.46039]
50	0.6	[2.45627, 2.66412]	[3.29934, 3.57735]	[1.57004, 1.57157]	[2.44297, 2.44496]
50	0.8	[2.51287, 2.61552]	[3.37937, 3.51857]	[1.56642, 1.56450]	[2.43632, 2.43287]
50	1.0	[2.57189, 2.57189]	[3.44761, 3.44761]	[1.58353, 1.58353]	[2.46734, 2.46734]
100	0.2	[2.49156, 2.92380]	[3.06563, 3.59558]	[1.69481, 1.69913]	[2.30911, 2.31523]
100	0.4	[2.52608, 2.84853]	[3.10048, 3.49632]	[1.72859, 1.72882]	[2.35010, 2.35073]
100	0.6	[2.56518, 2.78057]	[3.14433, 3.40965]	[1.73296, 1.72956]	[2.35893, 2.35378]
100	0.8	[2.66686, 2.77671]	[3.26224, 3.39763]	[1.74795, 1.74535]	[2.38185, 2.37815]
100	1.0	[2.70453, 2.70453]	[3.32435, 3.32435]	[1.70812, 1.70812]	[2.32687, 2.32687]

Table 2. Confidence intervals for λ and k based on MLE at various α levels

n	α	$\hat{\lambda}$		$\hat{k}$	
		Lower-bound Interval	Upper-bound Interval	Lower-bound Interval	Upper-bound Interval
30	0.2	[2.24272, 2.61107]	[3.27863, 3.83194]	[1.48369, 1.48165]	[2.64183, 2.62708]
30	0.4	[2.17308, 2.40882]	[3.24361, 3.65895]	[1.40921, 1.39146]	[2.49611, 2.49349]
30	0.6	[2.30873, 2.49722]	[3.34484, 3.61642]	[1.48207, 1.49231]	[2.65853, 2.64009]
30	0.8	[2.45152, 2.54727]	[3.55780, 3.74445]	[1.46573, 1.43189]	[2.57778, 2.57199]
30	1.0	[2.49570, 2.49570]	[3.59240, 3.59240]	[1.51283, 1.51283]	[2.64505, 2.64505]
50	0.2	[2.34170, 2.75545]	[3.13639, 3.69739]	[1.60275, 1.59479]	[2.44311, 2.44611]
50	0.4	[2.36445, 2.64212]	[3.15891, 3.54257]	[1.56677, 1.55098]	[2.43757, 2.42642]
50	0.6	[2.45855, 2.65167]	[3.29909, 3.57531]	[1.56123, 1.54065]	[2.40687, 2.40405]
50	0.8	[2.51774, 2.60623]	[3.38988, 3.52866]	[1.55600, 1.54540]	[2.40127, 2.39504]
50	1.0	[2.57781, 2.57781]	[3.45532, 3.45532]	[1.56178, 1.56178]	[2.43330, 2.43330]
100	0.2	[2.47992, 2.91976]	[3.05844, 3.60264]	[1.67170, 1.67856]	[2.29620, 2.28805]
100	0.4	[2.51850, 2.83551]	[3.11103, 3.48645]	[1.70046, 1.71138]	[2.33493, 2.34113]
100	0.6	[2.55635, 2.78449]	[3.15620, 3.41722]	[1.71621, 1.70177]	[2.34447, 2.33200]
100	0.8	[2.66713, 2.77045]	[3.25556, 3.39364]	[1.73924, 1.72009]	[2.36832, 2.34767]
100	1.0	[2.70276, 2.70276]	[3.32151, 3.32151]	[1.69384, 1.69384]	[2.30439, 2.30439]

Table 3. HPD intervals for λ and k based on MH method at various α levels

A numerical comparison of MSE and bias for the MLE, Lindley, and MH methods is reported in Table 4. For both λ and k , the MSE generally decreases when the sample size increases from 30 to 100. This indicates that the proposed procedures are consistent in the numerical sense: larger samples give more accurate estimates. The bias values also become smaller or remain close to zero in many cases.

The comparison based on MSE and bias indicates that the Bayesian methods are competitive with MLE. In many settings, the MH estimator gives smaller MSE, while Lindley's approximation gives values close to MH with less computational effort. Thus, Lindley's approximation is useful when a fast Bayesian approximation is desired, whereas MH is useful when a simulation-based posterior summary is preferred. The results show that all three methods are suitable for fuzzy Weibull estimation, but the Bayesian estimators provide a strong alternative to the classical MLE approach.

Real Data Analysis

The real-data analysis further illustrates the practical importance of this interpretation. The dataset used in this study is the NASA CMAPSS turbofan engine dataset. Specifically, the FD001 subset was considered. The data were obtained from Kaggle²⁸, and were originally developed by NASA's Prognostics Center of Excellence. Figure 6 displays the first 20 triangular fuzzy numbers from the turbofan engine data. Each triangular fuzzy number has a lower endpoint, a core value, and an upper

n	α	Method	$\hat{\lambda}$				$\hat{k}$			
			MSE(L)	MSE(U)	Bias(L)	Bias(U)	MSE(L)	MSE(U)	Bias(L)	Bias(U)
30	0.2	MLE	0.13237	0.15348	-0.24689	0.22278	0.13652	0.12840	0.12048	0.10904
30	0.2	Lindley	0.15979	0.11537	-0.30628	0.13941	0.09954	0.09567	-0.00057	-0.01512
30	0.2	MH	0.12570	0.13523	-0.24489	0.20050	0.09917	0.10119	0.04471	0.02867
30	0.4	MLE	0.12865	0.06000	-0.28808	0.06296	0.08742	0.08712	-0.00788	0.00606
30	0.4	Lindley	0.16293	0.05078	-0.34867	-0.01512	0.08578	0.08351	-0.11630	-0.10634
30	0.4	MH	0.12986	0.05623	-0.29579	0.04759	0.08697	0.08073	-0.07111	-0.06235
30	0.6	MLE	0.07510	0.06444	-0.16026	0.07382	0.08515	0.08515	0.14541	0.13951
30	0.6	Lindley	0.09548	0.05384	-0.22374	-0.00138	0.05359	0.05475	0.02202	0.01447
30	0.6	MH	0.07711	0.06117	-0.17396	0.05643	0.05661	0.05793	0.05322	0.05315
30	0.8	MLE	0.18682	0.22383	0.03396	0.15541	0.12921	0.13141	0.08868	0.08942
30	0.8	Lindley	0.17322	0.18948	-0.04037	0.07453	0.10096	0.10263	-0.03229	-0.03275
30	0.8	MH	0.17682	0.20664	0.01595	0.13302	0.09519	0.10304	0.00420	-0.00125
30	1.0	MLE	0.13513	0.13513	0.04949	0.04949	0.09618	0.09618	0.15245	0.15245
30	1.0	Lindley	0.12185	0.12185	-0.02471	-0.02471	0.06022	0.06022	0.02573	0.02573
30	1.0	MH	0.12509	0.12509	0.03370	0.03370	0.05971	0.05971	0.06375	0.06375
50	0.2	MLE	0.09632	0.10356	-0.24689	0.23146	0.04162	0.04222	0.04991	0.04442
50	0.2	Lindley	0.11444	0.07962	-0.28409	0.17925	0.03518	0.03658	-0.01864	-0.02634
50	0.2	MH	0.09781	0.09076	-0.25328	0.21532	0.03750	0.03599	0.00948	-0.00250
50	0.4	MLE	0.09368	0.05012	-0.24922	0.09333	0.03732	0.03345	0.02461	0.02333
50	0.4	Lindley	0.11213	0.04117	-0.28664	0.04522	0.03455	0.03141	-0.04257	-0.04563
50	0.4	MH	0.09031	0.04818	-0.24650	0.08030	0.03133	0.03110	-0.01799	-0.02339
50	0.6	MLE	0.04948	0.05631	-0.12220	0.12074	0.04075	0.03942	0.00650	0.00827
50	0.6	Lindley	0.06007	0.04520	-0.16373	0.07149	0.04028	0.03897	-0.06049	-0.06005
50	0.6	MH	0.04906	0.05120	-0.12533	0.10724	0.03986	0.03491	-0.03380	-0.04406
50	0.8	MLE	0.06074	0.06661	-0.05388	0.06704	0.05446	0.05268	0.00137	-0.00132
50	0.8	Lindley	0.06537	0.06041	-0.09719	0.01977	0.05297	0.05180	-0.06567	-0.06884
50	0.8	MH	0.05867	0.06414	-0.05712	0.05859	0.05387	0.04926	-0.03977	-0.04709
50	1.0	MLE	0.03524	0.03524	0.00975	0.00975	0.04584	0.04584	0.02543	0.02543
50	1.0	Lindley	0.03478	0.03478	-0.03529	-0.03529	0.04248	0.04248	-0.04300	-0.04300
50	1.0	MH	0.03364	0.03364	0.00094	0.00094	0.03892	0.03892	-0.01947	-0.01947
100	0.2	MLE	0.06945	0.09575	-0.22141	0.25969	0.01265	0.01274	0.00196	0.00718
100	0.2	Lindley	0.07768	0.08168	-0.24048	0.23304	0.01294	0.01274	-0.03094	-0.02707
100	0.2	MH	0.07286	0.09083	-0.22578	0.25177	0.01257	0.01213	-0.02356	-0.01866
100	0.4	MLE	0.05451	0.05510	-0.18672	0.17243	0.04570	0.04766	0.03934	0.03978
100	0.4	Lindley	0.06202	0.04666	-0.20651	0.14699	0.04187	0.04369	0.00490	0.00438
100	0.4	MH	0.05760	0.04958	-0.19806	0.16009	0.04111	0.04341	0.00584	0.00933
100	0.6	MLE	0.04749	0.03952	-0.14525	0.09511	0.03264	0.03098	0.04595	0.04167
100	0.6	Lindley	0.05314	0.03490	-0.16524	0.07137	0.02898	0.02766	0.01152	0.00672
100	0.6	MH	0.05029	0.03810	-0.15183	0.09077	0.02848	0.02701	0.01405	0.01505
100	0.8	MLE	0.02126	0.02891	-0.03545	0.08717	0.02953	0.02880	0.06490	0.06175
100	0.8	Lindley	0.02278	0.02490	-0.05687	0.06382	0.02484	0.02434	0.02973	0.02634
100	0.8	MH	0.02244	0.02700	-0.04228	0.07521	0.02515	0.02399	0.03716	0.02072
100	1.0	MLE	0.03209	0.03209	0.01444	0.01444	0.03089	0.03089	0.01749	0.01749
100	1.0	Lindley	0.03126	0.03126	-0.00818	-0.00818	0.02924	0.02924	-0.01659	-0.01659
100	1.0	MH	0.03061	0.03061	0.00781	0.00781	0.02783	0.02783	-0.01286	-0.01286

Table 4. MSE and bias for interval bounds of parameters λ and k by the methods at different α levels

endpoint, showing that the recorded lifetime is not treated as a single exact value. This representation is more realistic when the observed failure times are affected by imprecision.

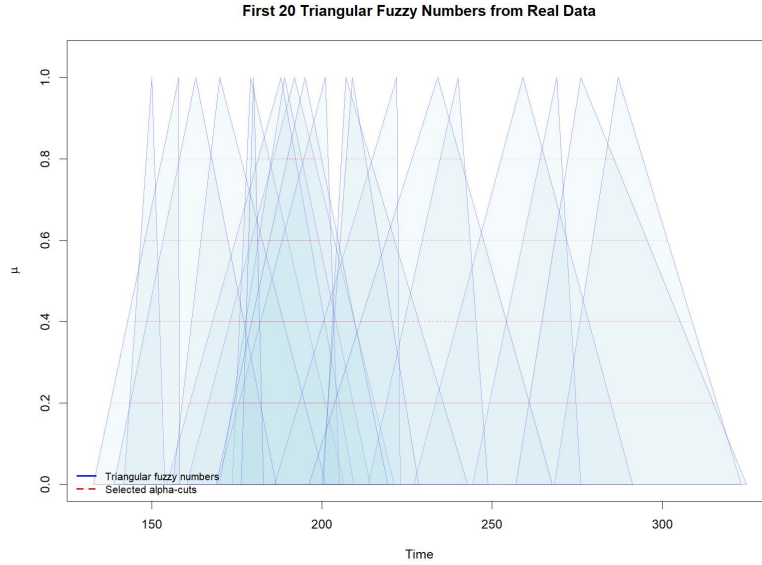


Figure 6. First 20 triangular fuzzy numbers for real data set

The parameter estimates of λ and κ for the MLE, Lindley, and MH methods across all α levels are presented in Table 5. For λ , the estimates from the lower endpoint values are smaller than those from the upper endpoint values, reflecting the natural ordering of the lifetime endpoints. At $\alpha = 1$, the lower and upper endpoint estimates coincide. For k , the lower endpoint estimates are often larger than the upper endpoint estimates, as observed in the simulation study. This again confirms that the Weibull shape parameter responds to the relative pattern of the data rather than only to the magnitude of the observations. Therefore, for real-data interpretation, the estimated k values should be arranged as minimum and maximum values when an ordered interval is required.

α	Method	$\hat{\lambda}$		$\hat{k}$	
		Lower	Upper	Lower	Upper
0.2	MLE	206.38937	244.31096	4.38239	4.19202
0.2	Lindley	180.76536	205.21175	3.76054	3.47538
0.2	MH	178.15912	194.89953	3.44840	2.83045
0.4	MLE	210.97819	239.44962	4.41338	4.25176
0.4	Lindley	184.56488	202.89776	3.77551	3.54368
0.4	MH	181.04968	196.75817	3.41685	3.03813
0.6	MLE	215.61672	234.61283	4.42726	4.30892
0.6	Lindley	188.19488	200.40987	3.77387	3.60936
0.6	MH	184.11022	193.91855	3.37565	3.10909
0.8	MLE	220.30078	229.80378	4.42514	4.36190
0.8	Lindley	191.64545	197.74728	3.75696	3.67094
0.8	MH	186.28690	192.34129	3.29384	3.17229
1.0	MLE	225.02582	225.02582	4.40871	4.40871
1.0	Lindley	194.90903	194.90903	3.72664	3.72664
1.0	MH	190.25001	190.25001	3.30396	3.30396

Table 5. Real data parameter estimates for λ and k by the methods at various α levels

Tables 6 and 7 represents real-data MLE confidence and MH HPD intervals provides a broader uncertainty description than ordinary crisp intervals. Since the lower endpoint and upper endpoints each produce a separate interval, the resulting output gives a lower-side and an upper-side intervals for each parameter. This is useful for decision makers because it identifies a range of possible parameter values under data imprecision. For example, in reliability planning, maintenance scheduling, or risk assessment, one may use the lower-side and upper-side intervals to compare conservative and reliability conclusions.

α	$\hat{\lambda}$		$\hat{k}$	
	Lower-bound Interval	Upper-bound Interval	Lower-bound Interval	Upper-bound Interval
0.2	[196.58635, 232.16854]	[216.19240, 256.45338]	[3.78782, 3.61928]	[4.97697, 4.76476]
0.4	[201.02485, 227.71525]	[220.93152, 251.18400]	[3.81263, 3.67032]	[5.01413, 4.83319]
0.6	[205.47365, 223.26747]	[225.75978, 245.95819]	[3.82263, 3.71896]	[5.03190, 4.89888]
0.8	[209.92965, 218.82569]	[230.67190, 240.78187]	[3.81898, 3.76383]	[5.03130, 4.95996]
1.0	[214.39021, 214.39021]	[235.66144, 235.66144]	[3.80330, 3.80330]	[5.01413, 5.01413]

Table 6. Real data confidence intervals for λ and k based on MLE at various α levels

α	$\hat{\lambda}$		$\hat{k}$	
	Lower-bound Interval	Upper-bound Interval	Lower-bound Interval	Upper-bound Interval
0.2	[166.24957, 170.84398]	[189.71925, 212.43781]	[2.89552, 2.20462]	[4.08018, 3.55517]
0.4	[165.93257, 181.85845]	[193.06435, 211.61311]	[2.84243, 2.49762]	[4.00839, 3.74869]
0.6	[171.91944, 175.19167]	[199.34431, 209.33624]	[2.68264, 2.38388]	[4.03627, 3.77155]
0.8	[168.16426, 181.26453]	[199.40525, 210.57079]	[2.64771, 2.59208]	[4.02453, 3.84932]
1.0	[177.12464, 177.12464]	[205.62196, 205.62196]	[2.73743, 2.73743]	[3.94981, 3.94981]

Table 7. Real data HPD intervals for λ and k based on MH method at various α levels

The reliability estimates for the real data are presented in Table 8 and Figure 7, corresponding to $\alpha = 0.6$ and 1.0, respectively. In all cases, the reliability starts at one when time is zero and decreases as time increases, which is consistent with lifetime-data theory. The lower and upper endpoint reliability estimates form a reliability band. This band represents the effect of fuzzy uncertainty on survival probability. For smaller α values, the band is wider, giving a larger range of possible reliability values. As α increases, the band becomes narrower. At $\alpha = 1$, the lower and upper reliability estimates coincide, and the fuzzy reliability analysis reduces to the ordinary crisp reliability analysis.

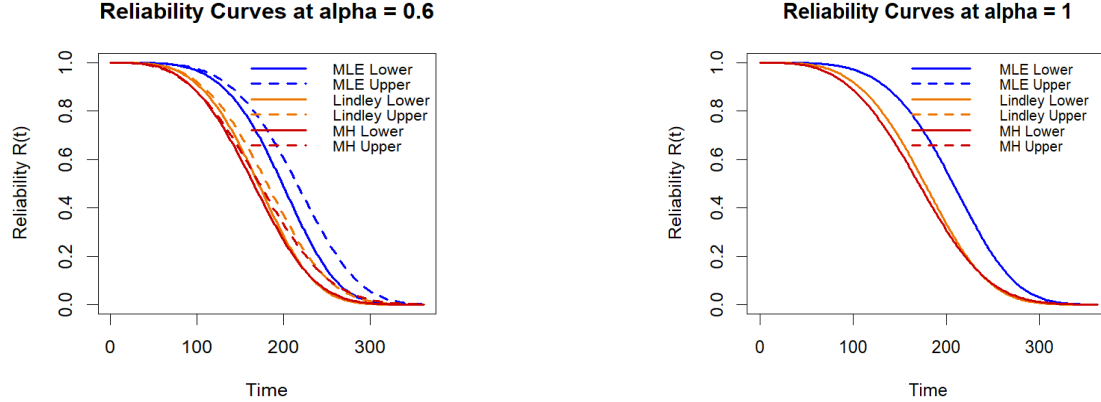


Figure 7. Reliability estimate plot for all methods at $\alpha = 0.6$ and $\alpha = 1$.

Overall, the numerical results show that the proposed fuzzy Weibull framework is meaningful for both estimation and decision making. In the simulation study, the true parameter values are generally contained within the endpoint-based fuzzy estimates, showing that the method captures the imposed uncertainty. The MSE and bias results show that the estimators improve with increasing sample size. The interval estimates show that fuzzy analysis gives more information than crisp interval estimation because each endpoint has its own interval. The real-data results show that the method provides interpretable parameter and reliability bands, which are useful when decisions must be made under imprecise lifetime information.

Conclusion and Future Work

This paper developed classical and Bayesian estimation procedures for the Weibull distribution under fuzzy lifetime observations. The fuzzy observations were represented by triangular fuzzy numbers and analysed through α -cuts. At each α -cut level, lower and upper endpoint samples were used to estimate the scale parameter λ , shape parameter k , and reliability function.

Time	MLE		Lindley		MH	
	Lower	Upper	Lower	Upper	Lower	Upper
$\alpha = 0.6$						
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
20	0.99997	0.99998	0.99979	0.99976	0.99944	0.99914
40	0.99942	0.99951	0.99710	0.99702	0.99423	0.99263
60	0.99653	0.99719	0.98668	0.98719	0.97750	0.97423
80	0.98764	0.99033	0.96079	0.96424	0.94168	0.93815
100	0.96716	0.97490	0.91197	0.92175	0.88005	0.88005
120	0.92788	0.94577	0.83247	0.85441	0.78965	0.79833
140	0.86233	0.89732	0.72032	0.75998	0.67208	0.69508
160	0.76527	0.82481	0.58099	0.64119	0.53597	0.57643
180	0.63722	0.72619	0.42872	0.49412	0.39527	0.45179
200	0.48750	0.60424	0.28352	0.35191	0.26590	0.33204
220	0.33433	0.46784	0.16429	0.22577	0.16084	0.22701
240	0.19979	0.33116	0.08142	0.13009	0.08619	0.14322
260	0.10072	0.21007	0.03362	0.06482	0.04029	0.08270
280	0.04130	0.11680	0.01125	0.02755	0.01617	0.04336
300	0.01323	0.05554	0.00296	0.00979	0.00548	0.01901
320	0.00316	0.02198	0.00059	0.00284	0.00154	0.00762
340	0.00054	0.00703	0.00009	0.00066	0.00035	0.00265
360	0.00006	0.00176	0.00001	0.00012	0.00007	0.00105
362	0.00005	0.00153	0.00001	0.00010	0.00006	0.00095
$\alpha = 1.0$						
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
20	0.99998	0.99998	0.99979	0.99979	0.99941	0.99941
40	0.99951	0.99951	0.99726	0.99726	0.99422	0.99422
60	0.99705	0.99705	0.98766	0.98766	0.97812	0.97812
80	0.98956	0.98956	0.96438	0.96438	0.94437	0.94437
100	0.97233	0.97233	0.92006	0.92006	0.88725	0.88725
120	0.93924	0.93924	0.84844	0.84844	0.80372	0.80372
140	0.88366	0.88366	0.74682	0.74682	0.69516	0.69516
160	0.80025	0.80025	0.61868	0.61868	0.56821	0.56821
180	0.68761	0.68761	0.47484	0.47484	0.43424	0.43424
200	0.55103	0.55103	0.33190	0.33190	0.30682	0.30682
220	0.40364	0.40364	0.20737	0.20737	0.19814	0.19814
240	0.26410	0.26410	0.11351	0.11351	0.11556	0.11556
260	0.15034	0.15034	0.05329	0.05329	0.06013	0.06013
280	0.07229	0.07229	0.02097	0.02097	0.02757	0.02757
300	0.02841	0.02841	0.00675	0.00675	0.01099	0.01099
320	0.00880	0.00880	0.00174	0.00174	0.00376	0.00376
340	0.00206	0.00206	0.00035	0.00035	0.00109	0.00109
360	0.00035	0.00035	0.00005	0.00005	0.00026	0.00026
362	0.00029	0.00029	0.00004	0.00004	0.00023	0.00023

Table 8. Reliability estimates for all methods at different values of α .

The estimation methods considered were MLE, Lindley's approximation, and the MH algorithm.

The simulation results show that the proposed approach successfully captures the uncertainty in fuzzy observations. For the scale parameter λ , the true value generally lies between the lower and upper endpoint estimates. For the shape parameter k , the true value is also well represented by the endpoint-based estimates, although the lower endpoint estimate may sometimes be larger than the upper endpoint estimate. This behaviour is justified because k is a shape parameter and depends on the relative dispersion and ageing pattern of the data. Hence, for k , the endpoint estimates should be interpreted through their minimum and maximum values when an ordered fuzzy interval is required.

The MSE and bias results confirm the good performance of the proposed methods. The MSE values generally decrease as the sample size increases, and the bias values remain reasonably small. The Bayesian methods are competitive with MLE, with MH often providing strong performance and Lindley's approximation giving similar results with simpler computation. The diagnostic plots confirm the numerical stability of the Newton-Raphson and MH procedures.

A major advantage of the proposed fuzzy framework is its richer interval interpretation. In crisp data analysis, each confidence or HPD interval has only two endpoints. In the fuzzy-data setting, the lower endpoint sample and the upper endpoint sample each produce their own interval. Thus, fuzzy interval estimation captures both statistical uncertainty and data imprecision. This gives a wider and more informative uncertainty band, which can be useful for reliability-based decision making. Decision makers can use these bands to assess conservative and optimistic estimates of model parameters and reliability probabilities.

The real-data application using turbofan engine failure times supports the practical usefulness of the proposed method. The fuzzy representation of the data leads to parameter and reliability bands rather than single crisp estimates. The reliability estimates decrease over time, as expected for survival probabilities. As α increases, the lower and upper endpoint estimates move closer, and at $\alpha = 1$ they coincide. This confirms that the proposed method reduces to ordinary crisp-data analysis when the fuzzy uncertainty vanishes.

Future work may extend the present study in several directions. The proposed methodology can be extended to censored and progressively censored data, as well as to competing-risk reliability models. In addition, other lifetime distributions, including the gamma, lognormal, inverse Weibull, and generalized Weibull distributions, may be investigated within the same fuzzy-data framework.

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Author contributions statement

All authors contributed to the study design, analysis, and preparation of the manuscript. All authors reviewed and approved the final version.

Conflict of interest

The authors declare that there are no conflicts of interest related to this study.

Data availability

The data used in this study were obtained from the Kaggle repository. The dataset is publicly available at: <https://www.kaggle.com/datasets/fareselgohary003/nasa-cmapss-turbofan-engine-rul-dataset/data>.