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The Nonlinear Self-excited Threshold Two-component Mixture Multiplicative Error Model

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Abstract

The main purpose of the present paper is to introduce an innovative model for non-negative time series, especially for financial data, which incorporates the mixture multiplicative error models with a self-excited threshold effect. The mixture distributions are applied to the effective handling of the multimodal, asymmetric, and skewed features. The Expectation-Maximization estimation algorithm is developed to estimate the model parameters, including an unknown threshold parameter. Results from the Monte Carlo simulation indicate that the estimates of the unknown parameters are converging towards their true values. Finally, the proposed process is effectively applied to model the producer price index, with goodness-of-fit measures demonstrating its superiority over other relevant models, thereby confirming the validity of the proposed modeling.

Keywords: Multiplicative error, Mixture model, EM algorithm, Threshold, Producer price index.

1 Introduction

Time series composed of non-negative random variables appear in many real-world contexts, including financial durations, trading volumes, squared returns, realized volatility and insurance claims.

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These series are frequently modeled using the multiplicative error model (MEM). The MEM effectively captures the distinctive characteristics of non-negative time series, as supported by extensive empirical studies. In the MEM framework, the dynamics of a non-negative variable are modeled as the product of two components: a unit-mean error term, and an autoregressive (AR) conditional expectation based on past information, (Cipollini and Gallo; 2025).

Engle (2002) presented the MEM as a prominent approach for analyzing non-negative time series. Consequently, models such as the generalized AR conditional heteroscedasticity (GARCH) (Bollerslev; 1986), conditional AR range (Chou; 2005), and AR conditional duration (ACD) can all be viewed as special cases of the MEM. The MEMs have also been successfully applied to model and predict various types of non-negative financial data, including trading volume, (Manganelli; 2005) and (Naimoli and Storti; 2019), absolute returns (Engle and Gallo; 2006), and realized volatility (Lanne; 2006) and (Cipollini et al.; 2021).

Brownlees and Gallo (2010) utilized a MEM within a risk management context, assessing forecast accuracy through value-at-risk measures to capture different forms of realized volatility. Otranto (2012) introduced a Markovian dynamic within a MEM to model the conditional expectation of realized volatility. This approach addresses challenges related to the gradually changing average volatility levels and varying dynamics across different regimes. They applied the model to a realized volatility index and demonstrated its advantages through more interpretable persistence, improved residual behavior, and better overall fit. Gallo and Otranto (2015) explored the features of a low-frequency component in MEMs using smooth transition and Markov switching frameworks, proposing the concept of shifting average volatility levels and comparing model behaviors.

Härdle et al. (2015) developed a locally adaptive MEM that allows parameters to vary over time, estimating them adaptively through a sequential testing method. Li et al. (2025) addressed the issue of outliers in MEMs by first examining their effects on least squares and bias-corrected weighted least squares estimators, finding that the effect on bias-corrected weighted least squares is transmitted through the least squares solution. Meintanis et al. (2025) proposed specification tests for the conditional mean function in vector MEMs, with test statistics that are straightforward to compute given an appropriate parameter estimator.

Mixture models are probabilistic frameworks used to represent sub-populations within a broader population, where each sub-population is defined by a distinct probability distribution. To characterize the properties of these sub-populations, it is assumed that each one adheres to a simple parametric distribution, commonly known as a component in mixture models. These models are advantageous when data points do not distinctly belong to a single distribution and can be re-

garded as a weighted combination of various component distributions. Due to their flexibility and computational simplicity in modeling complex distributions, finite mixture models are extensively employed to describe heterogeneous populations across many statistical fields.

The expectation-maximization (EM) algorithm is widely regarded as one of the most effective techniques for estimating mixture model parameters, and functions as an iterative process that serves as a maximum likelihood (ML) estimator. The EM algorithm for mixture models offers several attractive features, such as monotonic convergence and adherence to probabilistic constraints, making it a preferred method for parameter estimation in these models. The increasing acceptance of the EM algorithm stems from its proven effectiveness in estimating parameters in mixture models (see Yao (2013) and McLachlan et al. (2019)).

Deiri (2014) developed the moment and ML estimation methods for the parameters of the exponentiated-gamma distribution in the case of involving multiple outliers. Chatterjee et al. (2022) introduced a generalized EM algorithm specifically designed for Gaussian mixture models, where the maximization step is substituted with an increasing step that can be interpreted as a linear time-invariant system with a feedback nonlinearity. Bazyar et al. (2023) considered the application of the EM algorithm as well as Bayesian and E-Bayesian estimation approaches for the Moore and Bilikam family of distributions, including the Burr XII distribution as a special case, within the framework of progressive Type-II censoring.

Research on nonlinear time series and threshold models have garnered significant interest in finance and econometrics, as noted by Chen et al. (2011), Ramírez-Rondán (2020), and Wei et al. (2025). Chan and Kutoyants (2012) proposed estimating a threshold AR model, treating the threshold parameter as a random variable. Their findings indicate that the Bayesian estimator is consistent and moment convergent, with its limiting distribution characterized by a limiting likelihood ratio.

Liu and Chen (2020) investigated the threshold factor model for high-dimensional time series, assuming that the dynamics switch between regimes based on a threshold variable. Each threshold regime follows a factor model, and the regime-switching mechanism induces structural changes in the factor loading matrices. Hoseini Baladezaei et al. (2023) proposed the discrete Burr-Hatke distribution derived via a the balanced discretization approach that retains the partial moment feature. Consequently, they provided a new integer-valued autoregressive model based on the mixing Pegram and binomial thinning operators.

To estimate parameters of a two-regime threshold diffusion process, Yu et al. (2020) proposed an approximate ML method based on approximating the log-likelihood function of observations.

This method accommodates both linear and nonlinear forms for the drift and diffusion terms. To identify the most appropriate model among four candidates, three information criteria are employed, and a likelihood ratio test determines the presence of threshold effects. Tsai and Nikitin (2024) discussed the threshold effect in the Lévy processes and the approximate ML estimators of the parameters, showing that threshold effects can be identified by evaluating the likelihood ratio.

Pan and Pan (2024) introduced a multivariate GARCH-type model that simplifies parametrization through a network structure and is well-suited for certain types of high-dimensional data. Nonlinear threshold ARMA (TARMA) models extend the well-known TAR model by incorporating a regime-switching mechanism that specifies an ARMA sub-model within each regime. Angelini et al. (2024) developed supremum Lagrange multiplier tests for threshold effects in TARMA models and discussed the asymptotic distribution of the test statistics.

Structural breaks in the MEM parameters have been addressed by Zhang et al. (2001), who identified regime shifts in trade durations and proposed a threshold ACD specification within TARMA models. Perera and Koul (2017) developed a lack-of-fit test for fitting a two-phase threshold model for the conditional mean function in MEM, which can be adapted for GARCH models.

Modern approaches in non-negative time series analysis aim to model all unique features of real phenomena, resulting in improved modeling and forecasting accuracy. However, there is currently a lack of theoretical discussion when dealing with asymmetrical data, multimodality, heavy-tails, and nonlinear structures within the MEM paradigm. In the context of the threshold effect and MEMs, it is worth mentioning the threshold model facilitates different forms of persistence and conditional means in the sub-regimes through the threshold variable, and can capture features that are not fully considered by the linear MEMs. Hence, we introduce a novel self-excited threshold model belonging to the family of MEM with two-component mixture errors. The identical distributions are the most common candidates as the components of the mixture models. We present an innovative and adaptable approach for constructing mixture models involving non-identical gamma and log-normal distributions, such that the parameters of each model differ across regimes. The effectiveness of our model is shown through its application to both simulated and real-world data sets.

The paper is structured as follows. The MEM model based on the mixture error terms is presented in Section 2, and the new self-excited threshold MEM model is introduced, where errors follow a mixture distribution of gamma and log-normal. In Section 3, the EM estimation of the unknown parameters of the proposed model is discussed, as well as the unknown threshold parameter. Moreover, the nonlinearity test of the threshold model is discussed. The Monte Carlo simulations

of the proposed model are provided in Section 4, addressing both known and unknown threshold cases. Section 5 is dedicated to the applicability of the proposed model using the monthly producer price index series, comparing its performance with several other MEM models. The residual analysis confirms the adequacy of the proposed model in real data applications.

2 The Self-excited Threshold Multiplicative Mixture Error Model

Flexible mixture models are extensively used to explore the complex structure of some real data sets, accommodating multiple distinctive features of the phenomena, such as asymmetry and multimodality. In this section, the MEM model with mixture errors of two non-negative distributions, gamma and log-normal, is provided. Subsequently, the first-order self-excited threshold MEM incorporating mixture error terms, abbreviated as the SET-MMEM(1,1) is introduced, and several statistical properties of the proposed model are expressed.

2.1 The Multiplicative Two-Component Mixture Error Model

Suppose $\{X_t\}_{t=1,2,\ldots}$ is a non-negative time series process, the first-order mixture MEM, called MMEM(1,1), is represented as

$$X_t = \mu_t \varepsilon_t, \qquad X_0 = 0, \;\; t = 1, 2, \ldots, \tag{1}$$

where,

$$\mu_t = \eta + \alpha X_{t-1} + \beta \mu_{t-1}, \qquad \alpha, \beta \geq 0, \; \alpha + \beta < 1, \;\; \mu_1 = \eta. \tag{2}$$

The variable ε_t is as a positive random variable with unit mean and finite variance. Considering the MMEM(1,1), the two-component mixture model comprising the gamma distribution with parameters (λ, λ) and log-normal distribution with parameters $(\frac{-\sigma^2}{2}, \sigma^2)$ with weights $\omega_1 = \omega$ and $\omega_2 = 1 - \omega$, the probability distribution function (PDF) of ε is given as

$$\begin{aligned} f_{\varepsilon}(\varepsilon; \lambda, \sigma, \omega) &= \omega_1 f_1(\varepsilon; \lambda) + \omega_2 f_2(\varepsilon; \sigma) \\ &= \omega_1 \frac{\lambda^{\lambda} \varepsilon^{\lambda - 1} \exp(-\lambda \varepsilon)}{\Gamma(\lambda)} + \omega_2 \frac{1}{\sqrt{2\pi\sigma^2}\varepsilon} \exp\Big(\frac{-(\ln(\varepsilon) - \mu)^2}{2\sigma^2}\Big). \end{aligned}$$

It is essential to note that the parameters associated with the gamma and log-normal distributions are selected to meet the requirement of a unit mean. Therefore, the shape and scale

parameters of the gamma distribution are designated as λ, and the mean parameter of the log-normal distribution is defined as $\frac{-\sigma^2}{2}$, with $\lambda > 0$ and $\sigma > 0$, ensuring that the unit mean condition of the MMEM(1,1) model is satisfied. The variance of the error term is calculated as

$$\mathrm{Var}(\varepsilon) = \omega_1^2\big(\frac{1}{\lambda}\big) + \omega_2^2\big(\exp(\sigma^2) - 1\big).$$

The PDF of $\{X_t\}$, based on the two-component mixture distribution of gamma and log-normal for errors, is obtained as

$$X_t \sim f_{\varepsilon}\big(\frac{x_t}{\mu_t}; \lambda, \sigma, \omega\big)\frac{1}{\mu_t}.$$

2.2 The Self-excited Threshold MMEM(1,1)

Here, a new MMEM(1,1) model is introduced with a threshold parameter, where two different MMEM(1,1) models are assigned to each regime.

Consider Equation 1, the new SET-MMEM(1,1) process is defined as follows

$$X_t = \begin{cases} \mu_t\,\varepsilon_{1,t} & X_{t-1} < r, \\ \mu_t\,\varepsilon_{2,t} & X_{t-1} \geq r, \end{cases} = \mu_t\,\varepsilon_{1,t}\mathrm{I}_{t-1}(r) + \mu_t\,\varepsilon_{2,t}\big(1 - \mathrm{I}_{t-1}(r)\big), \qquad t = 1, 2, \ldots, \tag{3}$$

where r is the unknown threshold parameter, $\mathrm{I}_{t-1}(r) = \begin{cases} 1, & X_{t-1} < r \\ 0, & X_{t-1} < r \end{cases}$ is the identity function, and the PDF of the error terms $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are given respectively by

$$\begin{aligned} f_{\varepsilon_{1,t}}(\varepsilon) &= \omega_1 f_1(\varepsilon; \lambda_1) + \omega_2 f_2(\varepsilon; \sigma_1), \\ f_{\varepsilon_{2,t}}(\varepsilon) &= \omega_1 f_1(\varepsilon; \lambda_2) + \omega_2 f_2(\varepsilon; \sigma_2). \end{aligned}$$

where $\lambda_1 \neq \lambda_2$ and $\sigma_1 \neq \sigma_2$.

Proposition 2.1. *The h-step ahead conditional expectations of the SET-MMEM(1,1) model are provided as*

$$E(X_{t+h}|X_t) = \omega\frac{1 - (\alpha + \beta)^{h-1}}{1 - (\alpha + \beta)} + \big(\alpha + \beta\big)^{h-1}\mu_{t+1}, \tag{4}$$

Proof. Based on Equation 2, we have

$$\begin{aligned} E(X_{t+1}|X_t) &= \mathrm{I}_{t-1}(r)E(\mu_{t+1}\varepsilon_{1,t+1}|X_t) + \big(1 - \mathrm{I}_{t-1}(r)\big)E(\mu_{t+1}\varepsilon_{2,t+1}|X_t) = \mu_{t+1}, \\ E(X_{t+2}|X_t) &= E(\mu_{t+2}|X_t) = \eta + \alpha E(X_{t+1}|X_t) + \beta\mu_{t+1} = \eta + \big(\alpha + \beta\big)\mu_{t+1}, \end{aligned}$$

$$
\begin{aligned}
E(X_{t+3}|X_t) &= E(\mu_{t+3}|X_t) = \eta + \alpha E(X_{t+2}|X_t) + \beta E(\mu_{t+2}|X_t) = \eta(1 + (\alpha+\beta)) + \big(\alpha+\beta\big)^2 \mu_{t+1}, \\
E(X_{t+4}|X_t) &= E(\mu_{t+4}|X_t) = \eta + \alpha E(X_{t+3}|X_t) + \beta E(\mu_{t+3}|X_t) \\
&= \eta(1 + (\alpha+\beta) + (\alpha+\beta)^2) + \big(\alpha+\beta\big)^3 \mu_{t+1}.
\end{aligned}
$$

Hence, the h-step ahead conditional expectation of the SET-MMEM(1,1) is obtained recursively as

$$
\begin{aligned}
E(X_{t+h}|X_t) &= E(\mu_{t+h}|X_t) = \eta(1 + (\alpha+\beta) + (\alpha+\beta)^2 + \ldots + (\alpha+\beta)^{h-2}) + \big(\alpha+\beta\big)^{h-1} \mu_{t+1} \\
&= \eta \sum_{i=0}^{h-2} (\alpha+\beta)^i + \big(\alpha+\beta\big)^{h-1} \mu_{t+1} = \eta \frac{1-(\alpha+\beta)^{h-1}}{1-(\alpha+\beta)} + \big(\alpha+\beta\big)^{h-1} \mu_{t+1}, \qquad h \geq 2
\end{aligned}
$$

which completes the proof. □

Consequently, the one-step ahead conditional variance of the SET-MMEM(1,1) is shown as

$$
\begin{aligned}
\mathrm{Var}(X_{t+1}|X_t) &= \mathrm{I}_{t-1}(r)\mathrm{Var}(\mu_{t+1}\varepsilon_{1,t+1}|X_t) + \big(1 - \mathrm{I}_{t-1}(r)\big)\mathrm{Var}(\mu_{t+1}\varepsilon_{2,t+1}|X_t) \qquad (5) \\
&= \mu_{t+1}^2 \Big(\mathrm{I}_{t-1}(r) \big(\frac{\omega_1^2}{\lambda_1} + \omega_2^2 \big(\exp(\sigma_1^2) - 1 \big) \big) + \big(1 - \mathrm{I}_{t-1}(r)\big) \big(\frac{\omega_1^2}{\lambda_2} + \omega_2^2 \big(\exp(\sigma_2^2) - 1 \big) \big) \Big).
\end{aligned}
$$

3 The Estimation of the SET-MMEM(1,1) Model

The EM algorithm is an iterative procedure utilized to obtain ML estimates of parameters in statistical models, especially for incomplete data sets or data sets with latent variables. In this section, we discuss the estimation of unknown parameters of the SET-MMEM(1,1) model based on a realization $X_1, X_2, \ldots, X_n$. The unknown parameters are estimated based on the EM method, while the minimum of the maximized log-likelihood (MMLL) is considered to estimate the threshold parameter r. The nonlinearity test is performed for the SET-MMEM(1,1) process.

3.1 The EM Estimation of the SET-MMEM(1,1) Model

Here, we propose the EM algorithm to estimate the unknown parameters of the SET-MMEM(1,1) model. Let the vector of parameters as $\boldsymbol{\xi} = (\alpha, \beta, \eta, \lambda_1, \lambda_2, \sigma_1, \sigma_2, \omega)'$, the log-likelihood function given the observations $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)'$ is formulated as

$$
\begin{aligned}
\ell(\boldsymbol{\xi}|\boldsymbol{X}) &= \sum_{t=2}^{n} \mathrm{I}_{t-1}(r) \ln \Big(f_{\varepsilon_1}\big(\frac{x_t}{\mu_t}; \lambda_1, \sigma_1, \omega\big) \frac{1}{\mu_t} \Big) + \big(1 - \mathrm{I}_{t-1}(r)\big) \ln \Big(f_{\varepsilon_2}\big(\frac{x_t}{\mu_t}; \lambda_2, \sigma_2, \omega\big) \frac{1}{\mu_t} \Big) \qquad (6) \\
&= \sum_{t=2}^{n} \mathrm{I}_{t-1}(r) \ln \Big(\Big[\omega f_1\big(\frac{x_t}{\mu_t}; \lambda_1\big) + (1-\omega) f_2\big(\frac{x_t}{\mu_t}; \sigma_1\big) \Big] \frac{1}{\mu_t} \Big)
\end{aligned}
$$

$$+\sum_{t=2}^{n}\big(1-\mathrm{I}_{t-1}(r)\big)\ln\Big(\Big[\omega f_1\big(\frac{x_t}{\mu_t};\lambda_2\big)+(1-\omega)f_2\big(\frac{x_t}{\mu_t};\sigma_2\big)\Big]\frac{1}{\mu_t}\Big),$$

where $f_1\big(\frac{x_t}{\mu_t};.\big)$ and $f_2\big(\frac{x_t}{\mu_t};.\big)$ are the PDF of the gamma and log-normal distributions, respectively.

The corresponding score equations of the log-likelihood function, in Equation 6, do not have an explicit solution for the SET-MMEM(1,1); hence, the EM algorithm is adopted for ML estimation of the parameters.

Let us define the latent variables Z_{tj} as below

$$Z_{tj}=\begin{cases}1, & \text{if } t\text{-th observations belongs to } j\text{-th component,}\\ 0, & \text{otherwise,}\end{cases},\quad t=1,\ldots,n,\ j=1,2.$$

Consider the complete data sets as $\boldsymbol{X}_C=(\boldsymbol{X},\boldsymbol{Z})'$, the log-likelihood function given the complete data is represented as

$$\begin{aligned}\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)=\sum_{t=2}^{n}\sum_{j=1}^{2}Z_{tj}\Bigg(&\mathrm{I}_{t-1}(r)\ln\Big(\Big[\omega f_1\big(\frac{x_t}{\mu_t};\lambda_1\big)+(1-\omega)f_2\big(\frac{x_t}{\mu_t};\sigma_1\big)\Big]\frac{1}{\mu_t}\Big)\\&+\big(1-\mathrm{I}_{t-1}(r)\big)\ln\Big(\Big[\omega f_1\big(\frac{x_t}{\mu_t};\lambda_2\big)+(1-\omega)f_2\big(\frac{x_t}{\mu_t};\sigma_2\big)\Big]\frac{1}{\mu_t}\Big)\Bigg).\end{aligned}$$

In the E-step of the EM algorithm, we obtain the $E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)$ at the $(k+1)$-th iteration based on the $\boldsymbol{\xi}^{(k)}$, which requires $E\Big(Z_{tj}\big|\boldsymbol{X}_j,\boldsymbol{\xi}^{(k)}\Big)$, so

$$\begin{aligned}E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)=&\sum_{t=2}^{n}\sum_{j=1}^{2}E\Big(Z_{tj}\big|\boldsymbol{X},\boldsymbol{\xi}_j^{(k)}\Big)\mathrm{I}_{t-1}(r)\ln\Big(\hat{\omega}f_1\big(\frac{x_t}{\hat{\mu}_t};\hat{\lambda}_1\big)+(1-\hat{\omega})f_2\big(\frac{x_t}{\hat{\mu}_t};\sigma_1\big)\Big)\\&+\sum_{t=2}^{n}\sum_{j=1}^{2}E\Big(Z_{tj}\big|\boldsymbol{X},\boldsymbol{\xi}_j^{(k)}\Big)\big(1-\mathrm{I}_{t-1}(r)\big)\ln\Big(\hat{\omega}f_1\big(\frac{x_t}{\hat{\mu}_t};\hat{\lambda}_2\big)+(1-\hat{\omega})f_2\big(\frac{x_t}{\hat{\mu}_t};\hat{\sigma}_2\big)\Big)\\&-\sum_{t=2}^{n}\ln(\hat{\mu}_t).\end{aligned}\tag{7}$$

Finally, the maximum value of the conditional expectation Equation 7 is obtained in the M-step with respect to each parameter. The EM algorithm of the SET-MMEM(1,1) is summarized as follows.

Algorithm 1.

1. Set the initial value of the parameters as $\boldsymbol{\xi}^{(0)}=(\alpha^{(0)},\beta^{(0)},\omega^{(0)},\lambda_1^{(0)},\lambda_2^{(0)},\sigma_1^{(0)},\sigma_2^{(0)},\omega^{(0)})'$.
2. At the $(k+1)$-th iteration and $\hat{\mu}_t=\hat{\omega}^{(k)}+\hat{\alpha}^{(k)}X_{t-1}+\hat{\beta}^{(k)}\hat{\mu}_{t-1}$, obtain the conditional expectations

as

$$\hat{Z}_{t1}^{(k+1)} = E\big(Z_{t1}\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\big) = \frac{\hat{\omega}_1\Big(\mathrm{I}_1 f_1\big(\frac{x_t}{\hat{\mu}_t};\hat{\lambda}_1^{(k)}\big)+\mathrm{I}_2 f_1\big(\frac{x_t}{\hat{\mu}_t};\hat{\lambda}_2^{(k)}\big)\Big)}{\hat{\omega}_1\Big(\mathrm{I}_1 f_1\big(\frac{x_t}{\hat{\mu}_t};\hat{\lambda}_1^{(k)}\big)+\mathrm{I}_2 f_1\big(\frac{x_t}{\hat{\mu}_t};\hat{\lambda}_2^{(k)}\big)\Big)+\hat{\omega}_2\Big(\mathrm{I}_1 f_2\big(\frac{x_t}{\hat{\mu}_t};\hat{\sigma}_1^{(k)}\big)+\mathrm{I}_2 f_2\big(\frac{x_t}{\hat{\mu}_t};\hat{\sigma}_2^{(k)}\big)\Big)},$$

$$\hat{Z}_{t2}^{(k+1)} = E\big(Z_{t2}\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\big) = 1-\hat{Z}_{t1}^{(k+1)}, \qquad t=1,\ldots,n$$

where in the rest of the paper, for simplicity, we use the notations $\mathrm{I}_1 = \mathrm{I}_{t-1}(r)$ and $\mathrm{I}_2 = 1-\mathrm{I}_{t-1}(r)$.

3. At the $(k+1)$-th iteration, obtain the maximum of Equation 7 for each parameter as:

(i) Using $(\hat{\alpha}^{(k)},\hat{\beta}^{(k)},\hat{\eta}^{(k)},\hat{\lambda}_1^{(k)},\hat{\lambda}_2^{(k)},\hat{\sigma}_1^{(k)},\hat{\sigma}_2^{(k)})'$, the EM estimate of ω is

$$\hat{\omega}^{(k+1)} = \frac{1}{n-1}\sum_{t=2}^{n}\hat{Z}_{t1}^{(k+1)},$$

(ii) Using $(\hat{\beta}^{(k)},\hat{\eta}^{(k)},\hat{\lambda}_1^{(k)},\hat{\lambda}_2^{(k)},\hat{\sigma}_1^{(k)},\hat{\sigma}_2^{(k)},\hat{\omega}^{(k+1)})'$, the EM estimates of the parameter α, during iteration $(k+1)$, is determined by finding the root of the subsequent equation

$$\begin{aligned}\frac{\partial E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)}{\partial\alpha} &= -\sum_{t=2}^{n}\hat{Z}_{t1}^{(k)}\frac{X_{t-1}}{\hat{\mu}_t^2}\Big(\mathrm{I}_1\big((\hat{\lambda}_1^{(k)}-1)\hat{\mu}_t-\hat{\lambda}_1^{(k)}X_t\big)+\mathrm{I}_2\big((\hat{\lambda}_2^{(k)}-1)\hat{\mu}_t-\hat{\lambda}_2^{(k)}X_t\big)\Big)\\ &+\sum_{t=2}^{n}\hat{Z}_{t2}^{(k)}\frac{X_{t-1}}{\hat{\mu}_t}\Big(\mathrm{I}_1\big(1+\frac{1}{\hat{\sigma}_1^{2(k)}}\ln\big(\frac{X_t}{\hat{\mu}_t}\big)\big)+\mathrm{I}_2\big(1+\frac{1}{\hat{\sigma}_2^{2(k)}}\ln\big(\frac{X_t}{\hat{\mu}_t}\big)\big)\Big)\\ &-\sum_{t=2}^{n}\frac{X_{t-1}}{\hat{\mu}_t}=0.\end{aligned}$$

(iii) Using $(\hat{\alpha}^{(k+1)},\hat{\eta}^{(k)},\hat{\lambda}_1^{(k)},\hat{\lambda}_2^{(k)},\hat{\sigma}_1^{(k)},\hat{\sigma}_2^{(k)},\hat{\omega}^{(k+1)})'$, the EM estimates of the parameter β, during iteration $(k+1)$, is obtained as the root of the following equation

$$\begin{aligned}\frac{\partial E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)}{\partial\beta} &= -\sum_{t=2}^{n}\hat{Z}_{t1}^{(k)}\frac{\hat{\mu}_{t-1}}{\hat{\mu}_t^2}\Big(\mathrm{I}_1\big((\hat{\lambda}_1^{(k)}-1)\hat{\mu}_t-\hat{\lambda}_1^{(k)}X_t\big)+\mathrm{I}_2\big((\hat{\lambda}_2^{(k)}-1)\hat{\mu}_t-\hat{\lambda}_2^{(k)}X_t\big)\Big)\\ &+\sum_{t=2}^{n}\hat{Z}_{t2}^{(k)}\frac{\hat{\mu}_{t-1}}{\hat{\mu}_t}\Big(\mathrm{I}_1\big(1+\frac{1}{\hat{\sigma}_1^{2(k)}}\ln\big(\frac{X_t}{\hat{\mu}_t}\big)\big)+\mathrm{I}_2\big(1+\frac{1}{\hat{\sigma}_2^{2(k)}}\ln\big(\frac{X_t}{\hat{\mu}_t}\big)\big)\Big)\\ &-\sum_{t=2}^{n}\frac{\hat{\mu}_{t-1}}{\hat{\mu}_t}=0.\end{aligned}$$

(iv) Using $(\hat{\alpha}^{(k+1)},\hat{\beta}^{(k+1)},\hat{\lambda}_1^{(k)},\hat{\lambda}_2^{(k)},\hat{\sigma}_1^{(k)},\hat{\sigma}_2^{(k)},\hat{\omega}^{(k+1)})'$, the EM estimates of the parameter η, during iteration $(k+1)$, is determined by finding the root of the subsequent equation

$$\frac{\partial E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)}{\partial\eta} = -\sum_{t=2}^{n}\hat{Z}_{t1}^{(k)}\frac{1}{\hat{\mu}_t^2}\Big(\mathrm{I}_1\big((\hat{\lambda}_1^{(k)}-1)\hat{\mu}_t-\hat{\lambda}_1^{(k)}X_t\big)+\mathrm{I}_2\big((\hat{\lambda}_2^{(k)}-1)\hat{\mu}_t-\hat{\lambda}_2^{(k)}X_t\big)\Big)$$

$$+\sum_{t=2}^{n}\hat{Z}_{t2}^{(k)}\frac{1}{\hat{\mu}_t}\Big(\mathrm{I}_1\big(1+\frac{1}{\hat{\sigma}_1^{2(k)}}\ln\big(\frac{X_t}{\hat{\mu}_t}\big)\big)+\mathrm{I}_2\big(1+\frac{1}{\hat{\sigma}_2^{2(k)}}\ln\big(\frac{X_t}{\hat{\mu}_t}\big)\big)\Big)$$
$$-\sum_{t=2}^{n}\frac{1}{\hat{\mu}_t}=0.$$

(v) Using $(\hat{\alpha}^{(k+1)},\hat{\beta}^{(k+1)},\hat{\eta}^{(k+1)},\hat{\lambda}_2^{(k)},\hat{\sigma}_1^{(k)},\hat{\sigma}_2^{(k)},\hat{\omega}^{(k+1)})'$, the EM estimates of the parameter λ_1, during iteration $(k+1)$, is obtained as the root of the following equation

$$\frac{\partial E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)}{\partial\lambda_1}=\sum_{t=2}^{n}\hat{Z}_{t1}^{(k)}\mathrm{I}_1\Big(\ln(\lambda_1)+1+\ln\big(\frac{X_t}{\hat{\mu}_t}\big)-\frac{X_t}{\hat{\mu}_t}-\frac{\Gamma'(\lambda_1)}{\Gamma(\lambda_1)}\Big)=0,$$

where $\Gamma'(.)$ is the derivate of the function $\Gamma(.)$.

(vi) Using $(\hat{\alpha}^{(k+1)},\hat{\beta}^{(k+1)},\hat{\eta}^{(k+1)},\hat{\lambda}_1^{(k+1)},\hat{\sigma}_1^{(k)},\hat{\sigma}_2^{(k)},\hat{\omega}^{(k+1)})'$, the EM estimates of the parameter λ_2, during iteration $(k+1)$, is determined by finding the root of the subsequent equation

$$\frac{\partial E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)}{\partial\lambda_2}=\sum_{t=2}^{n}\hat{Z}_{t1}^{(k)}\mathrm{I}_2\Big(\ln(\lambda_2)+1+\ln\big(\frac{X_t}{\hat{\mu}_t}\big)-\frac{X_t}{\hat{\mu}_t}-\frac{\Gamma'(\lambda_2)}{\Gamma(\lambda_2)}\Big)=0,$$

(vii) Using $(\hat{\alpha}^{(k+1)},\hat{\beta}^{(k+1)},\hat{\eta}^{(k+1)},\hat{\lambda}_1^{(k)},\hat{\lambda}_2^{(k)},\hat{\sigma}_2^{(k)},\hat{\omega}^{(k+1)})'$, the EM estimates of the parameter σ_1, during iteration $(k+1)$,is obtained as the root of the following equation

$$\frac{\partial E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)}{\partial\sigma_1}=\sum_{t=2}^{n}\frac{\hat{Z}_{t2}^{(k)}\mathrm{I}_1}{\sigma_1^3}\Big(\ln^2\big(\frac{X_t}{\hat{\mu}_t}\big)-\frac{\sigma_1^4}{4}-\sigma_1^2\Big)=0,$$

(viii) Using $(\hat{\alpha}^{(k+1)},\hat{\beta}^{(k+1)},\hat{\eta}^{(k+1)},\hat{\lambda}_1^{(k)},\hat{\lambda}_2^{(k)},\hat{\sigma}_1^{(k+1)},\hat{\omega}^{(k+1)})'$, the EM estimates of the parameter σ_2, during iteration $(k+1)$, is determined by finding the root of the subsequent equation

$$\frac{\partial E\Big(\ell_C(\boldsymbol{\xi}|\boldsymbol{X}_C)\big|\boldsymbol{X},\boldsymbol{\xi}^{(k)}\Big)}{\partial\sigma_2}=\sum_{t=2}^{n}\frac{\hat{Z}_{t2}^{(k)}\mathrm{I}_2}{\sigma_2^3}\Big(\ln^2\big(\frac{X_t}{\hat{\mu}_t}\big)-\frac{\sigma_2^4}{4}-\sigma_2^2\Big)=0,$$

4. The parameters are estimated iteratively in the EM algorithm until convergence is reached with a margin of error of 0.001.

3.2 The Estimation of the Threshold Parameter

The estimation of the unknown threshold parameter is provided based on the approach of Möller et al. (2016). We obtain the MMLL estimation of the threshold parameter as the value that gives the minimum value of the maximized log-likelihood function $-\ell(\boldsymbol{\xi}\mid\boldsymbol{X})$ in (6).

The estimation algorithm is executed to compute the estimation of the parameters as the following steps.

1. Set the initial values of parameters as $(\hat{\alpha}^{(0)}, \hat{\beta}^{(0)}, \hat{\eta}^{(0)}, \hat{\lambda}_1^{(0)}, \hat{\lambda}_2^{(0)}, \hat{\sigma}_1^{(0)}, \hat{\sigma}_2^{(0)}, \hat{\omega}^{(0)})$.
2. Determine whether r is known, if unknown, estimate $\hat{r}$ by the MMLL method, and set $r = \hat{r}$.
3. For $k \geq 1$,

   a) Based on the parameters $(\hat{\alpha}^{(k-1)}, \hat{\beta}^{(k-1)}, \hat{\eta}^{(k-1)}, \hat{\lambda}_1^{(k-1)}, \hat{\lambda}_2^{(k-1)}, \hat{\sigma}_1^{(k-1)}, \hat{\sigma}_2^{(k-1)})$, compute $\hat{\omega}^{(k)}$.

   b) Based on the parameters $(\hat{\beta}^{(k-1)}, \hat{\eta}^{(k-1)}, \hat{\lambda}_1^{(k-1)}, \hat{\lambda}_2^{(k-1)}, \hat{\sigma}_1^{(k-1)}, \hat{\sigma}_2^{(k-1)}, \hat{\omega}^{(k)})$, compute $\hat{\alpha}^{(k)}$.

   c) Based on the parameters $(\hat{\alpha}^{(k)}, \hat{\eta}^{(k-1)}, \hat{\lambda}_1^{(k-1)}, \hat{\lambda}_2^{(k-1)}, \hat{\sigma}_1^{(k-1)}, \hat{\sigma}_2^{(k-1)}, \hat{\omega}^{(k)})$, compute $\hat{\beta}^{(k)}$.

   d) Based on the parameters $(\hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\lambda}_1^{(k-1)}, \hat{\lambda}_2^{(k-1)}, \hat{\sigma}_1^{(k-1)}, \hat{\sigma}_2^{(k-1)}, \hat{\omega}^{(k)})$, compute $\hat{\eta}^{(k)}$.

   e) Based on the parameters $(\hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\eta}^{(k)}, \hat{\lambda}_2^{(k-1)}, \hat{\sigma}_1^{(k-1)}, \hat{\sigma}_2^{(k-1)}, \hat{\omega}^{(k)})$, compute $\hat{\lambda}_1^{(k)}$.

   f) Based on the parameters $(\hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\eta}^{(k)}, \hat{\lambda}_1^{(k)}, \hat{\sigma}_1^{(k-1)}, \hat{\sigma}_2^{(k-1)}, \hat{\omega}^{(k)})$, compute $\hat{\lambda}_2^{(k)}$.

   g) Based on the parameters $(\hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\eta}^{(k)}, \hat{\lambda}_1^{(k)}, \hat{\lambda}_2^{(k)}, \hat{\sigma}_2^{(k-1)}, \hat{\omega}^{(k)})$, compute $\hat{\sigma}_1^{(k)}$.

   h) Based on the parameters $(\hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\eta}^{(k)}, \hat{\lambda}_1^{(k)}, \hat{\lambda}_2^{(k)}, \hat{\sigma}_1^{(k)}, \hat{\omega}^{(k)})$, compute $\hat{\sigma}_2^{(k)}$.

4. Set $k = k + 1$ and go to Step 3 until convergence is achieved. Given X_t, the iteration procedure can start with initial values and end when the error of the difference between $k + 1$ and k steps is less than 0.001, i.e. $\mid \hat{\boldsymbol{\xi}}^{(h+1)} - \hat{\boldsymbol{\xi}}^{(h)}) \mid < 0.001$.

3.3 Testing Nonlinearity of the SET-MMEM(1,1) Model

The nonlinearity of the model is checked by the likelihood ratio test (LRT) with the null hypothesis of the MMEM(1,1) with two-mixture errors versus the alternative hypothesis SET-MMEM(1,1), indicated as

$$\begin{cases} H_0 : \lambda_1 = \lambda_2 = \lambda, \text{ and } \sigma_1 = \sigma_2 = \sigma \\ H_1 : \text{otherwise,} \end{cases}$$

Let $\boldsymbol{\xi}_0 = (\alpha, \beta, \eta, \lambda, \sigma, \omega)'$ be the parameter set of the MMEM(1,1) model, $\ell_0(\boldsymbol{\xi}_0)$, $\ell_1(\boldsymbol{\xi})$ are respectively the log-likelihood function of MMEM(1,1) and SET-MMEM(1,1) models. Since the process $\{X_t\}$ is a Markov chain, under the null hypothesis and the standard assumptions, it is concluded that

$$\Delta = -2\Big(\ell_0(\boldsymbol{\xi}_0) - \ell_1(\boldsymbol{\xi})\Big) \xrightarrow{d} \chi^2(2),$$

where $\chi^2(2)$ is the chi-square distribution with 2 degree of freedom. Therefore, based on the LRT test, the nonlinearity of the SET-MMEM(1,1) process can be induced by the chi-square distribution.

4 Simulation Study

In this section, we present a simulation study to assess the effectiveness of the EM estimation methodology associated with the SET-MMEM(1,1) process. The parameter values are specified as follows

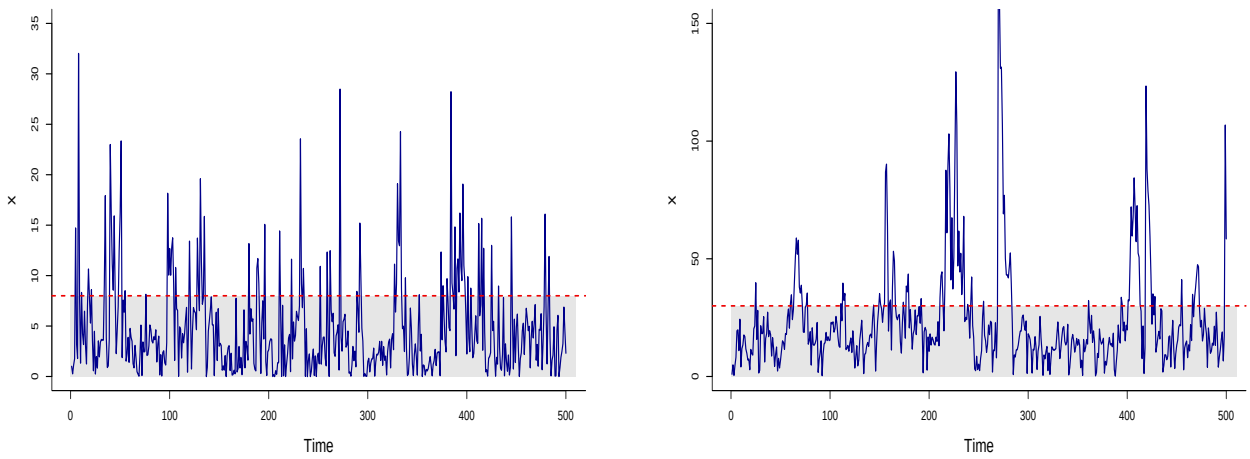
$$\begin{aligned} M_1 &= (\alpha, \beta, \eta, \lambda_1, \lambda_2, \sigma_1, \sigma_2, \omega) = (0.3, 0.6, 0.5, 0.5, 2.5, 0.25, 0.45, 0.7),\ r = 8, \\ M_2 &= (\alpha, \beta, \eta, \lambda_1, \lambda_2, \sigma_1, \sigma_2, \omega) = (0.6, 0.3, 3.5, 12, 5, 1.5, 0.5, 0.8),\ r = 30, \end{aligned}$$

with the gamma and log-normal mixtures distribution of error term. The simulation sample sizes are specified as 100, 200, 500, and 1000, respectively, with each simulation repeated 1000 times. Following Li and Tong (2016), to ensure the reliability of the results, the proportion of observations within a single regime relative to the entire sample must exceed 5%. Consequently, the threshold parameter r is selected to allocate at least 20% of the total sample observations to each regime.

Table 1 presents several statistical measures for the two regimes, where observations less than the threshold parameter are classified as regime 1 and those more than as regime 2, for a sample size of $n = 500$ under parameter sets M_1 and M_2. Additionally, Figure 1 illustrates the sample path plots corresponding to the considered parameter sets for $n = 500$. Notably, the sample path associated with the M_2 series more distinctly exhibits the presence of two regimes, attributable to the greater difference between the parameters λ_1, λ_2, and σ_1, σ_2.

Table 1: The descriptive statistics of the simulated data sets.

Parameter set	Regime	Min	Max	Mean	Variance	Skewness	Kurtosis	Median
Set M_1	Regime 1	0.003	7.888	2.951	4.711	0.4534	2.1637	2.496
	Regime 2	8.097	32.012	13.194	24.926	1.666	5.815	11.683
Set M_2	Regime 1	0.213	29.987	14.783	51.654	-0.082	2.383	15.021
	Regime 2	30.25	186.10	55.272	866.874	1.941	7.185	43.89



(a) (b)

Figure 1: The sample path of SET-MMEM(1,1), under (a) parameter set M_1 and (b) parameter set M_2.

The proposed EM algorithm was executed iteratively until the maximum change in parameter estimates between consecutive iterations fell below 0.001. The outcomes of this procedure are presented in Table 2 and Table 3, respectively. Additionally, the threshold estimation approach was evaluated using the MMLL method, with the corresponding results summarized in Table 3.

Table 2: The estimates and RMSE of parameters of SET-MMEM(1,1) model, without known r.

n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\omega}$
	Parameter set: M_1							
100	0.28276	0.63565	0.53518	0.52196	2.59488	0.24046	0.44251	0.75257
RMSE	(0.02996)	(0.03662)	(0.05504)	(0.03413)	(0.26164)	(0.01331)	(0.02405)	(0.08359)
200	0.28588	0.62047	0.47216	0.51025	2.57057	0.25092	0.44191	0.73065
RMSE	(0.01617)	(0.02221)	(0.04418)	(0.03138)	(0.24233)	(0.00799)	(0.02035)	(0.07262)
500	0.28736	0.60437	0.48511	0.50977	2.55979	0.24998	0.45117	0.72003
RMSE	(0.01442)	(0.01764)	(0.03293)	(0.02764)	(0.14645)	(0.00403)	(0.01642)	(0.04896)
1000	0.29447	0.60475	0.49207	0.50581	2.48074	0.25048	0.45021	0.71159
RMSE	(0.00936)	(0.01262)	(0.02102)	(0.01915)	(0.10117)	(0.00194)	(0.01497)	(0.03641)
	Parameter set: M_2							
100	0.56546	0.28651	3.62279	12.52398	5.22189	1.57005	0.46996	0.85329
RMSE	(0.05043)	(0.02151)	(0.37439)	(0.64881)	(0.28974)	(0.13093)	(0.04757)	(0.09763)
200	0.62327	0.28951	3.41636	12.4818	5.21481	1.56414	0.47992	0.84688
RMSE	(0.04323)	(0.01783)	(0.35077)	(0.60852)	(0.24785)	(0.11226)	(0.03947)	(0.08693)
500	0.58075	0.30707	3.56022	11.6743	4.85965	1.58535	0.50964	0.82512
RMSE	(0.03708)	(0.01466)	(0.26411)	(0.41926)	(0.15653)	(0.08579)	(0.01866)	(0.06594)
1000	0.58931	0.30478	3.54784	11.7964	5.10554	1.54936	0.50757	0.81008
RMSE	(0.02851)	(0.01006)	(0.17168)	(0.36361)	(0.12199)	(0.06984)	(0.01472)	(0.04031)

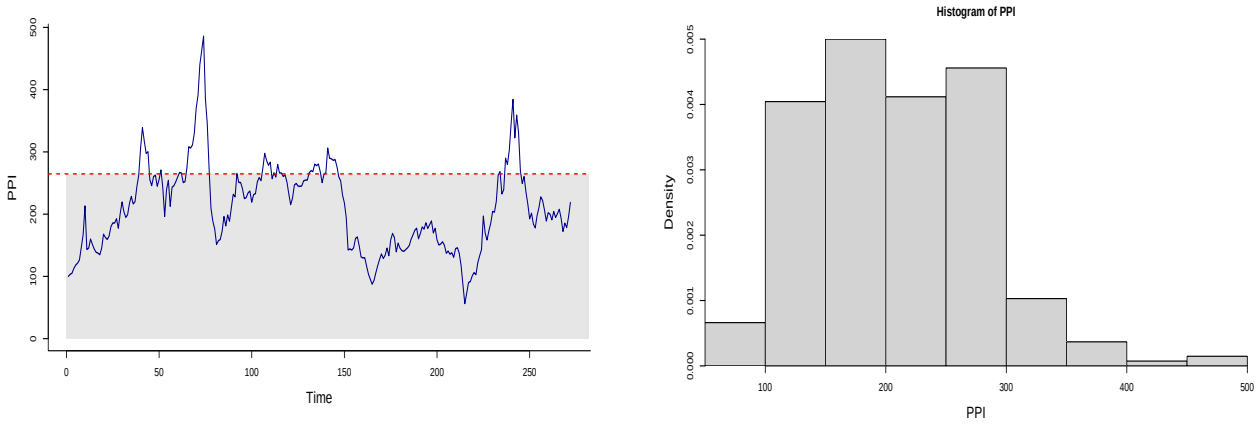
Table 2 represents the estimation of the parameters of the SET-MMEM(1,1) model, accompanied by the RMSEs indicated in parentheses, which leads to the consistency of the estimation procedure. Increasing the sample size corresponds to a reduction in the RMSE values for all unknown parameters. Table 3 provides the estimation of the parameters of the SET-MMEM(1,1) model, with RMSEs in parentheses, which corroborates the previous findings.

Table 3: The estimates and RMSE of parameters of SET-MMEM(1,1) model, without unknown r.

n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\omega}$	$\hat{r}$
	Parameter set: M_1								
100	0.27774	0.56953	0.47685	0.47112	2.61838	0.24189	0.48305	0.66291	8.65261
RMSE	(0.03425)	(0.04518)	(0.05764)	(0.03883)	(0.28128)	(0.01824)	(0.04342)	(0.09753)	(0.40871)
200	0.28046	0.57169	0.48309	0.48071	2.56932	0.24452	0.46305	0.67871	8.60188
RMSE	(0.02889)	(0.03532)	(0.04828)	(0.03434)	(0.22419)	(0.01209)	(0.03725)	(0.08845)	(0.38149)
500	0.29193	0.61119	0.48761	0.49605	2.46469	0.25297	0.45256	0.68359	8.27747
RMSE	(0.01825)	(0.02471)	(0.03665)	(0.02881)	(0.17987)	(0.00678)	(0.02665)	(0.07755)	(0.27491)
1000	0.29685	0.60891	0.48952	0.50266	2.47023	0.25043	0.45089	0.68942	8.21707
RMSE	(0.01174)	(0.01999)	(0.02953)	(0.02112)	(0.13138)	(0.00285)	(0.01854)	(0.05909)	(0.23317)
	Parameter set: M_2								
100	0.56499	0.31286	3.36708	12.67209	4.65949	1.43006	0.48043	0.86457	32.65408
RMSE	(0.06261)	(0.02249)	(0.38862)	(0.81636)	(0.39464)	(0.16006)	(0.05232)	(0.10496)	(0.76339)
200	0.57129	0.28711	3.62836	11.7828	4.71557	1.62587	0.48322	0.85867	32.1417
RMSE	(0.05707)	(0.01951)	(0.35474)	(0.74448)	(0.30801)	(0.12683)	(0.04675)	(0.09003)	(0.62297)
500	0.57664	0.30867	3.58551	12.2926	5.20345	1.57776	0.49025	0.83068	30.8431
RMSE	(0.04165)	(0.01805)	(0.28107)	(0.53625)	(0.21857)	(0.09942)	(0.02536)	(0.07112)	(0.44964)
1000	0.58404	0.30435	3.56122	12.24717	4.87314	1.56198	0.49471	0.82664	30.9475
RMSE	(0.02989)	(0.01257)	(0.20915)	(0.44375)	(0.16384)	(0.07421)	(0.01756)	(0.05123)	(0.35062)

5 Real Data Analysis

This section is dedicated to checking the application of the SET-MMEM(1,1) model using a real data set sourced from the federal reserve economic data (FRED) website, (https://fred.stlouisfed.org). Specifically, the model is applied to the monthly producer price index (PPI) for the crude petroleum and natural gas extraction industry. The PPI measures the average temporal change in the selling prices received by domestic producers for their output, thereby capturing price fluctuations from the perspective of producers. The PPI series spans observations from June 1, 2002, to January 1, 2025, resulting in a total of $n = 272$ observations (refer to Figure 2 (a)). Furthermore, Figure 2 (b) illustrates that the empirical distribution of the series exhibits asymmetry and a right-skewed pattern.



(a) (b)

Figure 2: (a) The time series plot, and (b) histogram of PPI.

Table 4 presents the descriptive statistics for the PPI data set. Analysis through skewness (right-skewed), kurtosis (leptokurtic), and Jarque-Bera tests indicates that the PPI data is non-normal. The augmented Dickey-Fuller (ADF) test confirms that the series is stationary, with the ADF test statistics -4.612 and p-value equals 0.039.

Table 4: The descriptive statistics for the PPI data set.

Series	Min	Max	Mean	Variance	Median	Skewness	Kurtosis	Jarque-Bera test
PPI	56.2	486.0	209.4	4869.1	202.7	0.64	0.91	28.0 (P-value = 8.154e-07)

The LRT statistic and its associated p-value for comparing the MMEM(1,1) model against the SET-MMEM(1,1) model are 21.33 and $2.325e-05$, respectively. We note that the LRT statistic exceeds the critical value of $\chi^2_{0.05}(1) = 5.99$, hence the statistic indicates nonlinearity of the real data set.

The performance of the SET-MMEM(1,1) in modeling the PPI data set was evaluated against several MEMs with established error distributions, such as exponential MEM under $\lambda = 1$ (EMEM(1,1)) with parameter (α, β, η), gamma MEM (GMEM(1,1)) with parameters $(\alpha, \beta, \eta, \lambda)$, log-normal (LN-MEM(1,1)) with parameters $(\alpha, \beta, \eta, \sigma)$, and MMEM(1,1) with parameters $(\alpha, \beta, \eta, \lambda, \sigma, \omega)$. The ML estimates of the parameters and the Akaike information criterion (AIC) values are reported in Table 5. Based on the comparative analysis of these results, the SET-MMEM(1,1) model emerges as the preferred candidate for modeling the PPI data.

Table 5: Comparison results.

Models	ML estimates	AIC
EMEM(1,1)	$\hat{\alpha} = 0.785,\ \hat{\beta} = 0.128,\ \hat{\eta} = 10.465$	3617.39
GMEM(1,1)	$\hat{\alpha} = 0.751,\ \hat{\beta} = 0.176,\ \hat{\eta} = 10.109,\ \hat{\lambda} = 1.568$	3590.08
LNMEM(1,1)	$\hat{\alpha} = 0.713,\ \hat{\beta} = 0.265,\ \hat{\eta} = 9.918,\ \hat{\sigma} = 0.294$	2996.24
MMEM(1,1)	$\hat{\alpha} = 0.681,\ \hat{\beta} = 0.306,\ \hat{\eta} = 8.924,\ \hat{\lambda} = 71.502,\ \hat{\sigma} = 0.147\ \hat{\omega} = 0.754$	2461.38
SET-MMEM(1,1)	$\hat{\alpha} = 0.643,\ \hat{\beta} = 0.323,\ \hat{\eta} = 7.394,\ \hat{\lambda}_1 = 37.248$ $\hat{\lambda}_2 = 62.035,\ \hat{\sigma}_1 = 0.11463,\ \hat{\sigma}_2 = 0.18079,\ \hat{\omega} = 0.821$	2441.26

The residual analysis is performed to evaluate the suitability of the proposed model, which indicates the compliance between the observed data and the fitted model. The standardized residuals are denoted as

$$r_t = \frac{X_{t+1} - E(X_{t+1}|X_t)}{\sqrt{\text{Var}(X_{t+1}|X_t)}},$$

where $E(X_{t+1}|X_t)$ and $\text{Var}(X_{t+1}|X_t)$ are provided in Equation 4 and Equation 5, respectively.

The Ljung-Box test applied to the residual series yielded a test statistic of 11.462 with an associated p-value of 0.0007. The normalized residual diagnostics, in Figure 3, indicate an approximately constant variance over time. The histogram of residuals suggests that prediction errors are approximately normally distributed, with a mean of -0.076, which is close to zero, and a variance of 0.883, which approximates unity. The autocorrelation function (ACF) plot in Figure 3-(a) reveals no apparent pattern or significant autocorrelation in the residuals, supporting the assumption of normality.

Specifically, the standardized Pearson residuals of the SET-MMEM(1,1) model exhibit a mean of 0.012 and a variance of 1.058, indicating values proximate to the ideal benchmarks. Figure 3 depicts the ACF of these standardized Pearson residuals, which are uncorrelated. These findings are corroborated by the Ljung-Box test applied to the PPI dataset, which produced a p-value of 0.487, thereby supporting the adequacy of the model.

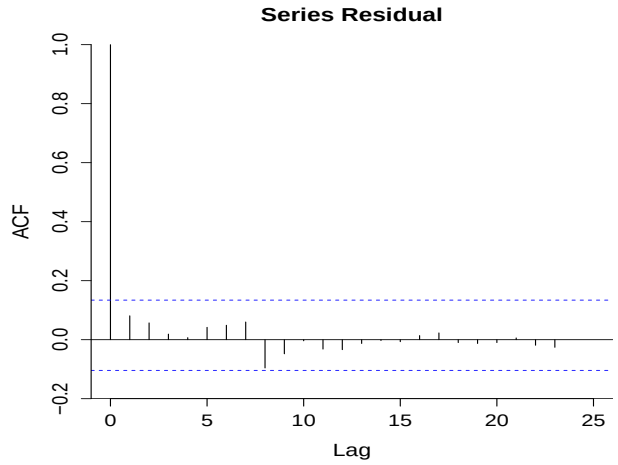


Figure 3: The ACF plot of the normalized residuals.

Conclusion

The multiplicative error modeling is a common and well-known approach for analyzing positive time series. However, it has certain limitations, as it cannot capture all specific characteristics of real-world series. To address this, a new MEM with the mixture errors is introduced to cover the multimodal and asymmetric structures. By incorporating a threshold variable, the proposed process divides the classical MMEM into a two-regime model, assigning a distinct process with specific parameters to each regime. This approach effectively captures the nonlinear structure of positive real sets. The unknown parameters of the proposed model are estimated through the EM-algorithm approach, while the threshold parameter is determined by minimizing the maximized log-likelihood function. Simulation results and residual analysis of real data indicate the applicability of the new model. Additionally, the AIC measure confirms the preference of the process compared to other classical MEMs.

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