

New class of Poisson mixture models and its statistical properties

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New class of Poisson mixture models and its statistical properties

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Abstract

Numerous Poisson mixture distributions are available in the literature in order to model over-dispersion found in many areas of study. In this paper, we introduce a new class of one parameter Poisson mixture distributions that includes several such existing Poisson mixture models as special cases. Specifically, the intensity parameter of the Poisson distribution is assumed to follow a finite mixture of gamma distributions. The proposed class provides greater modeling flexibility for count data exhibiting over-dispersion and heterogeneity. The mixture representation makes it easier to simulate random numbers from distributions that are otherwise difficult to simulate directly. Various statistical properties of the distributions within this class are derived and discussed. A general method of parameter estimation is proposed based on the principle of maximum likelihood and the Expectation–Maximization (EM) algorithm. The performance of the estimators is evaluated through a detailed simulation study. Finally, the practical applicability of the distributions belonging to the proposed class is demonstrated through the analysis of three real datasets.

Key words: Poisson mixture models, EM algorithm, Count data, over-dispersion, negative binomial, unimodality.

1 Introduction

Count data consists of data that represent the frequency of occurrence of events within a category or an interval. To analyze such count data, the most commonly used distribution is Poisson distribution because of its simplicity and flexibility. An important property of Poisson distribution is equi-dispersion. But in certain real-life situations count data may be over-dispersed. In such cases using Poisson distribution would be inappropriate and may lead to inaccurate inference. In this context, to overcome such limitations, the use of Poisson mixture models has been proposed. The Negative Binomial (NB) distribution, Poisson-Inverse Gaussian distribution, Poisson-Lognormal distribution etc. are some of the important distributions of this kind which arise as mixed Poisson models and are widely employed to accommodate over-dispersion in count data. Even though these distributions offer greater flexibility in modeling such over-dispersed data, the model complexity of these models are high due to the presence of multiple parameters. Hence there may be difficulties in estimation and other inference problems, particularly for small or moderately sized samples.

This has motivated the construction of Poisson mixture distributions with a single parameter, which is capable of handling over-dispersion while being more parsimonious, easier to analyze, and estimate. Among them we mention the Poisson-Lindley distribution introduced by Sankaran [1970], Poisson-Sujatha distribution proposed by Shanker and Fesshaye [2016], Poisson-Amarendra distribution described by Shanker [2016], Poisson-Akash distribution developed by Shanker [2017a], Poisson-Aradhana distribution developed by Shanker [2017b], Poisson-Shanker distribution introduced by Shanker [2017c], Poisson-Garima distribution proposed by Shanker [2017d], Poisson-Ishitha distribution introduced by Shukla and Shanker [2019], Poisson-Pranav distribution proposed by Hassan et al. [2020b], Poisson-Komal distribution introduced by Shanker et al. [2025], Poisson-Rama distribution invented by Shukla et al. [2025], Poisson-Xgamma distribution proposed by Para et al. [2020] and Poisson-Rani distribution developed by Ahmad et al. [2021]. Ahmad and Wani [2024] introduced a new Poisson mixture distribution where the mixing distribution follows a Chris-Jerry distribution. Borbye et al. [2024] studied Poisson-XRani distribution as an alternative model. Although many Poisson mixture distributions have been developed, they are mainly model-specific and lacks a unified framework. In particular, there is an absence of a general framework that simultaneously (i) accommodates many Poisson -mixture distributions and (ii) simplifies the general derivation of distributional properties. Motivated by these considerations, in this paper, we develop a general class of one-parameter Poisson mixture distributions derived from a unified mixing mechanism.

The paper is organized as follows. In Section 2, we introduce the new class of distributions. Section 3 describes some important statistical properties. We illustrate the properties for Poisson-Sujatha distribution. Additionally we introduce new members to the family such as Poisson-Akshaya, Poisson-Devya and Poisson-Shambhu distributions. The statistical properties of Poisson-Akshaya

distribution are then demonstrated. In the next section, we describe different parameter estimation methods namely the method of moment estimation and maximum likelihood estimation using EM algorithm for the general class of distributions. We demonstrate these methods using Poisson-Sujatha distribution and Poisson-Akshaya distribution. In Section 5, additionally we introduce a new class of zero inflated distributions and study the properties of this general class followed by a description of zero truncated version. Finally we demonstrate the applicability of methods of fitting new models using both simulated and real data sets. In the last section concluding remarks are included.

2 A new class of Poisson mixture models

We introduce a new one-parameter discrete random variable (rv) which is useful for modeling over-dispersed data. Let $X|\lambda$ follows a Poisson distribution with intensity parameter λ and λ follows a distribution which is a finite mixture of certain Gamma distributions. Then the resulting unconditional distribution constitutes a class of discrete distributions. In the following theorem, we identify the form of the distribution as a finite mixture of negative binomial distributions. Let us denote the NB rv with parameters i and $\frac{\theta}{\theta+1}$ as $NB(i, \frac{\theta}{\theta+1})$ and its probability mass function (PMF) by $f_{NB}(x; i, \frac{\theta}{\theta+1})$, given by

$$f_{NB}(x; i, \frac{\theta}{\theta+1}) = \binom{x+i-1}{x} \left(\frac{\theta}{\theta+1} \right)^i \left(\frac{1}{\theta+1} \right)^x, x = 0, 1, 2 \dots \theta > 0.$$

Also let $f_G(x; \theta, i)$ be the probability density function (PDF) of the Gamma distribution with shape parameter i and rate parameter $\theta > 0$, where we denote such a rv by $G(\theta, i)$.

Theorem 2.1. *Let $X|\lambda$ follows a Poisson distribution with parameter λ and λ is assumed to be a rv with a probability distribution which is a finite mixture of Gamma distributions such that the PDF of λ is $g(\lambda) = \sum_{i=1}^k \alpha_i f_G(\lambda; \theta, i)$, where $\alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1$. Then the unconditional distribution of X is specified by*

$$f(x; \theta) = \sum_{i=1}^k \alpha_i f_{NB} \left(x; i, \frac{\theta}{\theta+1} \right). \quad (1)$$

Proof. Let $X | \lambda \sim \text{Poisson}(\lambda)$ and λ have the mixture of Gamma densities

$$g(\lambda) = \sum_{i=1}^k \alpha_i f_G(\lambda; \theta, i), \quad \sum_{i=1}^k \alpha_i = 1, \alpha_i > 0.$$

Then

$$f(x) = \int_0^\infty p(x | \lambda) g(\lambda) d\lambda.$$

Substituting the kernels of the Poisson and Gamma densities gives

$$\begin{aligned} f(x) &= \sum_{i=1}^k \alpha_i \int_0^\infty \frac{e^{-\lambda} \lambda^x}{x!} \frac{\theta^i}{\Gamma(i)} \lambda^{i-1} e^{-\theta \lambda} d\lambda \\ &= \sum_{i=1}^k \frac{\alpha_i \theta^i}{\Gamma(i) x!} \int_0^\infty e^{-(\theta+1)\lambda} \lambda^{x+i-1} d\lambda. \end{aligned} \quad (2)$$

Using the Gamma–integral identity

$$\int_0^\infty e^{-a\lambda} \lambda^{r-1} d\lambda = \frac{\Gamma(r)}{a^r}, \quad a > 0, r > 0,$$

the integral in (2) becomes

$$\int_0^\infty e^{-(\theta+1)\lambda} \lambda^{x+i-1} d\lambda = \frac{\Gamma(x+i)}{(\theta+1)^{x+i}}.$$

Hence

$$\begin{aligned} f(x) &= \sum_{i=1}^k \alpha_i \frac{\theta^i}{\Gamma(i) x!} \frac{\Gamma(x+i)}{(\theta+1)^{x+i}} \\ &= \sum_{i=1}^k \alpha_i \binom{x+i-1}{x} \left(\frac{\theta}{\theta+1} \right)^i \left(\frac{1}{\theta+1} \right)^x. \end{aligned}$$

The last expression is the linear combination of PMF of a NB distribution with parameters $(i, \frac{\theta}{\theta+1})$. Therefore,

$$f(x) = \sum_{i=1}^k \alpha_i f_{NB} \left(x; i, \frac{\theta}{\theta+1} \right),$$

which completes the proof. $\square$

Let us denote this class by $\mathcal{E}$. Some of the members of this class are given in Table 1. The Table 1 illustrates that the class $\mathcal{E}$ comprises mixtures containing two, three, four, five and six components. The class of two-component mixtures includes the Poisson-Lindley, Poisson-Garima, Poisson-Shanker, Poisson-Fav-Jerry, Poisson-Komal, Poisson-Akash, Poisson-Xgamma, Poisson-Chris-Jerry, Poisson-Ishitha, Poisson-Pranav, Poisson-Rama, Poisson-Rani and Poisson-Xrani distributions. The Poisson-Sujatha and Poisson-Aradhana distributions represent three-component mixtures, while Poisson-Amarendra and Poisson-Akshaya belong to the four-component category. The Poisson-Devya distribution forms a five-component mixture, and Poisson-Shambhu represents a six-component distribution.

Table 1: Various members of the class $\mathcal{E}$

Sl No.	Distribution	α_1	α_2	α_3	α_4	α_5	α_6
1	Poisson-Lindley	$\frac{\theta}{\theta+1}$	$\frac{1}{\theta+1}$	0	0	0	0
2	Poisson-Garima	$\frac{\theta+1}{\theta+2}$	$\frac{1}{\theta+2}$	0	0	0	0
3	Poisson-Shanker	$\frac{\theta^2}{\theta^2+1}$	$\frac{1}{\theta^2+1}$	0	0	0	0
4	Poisson-Fav-Jerry	$\frac{2}{\theta^2+2}$	$\frac{\theta^2}{\theta^2+2}$	0	0	0	0
5	Poisson-Komal	$\frac{\theta^2+\theta}{\theta^2+\theta+1}$	$\frac{1}{\theta^2+\theta+1}$	0	0	0	0
6	Poisson-Akash	$\frac{\theta^2}{\theta^2+2}$	0	$\frac{2}{\theta^2+2}$	0	0	0
7	Poisson-Xgamma	$\frac{\theta}{\theta+1}$	0	$\frac{1}{\theta+1}$	0	0	0
8	Poisson-Chris-Jerry	$\frac{\theta}{\theta+2}$	0	$\frac{2}{\theta+2}$	0	0	0
9	Poisson-Ishitha	$\frac{\theta^3}{\theta^3+2}$	0	$\frac{2}{\theta^3+2}$	0	0	0
10	Poisson-Pranav	$\frac{\theta^4}{\theta^4+6}$	0	0	$\frac{6}{\theta^4+6}$	0	0
11	Poisson-Rama	$\frac{\theta^3}{\theta^3+6}$	0	0	$\frac{6}{\theta^3+6}$	0	0
12	Poisson-Rani	$\frac{\theta^5}{\theta^5+24}$	0	0	0	$\frac{24}{\theta^5+24}$	0
13	Poisson-XRani	$\frac{\theta^{10}+48\theta^5}{(\theta^5+24)^2}$	0	0	0	$\frac{576}{(\theta^5+24)^2}$	0
14	Poisson-Sujatha	$\frac{\theta^2}{\theta^2+\theta+2}$	$\frac{\theta}{\theta^2+\theta+2}$	$\frac{2}{\theta^2+\theta+2}$	0	0	0
15	Poisson-Aradhana	$\frac{\theta^2}{\theta^2+2\theta+2}$	$\frac{2\theta}{\theta^2+2\theta+2}$	$\frac{2}{\theta^2+2\theta+2}$	0	0	0
16	Poisson-Amarendra	$\frac{\theta^3}{\theta^3+\theta^2+2\theta+6}$	$\frac{\theta^2}{\theta^3+\theta^2+2\theta+6}$	$\frac{2\theta}{\theta^3+\theta^2+2\theta+6}$	$\frac{6}{\theta^3+\theta^2+2\theta+6}$	0	0
17	Poisson-Akshaya	$\frac{\theta^3}{\theta^3+3\theta^2+6\theta+6}$	$\frac{3\theta^2}{\theta^3+3\theta^2+6\theta+6}$	$\frac{6\theta}{\theta^3+3\theta^2+6\theta+6}$	$\frac{6}{\theta^3+3\theta^2+6\theta+6}$	0	0
18	Poisson-Devya	$\frac{\theta^4}{\theta^4+\theta^3+2\theta^2+6\theta+24}$	$\frac{\theta^3}{\theta^4+\theta^3+2\theta^2+6\theta+24}$	$\frac{2\theta^2}{\theta^4+\theta^3+2\theta^2+6\theta+24}$	$\frac{6\theta}{\theta^4+\theta^3+2\theta^2+6\theta+24}$	$\frac{24}{\theta^4+\theta^3+2\theta^2+6\theta+24}$	0
19	Poisson-Shambhu	$\frac{\theta^5}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}$	$\frac{\theta^4}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}$	$\frac{2\theta^3}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}$	$\frac{6\theta^2}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}$	$\frac{24\theta}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}$	$\frac{120}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}$

Remark 1. Recently Hanandeh and Al-Nasser [2021] introduced the Poisson-Gold family of distributions which includes Poisson-Lindley, Poisson-Sujatha, Poisson-Amarendra, Poisson-Devya and Poisson-Shambhu distributions etc. as special cases. For this family, the intensity parameter λ follows the Gold distribution proposed by Al-Talib et al. [2023] which is defined as a finite mixture of k independent Gamma distributions with common scale parameter θ and shape parameter j where $j = 1, 2, \dots, k$. The PDF of Gold distribution is given by

$$f_{GD}(x; \theta, k) = \begin{cases} \frac{1}{\sum_{j=1}^k \Gamma(j)\theta^{j-1}} \frac{x^{k-1}}{x-1} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x > 0, \theta > 0; k = 1, 2, \dots; x \neq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

We can see that this distribution is a finite mixture of Gamma distributions with mixing proportions having the form

$$\alpha_j = \frac{\Gamma(j)\theta^{j-1}}{\sum_{j=1}^k \Gamma(j)\theta^{j-1}}. \quad (4)$$

The PMF of Poisson-Gold distribution (PGD), (Hanandeh and Al-Nasser [2021]),

$$f_{PGD}(x; \theta, k) = \begin{cases} \frac{\sum_{j=1}^k \Gamma(x+j)(\frac{\theta}{\theta+1})^{x+j}}{\Gamma(x+1) \sum_{j=1}^k \Gamma(j)\theta^j}, & x = 0, 1, 2, \dots \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

can be represented as a finite mixture of $NB(j, \frac{1}{\theta+1})$, $j = 1, 2, \dots, k$ with mixing coefficients α_j as follows:

$$\begin{aligned} f_{PGD}(x; \theta, k) &= \frac{\sum_{j=1}^k \Gamma(x+j)(\frac{\theta}{\theta+1})^{x+j}}{\Gamma(x+1) \sum_{j=1}^k \Gamma(j)\theta^j} \\ &= \sum_{j=1}^k \frac{\Gamma(j)\theta^{j-1}}{\sum_{j=1}^k \Gamma(j)\theta^{j-1}} \frac{\Gamma(x+j)}{\Gamma(j)\Gamma(x+1)} \left(\frac{1}{\theta+1}\right)^j \left(\frac{\theta}{\theta+1}\right)^x \\ &= \sum_{j=1}^k \alpha_j f_{NB}\left(x; j, \frac{1}{\theta+1}\right). \end{aligned}$$

From the above facts we can conclude that PGD belongs to the family $\mathcal{E}$. Thus the PGD family becomes a subfamily of $\mathcal{E}$.

Remark 2. Poisson mixture distributions generated by mixing two or more different positive valued random variables other than gamma such as Xgamma, XLindley are also studied in the literature. For instance, Ahsan-ul Haq et al. [2022] introduced Poisson-XLindley (PXL) distribution, where the intensity parameter λ is generated as a special mixture of two distributions: exponential and Lindley. Later, Irshad et al. [2023] suggested a new Poisson mixture distribution where the λ follows a new XLindley distribution. Both the distributions are members of $\mathcal{E}$ where the respective PMFs can be expressed as

$$f_{PXL}(x; \theta) = \frac{\theta^2 (x + \theta^2 + 3(1 + \theta))}{(1 + \theta)^{4+x}}, \quad x = 0, 1, 2, \dots, \theta > 0,$$

and

$$f_{P_{XNL}}(x; \theta) = \frac{\theta(2\theta + \theta x + 1)}{2(\theta + 1)^{x+2}}, \quad \text{for } x = 0, 1, 2, \dots \text{ and } \theta > 0.$$

We note that for PXL distribution, the distribution is identified in the general representation given in Theorem 1 with $k = 3$, and $\alpha_1 = 1 - \frac{1}{(1+\theta)^2}$, $\alpha_2 = 0$, $\alpha_3 = \frac{1}{(1+\theta)^2}$. The PXNL distribution is recognized with $k = 2$ and $\alpha_1 = \alpha_2 = 1/2$. The Poisson Ramos-Louzada distribution (Alkhairy [2023]), Poisson Agueghwerido distribution (Alamri [2024]) and Poisson-Haq distribution (Alomair et al. [2025]) are all examples of other distributions. These distributions come under the family $\mathcal{E}$. The details are given in Appendix.

Remark 3. It is noted that the results in Theorem 1 is valid even if a single component is present in the general representation given in (1). In this case, there is no finite mixture representation so we will have only one component with $\alpha_k = 1$. For example, when $k = r$, and $\lambda \sim G(\theta, r)$ we get usual NB($r, p = \frac{\theta}{\theta+1}$). Ahsan-ul Haq [2024] studied Poisson Moment Exponential Distribution (PMExD), where the λ follows a $G(1/\theta, 2)$. Also, when $\frac{1}{\theta} = \delta$ and $\lambda \sim G(\delta, 2)$, or $G(2\theta, 2)$ then X follows Poisson-Ailmujia distribution (Hassan et al. [2020a]). Thus these two distributions are then belong to $\mathcal{E}$. However, the general results given in the following sections should be understood with appropriate modifications.

3 Statistical properties

In this section we present some statistical properties of the distributions belonging to the class $\mathcal{E}$.

3.1 Distribution Function

Let X be a rv belonging to the class $\mathcal{E}$ with PMF $f(x)$. Then its distribution function is given by

$$\begin{aligned} F(x) &= \sum_{t=0}^x f(t) \\ &= \sum_{t=0}^x \sum_{i=1}^k \alpha_i f_{NB}\left(t; i, \frac{\theta}{\theta+1}\right) \\ &= \sum_{i=1}^k \alpha_i I_{\frac{\theta}{\theta+1}}(i, x+1), \end{aligned} \tag{6}$$

where $I_x(a, b)$ is the regularized incomplete beta function.

As an illustration we derive the distribution function of Poisson-Lindley distribution using the above expression (6). We know that Poisson-Lindley distribution

is a mixture of $NB(1, \frac{\theta}{\theta+1})$ and $NB(2, \frac{\theta}{\theta+1})$ with mixing proportions $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively. Hence,

$$\begin{aligned}
F(x) &= \sum_{i=1}^2 \alpha_i I_{\frac{\theta}{\theta+1}}(i, x+1) \\
&= \frac{\theta}{\theta+1} I_{\frac{\theta}{\theta+1}}(1, x+1) + \frac{1}{\theta+1} I_{\frac{\theta}{\theta+1}}(2, x+1) \\
&= \frac{\theta}{\theta+1} \left[1 - \left(\frac{1}{\theta+1} \right)^{x+1} \right] + \frac{1}{\theta+1} \left[1 - \frac{x+2}{(\theta+1)^{x+1}} + \frac{x+1}{(\theta+1)^{x+2}} \right] \\
&= 1 - \frac{\theta^2 + 3\theta + \theta x + 1}{(\theta+1)^{x+3}}.
\end{aligned}$$

3.2 Moment Generating Function

Let X be a rv following a distribution which belongs to the class $\mathcal{E}$, then its Moment Generating Function (MGF) is given by

$$\begin{aligned}
M_X(t) &= \sum_x e^{tx} \left(\sum_{i=1}^k \alpha_i f_{NB}\left(x; i, \frac{\theta}{\theta+1}\right) \right) \\
&= \sum_{i=1}^k \alpha_i \left(\sum_x e^{tx} f_{NB}\left(x; i, \frac{\theta}{\theta+1}\right) \right) \\
&= \sum_{i=1}^k \alpha_i \left(\frac{\theta}{\theta+1 - e^t} \right)^i.
\end{aligned} \tag{7}$$

3.3 Factorial moments

The Factorial moment of a rv X belonging to the class $\mathcal{E}$ is obtained as

$$\begin{aligned}
E[X_{(r)}] &= E[X(X-1)(X-2)\dots(X-r+1)] \\
&= E[E[X_{(r)}|\lambda]] \\
&= \int_0^\infty \left[\sum_0^\infty x_{(r)} \frac{e^{-\lambda} \lambda^x}{x!} \right] g(\lambda) d\lambda \\
&= \frac{1}{\theta^r} \sum_{i=1}^k \alpha_i \frac{\Gamma(r+i)}{\Gamma i}.
\end{aligned}$$

3.4 Moments

Let X be a rv belonging to the class $\mathcal{E}$. Then it's moments are obtained as

$$E[X] = \sum_{i=1}^k \frac{i\alpha_i}{\theta}, \tag{8}$$

$$E[X^2] = \sum_{i=1}^k \frac{i\alpha_i}{\theta^2} (i + \theta + 1), \quad (9)$$

$$V[X] = \frac{1}{\theta^2} \left\{ \sum_{i=1}^k i\alpha_i (i + \theta + 1) - \left(\sum_{i=1}^k i\alpha_i \right)^2 \right\}. \quad (10)$$

Other raw moments can be obtained from factorial moments. Now as an illustration we compute the mean and variance of Poisson-Sujatha distribution and Poisson-Akshaya distribution. The PMF of Poisson-Sujatha distribution is given by

$$P[X = x] = f(x) = \frac{\theta^3}{\theta^2 + \theta + 2} \frac{x^2 + (\theta + 4)x + \theta^2 + 3\theta + 4}{(\theta + 1)^{x+3}}; x = 0, 1, 2, \dots; \theta > 0. \quad (11)$$

Note that Sujatha distribution is a three component mixture of Gamma distributions with common scale parameter θ and shape parameters 1, 2 and 3 with mixing proportions $\frac{\theta^2}{\theta^2 + \theta + 2}$, $\frac{\theta}{\theta^2 + \theta + 2}$ and $\frac{2}{\theta^2 + \theta + 2}$ respectively. From Theorem 2.1 we can see that Poisson-Sujatha distribution is a mixture of $NB(1, \frac{\theta}{\theta+1})$, $NB(2, \frac{\theta}{\theta+1})$ and $NB(3, \frac{\theta}{\theta+1})$, with the same mixing proportions respectively. By using the equations (8) and (10), we obtain the mean and variance of Poisson-Sujatha distribution as

$$\mu_1' = \frac{\theta^2 + 2\theta + 6}{\theta(\theta^2 + \theta + 2)}$$

and

$$\mu_2 = \frac{\theta^5 + 4\theta^4 + 14\theta^3 + 28\theta^2 + 24\theta + 12}{\theta^2(\theta^2 + \theta + 2)^2}$$

respectively.

Also Akshaya distribution is a four component mixture of Gamma distributions with common scale parameter θ and shape parameter 1, 2, 3 and 4 with mixing proportions $\frac{\theta^3}{\theta^3 + 3\theta^2 + 6\theta + 6}$, $\frac{3\theta^2}{\theta^3 + 3\theta^2 + 6\theta + 6}$, $\frac{6\theta}{\theta^3 + 3\theta^2 + 6\theta + 6}$ and $\frac{6}{\theta^3 + 3\theta^2 + 6\theta + 6}$ respectively. So from Theorem 2.1, we can see that a Poisson mixture of Akshaya distribution is a mixture of $NB(1, \frac{\theta}{\theta+1})$, $NB(2, \frac{\theta}{\theta+1})$, $NB(3, \frac{\theta}{\theta+1})$ and $NB(4, \frac{\theta}{\theta+1})$ with the same mixing proportions respectively. Thus we introduce a new Poisson mixture distribution namely Poisson-Akshaya distribution. The PMF is given by

$$f(x; \theta) = \frac{\theta^4 \left[(1 + \theta)^3 + (x + 1)(x + 2)(x + 3) + 3(x + 1)(\theta + 1)(x + \theta + 3) \right]}{(\theta^3 + 3\theta^2 + 6\theta + 6) (\theta + 1)^{x+4}},$$

$$x = 0, 1, 2, \dots, \theta > 0$$

$$f(x; \theta) = 0, \quad \text{otherwise.}$$

Hence by using the equations (8) and (10), we obtain the mean and variance of Poisson-Akshaya distribution as

$$\mu_1' = \frac{\theta^3 + 6\theta^2 + 18\theta + 24}{\theta(\theta^3 + 3\theta^2 + 6\theta + 6)},$$

and

$$\mu_2 = \frac{\theta^7 + 10\theta^6 + 54\theta^5 + 186\theta^4 + 408\theta^3 + 540\theta^2 + 432\theta + 144}{\theta^2(\theta^3 + 3\theta^2 + 6\theta + 6)^2}.$$

Next we establish the over dispersion property of the members of the class $\mathcal{E}$.

Theorem 3.1. *The distributions belonging to the class $\mathcal{E}$ exhibit overdispersion.*

Proof. Let X be a rv belonging to the class $\mathcal{E}$. Then its mean, $E[X]$ and variance, $V[X]$ are given by equation (8) and equation (10) respectively. Therefore,

$$\begin{aligned} \frac{V[X]}{E[X]} &= \frac{\frac{1}{\theta^2} \left\{ \sum_{i=1}^k i\alpha_i(i + \theta + 1) - (\sum_{i=1}^k i\alpha_i)^2 \right\}}{\sum_{i=1}^k \frac{i\alpha_i}{\theta}} \\ &= \frac{\frac{\sum_{i=1}^k i^2\alpha_i}{\theta^2}}{\frac{\sum_{i=1}^k i\alpha_i}{\theta}} + \frac{\frac{\sum_{i=1}^k i\alpha_i}{\theta}}{\frac{\sum_{i=1}^k i\alpha_i}{\theta}} + \frac{\frac{\sum_{i=1}^k i\alpha_i}{\theta^2}}{\frac{\sum_{i=1}^k i\alpha_i}{\theta}} - \frac{\frac{(\sum_{i=1}^k i\alpha_i)^2}{\theta^2}}{\frac{\sum_{i=1}^k i\alpha_i}{\theta}} \\ &= 1 + \frac{1}{\theta} + \frac{1}{\theta} \frac{\left\{ \sum_{i=1}^k i^2\alpha_i - (\sum_{i=1}^k i\alpha_i)^2 \right\}}{\sum_{i=1}^k i\alpha_i} \\ &> 1. \end{aligned}$$

□

Table 2 shows the mean, variance and index of dispersion for various members of the class $\mathcal{E}$. Next, we derive the conditions for which the distribution is log concave.

Theorem 3.2. *The distribution belonging to the class $\mathcal{E}$ is log-convex (log-concave) if $0 < R \leq 1/2$ ($1/2 < R < 1$), where*

$$R = \frac{\left(\sum_{i=1}^k a_i \right)^2}{\left(\sum_{i=1}^k a_i(x+i) \right) \left(\sum_{i=1}^k \frac{a_i}{x+i-1} \right)},$$

with $a_i = \alpha_i x(x+1)(x+2)\dots(x+i-1)t^i$, and $t = \frac{\theta}{\theta+1}$.

Proof. Let X be a rv following a distribution which belongs to the class $\mathcal{E}$. Then after some algebra, we obtain

$$\frac{f^2(x)}{f(x-1)f(x+1)} = \frac{A}{B.C}$$

where

$$\begin{aligned} A &= \left[\sum_{i=1}^k \alpha_i \frac{(x+i-1)!}{(i-1)!x!} \left(\frac{\theta}{\theta+1} \right)^i \left(\frac{1}{\theta+1} \right)^x \right]^2, \\ B &= \sum_{i=1}^k \alpha_i \frac{(x+i)!}{(i-1)!(x+1)!} \left(\frac{\theta}{\theta+1} \right)^i \left(\frac{1}{\theta+1} \right)^{x-1}, \\ C &= \sum_{i=1}^k \alpha_i \frac{(x+i-2)!}{(i-1)!(x-1)!} \left(\frac{\theta}{\theta+1} \right)^i \left(\frac{1}{\theta+1} \right)^{x+1}. \end{aligned}$$

On simplification, we get

$$\frac{f^2(x)}{f(x-1)f(x+1)} = \frac{x+1}{x} \frac{A'}{B' \cdot C'}$$

where

$$\begin{aligned} A' &= \left[\sum_{i=1}^k \alpha_i x(x+1)\dots(x+i-1) \left(\frac{\theta}{\theta+1} \right)^i \right]^2, \\ B' &= \sum_{i=1}^k \alpha_i x(x+1)\dots(x+i) \left(\frac{\theta}{\theta+1} \right)^i, \end{aligned}$$

and

$$C' = \sum_{i=1}^k \alpha_i x(x+1)\dots(x+i-2) \left(\frac{\theta}{\theta+1} \right)^i.$$

Let $t = \frac{\theta}{\theta+1}$ and $a_i = \alpha_i x(x+1)(x+2)\dots(x+i-1)t^i$. Then

$$\frac{f^2(x)}{f(x-1)f(x+1)} = \frac{x+1}{x} \frac{\left(\sum_{i=1}^k a_i \right)^2}{\left(\sum_{i=1}^k a_i(x+i) \right) \left(\sum_{i=1}^k \frac{a_i}{x+i-1} \right)} = \frac{x+1}{x} R.$$

Then by using Cauchy-Schwarz inequality we get $R < 1$. Therefore, for $0 < R \leq \frac{1}{2}$, $\frac{f^2(x)}{f(x-1)f(x+1)} = \frac{x+1}{x} R \leq 1$ and for $\frac{1}{2} < R < 1$, $\frac{f^2(x)}{f(x-1)f(x+1)} = \frac{x+1}{x} R > 1$. Hence for $0 < R \leq \frac{1}{2}$, the distribution is log-convex and for $\frac{1}{2} < R < 1$ the distribution is log-concave. $\square$

4 Estimation of Parameters

In this section we present the estimation of the parameter using two methods *i.e.* Method of moments and Method of maximum likelihood using Expectation Maximization (EM) algorithm. Note that the two methods given below are fairly general for the class $\mathcal{E}$. The method of MLE without using the EM algorithm may be directly available for certain members, but we suggest the EM algorithm as a general way of estimation for the members of $\mathcal{E}$.

Table 2: Index of Dispersion for various members of the class $\mathcal{E}$.

Sl No.	Distribution	Mean	Variance	Index of Dispersion
1	Poisson Lindley	$\frac{\theta+2}{\theta(\theta+1)}$	$\frac{\theta^3+4\theta^2+6\theta+2}{\theta^2(\theta+1)^2}$	$1 + \frac{\theta^2+4\theta+2}{\theta^2+3\theta^2+2\theta}$
2	Poisson Garima	$\frac{\theta+3}{\theta(\theta+2)}$	$\frac{\theta^3+6\theta^2+12\theta+7}{\theta^2(\theta+2)^2}$	$1 + \frac{\theta^2+6\theta+7}{\theta^2+5\theta^2+6\theta}$
3	Poisson Shanker	$\frac{\theta^2+2}{\theta(\theta^2+1)}$	$\frac{\theta^5+\theta^4+3\theta^3+4\theta^2+2\theta+2}{\theta^2(\theta^2+1)^2}$	$1 + \frac{\theta^4+4\theta^2+2}{\theta^2+3\theta^2+2\theta}$
4	Poisson Akash	$\frac{\theta^2+6}{\theta(\theta^2+2)}$	$\frac{\theta^5+\theta^4+8\theta^3+16\theta^2+12\theta+12}{\theta^2(\theta^2+2)^2}$	$1 + \frac{\theta^4+16\theta^2+12}{\theta^2+8\theta^2+12\theta}$
5	Poisson Ishitha	$\frac{\theta^3+6}{\theta(\theta^3+2)}$	$\frac{\theta^7+\theta^6+8\theta^5+16\theta^4+12\theta+12}{\theta^2(\theta^3+2)^2}$	$1 + \frac{\theta^6+16\theta^3+12}{\theta^2+8\theta^4+12\theta}$
6	Poisson Pranav	$\frac{\theta^4+24}{\theta(\theta^4+6)}$	$\frac{\theta^8+\theta^6+30\theta^5+84\theta^4+144\theta+144}{\theta^2(\theta^4+6)^2}$	$1 + \frac{\theta^6+84\theta^4+144}{\theta^2+30\theta^2+144\theta}$
7	Poisson Rama	$\frac{\theta^3+24}{\theta(\theta^3+6)}$	$\frac{\theta^7+\theta^6+30\theta^5+84\theta^4+144\theta+144}{\theta^2(\theta^3+6)^2}$	$1 + \frac{\theta^6+84\theta^4+144}{\theta^2+30\theta^2+144\theta}$
8	Poisson Sujatha	$\frac{\theta^2+2\theta+6}{\theta(\theta^2+\theta+2)}$	$\frac{\theta^5+4\theta^4+14\theta^3+28\theta^2+24\theta+12}{\theta^2(\theta^2+\theta+2)^2}$	$1 + \frac{\theta^4+1\theta^3+18\theta^2+12\theta+12}{\theta^2+3\theta^3+10\theta^2+10\theta^2+12\theta}$
9	Poisson Aradhana	$\frac{\theta^2+4\theta+6}{\theta(\theta^2+2\theta+2)}$	$\frac{\theta^5+7\theta^4+24\theta^3+44\theta^2+36\theta+12}{\theta^2(\theta^2+2\theta+2)^2}$	$1 + \frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta^2+6\theta^3+16\theta^2+20\theta^2+12\theta}$
10	Poisson Amarendra	$\frac{\theta^3+2\theta^2+6\theta+24}{\theta(\theta^3+\theta^2+2\theta+6)}$	$\frac{\theta^7+4\theta^6+14\theta^5+58\theta^4+144\theta^3+156\theta^2+240\theta+144}{\theta^2(\theta^3+\theta^2+2\theta+6)^2}$	$1 + \frac{\theta^6+4\theta^5+18\theta^4+96\theta^3+72\theta^2+96\theta}{\theta^2+3\theta^3+10\theta^2+40\theta^2+48\theta^2+144\theta}$
11	Poisson Akshaya	$\frac{\theta^3+6\theta^2+18\theta+24}{\theta(\theta^3+3\theta^2+6\theta+6)}$	$\frac{\theta^7+10\theta^6+4\theta^5+18\theta^4+408\theta^3+540\theta^2+432\theta+144}{\theta^2(\theta^3+3\theta^2+6\theta+6)^2}$	$1 + \frac{\theta^6+12\theta^5+66\theta^4+192\theta^3+288\theta^2+288\theta+144}{\theta^2+9\theta^3+42\theta^2+216\theta^2+252\theta^2+288\theta}$
12	Poisson Devya	$\frac{\theta^4+2\theta^3+6\theta^2+24\theta+120}{\theta(\theta^4+\theta^3+2\theta^2+6\theta+24)}$	$\frac{\theta^8+4\theta^7+14\theta^6+58\theta^5+288\theta^4+852\theta^3+1008\theta^2+2016\theta+4320\theta+2880}{\theta^2(\theta^4+\theta^3+2\theta^2+6\theta+24)^2}$	$1 + \frac{\theta^6+4\theta^5+18\theta^4+96\theta^3+600\theta^2+480\theta^2+720\theta^2+1440\theta+2880}{\theta^2+3\theta^3+10\theta^2+40\theta^2+192\theta^2+252\theta^2+528\theta^2+1296\theta^2+2880\theta}$
13	Poisson Shambhu	$\frac{\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}{\theta(\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120)}$	$\frac{\theta^{11}+4\theta^{10}+14\theta^9+58\theta^8+288\theta^7+1692\theta^6+5808\theta^5+7056\theta^4+15840\theta^3+44640\theta^2+120960\theta+86400}{\theta^2(\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120)^2}$	$1 + \frac{\theta^{10}+4\theta^9+18\theta^8+96\theta^7+600\theta^6+4320\theta^5+3600\theta^4+5760\theta^3+12960\theta^2+34560\theta+86400}{\theta^{11}+3\theta^{10}+10\theta^9+40\theta^8+192\theta^7+1092\theta^6+1488\theta^5+3456\theta^4+10080\theta^3+31680\theta^2+86400\theta}$

4.1 Method of moments

Let $x_1, x_2, \dots, x_n$ be a random sample from a distribution which is a member of the class $\mathcal{E}$ and let $\bar{x}$ be the sample mean. Then the moment estimator of θ is obtained by solving

$$E(X) = \bar{x}$$

i.e.

$$\sum_{i=1}^n i\alpha_i(\theta) - \theta\bar{x} = 0, \quad (12)$$

where $\alpha_i(\theta)$ is a nonlinear function of θ . So the moment estimator of θ is obtained by solving equation (12) numerically. For example, in Poisson-Sujatha distribution the moment estimator of θ is obtained by solving the equation

$$\theta^3\bar{x} + \theta^2(\bar{x} - 1) + 2\theta(\bar{x} - 1) - 6 = 0,$$

whereas in Poisson-Akshaya distribution it will be the solution of the equation

$$\theta^4\bar{x} + 3(\bar{x} - 1)\theta^3 + 6(\bar{x} - 1)\theta^2 + 6(\bar{x} - 3)\theta - 24 = 0.$$

4.2 Method of maximum likelihood using EM Algorithm

Utilizing the mixture representation, we propose a general method of computing maximum likelihood estimate (MLE) using EM algorithm. Let $(x_1, x_2, \dots, x_n)$ be a random sample from this distribution with corresponding latent variables $Z_1 = (z_{11}, z_{21}, \dots, z_{n1})$, $Z_2 = (z_{12}, z_{22}, \dots, z_{n2})$, ... $Z_k = (z_{1k}, z_{2k}, \dots, z_{nk})$ where

$$z_{ij} = \begin{cases} 1 & \text{if } x_i \text{ comes from } NB\left(j, \frac{\theta}{\theta+1}\right) \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

In this case the augmented likelihood function is given by

$$L(\theta|X, Z_1, Z_2, \dots, Z_k) = \prod_{i=1}^n \prod_{j=1}^k \left\{ \alpha_j f_{NB}(x_i; j, \frac{\theta}{\theta+1}) \right\}^{z_{ij}}$$

so that

$$\log L = \sum_{i=1}^n \sum_{j=1}^k z_{ij} \left\{ \log(\alpha_j) + \log \left[\binom{x_i+j-1}{j-1} \left(\frac{\theta}{\theta+1} \right)^j \left(\frac{1}{\theta+1} \right)^{x_i} \right] \right\}. \quad (14)$$

For maximising $\log L$ we solve the equation

$$\frac{\partial \log L}{\partial \theta} = 0$$

i. e.

$$\sum_{i=1}^n \sum_{j=1}^k \left(\theta(\theta+1) z_{ij} \frac{1}{\alpha_j} \frac{\partial \alpha_j}{\partial \theta} + j z_{ij} - \theta z_{ij} x_i \right) = 0.$$

Note that, the mixing coefficients α_j 's are functions of θ . According to EM algorithm, the MLE of θ , $\tilde{\theta}$ can be found iteratively alternating between two steps i.e, E-step and M-step. In E step we will find the expectation of $Z_{i1}, Z_{i2}, \dots, Z_{ik}$, $i = 1, 2, \dots, n$ under the current estimates of θ and in M-step we maximize $L(\theta|X, Z_1, Z_2, \dots, Z_k)$ with $(Z_{i1}, Z_{i2}, \dots, Z_{ik})$ fixed at their expected value from E-step. The iteration is continued till the estimated value of θ converges and the final iteration gives the MLE of θ , $\tilde{\theta}$. The $(m+1)^{\text{th}}$ iteration of EM algorithm is as follows.

E-step

Let $\theta^{(m)}$ be the estimate of θ in the m^{th} iteration and $\alpha_j^{(m)}$ be the corresponding estimate of α_j . Then

$$Z_{ij}^{(m+1)} = E \left[Z_{ij}^{(m+1)} \right] \quad (15)$$

$$= \frac{\alpha_j^{(m)} \binom{x_i+j-1}{j-1} \left(\frac{\theta^{(m)}}{\theta^{(m)}+1} \right)^j}{\sum_{j=1}^k \alpha_j^{(m)} \binom{x_i+j-1}{j-1} \left(\frac{\theta^{(m)}}{\theta^{(m)}+1} \right)^j}. \quad (16)$$

In this step we will compute $z_{i1}^{(m+1)}, z_{i2}^{(m+1)}, \dots, z_{ik}^{(m+1)}$ for $i = 1, 2, \dots, n$.

M-Step

Let

$$\begin{aligned} Z_1^{(m+1)} &= (z_{11}^{(m+1)}, z_{21}^{(m+1)}, \dots, z_{n1}^{(m+1)}) \\ Z_2^{(m+1)} &= (z_{12}^{(m+1)}, z_{22}^{(m+1)}, \dots, z_{n2}^{(m+1)}) \end{aligned}$$

$$\begin{aligned} & \vdots \\ Z_k^{(m+1)} &= (z_{1k}^{(m+1)}, z_{2k}^{(m+1)}, \dots, z_{nk}^{(m+1)}). \end{aligned}$$

In this step we will find $\tilde{\theta}^{(m+1)}$ by maximizing $L(\theta|X, Z_1^{(m+1)}, Z_2^{(m+1)}, \dots, Z_k^{(m+1)})$ i.e., we solve the equation

$$\sum_{i=1}^n \sum_{j=1}^k \left\{ \theta(\theta+1) \frac{z_{ij}^{(m+1)}}{\alpha_i} \frac{\partial \alpha_i}{\partial \theta} + j z_{ij}^{(m+1)} - \theta z_{ij}^{(m+1)} x_i \right\} = 0,$$

to get the estimate of θ i.e., $\tilde{\theta}^{(m+1)}$. The iteration is continued till the estimated value of θ converges and the final iteration gives the MLE of θ , $\tilde{\theta}$.

As an illustration we give the details of estimation of the parameter θ of Poisson-Akshaya distribution using EM algorithm. Since this distribution is a four component mixture, we have $k = 4$. Let $(x_1, x_2, \dots, x_n)$ be a sample from Poisson-Akshaya Distribution with corresponding latent variables $Z_1 = (z_{11}, z_{21}, \dots, z_{n1})$, $Z_2 = (z_{12}, z_{22}, \dots, z_{n2})$, $Z_3 = (z_{13}, z_{23}, \dots, z_{n3})$ and $Z_4 = (z_{14}, z_{24}, \dots, z_{n4})$. In this case the logarithm of the augmented likelihood function is obtained by substituting for $\alpha_1, \alpha_2, \alpha_3$ and α_4 in equation (14) and is given by

$$\begin{aligned} \log L &= \sum_{i=1}^n \left[4 \log \theta + z_{i2} \log 3 + (z_{i3} + z_{i4}) \log 6 - \log(\theta^3 + 3\theta^2 + 6\theta + 6) \right. \\ &\quad + z_{i2} \log \binom{x_i+1}{1} + z_{i3} \log \binom{x_i+2}{2} + z_{i4} \log \binom{x_i+3}{3} \\ &\quad \left. - (1 + z_{i2} + 2z_{i3} + 3z_{i4} + x_i) \log(\theta+1) \right]. \end{aligned}$$

The $(m+1)^{th}$ iteration of EM algorithm is as follows.

E-step

Let the estimate of θ in the m^{th} iteration be $\tilde{\theta}^{(m)}$. In this step we find for $j = 1, 2, 3, 4$

$$\begin{aligned} Z_{ij}^{(m+1)} &= P[Z_{ij}|X_i, \tilde{\theta}^{(m)}] \\ &= \frac{\alpha_j \binom{x_i+j-1}{j-1} \left(\frac{\tilde{\theta}^{(m)}}{\tilde{\theta}^{(m)}+1} \right)^j}{\sum_{j=1}^4 \alpha_j \binom{x_i+j-1}{j-1} \left(\frac{\tilde{\theta}^{(m)}}{\tilde{\theta}^{(m)}+1} \right)^j}. \end{aligned}$$

M-Step

Let

$$\begin{aligned} Z_1^{(m+1)} &= (z_{11}^{(m+1)}, z_{21}^{(m+1)}, \dots, z_{n1}^{(m+1)}) \\ Z_2^{(m+1)} &= (z_{12}^{(m+1)}, z_{22}^{(m+1)}, \dots, z_{n2}^{(m+1)}) \end{aligned}$$

$$Z_3^{(m+1)} = (z_{13}^{(m+1)}, z_{23}^{(m+1)}, \dots, z_{n3}^{(m+1)})$$

$$Z_4^{(m+1)} = (z_{14}^{(m+1)}, z_{24}^{(m+1)}, \dots, z_{n4}^{(m+1)})$$

In this step we get $\tilde{\theta}^{(m+1)}$ by maximizing $L(\theta|X, Z_1^{(m+1)}, Z_2^{(m+1)}, Z_3^{(m+1)}, Z_4^{(m+1)})$. i.e, by solving

$$\frac{\partial \log L}{\partial \theta} = \sum_{i=1}^n \left\{ \frac{4}{\theta} - \frac{3\theta^2 + 6\theta + 6}{\theta^3 + 3\theta^2 + 6\theta + 6} - \frac{1 + z_{i2}^{(m+1)} + 2z_{i3}^{(m+1)} + 3z_{i4}^{(m+1)} + x_i}{\theta + 1} \right\} = 0. \quad (17)$$

The iteration is continued till the estimated value of θ converges and the final iteration gives the MLE of θ , $\tilde{\theta}$.

5 Modified distributions

In this section, we discuss two modifications of the new mixture distributions, namely zero inflated and zero truncated distributions.

5.1 Zero inflated distributions

Count data are often over-dispersed and one of the main reason for over-dispersion is the presence of excess number of zeros. In order to model these kind of data appropriately, zero inflated distributions are used. In zero inflated distributions zeros are arising from two sources, one from a degenerate distribution and the other from a standard count distribution such as Poisson, Poisson-Lindley, Negative binomial distributions etc. Thus the zero inflated models are actually a mixture of a degenerate distribution, degenerate at zero and a standard count distribution, so that the model is capable of capturing over-dispersion and zero inflation.

In this section we introduce a new class of zero inflated distributions namely $\mathcal{E}^*$. A member of this class is a mixture of a degenerate distribution, degenerate at zero and a count distribution which is a member of the class $\mathcal{E}$. Also we give some important statistical properties of distributions belonging to the new class $\mathcal{E}^*$.

Let Y be a discrete r.v following a distribution belonging to $\mathcal{E}^*$. Then its PMF is of the form

$$f^*(y; \theta) = \begin{cases} \pi + (1 - \pi)f(0) & \text{for } y = 0 \\ (1 - \pi)f(y) & \text{for } y = 1, 2, 3, \dots \end{cases} \quad (18)$$

where $f(\cdot)$ is the PMF of a distribution belonging to the class $\mathcal{E}$. The distribution function of Y is obtained as

$$F_Y(y) = \pi + (1 - \pi)F_X(y) = \pi + (1 - \pi) \sum_{i=0}^n \alpha_i I_{\frac{1}{\theta+1}}(i, x+1)$$

where $F_X(\cdot)$ is the distribution of the standard count distribution which belongs to $\mathcal{E}$ and $I_x(a, b)$ is the regularized incomplete beta function. The mean and variance of the rv Y is given by

$$E(Y) = \mu^* = (1 - \pi)\mu = (1 - \pi) \sum_{i=1}^k \frac{i\alpha_i}{\theta}$$

where μ is the mean of the count distribution which belongs to $\mathcal{E}$.

$$V(Y) = \mu_2^* = (1 - \pi)(\mu_2 + \pi\mu^2) = (1 - \pi) \left[\sum_{i=1}^k \frac{i\alpha_i(i + \theta + 1)}{\theta^2} - (1 - \pi) \left(\sum_{i=1}^k \frac{i\alpha_i}{\theta} \right)^2 \right]$$

where μ_2 and μ are the mean and variance of the count distribution belonging to $\mathcal{E}$. The r^{th} factorial moment of Y is given by the expression

$$\mu_{(r)}^* = (1 - \pi)\mu_{(r)} = (1 - \pi) \frac{1}{\theta^r} \sum_{i=1}^k \alpha_i \frac{\Gamma(r + i)}{\Gamma i}$$

where $\mu_{(r)}$ is the r^{th} factorial moment of the baseline count distribution. Also the MGF of Y is obtained as

$$M_Y(t) = \pi + (1 - \pi)M_X(t) = \pi + (1 - \pi) \sum_{i=1}^n \alpha_i \left(\frac{\theta}{\theta + 1 - e^t} \right)^i$$

where $M_X(t)$ is the MGF of the baseline count distribution.

5.2 Estimation of parameters

For the parametric estimation of the distributions belonging to $\mathcal{E}^*$, we use two methods, i.e. Method of moments and Method of maximum likelihood using EM algorithm.

5.2.1 Method of Moments

Let $y_1, y_2, \dots, y_n$ be a random sample belonging to a distribution which is a member of the class $\mathcal{E}^*$ and let $m_1 = \bar{y}$ and $m_2 = \frac{1}{n} \sum_{i=1}^n y_i * (y_i - 1)$. We have

$$E(Y) = \mu^* = (1 - \pi) \sum_{i=1}^k \frac{i\alpha_i(\theta)}{\theta}$$

and

$$E(Y(Y - 1)) = (1 - \pi) \frac{1}{\theta^2} \sum_{i=1}^k \frac{\alpha_i(\theta)\Gamma(2 + i)}{\Gamma i}.$$

The moment estimators of π and θ are obtained by solving the moment equations

$$E(Y) = m_1, \tag{19}$$

and

$$E(Y(Y-1)) = m_2. \quad (20)$$

By solving equations(19) and(20) we get the equation

$$\frac{\sum_{i=1}^k i^2 \alpha_i(\theta)}{\sum_{i=1}^k i \alpha_i(\theta)} = \frac{m_2}{m_1} \theta - 1. \quad (21)$$

This equation is solved numerically to obtain the moment estimator of θ , i.e, $\hat{\theta}$. From equation(19) we get

$$\pi = 1 - \frac{m_1 \theta}{\sum_{i=1}^k i \alpha_i(\theta)}. \quad (22)$$

By substituting the moment estimator of θ in (22) we get the moment estimator of π as,

$$\hat{\pi} = 1 - \frac{m_1 \hat{\theta}}{\sum_{i=1}^k i \alpha_i(\hat{\theta})}. \quad (23)$$

5.2.2 Method of maximum likelihood using EM algorithm

For the iterative computation of MLE, we can use the EM algorithm. A rv whose distribution belonging to $\mathcal{E}^*$ can be represented by two independent latent variables i.e., a Bernoulli variable Z and a rv X which belongs to $\mathcal{E}$. Let

$$(Y_1, Y_2, \dots, Y_n)$$

be a random sample from a Zero inflated distribution belonging to $\mathcal{E}^*$ with corresponding latent variables $Z = (Z_1, Z_2, \dots, Z_n)$ and $X = (X_1, X_2, \dots, X_n)$. Also we have the mixture representation as

$$Y_i = Z_i V + (1 - Z_i) X_i. \quad (24)$$

Where V is a degenerate rv at zero. So if we could observe Z_i and hence X_i then by relation (24), the augmented likelihood function with augmented data (Z, X) would be

$$L_c(\pi, \theta | Y, Z) = \left[\prod_{i=1}^n \pi^{Z_i} (1 - \pi)^{1-Z_i} \right] \left[\prod_{i=1}^n f(x_i) \right]^{1-Z_i} \quad (25)$$

$$\triangleq L_c(\pi | Y, Z) L_c(\theta | Y, Z). \quad (26)$$

So the maximization of $L_c(\pi, \theta | Y, Z)$ can be done by maximizing $L_c(\pi | Y, Z)$ and $L_c(\theta | Y, Z)$ separately.

E-step

Let $(\pi^{(k)}, \theta^{(k)})$ be the estimates obtained in the k^{th} step. Then

$$Z_i^{(k+1)} = \begin{cases} \frac{\pi^{(k)}}{\pi^{(k)} + (1 - \pi^{(k)})f(0)} & ; Y_i = 0 \\ 0 & ; \text{otherwise} \end{cases}$$

M-step

Let $Z^{(k+1)} = (Z_1^{(k+1)}, Z_2^{(k+1)}, \dots, Z_n^{(k+1)})$. By maximising $L_c(\pi|Y, Z^{(k+1)})$, we will get the MLE of π as

$$\pi^{(k+1)} = \frac{1}{n} \sum_{i=1}^n Z_i^{(k+1)}.$$

The MLE of θ is obtained by solving

$$\sum_{i=1}^n \left(1 - Z_i^{(k+1)}\right) \frac{\sum_{j=1}^k \binom{x+j-1}{j-1} \left(\frac{\theta}{\theta+1}\right)^j \left[\frac{j\alpha_j(\theta)}{\theta} + \alpha_j(\theta)(x+j)(\theta+1) + \alpha_j'(\theta) \right]}{\sum_{j=1}^k \alpha_j(\theta) \binom{x+j-1}{j-1} \left(\frac{\theta}{\theta+1}\right)^j} = 0,$$

where $\alpha_j'(\theta)$ denote the derivative of $\alpha_j(\theta)$ with respect to θ .

5.3 Zero Truncated distributions

Let X be a rv which follows a distribution belonging to the class $\mathcal{E}$, where the PMF $f(x)$ is given by the equation (1). Then the zero truncated version X^* is defined on $\{1, 2, \dots\}$, with PMF

$$f^*(x; \theta) = \begin{cases} \frac{f(x)}{1-f(0)} & ; \text{for } x = 1, 2, 3, \dots \\ 0 & ; \text{otherwise,} \end{cases} \quad (27)$$

where

$$f(x) = \left(\frac{1}{\theta+1}\right)^x \sum_{i=1}^k \alpha_i \binom{x+i-1}{i-1} \left(\frac{\theta}{\theta+1}\right)^i,$$

and

$$f(0) = \sum_{i=1}^k \alpha_i \left(\frac{\theta}{\theta+1}\right)^i.$$

The distribution function of X^* is obtained as

$$\begin{aligned} F^*(x) &= \frac{F_X([x]) - f(0)}{1 - f(0)} \\ &= \frac{\sum_{i=1}^k \alpha_i \left\{ I_{\frac{1}{\theta+1}}(i, [x] + 1) - \left(\frac{\theta}{\theta+1}\right)^i \right\}}{1 - \sum_{i=1}^k \alpha_i \left(\frac{\theta}{\theta+1}\right)^i}. \end{aligned}$$

Also the mean and variance of X^* is given by

$$E(X^*) = \frac{\mu}{1 - f(0)} = \frac{\sum_{i=1}^k i\alpha_i}{\theta \left\{ 1 - \sum_{i=1}^k \alpha_i \left(\frac{\theta}{\theta+1}\right)^i \right\}},$$

$$V(X^*) = \frac{\mu_2}{1 - f(0)} - \frac{f(0)\mu^2}{(1 - f(0))^2},$$

where μ_2 and μ are the mean and variance of the count distribution belonging to $\mathcal{E}$. The MGF of X^* is obtained as

$$M_{X^*}(t) = \frac{M_X(t) - f(0)}{1 - f(0)} = \frac{\sum_{i=1}^k \alpha_i \left(\frac{\theta}{\theta+1}\right)^i \left[\left(\frac{\theta+1}{\theta+1-e^t}\right)^i - 1\right]}{1 - \sum_{i=1}^k \alpha_i \left(\frac{\theta}{\theta+1}\right)^i},$$

where $M_X(t)$ is the MGF of the count distribution belonging to $\mathcal{E}$.

6 Simulation study

We performed a simulation study to assess the finite sample performance of the above mentioned estimators in Sections 4 and 5. First, we have considered two distributions, namely, Poisson-Sujatha distribution and Poisson-Akshaya distribution. Since these distributions can be expressed as mixtures, the simulation from these models are easy. We consider different values for the parameter θ and simulate observations of varying sizes like $n = 100, 200, 500, 1000$. Then the estimates of the parameter θ were calculated using the two methods, Method of moments and Method of maximum likelihood using EM algorithm. The procedure is repeated for $N = 1000$ times and the arithmetic mean of the 1000 estimates is calculated and is considered as the final estimate of the parameter for each method. To assess the variability of estimates we have calculated the Mean squared error (MSE). Let $\hat{\theta}$ and $\tilde{\theta}$ denote respectively the estimators of θ under Method of moment and Maximum likelihood estimation using EM algorithm. $MSE_{\hat{\theta}}$ and $MSE_{\tilde{\theta}}$ denotes the corresponding MSE. The results are displayed in Table 3 and Table 4. The table clearly illustrates that increasing the sample size improves the accuracy of the estimates which results in a steady decline in MSE.

Additionally we have conducted a simulation study for assessing the performances of proposed estimators in zero-inflated models. We have considered Zero Inflated Poisson-Garima (ZIPG) distribution. For different values of π and θ we generated samples of various sizes like $n = 100, 200, 1000$ and the estimates of the parameters using method of moments and method of maximum likelihood were calculated. We have repeated this procedure for $N = 1000$ times and the average of the estimates is considered as the estimate of the parameters. Also we have calculated the root mean squared error (RMSE) to assess the performance of these estimators. Let $\tilde{\Lambda} = (\tilde{\pi}, \tilde{\theta})'$, $\hat{\Lambda} = (\hat{\pi}, \hat{\theta})'$ denote respectively the estimators of $\Lambda = (\pi, \theta)'$ under the method of moments and method of maximum likelihood estimation. The results are displayed in Table 5. From the table we can see that as the sample sizes increases the estimates becomes more closer to the parameters and hence bias and RMSE decrease.

Table 3: Parameter estimates and MSE of Poisson-Sujatha Distribution

θ	n	$\hat{\theta}$	$\text{MSE}_{\hat{\theta}}$	$\tilde{\theta}$	$\text{MSE}_{\tilde{\theta}}$
0.1	100	0.1002	0.000037	0.1002	0.000036
	200	0.1001	0.000018	0.1001	0.000017
	500	0.1001	0.000008	0.10002	0.000007
	1000	0.10009	0.000003	0.10008	0.000003
0.5	100	0.5026	0.0015	0.5019	0.0011
	200	0.5015	0.0007	0.5011	0.0005
	500	0.4998	0.0003	0.4997	0.0002
	1000	0.4999	0.0001	0.4998	0.0001
1	100	1.0071	0.01006	1.0036	0.00582
	200	1.0055	0.0049	1.0033	0.0029
	500	1.0023	0.0021	1.0014	0.0013
	1000	1.0027	0.0009	1.0018	0.0006
4	100	4.1207	0.7712	4.0847	0.4466
	200	4.0752	0.327	4.0538	0.1905
	500	4.0462	0.1318	4.0339	0.0774
	1000	4.0261	0.0594	4.0194	0.0348
10	100	10.8546	16.4162	10.7475	12.8942
	200	10.4782	7.1107	10.4227	5.581
	500	10.1877	2.1683	10.1659	1.7013
	1000	10.0643	1.0479	10.0567	0.8222

7 Applications

In this section, we illustrate the applicability of the methods described for fitting the distributions belonging to $\mathcal{E}$. We consider three different data sets.

Data set-I

A data on the number of European Corn Borer Larvae *Pyrausta nubilalis* (Hbn.), in Field Corn was provided by McGuire et al. [1957]. This data consists of 324 observations. The bar-plot of the data is exhibited in Figure 1. We have fitted the Poisson-Akshaya distribution, Poisson-Sujatha distribution and Poisson-Lindley distribution for the sake of comparison. The results obtained are encapsulated in the Table 6. From the Table, we can infer that Poisson-Akshaya

model has the lowest AIC and BIC among the competing models.

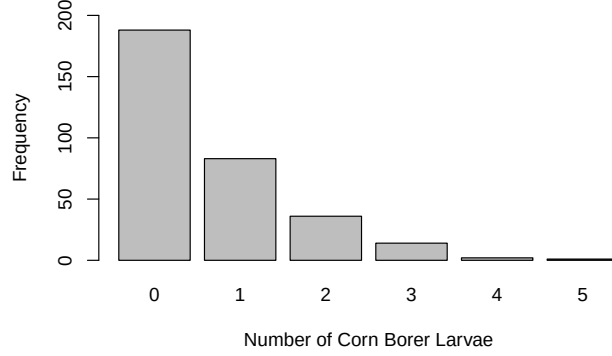


Figure 1: Barplot on European Corn Borer Larvae data.

Dataset-II

A count data on thunderstorm events at Cape Kennedy, Florida for the month of June within eleven years period; from 1957- 1967 is provided by Falls et al. [1971]. The data consists of the observed frequencies of days that experienced X thunderstorms for the month of June during the years 1957 to 1967. The total frequency is 330. For this data also, we have fitted the Poisson Akshaya distribution, Poisson Sujatha distribution and Poisson Lindley distribution for the sake of comparison. The results obtained are encapsulated in Table 7. From the table, we can infer that Poisson Sujatha distribution performs better than the other compared distributions in terms of AIC and BIC.

Data set-III

A data set consisting of weekly number of syphilis cases in the United States from 2007 to 2010 is obtained from **R** package **ZIM** by Yang et al. [2018]. The data consists of 209 observations of which 77.03 % are zeroes. Figure 3 gives the barplot of the data. Since this data consists of excessive zeros, we fitted ZIP, ZIPG, ZIPL and ZIPSh for the data. This data set has mean 0.9760 and variance 6.6773, implying a clear presence of overdispersion. The results obtained are summarized in Table 8. The values of AIC, BIC and χ^2 values are minimum for ZIPG model. Comparing to other models, the DI implied by ZIPG model is closer to empirical DI, indicating the suitability of ZIPG model to this data set. It is obvious that on an average, the ZIPG models fits well the data.

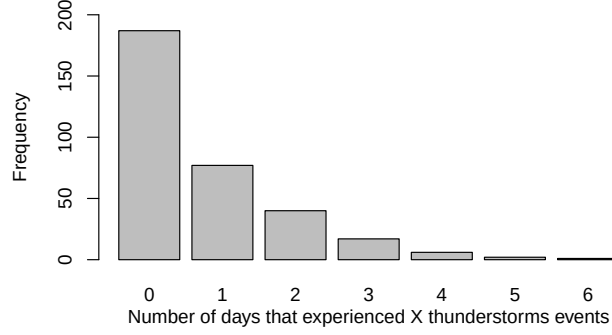


Figure 2: Barplot on number of days that experienced thunder storms.

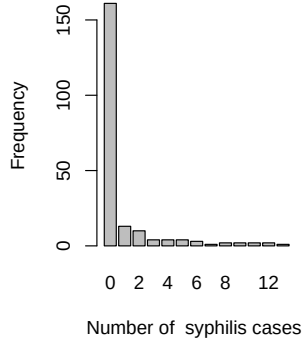


Figure 3: Barplot of weekly number of syphilis cases

8 Conclusion

In this paper, we have reviewed several one parameter Poisson mixture distributions where the intensity parameter of Poisson is allowed to vary according to certain continuous probability law. By introducing a new family of Poisson mixture distributions generated by a finite mixture of gamma random variables, we have constructed a large family of discrete distributions on $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Statistical properties such as distribution function, MGF, factorial moments were then studied. A general EM algorithm based estimation method is developed,

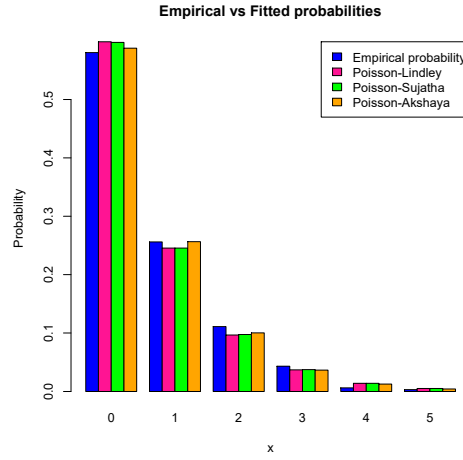


Figure 4: Bar Plot of fitted probabilities of European Corn Borer Larvae dataset

and its uses are illustrated with real and simulated data sets.

The current work can be extended in many directions. A possible direct generalization is to develop a new class of Poisson mixture distributions with two parameter mixing distributions such as two parameter Lindley, two parameter garima etc. Another direction is to develop multivariate Poisson-mixture models. An important application of the mixture models is to apply in the context of regression and count time series analysis. Studies in these directions are ongoing and are expected to be published elsewhere.

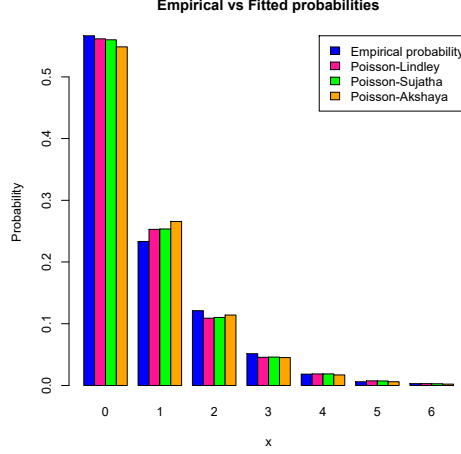


Figure 5: Bar Plot of fitted probabilities of number of days that experienced Thunder storms

Appendix

In this appendix we show the details of various distributions mentioned in the main text.

1. The Poisson-Haq distribution (PHD) is specified by the PMF,

$$f_{PHD}(x; \theta) = \frac{\theta^2}{(1+\theta)^{x+3}} \left[\frac{2(2+\theta)(1+\theta)^2 + \theta(2+x)(1+x)}{2(1+\theta)^2} \right], \quad x = 0, 1, \dots, \theta > 0.$$

which is obtained as mixing with Haq distribution given by

$$f(x) = \left(\frac{\theta}{1+\theta} \right)^2 \left(2 + \theta + \frac{\theta x^2}{2} \right) e^{-\theta x}, \quad x > 0, \theta > 0.$$

This can be written as

$$f(x) = \left(\frac{\theta}{1+\theta} \right)^2 \left(2 + \theta + \frac{\theta x^2}{2} \right) e^{-\theta x}, \quad x > 0, \theta > 0. \quad (28)$$

$$= \frac{(\theta+2)\theta}{(1+\theta)^2} \theta e^{-\theta x} + \frac{1}{(1+\theta)^2} \frac{\theta^3}{2} e^{-\theta x} x^2 \quad (29)$$

which is mixture of $G(\theta, 1)$ and $G(\theta, 3)$ with $\alpha_1 = \frac{(\theta+2)\theta}{(1+\theta)^2}$, $\alpha_2 = 0$, $\alpha_3 = \frac{1}{(1+\theta)^2}$. Hence, the PHD is a member of $\mathcal{E}$ with $k = 3$.

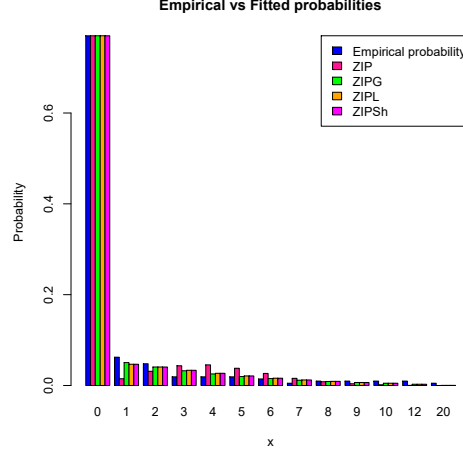


Figure 6: Bar Plot of fitted probabilities of weekly number of syphilis cases

2. The Poisson Agu Eghwerido (PAE) distribution is specified by the PMF

$$f_{PAE}(x; \theta) = \frac{\theta^3}{(\theta^6 + 8)} \frac{(4x^2 + 12x + \theta^4(1 + \theta)^2 + 8)}{(1 + \theta)^{x+3}}; x = 0, 1, 2, 3, \dots, \theta > 0.$$

This is obtained by mixing Poisson with Agu-Eghwerido distribution given by

$$f_{AE}(x) = \frac{\theta^3}{\theta^6 + 8} (\theta^4 + 4x^2) e^{-\theta x}, x > 0, \theta > 0 \quad (30)$$

$$= \frac{\theta^6}{\theta^6 + 8} \theta e^{-\theta x} + \frac{8}{\theta^6 + 8} \frac{\theta^3}{2} e^{-\theta x} x^2, \quad (31)$$

which is a mixture of $G(\theta, 1)$ and $G(\theta, 3)$ with $\alpha_1 = \frac{\theta^6}{\theta^6 + 8}, \alpha_2 = 0, \alpha_3 = 1 - \alpha_1$, implying the PAE distribution is a member of $\mathcal{E}$.

3. The Poisson Ramos-Louzada (PRL) distribution with the PMF

$$f_{PRL}(x; \theta) = \frac{(1 + \frac{1}{\theta})^{-x} (x - 1 + \theta(\theta - 1))}{(\theta - 1)(1 + \theta)^2}; x = 0, 1, 2, 3, \dots \& \theta \geq 2,$$

is derived as the mixing with the Ramos-Louzada (PRL) distribution hav-

ing the PDF,

$$f(x; \theta) = \frac{(\theta^2 - 2\theta + x)}{\theta^2(\theta - 1)} e^{-\left(\frac{x}{\theta}\right)}, \quad \theta \geq 2, x > 0. \quad (32)$$

$$= \frac{\theta^2 - 2\theta}{\theta^2(\theta - 1)} e^{-x/\theta} + \frac{1}{\theta^2(\theta - 1)} x e^{-x/\theta} \quad (33)$$

$$= \frac{\theta - 2}{\theta - 1} \frac{1}{\theta} e^{-x/\theta} + \frac{1}{\theta - 1} \frac{1}{\theta^2} e^{-x/\theta}. \quad (34)$$

With the restriction $\theta \geq 2$, the above PDF is a mixture of $G(1/\theta, 1)$ and $G(1/\theta, 2)$ where $\alpha_1 = \frac{\theta-2}{\theta-1}, \alpha_2 = 1 - \alpha_1$, ensuring the membership of PRL distribution in $\mathcal{E}$.

4. The Poisson Komal distribution (PKD) is specified by the PMF

$$f_{PKD}(x; \theta) = \frac{\theta^2}{(\theta^2 + \theta + 1)} \left(\frac{1}{(\theta + 1)^x} + \frac{x + 1}{(\theta + 1)^{x+2}} \right), x = 0, 1, 2, \dots, \theta > 0.$$

This is a mixing with the Komal distribution with PDF

$$f(x; \theta) = \frac{\theta^2}{\theta^2 + \theta + 1} (1 + \theta + x) e^{-\theta x} \quad x > 0 \quad \theta > 0. \quad (35)$$

$$= \frac{\theta^2 + \theta}{\theta^2 + \theta + 1} \theta e^{-\theta x} + \frac{1}{\theta^2 + \theta + 1} \theta^2 e^{-\theta x}, \quad (36)$$

a mixture of $G(\theta, 1)$ and $G(\theta, 2)$ with mixing coefficients $\alpha_1 = \frac{\theta^2 + \theta}{\theta^2 + \theta + 1}, \alpha_2 = 1 - \alpha_1$.

5. The PMF of Poisson Xgamma distribution (PXG) is specified by the

$$f_{PXG}(x; \theta) = \frac{\theta^2}{2(1 + \theta)^{x+4}} (2(1 + \theta)^2 + \theta(x + 2)(x + 1)), x = 0, 1, 2, \dots, \theta > 0.$$

This is obtained by mixing with the Xgamma distribution with PDF

$$f(x; \theta) = \frac{\theta^2}{\theta + 1} \left(1 + \frac{\theta}{2} x^2 \right) e^{-\theta x} \quad x > 0 \quad \theta > 0. \quad (37)$$

$$= \frac{\theta}{\theta + 1} \theta e^{-\theta x} + \frac{1}{\theta + 1} \frac{\theta^3}{\Gamma 3} x^2 e^{-\theta x}, \quad (38)$$

a mixture of $G(\theta, 1)$ and $G(\theta, 3)$ with mixing coefficients $\alpha_1 = \frac{\theta}{\theta + 1}, \alpha_2 = 0, \alpha_3 = 1 - \alpha_1$.

6. The Poisson Xrani (PXR) distribution with the PMF

$$f_{PXR}(x; \theta) = \frac{\theta^5}{(\theta^5 + 24)^2(\theta + 1)^{x+5}} ((\theta^6 + 48\theta)(\theta + 1)^4 + 24(x + 1)(x + 2)(x + 3)(x + 5)), \quad x = 0, 1, \dots, \theta > 0$$

is derived as the mixing with the Xrani distribution having the PDF,

$$f(x; \theta) = \frac{\theta^5(\theta^6 + 48\theta + 24x^4)}{(\theta^5 + 24)^2} e^{-\theta x}, \quad \theta > 0, x > 0. \quad (39)$$

$$= \frac{\theta^{10} + 48\theta^5}{(\theta^5 + 24)^2} \theta e^{-\theta x} + \frac{576}{(\theta^5 + 24)^2} \frac{\theta^5}{\Gamma 5} x^4 e^{-\theta x} \quad (40)$$

This is a mixture of $G(\theta, 1)$ and $G(\theta, 5)$ where $\alpha_1 = \frac{\theta^{10} + 48\theta^5}{(\theta^5 + 24)^2}$, $\alpha_5 = 1 - \alpha_1$, ensuring membership of the PXR distribution in $\mathcal{E}$.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Table 4: Parameter estimates and MSE of Poisson-Akshaya Distribution

θ	n	$\hat{\theta}$	$\text{MSE}_{\hat{\theta}}$	$\tilde{\theta}$	$\text{MSE}_{\tilde{\theta}}$
0.1	100	0.10005	0.000027	0.100043	0.00003
	200	0.100097	0.000014	0.100092	0.00001
	500	0.100098	0.000006	0.100095	0.000006
	1000	0.100049	0.000003	0.10005	0.000003
0.5	100	0.501837	0.001158	0.50135	0.00089
	200	0.500975	0.00057	0.50072	0.00044
	500	0.500167	0.00024	0.50008	0.000182
	1000	0.500476	0.000121	0.50039	0.000093
1	100	1.00508	0.00759	1.0027	0.00468
	200	1.00262	0.00354	1.0014	0.0022
	500	1.00235	0.001494	1.0016	0.00092
	1000	1.0024	0.00072	1.0018	0.00045
5	100	5.1895	1.1736	5.1211	0.5911
	200	5.1153	0.5079	5.0757	0.2592
	500	5.0659	0.2075	5.0447	0.1065
	1000	5.0198	0.0982	5.0128	0.0508
10	100	11.0431	15.5202	10.7985	9.5589
	200	10.4595	5.511	10.3516	3.4514
	500	10.2173	1.9111	10.168	1.2104
	1000	10.1197	0.9617	10.0924	0.61129

Table 5: Parameter estimates and their RMSE of ZIPG model

			MOM		RMSE		MLE		RMSE	
n	θ	π	$\tilde{\theta}$	$\tilde{\pi}$	$\tilde{\theta}$	$\tilde{\pi}$	$\hat{\theta}$	$\hat{\pi}$	$\hat{\theta}$	$\hat{\pi}$
100	0.2	0.2	0.2369	0.0243	0.0374	0.1758	0.2241	0.2309	0.0241	0.0308
200			0.1721	0.2775	0.0279	0.0775	0.2139	0.1566	0.0138	0.0436
1000			0.2004	0.2532	0.0004	0.0532	0.2106	0.1807	0.0105	0.0192
100	0.2	0.5	0.2096	0.4198	0.0096	0.0802	0.2249	0.4116	0.0249	0.0883
200			0.1707	0.5174	0.0293	0.0173	0.2289	0.4669	0.0289	0.0332
1000			0.1974	0.5344	0.0026	0.0346	0.2131	0.5010	0.0130	0.0010
100	0.2	0.7	0.2049	0.6575	0.0049	0.0425	0.2569	0.6772	0.0566	0.0228
200			0.1767	0.6290	0.0232	0.0707	0.2144	0.7189	0.0141	0.0190
1000			0.2044	0.7155	0.0044	0.0155	0.2171	0.6982	0.0170	0.0018
100	0.4	0.7	0.4431	0.6777	0.1212	0.0768	0.4185	0.6963	0.0883	0.0548
200			0.4198	0.6889	0.0837	0.0548	0.4085	0.6978	0.0648	0.0387
1000			0.4046	0.6979	0.0346	0.0245	0.4020	0.7000	0.0265	0.0173
100	0.8	0.2	1.0900	0.0711	0.2972	0.1288	0.8915	0.2519	0.0917	0.0520
200			0.7042	0.3856	0.0959	0.1857	0.8126	0.1244	0.0126	0.0755
1000			0.8217	0.1877	0.0217	0.0122	0.8147	0.2361	0.0148	0.0361
100	0.8	0.5	0.9431	0.4228	0.1432	0.0775	0.9374	0.5176	0.1374	0.0173
200			0.7586	0.5963	0.0412	0.0964	0.8700	0.3805	0.0700	0.1196
1000			0.8926	0.4621	0.0927	0.0374	0.7803	0.5094	0.0197	0.0094
100	0.8	0.7	1.0030	0.6309	0.2040	0.0693	1.0750	0.7159	0.2750	0.0173
200			0.6375	0.7758	0.1625	0.0755	0.8778	0.5961	0.0781	0.1039
1000			0.9161	0.6739	0.1162	0.0261	0.8206	0.7160	0.0207	0.0158

Table 6: Fitted distributions on Dataset-I, along with their expected frequencies, estimates of parameter, Standard error of the estimate and information statistics.

Distribution of number of European Corn Borer				
Larvae <i>Pyrausta nubilalis</i> , in Field Corn				
x	f	Fitted Distributions with expected frequencies.		
		Poisson-Lindley	Poisson-Sujatha	Poisson-Akshaya
0	188	194	193	190
1	83	80	80	83
2	36	31	32	34
3	14	12	12	12
4	2	5	5	4
5	1	2	2	1
Total	324	324	324	324
Estimates	θ	2.0433	2.4717	3.0914
Standard Error		0.1513	0.0466	0.1795
AIC		716.09	715.84	714.28
BIC		719.88	719.62	718.06
χ^2		3.7236	3.3611	1.272
d.f		3	3	3
p-value		0.2929	0.3392	0.7357
DI	1.3083	1.5707	1.559	1.4634

Table 7: Fitted distributions on Dataset-II, along with their expected frequencies, estimates of parameter, Standard error of the estimate and information statistics.

Distribution on number of days that experienced Thunder storms				
x	f	Fitted Distributions with expected frequencies.		
		Poisson-Lindley	Poisson-Sujatha	Poisson-Akshaya
0	187	186	185	181
1	77	83	84	87
2	40	37	37	38
3	17	15	15	15
4	6	6	6	6
5	2	2	2	2
6	1	1	1	1
Total	330	330	330	330
Estimates	θ	1.8043	2.2299	2.8027
Standard Error		0.1257	0.0468	0.1526
AIC		790.88	790.63	791.13
BIC		794.68	794.43	794.93
χ^2		0.9490	1.1149	1.7203
d.f		3	3	3
p-value		0.8136	0.7735	.6324
DI	1.5597	1.6480	1.6231	1.51125

Table 8: Fitted distributions on Dataset-III, along with their expected frequencies, estimates of parameters and information statistics.

Distribution of weekly number of syphilis cases					
		Fitted Distributions with expected frequencies			
x	f	ZIP	ZIPG	ZIPL	ZIPSh
0	161	161	161	161	155
1	13	3	11	10	12
2	10	7	9	9	11
3	4	9	7	7	8
4	4	9	5	6	7
5	4	8	4	5	5
6	3	6	3	3	4
7	1	3	2	3	3
8	2	2	2	2	2
9	2	1	2	1	1
10	2	0	2	1	1
12	2	0	1	1	0
20	1	0	0	0	0
Total	209	209	209	209	209
Estimates	p_1	0.7667	0.7095	0.7211	0.6785
	θ	4.19	0.4194	0.4802	0.5664
AIC		512.5916	452.5824	453.8479	456.0795
BIC		519.2763	459.2671	460.5325	462.7641
χ^2		46.34127	2.0716	7.0302	4.2559
d.f		5	4 32	5	4
p-value		< .001	0.7226	0.2184	0.3725
DI	6.8410	4.209	6.1837	5.8709	5.0556

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