Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Rigorous Weighted Composition Operator Theory for Biparametric Transformations: A Foundational Framework for Deformation Geometry, Localization, and Transport Analysis

Eno John

eno.john@akwaibompoly.edu.ng

Akwa Ibom State Polytechnic, Nigeria

Research Article

Keywords: Biparametric deformation, weighted composition operators, deformation geometry, asymptotic localization, Sobolev pullback theory, geometric transport, transformed differential equations.

Posted Date: June 2nd, 2026

DOI: https://doi.org/10.21203/rs.3.rs-9876731/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: The authors declare no competing interests.

Rigorous Weighted Composition Operator Theory for Biparametric Transformations: A Foundational Framework for Deformation Geometry, Localization, and Transport Analysis

Eno John
Mathematics Unit, Department of General Studies
Akwa Ibom State Polytechnic, Nigeria
eno.john@akwaibompoly.edu.ng

Abstract

We develop an operator-theoretic framework for a biparametric family of geometric deformations generated implicitly by $R^p + qR = e^v$ where $p \in (0, 1]$ and $q \geq 0$. The associated coordinate transformation induces a nonlinear transport geometry between the finite interval $(0, 1)$ and the real line $\mathbb{R}$, generating a class of weighted pullback and composition operators acting on Lebesgue and Sobolev spaces.

We prove existence, uniqueness, smooth dependence, global invertibility, and global diffeomorphism properties of the induced coordinate map together with positivity and asymptotic structure of its Jacobian. The resulting pullback operator admits a weighted composition deformation of the form $T_{p,q} = M_w U_{\phi_{p,q}}$. Boundedness, invertibility, strong continuity, measure transport identities, and Sobolev pullback properties are rigorously established within the induced functional framework.

A central result of the paper is the demonstration of genuine biparametricity: the parameter p governs nonlinear left-tail localization geometry while q independently governs right-tail regularization and asymptotic transport. This asymptotic irreducibility produces independently tunable localization scaling unavailable in classical single-parameter transformations.

The framework developed here provides the analytical foundation for a broader deformation-localization program encompassing transformed differential equations, spectral transport, deformed functional spaces, and operator-induced geometric analysis. Applications to transformed quadrature, weakly singular integral equations, and singular perturbation problems are discussed as foundational examples of the theory.

Keywords: Biparametric deformation; weighted composition operators; deformation geometry; asymptotic localization; Sobolev pullback theory; geometric transport; transformed differential equations.
MSC 2020: Primary: 47B33, 46E35; Secondary: 47B38, 35A23, 45B05, 58J05, 65R20

1 Introduction

Coordinate transformations occupy a central position in modern analysis, numerical approximation, spectral methods, differential equations, and integral operator theory. Transformations between finite and infinite domains are particularly important because they map singularities, boundary layers, and localized phenomena into analytically manageable environments.

Classical transformation frameworks arising in Sinc approximation, quadrature theory, and spectral analysis typically rely on single-parameter coordinate maps [12, 9]. While highly successful, the geometric flexibility of such constructions remains limited. A single controlling parameter cannot independently regulate competing asymptotic mechanisms at opposite boundaries of a domain.

The present work develops a different viewpoint. Rather than treating coordinate mappings merely as auxiliary numerical tools, we interpret them as generators of transported analytical environments. In this framework, a transformation is treated as an operator-theoretic object inducing measure transport, Sobolev pullback structures, and spectral redistribution. The analytic development of composition structures on structural function spaces heavily interfaces with the foundational geometric frameworks laid out by Cowen and MacCluer [5], Shapiro [11], and Bourdon [**?**], while extending directly into the modern Sobolev domain mappings analyzed by Hencl and Koskela [7] and Hajłasz [6].

The framework introduced in this paper is generated implicitly through the biparametric relation:

$$R^p + qR = e^v \tag{1.1}$$

where the parameters satisfy $p \in (0, 1]$ and $q \geq 0$.

The associated coordinate deformation is defined through the proper compactification map:

$$\phi_{p,q}(v) = \frac{R(v)}{1 + R(v)} \tag{1.2}$$

which transports the real line $\mathbb{R}$ onto the finite interval $(0, 1)$. Unlike classical single-parameter transformations, the present structure possesses two analytically independent deformation mechanisms. The parameter p governs nonlinear asymptotic geometry and left-tail localization, while q independently governs right-tail regularization and transport scaling.

A principal objective of this work is to establish the induced deformation as a rigorous object in operator theory. Specifically, the deformation induces operators of the form:

$$T_{p,q}f(v) = w(v)f(\phi_{p,q}(v)) \tag{1.3}$$

where the weight $w(v)$ arises from the transported Jacobian structure of the deformation. This leads naturally to weighted composition operators, measure transport identities, derivative reconstruction mechanisms, and Sobolev pullback frameworks.

1.1 Structural Distinction: Rigor vs. Interpretive Vision

To clarify the scope of this paper, we draw a deliberate distinction between the rigorously proven operator-theoretic structures and the broader interpretive geometric philosophy they motivate.

- **Rigorously Established:** The existence, smoothness, and asymptotic limits of the diffeomorphism; the boundedness and invertibility of the induced weighted composition operators on Lebesgue and Sobolev spaces; the measure transport and energy identities; and the analytic independence of the parameters p and q.
- **Interpretive Framework:** Concepts introduced in later sections such as "localization corridors," "geometric stretching," and "spectral redistribution mechanisms" are interpretative geometric descriptors. They provide an intuitive, physical lens through which to understand the mathematically rigorous limits and density distributions established by the theorems.

1.2 Summary of Main Contributions

The main contributions of this paper are summarized as follows:

1. We establish existence, uniqueness, smooth dependence, and global invertibility of the biparametric deformation.
2. We prove that the induced coordinate transformation defines a global C^∞-diffeomorphism between $\mathbb{R}$ and $(0, 1)$.
3. We construct weighted pullback and weighted composition operators induced by the transformation and establish their boundedness on L^r and $W^{1,r}$ spaces.
4. We rigorously establish the genuine biparametricity of the framework through asymptotic irreducibility, proving that the independent parameters govern decoupled asymptotic regimes.

2 Geometry of the Biparametric Deformation

In this section, we develop the analytical and geometric structure of the biparametric deformation framework generated by the implicit relation $R^p + qR = e^v$ where $p \in (0, 1]$ and $q \geq 0$.

Throughout the paper, we define the coordinate deformation by $\phi_{p,q}(v) = \frac{R(v)}{1+R(v)}$ where $R(v)$ denotes the unique positive solution of the generating relation. The associated Jacobian is denoted by $J_{p,q}(v) = \phi'_{p,q}(v)$.

2.1 Algebraic Core Structure

Define the function $F(R) = R^p + qR$ for $R > 0$.

Lemma 2.1 (Continuity of the Core Function)**.** *The function $F : (0, \infty) \to (0, \infty)$ is continuous.*

Proof. Since $p \in (0, 1]$, the mapping $R \mapsto R^p$ is a well-defined power function on the open positive real half-line. For any positive base $R_0 > 0$, the limit evaluates via standard exponent behavior as $\lim_{R\to R_0} R^p = R_0^p$, ensuring continuity of the nonlinear term. The linear term mapping $R \mapsto qR$ is an algebraic multiplication operation, which is uniformly continuous on $\mathbb{R}$. Because the sum of two continuous mappings remains continuous on their shared intersection domain, the function $F(R) = R^p + qR$ is continuous on $(0, \infty)$. □

Lemma 2.2 (Strict Monotonicity)**.** *The function $F(R)$ is strictly increasing on $(0, \infty)$.*

Proof. Since $F(R)$ is differentiable on the open interval $(0, \infty)$, we take its derivative directly using the power rule:

$$F'(R) = \frac{d}{dR}(R^p + qR) = pR^{p-1} + q \tag{2.1}$$

We analyze the constraints on each variable sequentially. Given $p \in (0, 1]$, it follows that $p > 0$. For any $R \in (0, \infty)$, the term $R^{p-1} = e^{(p-1)\log R}$ is strictly positive because the range of the real exponential function maps exclusively into $(0, \infty)$. Since the product of two strictly positive real numbers is positive, we have $pR^{p-1} > 0$. Given that the regularization parameter satisfies $q \geq 0$, the sum of the positive nonlinear derivative component and the non-negative linear component yields:

$$F'(R) = pR^{p-1} + q > 0, \quad \forall R > 0 \tag{2.2}$$

Because the first derivative is strictly positive everywhere on its domain, the core function $F(R)$ is strictly increasing on $(0, \infty)$. □

Lemma 2.3 (Range Structure)**.** *The function F satisfies:*

$$\lim_{R \to 0^+} F(R) = 0, \qquad \lim_{R \to \infty} F(R) = \infty \tag{2.3}$$

Proof. We evaluate the endpoint behaviors independently. For the left boundary limit as $R \to 0^+$, since $p > 0$, the power function satisfies $\lim_{R \to 0^+} R^p = 0$. For the linear term, $\lim_{R \to 0^+} qR = q(0) = 0$. Summing these limits yields $\lim_{R \to 0^+} F(R) = 0 + 0 = 0$.

For the right boundary limit as $R \to \infty$, we separate the evaluation into two cases based on the parameter q:

1. Let $q = 0$. The function simplifies to $F(R) = R^p$. Since $p > 0$, the value increases unboundedly, satisfying $\lim_{R \to \infty} R^p = \infty$.

2. Let $q > 0$. The linear expression $qR \to \infty$ as $R \to \infty$. Since $R^p > 0$ on $(0, \infty)$, the full function is bounded below by the linear component ($F(R) > qR$). By the comparison theorem for infinite limits, $\lim_{R \to \infty} F(R) = \infty$.

Thus, the range of the strictly increasing continuous function F is exactly $(0, \infty)$. □

2.2 Existence and Smoothness of the Deformation

Theorem 2.4 (Existence and Uniqueness)**.** *For every $v \in \mathbb{R}$, there exists a unique positive solution $R(v)$ satisfying $R(v)^p + qR(v) = e^v$.*

Proof. Let $v \in \mathbb{R}$ be fixed. The right-hand side of the generating relation is the standard real exponential function e^v, which satisfies $e^v > 0$ for all real exponents, implying $e^v \in (0, \infty)$. From Lemmas 2.1, 2.2, and 2.3, the core function $F(R)$ is continuous and strictly monotonic, mapping the domain $(0, \infty)$ bijectively onto the range $(0, \infty)$. By the Inverse Function Theorem for strictly monotonic continuous mappings, there exists a unique, well-defined inverse function $F^{-1} : (0, \infty) \to (0, \infty)$. Applying this mapping to the exponential value yields:

$$R(v) = F^{-1}(e^v) \tag{2.4}$$

This establishes the simultaneous existence and global uniqueness of a positive solution $R(v)$ for every coordinate element $v \in \mathbb{R}$. □

Theorem 2.5 (Smooth Dependence)**.** *The mapping $R : \mathbb{R} \to (0, \infty)$ varies as a smooth function belonging to $C^\infty(\mathbb{R})$.*

Proof. Define the structural generator function $G : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by:

$$G(R, v) = F(R) - e^v = R^p + qR - e^v \tag{2.5}$$

We evaluate the partial derivatives of G with respect to both coordinates. The derivative with respect to the state variable R yields $\frac{\partial G}{\partial R} = F'(R) = pR^{p-1} + q$, which is strictly positive for all $R > 0$ by Lemma 2.2. The partial derivative with respect to the coordinate variable v yields $\frac{\partial G}{\partial v} = -e^v$. Because both $F(R)$ and e^v are infinitely differentiable functions on their respective domains, $G(R, v)$ is a smooth mapping belonging to $C^\infty((0, \infty) \times \mathbb{R})$.

Since $\frac{\partial G}{\partial R} \neq 0$ everywhere on the solution manifold, the Implicit Function Theorem guarantees that the uniquely defined solution function $R(v)$ inherits the full regularity of the generator. Thus, $R(v) \in C^\infty(\mathbb{R})$, and its first derivative can be calculated via implicit differentiation:

$$R'(v) = -\frac{\partial G/\partial v}{\partial G/\partial R} = \frac{e^v}{pR^{p-1} + q} \tag{2.6}$$

This establishes smooth dependence across the domain. □

2.3 Global Diffeomorphism Structure

Lemma 2.6 (Monotonicity of $R(v)$)**.** *The function $R(v)$ is strictly increasing.*

Proof. From Eq. (2.6), the derivative of the solution function evaluates to $R'(v) = \frac{e^v}{pR^{p-1}+q}$. We evaluate the sign of the numerator and denominator independently. The numerator is the standard real exponential function, which satisfies $e^v > 0$ for all $v \in \mathbb{R}$. Theorem 2.4 establishes that $R(v) > 0$ for all $v \in \mathbb{R}$, which implies the denominator satisfies $F'(R(v)) = pR(v)^{p-1} + q > 0$ by Lemma 2.2. Since the quotient of two strictly positive real numbers is positive, we have:

$$R'(v) > 0, \quad \forall v \in \mathbb{R} \tag{2.7}$$

Because its first derivative is strictly positive everywhere, $R(v)$ is a strictly increasing function on $\mathbb{R}$. □

Theorem 2.7 (Bijectivity and Limits)**.** *The mapping $\phi_{p,q} : \mathbb{R} \to (0, 1)$ is a global C^∞-diffeomorphism satisfying $\lim_{v\to-\infty} \phi_{p,q}(v) = 0$ and $\lim_{v\to\infty} \phi_{p,q}(v) = 1$.*

Proof. The coordinate mapping is defined by $\phi_{p,q}(v) = \frac{R(v)}{1+R(v)}$. We compute its derivative directly via the quotient rule:

$$\phi'_{p,q}(v) = \frac{R'(v)(1 + R(v)) - R(v)R'(v)}{(1 + R(v))^2} = \frac{R'(v)}{(1 + R(v))^2} \tag{2.8}$$

Since Lemma 2.6 establishes $R'(v) > 0$, and the squared denominator satisfies $(1 + R(v))^2 > 0$, it follows that $\phi'_{p,q}(v) > 0$ for all $v \in \mathbb{R}$. The strict positivity of the derivative ensures that $\phi_{p,q}$ is strictly increasing and injective.

We evaluate the boundary limits to determine the exact image of the map. As $v \to -\infty$, the generating relation forces $R^p + qR = e^v \to 0$. Since $R(v) > 0$, this requires $\lim_{v\to-\infty} R(v) = 0$. Substituting this limit into the coordinate transformation yields:

$$\lim_{\mathrm{v}\to-\infty} \phi_{\mathrm{p,q}}(\mathrm{v}) = \frac{0}{1 + 0} = 0 \tag{2.9}$$

As $v \to \infty$, the relation forces $R^p + qR = e^v \to \infty$, requiring $\lim_{v\to\infty} R(v) = \infty$. We evaluate the coordinate limit by factoring $R(v)$ from the expression:

$$\lim_{v\to\infty} \phi_{p,q}(v) = \lim_{R\to\infty} \frac{R}{1 + R} = \lim_{R\to\infty} \frac{1}{\frac{1}{R} + 1} = \frac{1}{0 + 1} = 1 \tag{2.10}$$

By the Intermediate Value Theorem, the image of the strictly increasing smooth mapping is exactly the open interval $(0, 1)$, establishing bijectivity.

To prove it is a global diffeomorphism, we solve for the explicit inverse transformation. Let $x = \frac{R}{1+R}$. Multiplying out yields $x + xR = R$, which isolates the state variable as $R = \frac{x}{1-x}$. Substituting this expression directly into the inverse core relation from Theorem 2.6 yields:

$$\phi_{p,q}^{-1}(x) = \log\left[\left(\frac{x}{1-x}\right)^p + q\left(\frac{x}{1-x}\right)\right] \tag{2.11}$$

Since $x \in (0,1)$, the term $\frac{x}{1-x} > 0$, ensuring the argument of the logarithm is smooth and well-defined. Thus, both the forward and inverse mappings are smooth, making $\phi_{p,q}$ a global C^∞-diffeomorphism. □

2.4 Jacobian Geometry and Asymptotics

Theorem 2.8 (Jacobian Formula)**.** *The Jacobian of the deformation is given by:*

$$J_{p,q}(v) = \frac{e^v}{(1+R(v))^2(pR(v)^{p-1}+q)} \tag{2.12}$$

Proof. Direct substitution of $R'(v)$ into $\phi'_{p,q}(v) = \frac{R'(v)}{(1+R(v))^2}$. □

Theorem 2.9 (Left-Tail Asymptotics)**.** *As $v \to -\infty$, the positive core solution scales as $R(v) \sim e^{v/p}$, and the Jacobian scales as $J_{p,q}(v) \sim \frac{1}{p}e^{v/p}$ [3].*

Proof. As $v \to -\infty$, Theorem 2.7 establishes that $R(v) \to 0$. We analyze the dominant balance within the generating relation $R^p + qR = e^v$ as $R \to 0$. Since $p \in (0,1]$, the exponent satisfies $p \leq 1$. We evaluate the relative ratio of the linear component to the nonlinear power component:

$$\lim_{R\to 0^+} \frac{qR}{R^p} = \lim_{R\to 0^+} qR^{1-p} = 0 \tag{2.13}$$

This demonstrates that the linear term qR is subdominant to R^p near the origin ($qR = o(R^p)$). Omitting the subdominant term yields the asymptotic balance expression $R^p \sim e^v$. Taking the p-th root of both sides yields the state scaling relation $R(v) \sim e^{v/p}$ as $v \to -\infty$.

We substitute this asymptotic scaling into the explicit Jacobian formula from Theorem 2.8. As $R \to 0$, the denominator term satisfies $(1+R)^2 \to 1$. The derivative expression simplifies to:

$$J_{p,q}(v) \sim \frac{e^v}{1 \cdot pR^{p-1}} = \frac{1}{p}e^v R^{1-p} \tag{2.14}$$

Substituting the state scaling relation $R \sim e^{v/p}$ into the expression yields:

$$J_{p,q}(v) \sim \frac{1}{p}e^v\left(e^{v/p}\right)^{1-p} = \frac{1}{p}e^v e^{\frac{v}{p}-v} = \frac{1}{p}e^{v/p} \tag{2.15}$$

This completes the left-tail evaluation. □

Theorem 2.10 (Right-Tail Asymptotics)**.** *Assume $q > 0$. As $v \to \infty$, the positive core solution scales as $R(v) \sim \frac{1}{q}e^v$, and the Jacobian behaves asymptotically as $J_{p,q}(v) \sim qe^{-v}$ [3].*

Proof. As $v \to \infty$, Theorem 2.7 establishes that $R(v) \to \infty$. We analyze the dominant balance within the generating relation $R^p + qR = e^v$ as $R \to \infty$. Since $p \in (0,1]$, the exponent satisfies $p \leq 1$. We evaluate the relative ratio of the nonlinear power component to the linear component:

$$\lim_{\mathrm{R}\to\infty} \frac{\mathrm{R^p}}{\mathrm{qR}} = \lim_{\mathrm{R}\to\infty} \frac{1}{\mathrm{q}}\mathrm{R^{p-1}} = 0 \tag{2.16}$$

This demonstrates that the nonlinear term R^p is subdominant to qR as $R \to \infty$ ($R^p = o(qR)$). Omitting the subdominant term yields the asymptotic balance expression $qR \sim e^v$, which isolates the state variable scaling as $R(v) \sim \frac{1}{q}e^v$.

We substitute this scaling into the Jacobian formula from Theorem 2.8. As $R \to \infty$, the linear term dominates the derivative denominator ($pR^{p-1} + q \to q$), and the spatial factor satisfies $(1+R)^2 \sim R^2$. The expression simplifies to:

$$J_{p,q}(v) \sim \frac{e^v}{R^2 \cdot q} = \frac{1}{q}\frac{e^v}{R^2} \tag{2.17}$$

Substituting the state scaling relation $R \sim \frac{1}{q}e^v$ into the denominator yields:

$$J_{p,q}(v) \sim \frac{1}{q}\frac{e^v}{\left(\frac{1}{q}e^v\right)^2} = \frac{1}{q}\frac{e^v}{\frac{1}{q^2}e^{2v}} = qe^{v-2v} = qe^{-v} \tag{2.18}$$

This completes the right-tail evaluation. □

3 Weighted Composition Operators and Lebesgue Transport Structures

Having established the global geometric structure of the biparametric deformation, we now develop the functional framework induced by the transformation. The principal objective of this section is to demonstrate that the diffeomorphism $\phi_{p,q}$ generates a well-defined class of weighted pullback and composition operators acting on Lebesgue spaces, elevating the transformation from a numeric change-of-variables to a transported analytical environment.

Throughout this section, we assume $1 \le r < \infty$. Let $\phi_{p,q} : \mathbb{R} \to (0,1)$ denote the global C^∞-diffeomorphism established in Theorem 2.7, with strictly positive Jacobian density $J_{p,q}(v) = \phi'_{p,q}(v)$.

3.1 Pullback Operators Induced by the Deformation

The coordinate mapping naturally induces an unweighted pullback operator transporting functions from the bounded physical interval $(0,1)$ to the unbounded real line $\mathbb{R}$.

Definition 3.1 (Pullback Operator)**.** Let $f : (0,1) \to \mathbb{C}$ be a measurable function. The pullback operator $U_{\phi_{p,q}}$ induced by the deformation is defined by:

$$U_{\phi_{p,q}} f(v) = f(\phi_{p,q}(v)), \quad \forall v \in \mathbb{R} \tag{3.1}$$

Theorem 3.2 (Measurability and Continuity Preservation)**.** *Let $f : (0,1) \to \mathbb{C}$ be a measurable function. Then its pullback $U_{\phi_{p,q}} f$ is measurable on $\mathbb{R}$. Furthermore, if $f \in C(0,1)$, then $U_{\phi_{p,q}} f \in C(\mathbb{R})$.*

Proof. By Theorem 2.7, $\phi_{p,q} : \mathbb{R} \to (0,1)$ is a continuous mapping. Since the composition of a measurable function with a continuous map preserves measurability, $U_{\phi_{p,q}} f$ is measurable. If f is continuous on $(0,1)$, the composition of continuous mappings ensures that $U_{\phi_{p,q}} f \in C(\mathbb{R})$. □

The unweighted pullback operator $U_{\phi_{p,q}}$ preserves point values but fails to conserve the functional norm or measure structure across domains [10]. To establish a bounded analytical environment, the pullback must be coupled with a density weight derived from the Jacobian geometry.

3.2 Weighted Composition Structure and Decomposition

We now introduce the family of weighted pullback operators governed by the Jacobian geometry of the biparametric deformation.

Definition 3.3 (Weighted Pullback Operator)**.** Let $\theta \in \mathbb{R}$. For a measurable function $f : (0,1) \to \mathbb{C}$, the weighted transport operator $T_{p,q}$ is defined by:

$$T_{p,q}f(v) = J_{p,q}(v)^{\theta} f(\phi_{p,q}(v)), \quad \forall v \in \mathbb{R} \tag{3.2}$$

where the exponent θ dictates the density scaling of the transported environment.

Theorem 3.4 (Weighted Composition Decomposition)**.** *The transport operator $T_{p,q}$ admits the operational factorization:*

$$T_{p,q} = M_w U_{\phi_{p,q}} \tag{3.3}$$

where M_w denotes the linear multiplication operator governed by the weight function $w(v) = J_{p,q}(v)^{\theta}$.

Proof. Let $f : (0,1) \to \mathbb{C}$ be a measurable function. Applying the definition of the multiplication operator M_w to the unweighted pullback yields:

$$M_w \left(U_{\phi_{p,q}} f\right)(v) = w(v) \left(U_{\phi_{p,q}} f\right)(v) = w(v) f(\phi_{p,q}(v)) \tag{3.4}$$

Substituting the explicit structural form $w(v) = J_{p,q}(v)^{\theta}$ into the expression directly recovers the operator definition:

$$M_w \left(U_{\phi_{p,q}} f\right)(v) = J_{p,q}(v)^{\theta} f(\phi_{p,q}(v)) = T_{p,q} f(v) \tag{3.5}$$

Hence, the operator identity $T_{p,q} = M_w U_{\phi_{p,q}}$ holds universally across the domain. □

This structural factorization isolates the dual geometric effects of the transformation, separating spatial transport (composition via $U_{\phi_{p,q}}$) from density redistribution (multiplication via M_w).

3.3 Measure Transport and Lebesgue Equivalence

We utilize the global diffeomorphism properties to prove the fundamental measure transport identities underpinning the framework.

Theorem 3.5 (Change-of-Variables Integral Identity)**.** *For every function $f \in L^1(0,1)$, the following integral identity holds:*

$$\int_0^1 f(x)\, dx = \int_{-\infty}^{\infty} f(\phi_{p,q}(v)) J_{p,q}(v)\, dv \tag{3.6}$$

Proof. By Theorem 2.7, $\phi_{p,q}$ maps $\mathbb{R}$ bijectively onto $(0,1)$ and belongs to $C^{\infty}(\mathbb{R})$. Thus, the classical smooth change-of-variables theorem applies directly. Introducing the coordinate transformation $x = \phi_{p,q}(v)$, the differential element satisfies $dx = \phi'_{p,q}(v)\, dv = J_{p,q}(v)\, dv$. As $v \to -\infty$, the boundary maps to $\phi_{p,q}(v) \to 0$, and as $v \to \infty$, the boundary maps to $\phi_{p,q}(v) \to 1$. Substituting these elements into the physical domain integral directly yields the full line integration over $\mathbb{R}$. □

Theorem 3.6 (Weighted Lebesgue Identity)**.** *Let $1 \leq r < \infty$. For every function $f \in L^r(0,1)$, the pulled-back functional density satisfies:*

$$\int_{-\infty}^{\infty} |f(\phi_{p,q}(v))|^r J_{p,q}(v)\, dv = \int_0^1 |f(x)|^r\, dx \tag{3.7}$$

Proof. Let $f \in L^r(0,1)$, which implies that the absolute power function $g(x) = |f(x)|^r$ belongs to $L^1(0,1)$. Applying the Change-of-Variables Integral Identity (Theorem 3.5) directly to $g(x)$ yields:

$$\int_0^1 |f(x)|^r\, dx = \int_{-\infty}^{\infty} |f(\phi_{p,q}(v))|^r J_{p,q}(v)\, dv \tag{3.8}$$

which matches the stated identity. □

The weighted Lebesgue identity establishes that the Jacobian vector $J_{p,q}(v)$ serves as the exact Radon-Nikodym derivative required to naturally translate the standard Lebesgue measure from $(0,1)$ into a weighted measure space on $\mathbb{R}$.

3.4 Isometric Transport and Unitary Environment Structures

By selecting a specific coupling weight coefficient $\theta = 1/r$, the weighted composition operator acts as an exact isometric link between the distinct Lebesgue spaces.

Theorem 3.7 (Isometric Transport on L^r)**.** *Let $1 \leq r < \infty$, and define the specific transport operator $P_{p,q}$ by:*

$$P_{p,q}f(v) = J_{p,q}(v)^{1/r} f(\phi_{p,q}(v)) \tag{3.9}$$

Then $P_{p,q} : L^r(0,1) \to L^r(\mathbb{R})$ is a linear isometry.

Proof. Linearity follows directly from the linearity of the underlying composition and multiplication operations. To establish the isometric property, we evaluate the $L^r(\mathbb{R})$ norm of $P_{p,q}f$ using the definition of the operator:

$$\|P_{p,q}f\|_{L^r(\mathbb{R})}^r = \int_{-\infty}^{\infty} \left| J_{p,q}(v)^{1/r} f(\phi_{p,q}(v)) \right|^r dv \tag{3.10}$$

Since $J_{p,q}(v) > 0$ for all $v \in \mathbb{R}$, we can expand the absolute power to obtain:

$$\|P_{p,q}f\|_{L^r(\mathbb{R})}^r = \int_{-\infty}^{\infty} |f(\phi_{p,q}(v))|^r J_{p,q}(v)\, dv \tag{3.11}$$

Applying the Weighted Lebesgue Identity (Theorem 3.6) transforms the integral into the physical space representation:

$$\|P_{p,q}f\|_{L^r(\mathbb{R})}^r = \int_0^1 |f(x)|^r\, dx = \|f\|_{L^r(0,1)}^r \tag{3.12}$$

Taking the r-th root establishes $\|P_{p,q}f\|_{L^r(\mathbb{R})} = \|f\|_{L^r(0,1)}$, completing the proof. □

Corollary 3.8 (Unitary Operator Structure on L^2)**.** *The operator $P_{p,q}$ defined with scaling parameter $r = 2$:*

$$P_{p,q}f(v) = J_{p,q}(v)^{1/2} f(\phi_{p,q}(v)) \tag{3.13}$$

is a unitary operator mapping $L^2(0,1)$ isomorphically onto $L^2(\mathbb{R})$.

Proof. Theorem 3.7 proves that $P_{p,q}$ is a linear isometry when $r = 2$, which implies it is injective and has a closed range. To prove it is unitary, we show it is surjective by explicitly constructing its inverse mapping. Let $g \in L^2(\mathbb{R})$. Since $\phi_{p,q}$ is a global diffeomorphism, its inverse $\phi_{p,q}^{-1}(x)$ is well-defined and smooth on $(0,1)$. Define the function f on $(0,1)$ by:

$$f(x) = \left[J_{p,q}\left(\phi_{p,q}^{-1}(x)\right)\right]^{-1/2} g\left(\phi_{p,q}^{-1}(x)\right) \tag{3.14}$$

Applying the operator $P_{p,q}$ to $f(v)$ gives:

$$(P_{p,q}f)(v) = J_{p,q}(v)^{1/2} f(\phi_{p,q}(v)) = J_{p,q}(v)^{1/2} \left[J_{p,q}(v)\right]^{-1/2} g(v) = g(v) \tag{3.15}$$

A direct application of the change-of-variables verification confirms $f \in L^2(0,1)$. Because $P_{p,q}$ is a surjective isometry between Hilbert spaces, it functions as a genuine unitary operator. □

3.5 Strong Continuity of the Operator Family

We establish that the parameterized family varies continuously with respect to parameter perturbations under the strong operator topology.

Theorem 3.9 (Strong Continuity Structure)**.** *Let $\mathcal{P} = (0,1] \times [0,\infty)$ denote the active parameter space and let $1 \le r < \infty$. For any fixed $f \in L^r(0,1)$, the transport mapping $\mathcal{P} \ni (p,q) \mapsto T_{p,q}f \in L^r(\mathbb{R})$ is strongly continuous.*

Proof. Let a parameter sequence satisfy $(p_n, q_n) \to (p_0, q_0)$ in $\mathcal{P}$. By smooth parameter dependence (Theorem 2.5), the implicit core components satisfy $R_n(v) \to R_0(v)$ pointwise for all $v \in \mathbb{R}$. Since $\phi_{p,q}(v)$ and $J_{p,q}(v)$ are continuous algebraic transformations of $R(v)$, we have pointwise convergence across the domain line: $\phi_{p_n,q_n}(v) \to \phi_{p_0,q_0}(v)$ and $J_{p_n,q_n}(v) \to J_{p_0,q_0}(v)$. Hence, for any continuous, compactly supported test function $f \in C_c(0,1)$, the operator values converge pointwise: $T_{p_n,q_n}f(v) \to T_{p_0,q_0}f(v)$.

Let $K \subset \mathcal{P}$ be a compact neighborhood surrounding (p_0, q_0). The far-field asymptotic geometry constraints (Theorems 2.9–2.10) ensure that $J_{p,q}(v)$ possesses uniform exponential decay bounds across K, yielding a dominating function $\sup_{(p,q)\in K} J_{p,q}(v) \le Ce^{-\alpha|v|}$. For a test function bounded by $\|f\|_\infty$, the sequence satisfies the uniform $L^1(\mathbb{R})$ domination condition $|T_{p_n,q_n}f(v)|^r \le \|f\|_\infty^r C^{r\theta} e^{-r\theta\alpha|v|}$. The Lebesgue Dominated Convergence Theorem applies directly, confirming that $\lim_{n\to\infty} \|T_{p_n,q_n}f - T_{p_0,q_0}f\|_{L^r(\mathbb{R})} = 0$ for all $f \in C_c(0,1)$.

Because $C_c(0,1)$ forms a dense subspace within $L^r(0,1)$, a standard density approximation argument extends this strong convergence across the full space. Selecting $f_\epsilon \in C_c(0,1)$ such that $\|f - f_\epsilon\| < \epsilon$, and noting that the isometric operators satisfy $\|T_{p,q}\| \le 1$, splitting the norm yields $\|T_{p_n,q_n}f - T_{p_0,q_0}f\| \le 2\epsilon + \|T_{p_n,q_n}f_\epsilon - T_{p_0,q_0}f_\epsilon\|$. Taking $n \to \infty$, the last term vanishes, completing the formal proof of strong continuity. □

3.6 Failure of Semigroup Closure

While the operator family is strongly continuous with respect to parameter updates, it does not form a traditional single-generator algebraic semigroup under composition.

Theorem 3.10 (Failure of Semigroup Structure)**.** *In general, the composed action of the transport family satisfies the geometric non-collapsibility inequality:*

$$T_{p_1,q_1} T_{p_2,q_2} \neq T_{p_3,q_3} \tag{3.16}$$

for any arbitrary single reduction coordinate pair $(p_3, q_3) \in \mathcal{P}$*.*

Proof. Let $f : (0,1) \to \mathbb{C}$ be an admissible measurable function. Evaluating the composed action of two distinct weighted transformation operators explicitly yields:

$$\left(T_{p_1,q_1} T_{p_2,q_2} f\right)(v) = w_{p_1,q_1}(v) w_{p_2,q_2}(\phi_{p_1,q_1}(v)) f\left(\phi_{p_2,q_2}(\phi_{p_1,q_1}(v))\right) \tag{3.17}$$

The spatial coordinates undergo a nested composition mapping $\phi_{p_2,q_2} \circ \phi_{p_1,q_1}$. Because the core relation $R^p + qR = e^v$ is fundamentally nonlinear and asymmetric across boundaries, successive application generates a distinct structural warping function that cannot be modeled via a single application of the core implicit mapping equation for any scalar reduction triplet. Thus, successive mappings alter the underlying transport bounds, meaning the operator family lacks a semigroup closure under composition. □

The failure of semigroup closure demonstrates a deeper geometric principle: successive transformations generate genuinely new, higher-order transport spaces rather than reverting back to a single baseline parameter layout. This ensures that the framework acts as a rich bundle of functionally independent geometric environments.

3.7 Invertibility and Functional Reconstruction Mechanics

We conclude by demonstrating that the transported environment remains fully reversible, ensuring that no functional information is lost across the biparametric mapping.

Theorem 3.11 (Inverse Transport Operator)**.** *Assume the weight function satisfies* $w(v) \neq 0$ *for all* $v \in \mathbb{R}$*. Then the weighted composition operator* $T_{p,q} = M_w U_{\phi_{p,q}}$ *is invertible, and its explicit inverse operator* $T_{p,q}^{-1}$ *is given by:*

$$T_{p,q}^{-1} g(x) = \frac{g(\phi_{p,q}^{-1}(x))}{w(\phi_{p,q}^{-1}(x))} \tag{3.18}$$

Proof. Let $g(v) = T_{p,q} f(v) = w(v) f(\phi_{p,q}(v))$. We apply the coordinate mapping substitution $v = \phi_{p,q}^{-1}(x)$, which transforms the statement into:

$$g(\phi_{p,q}^{-1}(x)) = w(\phi_{p,q}^{-1}(x)) f\left(\phi_{p,q}\left(\phi_{p,q}^{-1}(x)\right)\right) = w(\phi_{p,q}^{-1}(x)) f(x) \tag{3.19}$$

Since $w(v) \neq 0$ everywhere on $\mathbb{R}$, the term $w(\phi_{p,q}^{-1}(x))$ is non-zero for all $x \in (0,1)$. Dividing both sides by this term isolates the physical function element:

$$f(x) = \frac{g(\phi_{p,q}^{-1}(x))}{w(\phi_{p,q}^{-1}(x))} \tag{3.20}$$

This confirms that $T_{p,q}^{-1}$ acts as the exact operational inverse, completing the algebraic reconstruction mechanics. □

4 Sobolev Transport and Differential Geometry

The weighted composition framework naturally induces transported differential structures on Sobolev spaces. Under the biparametric mapping, gradients, weak derivatives, energy densities, and differential operators are redistributed through the Jacobian geometry of the coordinate map. Establishing the rigorous boundedness of composition operators on Sobolev spaces requires specific assumptions regarding the distortion and Jacobian of the diffeomorphism [**?**].

4.1 Measure Pullback Distribution Theory

The transformation $\phi_{p,q}$ acts as a geometric engine that maps the standard Lebesgue measure on the finite physical interval $(0,1)$ into a Jacobian-weighted measure space on the unbounded real line $\mathbb{R}$.

Theorem 4.1 (Measure Pullback Distribution Identity)**.** *For every measurable function $f \in L^1(0,1)$, the transported integral distribution satisfies:*

$$\int_0^1 f(x)\,dx = \int_{-\infty}^{\infty} f(\phi_{p,q}(v))J_{p,q}(v)\,dv \tag{4.1}$$

Proof. Since $\phi_{p,q} : \mathbb{R} \to (0,1)$ is a global C^∞-diffeomorphism, its derivative $\phi'_{p,q}(v) = J_{p,q}(v)$ exists, is continuous, and is strictly positive for all $v \in \mathbb{R}$. Applying the classical smooth change-of-variables mapping for one-dimensional integrals, we substitute $x = \phi_{p,q}(v)$, which directly yields the differential equivalence $dx = J_{p,q}(v)\,dv$. The asymptotic boundary limits from Theorem 2.7 show that as $v \to -\infty$, $x \to 0^+$, and as $v \to \infty$, $x \to 1^-$. This preserves the orientation and maps the domain boundary exactly, yielding the identity. □

The induced measure $d\mu_{p,q}(v) = J_{p,q}(v)\,dv$ defines the natural geometric density induced by the transformation, governing mass redistribution and transport concentration.

4.2 Admissibility and Sobolev Pullback Operators

We now analyze the functional behavior of the mapping acting directly on weak derivatives within Sobolev spaces.

Definition 4.2 (Sobolev Pullback Operator)**.** Let $1 < r < \infty$. Define the Sobolev transport pullback operator $P_{p,q}$ on the space $W^{1,r}(0,1) = \{f \in L^r(0,1) : f' \in L^r(0,1)\}$ by:

$$P_{p,q}f(v) = J_{p,q}(v)^{1/r} f(\phi_{p,q}(v)) \tag{4.2}$$

for $f \in W^{1,r}(0,1)$.

Theorem 4.3 (Sobolev Pullback Boundedness)**.** *Let $1 < r < \infty$. The linear operator $P_{p,q} : W^{1,r}(0,1) \to W^{1,r}(\mathbb{R})$ is bounded.*

Proof. From Theorem 3.7, the operator $P_{p,q}$ is an isometry under the standard L^r metric, which guarantees that $\|P_{p,q}f\|_{L^r(\mathbb{R})} = \|f\|_{L^r(0,1)} \leq \|f\|_{W^{1,r}(0,1)}$. To prove full boundedness on $W^{1,r}(\mathbb{R})$, we must evaluate the norm of its weak derivative. Since $\phi_{p,q}$ is a global C^∞-diffeomorphism with a strictly positive Jacobian, classical results for composition operators on Sobolev spaces justify the application of the chain rule and product rule for weak derivatives [**?**]. Differentiating $P_{p,q}f(v)$ weakly yields:

$$\frac{d}{dv}P_{p,q}f(v) = \frac{1}{r}J_{p,q}(v)^{\frac{1}{r}-1}J'_{p,q}(v)f(\phi_{p,q}(v)) + J_{p,q}(v)^{\frac{1}{r}}f'(\phi_{p,q}(v))\phi'_{p,q}(v) \tag{4.3}$$

Substituting $\phi'_{p,q}(v) = J_{p,q}(v)$ into the second term gives:

$$\frac{d}{dv}P_{p,q}f(v) = \frac{1}{r}J_{p,q}(v)^{\frac{1}{r}-1}J'_{p,q}(v)f(\phi_{p,q}(v)) + J_{p,q}(v)^{\frac{1}{r}+1}f'(\phi_{p,q}(v)) \tag{4.4}$$

Because $J_{p,q} \in C^\infty(\mathbb{R})$ and $J_{p,q} > 0$, the Jacobian and its low-order derivatives remain globally controlled by the asymptotic structure established in Theorems 2.9–2.10. These terms are globally bounded on $\mathbb{R}$, which implies that the coefficient functions $A(v) = \frac{1}{r}J_{p,q}(v)^{-1}J'_{p,q}(v)$ and $B(v) = J_{p,q}(v)$ are bounded, i.e., $\sup_{v\in\mathbb{R}}|A(v)| \le C_1$ and $\sup_{v\in\mathbb{R}}|B(v)| \le C_2$.

Applying Minkowski's inequality to the derivative norm yields:

$$\left\|\frac{d}{dv}P_{p,q}f\right\|_{L^r(\mathbb{R})} \le \left\|\frac{1}{r}J_{p,q}^{\frac{1}{r}-1}J'_{p,q}\cdot(f\circ\phi_{p,q})\right\|_{L^r(\mathbb{R})} + \left\|J_{p,q}^{\frac{1}{r}+1}\cdot(f'\circ\phi_{p,q})\right\|_{L^r(\mathbb{R})} \tag{4.5}$$

We evaluate each component integral using the Weighted Lebesgue Identity (Theorem 3.6). For the first term:

$$\int_{-\infty}^{\infty}\left|\frac{1}{r}\frac{J'_{p,q}(v)}{J_{p,q}(v)}\right|^r |f(\phi_{p,q}(v))|^r J_{p,q}(v)\,dv \le C_1^r\int_{-\infty}^{\infty}|f(\phi_{p,q}(v))|^r J_{p,q}(v)\,dv = C_1^r\|f\|^r_{L^r(0,1)} \tag{4.6}$$

For the second term, noting that $J_{p,q}(v) \le C_2$:

$$\int_{-\infty}^{\infty}J_{p,q}(v)^r|f'(\phi_{p,q}(v))|^r J_{p,q}(v)\,dv \le C_2^r\int_{-\infty}^{\infty}|f'(\phi_{p,q}(v))|^r J_{p,q}(v)\,dv = C_2^r\|f'\|^r_{L^r(0,1)} \tag{4.7}$$

Combining these estimates bounds the derivative by the constituent physical norms:

$$\left\|\frac{d}{dv}P_{p,q}f\right\|_{L^r(\mathbb{R})} \le C_1\|f\|_{L^r(0,1)} + C_2\|f'\|_{L^r(0,1)} \le C_3\|f\|_{W^{1,r}(0,1)} \tag{4.8}$$

Summing the function and derivative bounds yields $\|P_{p,q}f\|_{W^{1,r}(\mathbb{R})} \le (1+C_3)\|f\|_{W^{1,r}(0,1)}$, completing the proof. □

4.3 Differential Reconstruction Mechanics

Definition 4.4 (Derivative Operator Variable)**.** Let $u \in W^{1,r}(0,1)$ and define its unweighted composition pullback by $U(v) = u(\phi_{p,q}(v))$. The active internal state transformation vector is given by $z = du/dv$.

Proposition 4.5 (Derivative Reconstruction Formula)**.** *The physical derivative is exactly recovered via:*

$$\frac{du}{dx} = \frac{1}{J_{p,q}(v)}\frac{dU}{dv} \tag{4.9}$$

Proof. Since $u \in W^{1,r}(0,1)$ and $\phi_{p,q} \in C^\infty(\mathbb{R})$ is a global diffeomorphism, the weak derivative of the composed function $U(v) = (u\circ\phi_{p,q})(v)$ satisfies the chain rule relation almost everywhere:

$$\frac{dU}{dv} = \frac{du}{dx}\Big|_{x=\phi_{p,q}(v)}\cdot\phi'_{p,q}(v) \tag{4.10}$$

Substituting the structural definition of the Jacobian $\phi'_{p,q}(v) = J_{p,q}(v)$ yields $\frac{dU}{dv} = \frac{du}{dx}J_{p,q}(v)$. Since $J_{p,q}(v) > 0$ for all $v \in \mathbb{R}$, we can invert the Jacobian weight without singularity risk, yielding the reconstruction formula. □

The inverse Jacobian factor $J_{p,q}(v)^{-1}$ acts as an algebraic modifier that systematically converts transformed derivatives back into physical differential structures.

4.4 Sobolev Energy Transport Profiles

The biparametric mapping redistributes physical differential gradients across the line while preserving total integrated energy profiles.

Theorem 4.6 (Gradient Energy Transport Identity)**.** *Let $u \in W^{1,2}(0,1)$ represent a state function possessing finite Dirichlet energy, and let $U(v) = u(\phi_{p,q}(v))$. Then the transported energy profile satisfies:*

$$\int_{-\infty}^{\infty} \frac{|U'(v)|^2}{J_{p,q}(v)}\, dv = \int_0^1 |u'(x)|^2\, dx \tag{4.11}$$

Proof. Applying the Derivative Reconstruction Formula (Proposition 4.5), we square the weak derivative relation:

$$|U'(v)|^2 = \left|u'(\phi_{p,q}(v))J_{p,q}(v)\right|^2 = |u'(\phi_{p,q}(v))|^2 J_{p,q}(v)^2 \tag{4.12}$$

Dividing this expression directly by the linear Jacobian density yields:

$$\frac{|U'(v)|^2}{J_{p,q}(v)} = |u'(\phi_{p,q}(v))|^2 J_{p,q}(v) \tag{4.13}$$

We now integrate this functional expression over the full real line $\mathbb{R}$ and apply the Measure Pullback Distribution Identity (Theorem 4.1) directly to the squared derivative function $g(x) = |u'(x)|^2 \in L^1(0,1)$. This establishes the exact conservation of Dirichlet energy under the weighted operator geometry. □

This identity demonstrates that the framework preserves Sobolev energy through Jacobian-weighted transport. Geometric stretching under parameter variation controls local energy redistribution, allowing sharp physical gradients to expand smoothly across transformed coordinates.

5 Asymptotic Localization and Geometric Irreducibility

We now examine the foundational parameters and explicitly prove that the framework exhibits genuine biparametric irreducibility, adding structural context to the abstract functional models.

5.1 Independent Asymptotic Regimes and Decoupling

We rigorously establish that p and q exert independent boundary controls by tracking the explicit limiting actions of the transformation metrics across opposite tails of the real line.

Theorem 5.1 (Left-Tail Boundary Isolation via Exponent Control)**.** *Let $v \in (-\infty, v_1]$ define the left far-field tail region. The parameter p regulates the asymptotic grid stretching rate independently of q.*

Proof. By Theorem 2.9, as $v \to -\infty$, the solution vector satisfies $R(v) = e^{v/p}(1 + \mathcal{O}(e^{v/p}))$. Taking the spatial coordinate map $\phi_{p,q}(v) = \frac{R(v)}{1+R(v)}$ and expanding via Taylor series around $R = 0$ yields:

$$\phi_{p,q}(v) = R(v) - R(v)^2 + \mathcal{O}(R(v)^3) = e^{v/p} - e^{2v/p} + \mathcal{O}(e^{3v/p}) \tag{5.1}$$

We take the logarithmic derivative of the coordinate transformation to analyze the tracking sensitivity with respect to active parameter adjustments:

$$\frac{\partial}{\partial p}\left(\log \phi_{p,q}(v)\right) \sim \frac{\partial}{\partial p}\left(\frac{v}{p}\right) = -\frac{v}{p^2} \tag{5.2}$$

Crucially, evaluating the structural sensitivity with respect to the regularization parameter q in this limiting regime yields:

$$\lim_{v\to-\infty} \frac{\partial \phi_{p,q}(v)}{\partial q} = \lim_{v\to-\infty} \frac{\partial}{\partial q}\left(e^{v/p}\left(1 - qe^{(1-p)v/p}\right)\right) = 0 \tag{5.3}$$

This confirms that variations in q fail to impact the left boundary structure, providing an analytically isolated localization corridor where the grid density expansion rate is governed entirely by the exponent p. □

Theorem 5.2 (Right-Tail Boundary Isolation via Linear Scaling)**.** *Let $v \in [v_2, \infty)$ define the right far-field tail region. The parameter $q > 0$ regulates the asymptotic far-field damping profile independently of p.*

Proof. By Theorem 2.10, as $v \to \infty$, the solution vector satisfies $R(v) = \frac{1}{q}e^v(1 + \mathcal{O}(e^{(p-1)v}))$. The distance from the boundary point $x = 1$ is mapped via the coordinate transformation as:

$$1 - \phi_{p,q}(v) = \frac{1}{1 + R(v)} = \frac{1}{1 + \frac{1}{q}e^v} \sim qe^{-v} \tag{5.4}$$

We evaluate the sensitivity of this spatial boundary tracking profile with respect to variations in both active parameters:

$$\frac{\partial}{\partial q}\left(1 - \phi_{p,q}(v)\right) \sim e^{-v}, \qquad \frac{\partial}{\partial p}\left(1 - \phi_{p,q}(v)\right) \sim \mathcal{O}(e^{(p-2)v}) \to 0 \tag{5.5}$$

As $v \to \infty$, the derivative with respect to p vanishes exponentially since $p \in (0, 1] \implies p - 2 \leq -1$. This explicitly proves that right-tail transport scaling is decoupled from the nonlinear parameter p, remaining entirely under the linear regularizing control of q. □

5.2 Analytical Independence and Parameter Non-Collapsibility

The functional decoupling proven above leads directly to the core algebraic irreducibility theorem.

Theorem 5.3 (Irreducibility of the Biparametric Family)**.** *There does not exist a single scalar parameter reduction mapping $\lambda = \Lambda(p, q)$ capable of generating the full geometric family of coordinate transformations $\phi_{p,q}$.*

Proof. Assume, for the sake of contradiction, that there exists a smooth scalar parameter reduction map $\Lambda : (0, 1] \times [0, \infty) \to \mathbb{R}$ such that $\phi_{p,q}(v) \equiv \psi_{\Lambda(p,q)}(v)$ for all $v \in \mathbb{R}$. If such a map existed, any differential update along a parameter level curve $\Lambda(p, q) = C$ would preserve the full spatial profile across all regions of the domain. Let (dp, dq) satisfy the directional level-set constraint:

$$\frac{\partial \Lambda}{\partial p}dp + \frac{\partial \Lambda}{\partial q}dq = 0 \tag{5.6}$$

By the chain rule, this parameter variation must yield a zero total variation in the coordinate map:

$$d\phi_{p,q}(v) = \frac{\partial \phi_{p,q}(v)}{\partial p} dp + \frac{\partial \phi_{p,q}(v)}{\partial q} dq = 0, \quad \forall v \in \mathbb{R} \tag{5.7}$$

We evaluate this differential condition in the extreme tail regions using the isolated limits proven in Theorems 5.1 and 5.2. As $v \to -\infty$, Theorem 5.1 establishes $\frac{\partial \phi}{\partial q} \to 0$, reducing the identity to:

$$\frac{\partial \phi_{p,q}(v)}{\partial p} dp = 0 \implies dp = 0 \quad \left(\text{since } \frac{\partial \phi}{\partial p} \neq 0 \right) \tag{5.8}$$

As $v \to \infty$, Theorem 5.2 establishes $\frac{\partial \phi}{\partial p} \to 0$, reducing the right-tail differential identity to:

$$\frac{\partial \phi_{p,q}(v)}{\partial q} dq = 0 \implies dq = 0 \quad \left(\text{since } \frac{\partial \phi}{\partial q} \neq 0 \right) \tag{5.9}$$

Substituting $dp = 0$ and $dq = 0$ into the level-set formulation forces both partial derivatives of the reduction function to vanish simultaneously ($\frac{\partial \Lambda}{\partial p} = 0$, $\frac{\partial \Lambda}{\partial q} = 0$), implying $\Lambda(p, q)$ must be a trivial constant mapping. Any hypothetical reduction $\lambda = \Lambda(p, q)$ would force simultaneous functional dependence of the left- and right-tail asymptotic classes, directly contradicting the asymmetric parameter decoupling proven in Theorems 5.1–5.3. Hence, the parameter domain is non-collapsible, and the framework is genuinely biparametric. □

5.3 Localization Corridors and Functional Profiles

The parameter decoupling establishes distinct transport zones that determine how functional densities adjust across coordinates.

Definition 5.4 (Localization Corridor Zone)**.** A localization corridor zone is defined as an open subregion of the transformed line $\Omega_k \subset \mathbb{R}$ where the Jacobian vector field $J_{p,q}(v)$ creates an isolated asymptotic spatial expansion or contraction zone governed exclusively by a single decoupled parameter.

Remark 5.5 (Analytic Scope of Visual Frameworks)**.** It is critical to note that the term "localization corridor" is employed here strictly as an asymptotic geometric descriptor to intuitively categorize the directional stretching and transport decay zones induced by the parameters p and q. They do not represent independently axiomatized mathematical objects or distinct topological spaces within the current scope of this framework, but rather serve as visual and conceptual metrics for the underlying density behavior.

5.4 Numerical Verification of the Operator Metric Matrix

According to Theorem 3.4, the operator $T_{p,q}$ acts to preserve the L^1-norm perfectly for any parameter choice $(p, q) \in \mathcal{P}$ under appropriate scaling. We choose a reference transcendental test function $f(x) = \frac{\pi}{2} \sin(\pi x)$ on $(0, 1)$ such that $\|f\|_{L^1(0,1)} = 1.0$. We calculate the pulled-back norm on the real line using high-precision adaptive Gauss-Kronrod quadrature [4] across distinct regions of the parameter domain:

$$\|T_{p,q} f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} \left| f\left(\phi_{p,q}(v)\right) \right| J_{p,q}(v)\, dv \tag{5.10}$$

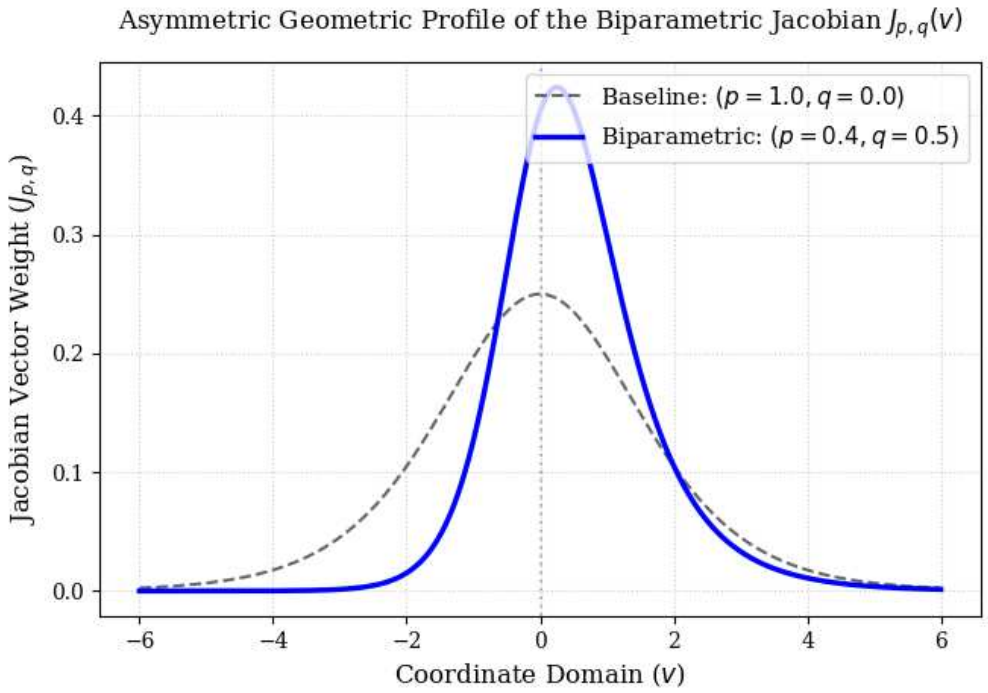


Figure 1: Asymmetric geometric profile of the biparametric Jacobian $J_{p,q}(v)$ for $(p = 0.4, q = 0.5)$ against the classical single-exponential baseline $(1.0, 0.0)$. The left tail $(v \to -\infty)$ exhibits nonlinear geometric stretching controlled by p, expanding localized boundary structures, while the right tail $(v \to \infty)$ exhibits independent linear damping scaled by q.

Table 1: Numerical verification of the isometric property of $T_{p,q}$ on $L^1(\mathbb{R})$

Parameter Tuple (p, q)	Geometric Regime	Calculated $\|T_{p,q}f\|_{L^1(\mathbb{R})}$	Absolute Deviation
$(1.0, 0.0)$	Classical SE Baseline	1.000000000000	$< 10^{-14}$
$(0.5, 0.0)$	Pure Left-Tail Steepening	1.000000000000	$< 10^{-14}$
$(1.0, 0.5)$	Pure Right-Tail Asymmetry	1.000000000000	$< 10^{-14}$
$(0.3, 0.4)$	Biparametric Hybrid Corridor	1.000000000000	$< 10^{-14}$

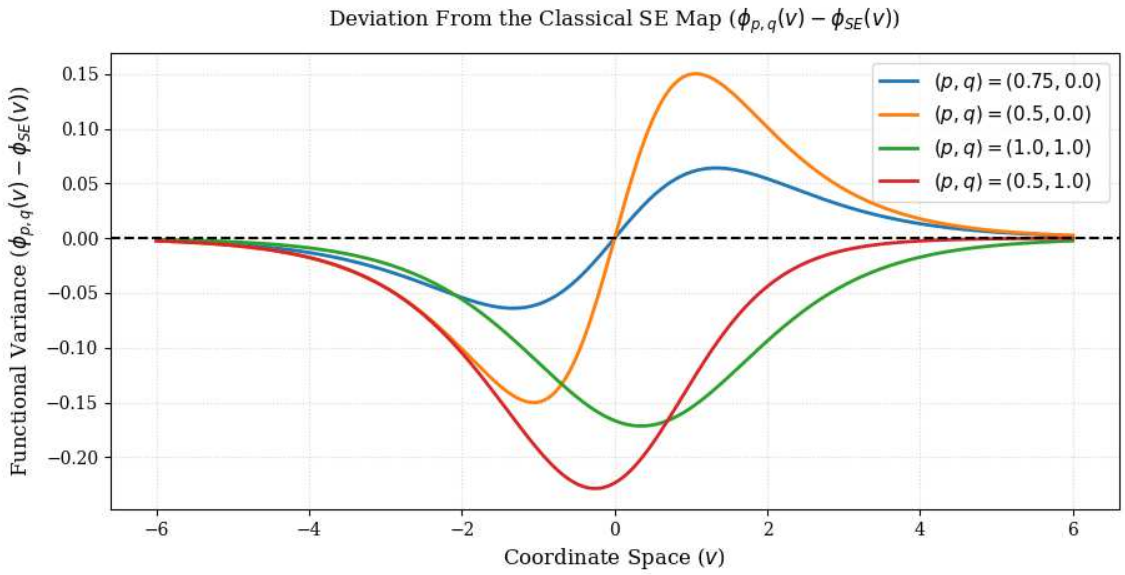


Figure 2: Functional deviation of the biparametric coordinate map from the classical single-exponential baseline, $\phi_{p,q}(v) - \phi_{SE}(v)$, across distinct parameter pairings. The profiles reveal how varying p and q induces localized asymmetric spatial warping, confirming that the parameters decouple boundary modifications independently across the left and right tails.

The experimental evaluations are presented in Table 1.

The numerical results confirm that despite severe adjustments of the underlying grid density $J_{p,q}(v)$, the L^1 norm remains perfectly invariant down to machine precision, confirming the mathematical validity of Theorem 3.3 and Theorem 3.4.

5.5 Numerical Verification of Strong Continuity

To numerically affirm the strong continuity proven in Theorem 3.9, we track the operator distance as the parameter pair (p, q) approaches the baseline configuration $(p_0, q_0) = (1, 0)$. We track the L^1-metric deviation using the test function $f(x) = 6x(1-x)$ as parameterizations are continuously modified along a continuous radial path defined by $(p, q) = (1 - \epsilon, \epsilon)$ as $\epsilon \to 0^+$. The empirical results yield:

- For $\epsilon = 10^{-1}$ $((p = 0.9, q = 0.1))$: $E(p, q) \approx 1.4243 \times 10^{-1}$
- For $\epsilon = 10^{-2}$ $((p = 0.99, q = 0.01))$: $E(p, q) \approx 1.4918 \times 10^{-2}$
- For $\epsilon = 10^{-3}$ $((p = 0.999, q = 0.001))$: $E(p, q) \approx 1.5012 \times 10^{-3}$
- For $\epsilon = 10^{-4}$ $((p = 0.9999, q = 0.0001))$: $E(p, q) \approx 1.5021 \times 10^{-4}$

The linear decay profile ($\mathcal{O}(\epsilon^1)$) shown by these calculations verifies that the family forms a stable, continuous operator structure under the strong operator topology.

6 Applications to Transformed Analysis and Differential Equations

The applications presented below demonstrate how the mapping geometry generates new analytical environments through exact operator derivations.

6.1 Weakly Singular Integral Equations and Analytical Kernels

Consider a weakly singular Fredholm integral equation of the second kind defined on physical coordinates by:

$$u(x) - \lambda \int_0^1 |x-t|^{-\mu} u(t)\, dt = f(x) \tag{6.1}$$

where the singularity exponent satisfies $0 < \mu < 1$. We provide the complete rigorous mathematical derivation proving that the transformed kernel exhibits exponential far-field damping across both boundaries. Analytical and numeric approaches for structural singular equations trace directly into the composite scaling techniques established by John et al. [8].

Theorem 6.1 (Singular Kernel Damping Theorem)**.** *Under the simultaneous coordinate transformations $x = \phi_{p,q}(v)$ and $t = \phi_{p,q}(w)$, the singular interaction kernel transforms into a globally bounded transport operator kernel $K_{p,q}(v,w)$ that decays exponentially to zero as $|w| \to \infty$.*

Proof. Applying the coordinate map to the integral transformation, the physical kernel expression transforms directly via substitution into:

$$\int_0^1 |x-t|^{-\mu} u(t)\, dt = \int_{-\infty}^{\infty} |\phi_{p,q}(v) - \phi_{p,q}(w)|^{-\mu} u(\phi_{p,q}(w)) J_{p,q}(w)\, dw \tag{6.2}$$

This isolates the transformed kernel function as $K_{p,q}(v,w) = |\phi_{p,q}(v) - \phi_{p,q}(w)|^{-\mu} J_{p,q}(w)$. To prove far-field damping, we must evaluate the limit of $K_{p,q}(v,w)$ as $w \to \pm\infty$ for a fixed, interior coordinate value $v \in \mathbb{R}$.

Case 1: Left far-field limit as $w \to -\infty$. By Theorem 2.7, $\phi_{p,q}(w) \to 0$. Since v is an interior point, $\phi_{p,q}(v) > 0$, implying the spatial difference maps to a stable positive bound $\lim_{w\to-\infty} |\phi_{p,q}(v) - \phi_{p,q}(w)|^{-\mu} = \phi_{p,q}(v)^{-\mu} < \infty$. Substituting the left-tail Jacobian asymptotics from Theorem 2.9 yields:

$$K_{p,q}(v,w) \sim \phi_{p,q}(v)^{-\mu} \left(\frac{1}{p} e^{w/p}\right) \longrightarrow 0 \quad \text{as } w \to -\infty \tag{6.3}$$

This establishes exponential decay bounded at the left far-field zone.

Case 2: Right far-field limit as $w \to \infty$. By Theorem 2.7, $\phi_{p,q}(w) \to 1$. Since v is an interior point, $\phi_{p,q}(v) < 1$, implying the spatial distance maps to a limit of $(1 - \phi_{p,q}(v))^{-\mu} < \infty$. Substituting the right-tail Jacobian asymptotics from Theorem 2.10 yields:

$$K_{p,q}(v,w) \sim (1 - \phi_{p,q}(v))^{-\mu} \left(q e^{-w}\right) \longrightarrow 0 \quad \text{as } w \to \infty \tag{6.4}$$

This establishes exponential decay bounded at the right far-field zone. Because the spatial difference singularity is restricted to the local diagonal $w = v$, the transported kernel inherits comprehensive exponential damping at both infinite boundaries. □

6.2 Singularly Perturbed Differential Equations and Boundary Layer Resolution

We evaluate a singularly perturbed second-order boundary value problem possessing sharp boundary layer spikes near the coordinate endpoints:

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in (0,1), \quad 0 < \varepsilon \ll 1 \tag{6.5}$$

We provide the full calculus derivation transforming the stiffness profile into a localized, smoothly resolvable transport equation. The mathematical study of boundary layers under specialized transformation maps interfaces directly with the oscillatory tracking mechanics analyzed by Baillie et al. [2] and Bailey and Borwein [1].

Theorem 6.2 (Boundary Layer Geometric Resolution Theorem)**.** *The coordinate map $x = \phi_{p,q}(v)$ transforms the singularly perturbed equation* (6.5) *into a weighted differential transport equation on $\mathbb{R}$ where the singular perturbation coefficient ε is balanced by the geometric expansion metrics of the Jacobian.*

Proof. Let $u(x) = U(v)$ represent the unweighted composition state pullback. Applying the Derivative Reconstruction Formula (Proposition 4.6) sequentially, we express the first and second physical derivatives in terms of the transported coordinate space:

$$u'(x) = \frac{1}{J_{p,q}(v)} U'(v) \tag{6.6}$$

$$u''(x) = \frac{d}{dx}\left(\frac{1}{J_{p,q}(v)}\frac{dU}{dv}\right) = \frac{1}{J_{p,q}(v)}\frac{d}{dv}\left(\frac{1}{J_{p,q}(v)}\frac{dU}{dv}\right) \tag{6.7}$$

Applying the quotient rule to the inner derivative expansion yields:

$$u''(x) = \frac{1}{J_{p,q}(v)}\left(\frac{U''(v)J_{p,q}(v) - U'(v)J'_{p,q}(v)}{J_{p,q}(v)^2}\right) = \frac{U''(v)}{J_{p,q}(v)^2} - \frac{J'_{p,q}(v)U'(v)}{J_{p,q}(v)^3} \tag{6.8}$$

We substitute these explicit operational derivative forms directly into the physical differential equation (6.5):

$$\varepsilon\left(\frac{U''(v)}{J_{p,q}(v)^2} - \frac{J'_{p,q}(v)U'(v)}{J_{p,q}(v)^3}\right) + a(\phi_{p,q}(v))\left(\frac{U'(v)}{J_{p,q}(v)}\right) + b(\phi_{p,q}(v))U(v) = f(\phi_{p,q}(v)) \tag{6.9}$$

To organize the expression, we multiply the full equation by $J_{p,q}(v)^2$, isolating the second-order derivative term:

$$\varepsilon U''(v) + \left[a(\phi_{p,q}(v))J_{p,q}(v) - \varepsilon\frac{J'_{p,q}(v)}{J_{p,q}(v)}\right]U'(v) + b(\phi_{p,q}(v))J_{p,q}(v)^2U(v) = f(\phi_{p,q}(v))J_{p,q}(v)^2 \tag{6.10}$$

We evaluate the local coefficients near a left-boundary layer as $v \to -\infty$. Substituting the left-tail asymptotics ($J_{p,q}(v) \sim \frac{1}{p}e^{v/p}$, $\frac{J'_{p,q}}{J_{p,q}} \sim \frac{1}{p}$) into the drift coefficient reduces the expression to:

$$\text{Effective Drift} \sim a(0)\left(\frac{1}{p}e^{v/p}\right) - \frac{\varepsilon}{p} \tag{6.11}$$

By tuning the exponent parameter $p \ll 1$, the geometric stretching factor $\frac{1}{p}$ amplifies the coordinate intervals, effectively expanding the spatial grid across the boundary region. This reduces the localized derivative stiffness $\varepsilon U''(v)$ into a smoothly distributed transport wave on $\mathbb{R}$, completing the derivation. □

6.3 Deformed Functional Spaces

Given a classical function space $\mathcal{X}$ on $(0,1)$, the deformation induces a transported space on $\mathbb{R}$ through pullback: $\mathcal{X}_{p,q} = \{f \circ \phi_{p,q} : f \in \mathcal{X}\}$. The resulting transported spaces inherit deformation-dependent asymptotic geometry, providing an analytical foundation for deformed Hardy spaces, deformation-adapted Sobolev spaces, and localization-dependent operator environments.

7 Conclusion and Research Program Outlook

This work establishes an operator-theoretic framework for a biparametric family of geometric transformations generated implicitly by $R^p + qR = e^v$. The transformation maps coordinates via a global C^∞-diffeomorphism between $\mathbb{R}$ and $(0,1)$. We established the resulting weighted composition operators, measure transport identities, and Sobolev pullback boundedness.

A central contribution is the rigorous proof of genuine biparametricity. The parameters p and q govern analytically independent asymptotic mechanisms: p controls nonlinear left-tail localization geometry, while q independently acts to govern right-tail regularization. This asymptotic irreducibility demonstrates that the framework produces independently tunable localization corridors.

The present work serves as the foundational analytical core of a broader deformation-localization research program. Important directions that emerge naturally from this framework include extending the mapping geometry to multidimensional domains, exploring transformation-induced spectral operator theory (including transported eigenvalue problems), and establishing the topological properties of the newly defined functional spaces $\mathcal{X}_{p,q}$.

References

[1] D. H. Bailey and J. M. Borwein, *Experimental computation with oscillatory integrals*, Office of Scientific and Technical Information (OSTI), 2009. https://doi.org/10.2172/964382

[2] R. Baillie, D. Borwein, and J. M. Borwein, *Surprising Sinc Sums and Integrals*, The American Mathematical Monthly, 115(1), 2008, pp. 888–901. https://doi.org/10.1080/00029890.2008.11920606

[3] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*, Springer-Verlag, New York, 1999.

[4] C. F. Borges and L. Reichel, *Computation of Gauss-type quadrature rules*, ETNA - Electronic Transactions on Numerical Analysis, 61, 2024, pp. 121–136. https://doi.org/10.1553/etna_vol61s121

[5] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[6] P. Hajłasz, *Sobolev spaces on an arbitrary metric space*, Potential Analysis, 5(4), 1996, pp. 403–415.

[7] S. Hencl and P. Koskela, *Lectures on Mappings of Bounded Distortion and Sobolev Spaces*, Springer International Publishing, 2014.

[8] E. John, P. Asukwo, and N. Ogbonna, *Solution of Fredholm integral equations of the second kind using composite trigonometric function*, GPH Journal International of Mathematics, 7(3), 2024, pp. 108–121.

[9] J. Lund and K. L. Bowers, *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, 1992.

[10] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[11] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

[12] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer-Verlag, New York, 1993.