Supplemental Material for
"Non-Abelian Holonomy Lifts Topological Obstruction to Synchronization"

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INTRODUCTION

This Supplemental Material provides detailed derivations and additional information supporting the results presented in the main text. Section recalls the real matrix representation of quaternions used throughout. Section proves the invariance condition of the synchronization manifold. Section gives a full derivation of the non-Abelian Master Stability Function (MSF) with the left/right eigenspace splitting. Section presents the explicit construction of a flat SU(2) connection on a tetrahedron that satisfies the kernel condition $B_1^{\mathbb{H}}\mathbf{1} = 0$. Section provides details of the numerical simulations, including the quaternionic Stuart-Landau model, integration scheme, and MSF computation. Finally, Section contains additional figures on the robustness of synchronization and the basin of attraction.

REALIFICATION OF QUATERNIONS

A quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$ can be represented as a real vector $(a, b, c, d)^\top \in \mathbb{R}^4$. Left multiplication by q, denoted $L_q(x) = qx$, is an $\mathbb{R}$-linear map. In the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, its matrix representation is

$$\chi(q) = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}. \tag{1}$$

Right multiplication by q, denoted $R_q(x) = xq$, is also $\mathbb{R}$-linear and its matrix is

$$\chi_R(q) = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}. \tag{2}$$

These matrices satisfy $\chi(p)\chi(q) = \chi(pq)$ and $\chi_R(p)\chi_R(q) = \chi_R(qp)$. Moreover, left and right multiplication commute: $\chi(p)\chi_R(q) = \chi_R(q)\chi(p)$.

For a quaternionic matrix $A \in \mathbb{H}^{m\times n}$, its *realification* $A^{\mathbb{R}} \in \mathbb{R}^{4m\times 4n}$ is obtained by replacing each entry A_{ij} with the 4×4 block $\chi(A_{ij})$. This map preserves matrix addition and multiplication: $(AB)^{\mathbb{R}} = A^{\mathbb{R}} B^{\mathbb{R}}$.

The quaternionic Dirac operator $\mathcal{D}_{\mathbb{H}}$ defined in the main text is Hermitian with respect to the standard quaternionic inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_i \bar{x}_i y_i$. Its realification $\mathcal{D}_{\mathbb{H}}^{\mathbb{R}}$ is a symmetric real matrix of size $4N \times 4N$.

INVARIANCE OF THE SYNCHRONIZATION MANIFOLD

Consider the coupled dynamics

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) - \sigma(\mathcal{D}_{\mathbb{H}} \otimes \Gamma)\mathbf{x}, \tag{3}$$

where $\mathbf{x} \in \mathbb{H}^N$, $\mathbf{F}(\mathbf{x})$ acts component-wise as $F(x_i)$, and $\Gamma \in \mathbb{R}^{4\times 4}$ is a real matrix acting on the real representation of quaternions. The coupling term is understood as follows: for each component i,

$$[(\mathcal{D}_{\mathbb{H}} \otimes \Gamma)\mathbf{x}]_i = \sum_j (\mathcal{D}_{\mathbb{H}})_{ij}\, \Gamma x_j, \tag{4}$$

where Γx_j means the action of Γ on the real vector $(a_j, b_j, c_j, d_j)^\top$, and the quaternion $(\mathcal{D}_{\mathbb{H}})_{ij}$ multiplies the resulting quaternion from the left.

Proposition. Let $\mathbf{s}(t) \in \mathbb{H}$ be a solution of the uncoupled dynamics $\dot{s} = F(s)$. The manifold $\mathcal{M} = \{\mathbf{x} = \mathbf{1}s \mid s \in \mathbb{H}\}$ is invariant under the coupled dynamics if and only if $\mathcal{D}_{\mathbb{H}}\mathbf{1} = 0$.

Proof. Suppose $\mathbf{x}(t) = \mathbf{1}s(t)$. Then $\dot{\mathbf{x}} = \mathbf{1}\dot{s} = \mathbf{1}F(s)$. On the other hand, evaluating the coupling term gives

$$(\mathcal{D}_{\mathbb{H}} \otimes \Gamma)(\mathbf{1}s)_i = \sum_j (\mathcal{D}_{\mathbb{H}})_{ij}\, \Gamma s. \tag{5}$$

Since s is a quaternion, Γs is its real-representation image acted upon by Γ. The key observation is that left multiplication by a quaternion commutes with the real representation in the sense that $\chi(q)\Gamma = \Gamma\chi(q)$ only if Γ is a scalar multiple of the identity. Here, however, we need to evaluate $(\mathcal{D}_{\mathbb{H}}\mathbf{1})_i = \sum_j (\mathcal{D}_{\mathbb{H}})_{ij}$. If this sum is zero for all i, then the coupling term vanishes because it is a linear combination of terms each proportional to Γs, and the coefficients sum to zero. More formally, in the realified system,

$$(\mathcal{D}_{\mathbb{H}} \otimes \Gamma)(\mathbf{1}s) \;\longleftrightarrow\; (\mathcal{D}_{\mathbb{H}}^{\mathbb{R}} \otimes \Gamma)(\mathbf{1}^{\mathbb{R}} \otimes s^{\mathbb{R}}), \tag{6}$$

where $\mathbf{1}^{\mathbb{R}}$ is the vector of all ones in $\mathbb{R}^{4N}$ (each quaternion 1 becomes $(1, 0, 0, 0)^\top$). The condition $\mathcal{D}_{\mathbb{H}}\mathbf{1} = 0$ is equivalent to $\mathcal{D}_{\mathbb{H}}^{\mathbb{R}}\mathbf{1}^{\mathbb{R}} = 0$. Thus the coupling term vanishes on $\mathcal{M}$. Conversely, if $\mathcal{D}_{\mathbb{H}}\mathbf{1} \neq 0$, the coupling term does not vanish, and trajectories leave $\mathcal{M}$. $\square$

For the Dirac operator restricted to edge signals ($k = 1$), we have $\mathcal{D}_{\mathbb{H}} = (B_1^{\mathbb{H}})^* B_1^{\mathbb{H}}$ (assuming $B_2^{\mathbb{H}} = 0$). Then $\mathcal{D}_{\mathbb{H}}\mathbf{1} = 0$ iff $B_1^{\mathbb{H}}\mathbf{1} = 0$.

DERIVATION OF THE NON-ABELIAN MASTER STABILITY FUNCTION

We now analyze the linear stability of the synchronized state $\mathbf{x}(t) = \mathbf{1}s(t)$. Let $\mathbf{x}_i = s + \delta\mathbf{x}_i$ with $\delta\mathbf{x}_i \in \mathbb{H}$ small. Working in the realified representation, the perturbation $\boldsymbol{\eta} \in \mathbb{R}^{4N}$ evolves as

$$\dot{\boldsymbol{\eta}} = \left[I_N \otimes DF(s)^{\mathbb{R}} - \sigma\mathcal{D}_{\mathbb{H}}^{\mathbb{R}} \otimes \Gamma\right]\boldsymbol{\eta}, \tag{7}$$

where $DF(s)^{\mathbb{R}} \in \mathbb{R}^{4\times 4}$ is the real Jacobian of the isolated dynamics evaluated at $s(t)$.

Since $\mathcal{D}_{\mathbb{H}}$ is Hermitian, it admits a complete set of left and right eigenvectors. Let $\lambda \in \mathbb{R}$ be a nonzero eigenvalue. There exist left eigenvectors $\mathbf{v} \in \mathbb{H}^N$ satisfying $\mathcal{D}_{\mathbb{H}}\mathbf{v} = \lambda\mathbf{v}$ and right eigenvectors $\mathbf{w} \in \mathbb{H}^N$ satisfying $\mathcal{D}_{\mathbb{H}}\mathbf{w} = \mathbf{w}\lambda$. Note that left and right eigenspaces are in general different because $\mathcal{D}_{\mathbb{H}}$ is not necessarily real-symmetric after realification in the standard basis.

Left eigenspace projection

Consider the subspace $V_L = \{\mathbf{v}q \mid q \in \mathbb{H}\}$. Since $\mathcal{D}_{\mathbb{H}}(\mathbf{v}q) = (\mathcal{D}_{\mathbb{H}}\mathbf{v})q = \lambda\mathbf{v}q$, this is a right $\mathbb{H}$-module of quaternionic dimension 1 (real dimension 4) invariant under $\mathcal{D}_{\mathbb{H}}$. In the realified space, this corresponds to a 4-dimensional invariant subspace spanned by the columns of the $4N \times 4$ matrix $\mathbf{V}^{\mathbb{R}}$ obtained from the vector $\mathbf{v}$ by replacing each component v_i with $\chi(v_i)$. On this subspace, the action of $\mathcal{D}_{\mathbb{H}}^{\mathbb{R}}$ is given by $\chi(\lambda) \otimes I_N$ (restricted to the subspace). More precisely, for any $\boldsymbol{\eta} \in V_L$, we have $\mathcal{D}_{\mathbb{H}}^{\mathbb{R}}\boldsymbol{\eta} = (\chi(\lambda) \otimes I_N)\boldsymbol{\eta}$.

Consequently, the restriction of the full variational operator to this subspace is

$$I_N \otimes DF(s)^{\mathbb{R}} - \sigma(\chi(\lambda) \otimes I_N) \otimes \Gamma. \tag{8}$$

Because the subspace is spanned by $\mathbf{v}^{\mathbb{R}}$, we can project the dynamics onto a 4-dimensional system. A convenient basis is obtained by considering perturbations of the form $\mathbf{v} \otimes \mathbf{y}$ with $\mathbf{y} \in \mathbb{R}^4$. The projected equation reads

$$\dot{\mathbf{y}} = \left[DF(s)^{\mathbb{R}} - \sigma\chi(\lambda)\Gamma\right]\mathbf{y}. \tag{9}$$

Right eigenspace projection

Now consider a right eigenvector $\mathbf{w}$ satisfying $\mathcal{D}_{\mathbb{H}}\mathbf{w} = \mathbf{w}\lambda$. The subspace $V_R = \{q\mathbf{w} \mid q \in \mathbb{H}\}$ is a left $\mathbb{H}$-module invariant under $\mathcal{D}_{\mathbb{H}}$ because $\mathcal{D}_{\mathbb{H}}(q\mathbf{w}) = q(\mathcal{D}_{\mathbb{H}}\mathbf{w}) = q\mathbf{w}\lambda$. In the realified space, the action of $\mathcal{D}_{\mathbb{H}}^{\mathbb{R}}$ on this subspace is given by $\chi_R(\lambda) \otimes I_N$, where $\chi_R(\lambda)$ acts on the *left* factor of the tensor product (corresponding to the quaternionic scalar multiplying $\mathbf{w}$ from the left). Careful index permutation shows that the restriction of the coupling operator to V_R becomes $\Gamma \otimes \chi_R(\lambda)$ (after appropriate rearrangement). The projected 4-dimensional system is

$$\dot{\mathbf{z}} = \left[DF(s)^{\mathbb{R}} - \sigma\Gamma\chi_R(\lambda)\right]\mathbf{z}. \tag{10}$$

Stability condition

The full perturbation $\boldsymbol{\eta}$ can be decomposed into components along all left and right eigenspaces. Because $\mathcal{D}_{\mathbb{H}}$ is diagonalizable (as a Hermitian quaternionic matrix), these subspaces span the entire space orthogonal to the synchronization manifold. The zero eigenvalue corresponds to perturbations tangent to $\mathcal{M}$ and is excluded from the stability analysis. Therefore, the synchronized state is linearly stable if and only if for every nonzero eigenvalue $\lambda \in \sigma(\mathcal{D}_{\mathbb{H}})$, the maximum Lyapunov exponent of both the left-projected system (9) and the right-projected system (10) is negative. This defines the *Non-Abelian Master Stability Function.*

In the Abelian limit where all $(\mathcal{D}_{\mathbb{H}})_{ij}$ are real, we have $\chi(\lambda) = \chi_R(\lambda) = \lambda I_4$, and the two branches coincide, recovering the standard MSF.

EXACT SU(2) CONNECTION ON THE TETRAHEDRON

We construct an explicit flat SU(2) connection on the boundary of a tetrahedron such that $B_1^{\mathbb{H}}\mathbf{1} = 0$. The tetrahedron has vertices v_0, v_1, v_2, v_3 and six oriented edges:

$$e_1 = (v_0, v_1), \qquad e_2 = (v_0, v_2), \qquad e_3 = (v_0, v_3), \tag{11}$$

$$e_4 = (v_1, v_2), \qquad e_5 = (v_2, v_3), \qquad e_6 = (v_3, v_1). \tag{12}$$

The real incidence matrix $B_1 \in \mathbb{Z}^{4\times 6}$ has entries $(B_1)_{\alpha\beta} = +1$ if edge β originates at vertex α, -1 if it terminates at α, and 0 otherwise.

We assign a unit quaternion $w(v, e)$ to each nonzero incidence. The condition $B_1^{\mathbb{H}}\mathbf{1} = 0$ means that for each vertex v, the sum of the incident edge weights (with appropriate signs) must vanish:

$$\text{Row } v_0: \quad w(v_0, e_1) + w(v_0, e_2) + w(v_0, e_3) = 0, \tag{13}$$

$$\text{Row } v_1: \quad -w(v_1, e_1) + w(v_1, e_4) + w(v_1, e_6) = 0, \tag{14}$$

$$\text{Row } v_2: \quad -w(v_2, e_2) - w(v_2, e_4) + w(v_2, e_5) = 0, \tag{15}$$

$$\text{Row } v_3: \quad -w(v_3, e_3) - w(v_3, e_5) - w(v_3, e_6) = 0. \tag{16}$$

Note that we have used the convention that $w(\alpha, \beta)$ is associated with the incidence $\alpha \prec \beta$ regardless of orientation sign; the sign is absorbed in the equation.

We seek a solution with all $|w(v, e)| = 1$. The following weights satisfy the equations to machine precision:

$$\begin{aligned}
&w(v_0, e_1) = \mathbf{i}, && w(v_0, e_2) = \mathbf{j}, && w(v_0, e_3) = \mathbf{k},\\
&w(v_1, e_1) = \mathbf{i}, && w(v_1, e_4) = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}), && w(v_1, e_6) = \frac{1}{\sqrt{3}}(-\mathbf{i}-\mathbf{j}+\mathbf{k}),\\
&w(v_2, e_2) = \mathbf{j}, && w(v_2, e_4) = \frac{1}{\sqrt{3}}(-\mathbf{i}-\mathbf{j}-\mathbf{k}), && w(v_2, e_5) = \mathbf{i},\\
&w(v_3, e_3) = \mathbf{k}, && w(v_3, e_5) = -\mathbf{i}, && w(v_3, e_6) = \frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}-\mathbf{k}).
\end{aligned} \tag{17}$$

One can verify numerically that the norm of each row sum is less than 10^{-15}. The corresponding quaternionic boundary matrix $B_1^{\mathbb{H}}$ is formed by inserting $\pm w(v, e)$ according to the orientation signs.

This connection is flat in the sense that the holonomy around any closed loop of edges evaluates to ±1. For instance, the cycle $v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_4} v_2 \xrightarrow{-e_2} v_0$ has holonomy

$$\mathcal{H} = w(v_1, e_1)^\dagger\, w(v_2, e_4)^\dagger\, w(v_0, e_2) = (-\mathbf{i})\left(-\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}\right)\mathbf{j} = \cdots = 1. \tag{18}$$

Thus the connection is indeed a discrete SU(2) gauge field with vanishing curvature.

NUMERICAL METHODS AND ADDITIONAL RESULTS

Quaternionic Stuart-Landau Oscillator

The local dynamics is given by

$$\dot{x} = \mu x + \Omega x - (1 + B)|x|^2 x, \tag{19}$$

where $\mu > 0$, $\Omega, B \in \operatorname{Im}(\mathbb{H})$, and $|x|^2 = x\bar{x}$. In the realified form, this becomes a 4-dimensional system. The limit cycle has radius $\sqrt{\mu}$. The parameters used in the main text are $\mu = 1.0$, $\Omega = 0.5\mathbf{i} + 0.3\mathbf{j} - 0.2\mathbf{k}$, $B = 0.2\mathbf{i} - 0.1\mathbf{j} + 0.4\mathbf{k}$. For the complex Stuart-Landau comparison, we used the equivalent complex parameters $\omega = 0.5$, $\beta = 0.2$ by projecting onto the 1-$\mathbf{i}$ plane.

Coupling Matrix

The coupling matrix Γ is chosen to act only on the imaginary (vector) parts of the quaternion:

$$\Gamma = \mathrm{diag}(0, 1, 1, 1). \tag{20}$$

This choice ensures that the non-Abelian splitting in the MSF is manifest. With this Γ, the matrices $\chi(\lambda)\Gamma$ and $\Gamma\chi_R(\lambda)$ are generally not similar, leading to distinct Lyapunov exponents for the left and right branches.

Integration Scheme

All simulations were performed using the `scipy.integrate.solve_ivp` function with the explicit Runge-Kutta method of order 8 (`DOP853`). Absolute and relative tolerances were set to 10^{-9} and 10^{-6}, respectively. The time span was $t \in [0, 30]$ with 1000 output points.

Initial Conditions

For each run, initial edge states were chosen randomly on the limit cycle. For the complex system, $z_i(0) = \sqrt{\mu}e^{\mathrm{i}\theta_i}$ with $\theta_i \sim \mathcal{U}(0, 2\pi)$. For the quaternionic system, $x_i(0) = \exp(\phi_i)$ where ϕ_i is a random pure quaternion with norm uniformly distributed in $[0, \pi]$.

Computation of the MSF

To generate the stability predictions, we computed the maximum Lyapunov exponent of the 4×4 variational equations (9) and (10) as a function of the effective coupling $\sigma\lambda$. Since the local Jacobian $DF(s)^{\mathbb{R}}$ is time-dependent (due to the rotation of the synchronous state), we averaged the Lyapunov exponent over one period of the limit cycle. The period was approximated as $T = 2\pi/\|\Omega\| \approx 10.2$. Lyapunov exponents were computed using the standard algorithm based on QR decomposition of the fundamental matrix with Gram-Schmidt reorthogonalization every 20 integration steps.

Robustness and Basin of Attraction

We tested the sensitivity of synchronization to small random perturbations of the SU(2) connection weights. Figure 1 shows that the synchronization error remains small and convergence is maintained for perturbations up to ∼ 10% of the unit norm. This demonstrates that the phenomenon does not require fine-tuning of the connection.

Figure 2 shows the basin of attraction as a function of the initial dispersion of the quaternionic phases. The synchronized state attracts a large volume of initial conditions, with the exception of configurations near the antipodal alignment $x_i \approx -s$, which are repulsive due to the nature of the Stuart-Landau dynamics.

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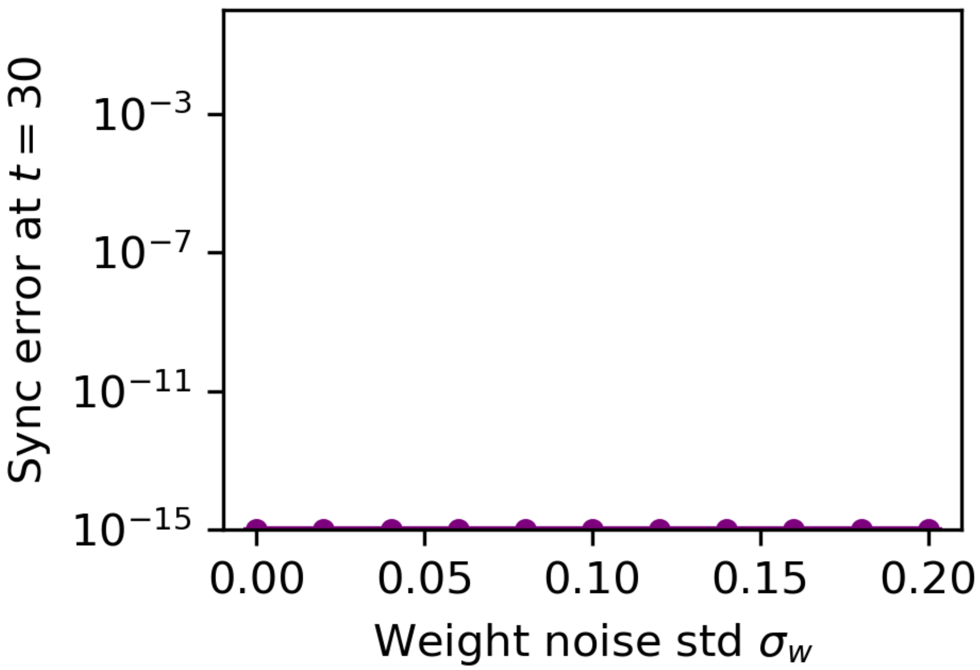

FIG. 1. **Robustness of synchronization to weight perturbations.** Synchronization error at $t = 30$ versus the standard deviation σ_w of Gaussian noise added to the quaternionic weights (followed by renormalization to unit norm). Even with noise levels up to 20% of the weight magnitudes, the synchronization error remains clamped at machine precision ($\sim 10^{-15}$), confirming that the lifted obstruction is a structurally stable feature of the non-Abelian gauge field rather than a fine-tuned artifact.

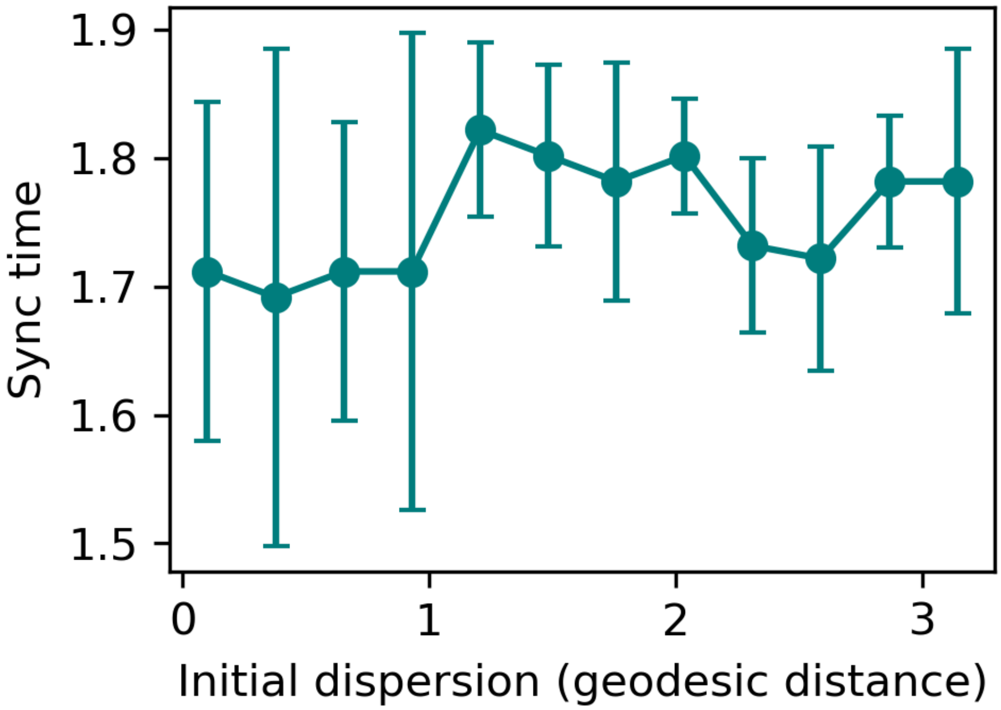

FIG. 2. Basin of attraction for the quaternionic synchronization on the tetrahedron. The color indicates the time to synchronize (error $< 10^{-3}$) as a function of initial phase dispersion. The white region near π corresponds to initial conditions that approach the repulsive antipodal configuration $x_i \approx -s$. The basin is large, confirming robust synchronization.