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An Inverse Problem for a Semilinear Integro-Differential Pseudoparabolic Equation

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Abstract

This paper investigates a semilinear integro-differential pseudoparabolic equation in a bounded domain. In the inverse problem, the unknown source function is reconstructed using an additional integral condition. First, the problem is discretized in time and is reduced to an elliptic problem at each time level. Then, by applying the Lax–Milgram theorem, the existence and uniqueness of the solution at each time step are established. Second, fundamental energy estimates for the unknown functions are derived. Based on these estimates, the existence and uniqueness of the solution over the entire time interval are proved using Rothe's method. After that, the elliptic problem is discretized in space using the Legendre pseudospectral method. The fully discretized scheme leads to a symmetric linear algebraic system, which is advantageous from a computational point of view. Finally, the effectiveness of the constructed algorithm is verified using a numerical method.

Keywords: semilinear parabolic equation, Rothe method, inverse source problem, existence, uniqueness, Legendre pseudospectral method

Mathematics Subject Classification: 35K90 , 34M50

1 Introduction

Many physical processes can be modeled by partial differential equations. This, in turn, leads to the need to study pseudoparabolic differential equations under appropriate boundary conditions. Such types of differential equations arise in the investigation of various physical and engineering models, for example, the motion of homogeneous fluids in fissured rocks, moisture transfer in soils, transport phenomena in porous media, diffusion of trapped radiation in gases, and similar processes. Many researchers have studied pseudoparabolic differential equations and their physical properties; see, for example, references [1–6].

In [7, 8], inverse problems involving the determination of a kernel function in multidimensional third-order integro-differential pseudoparabolic equations were analyzed. These studies proved the global existence and uniqueness of solutions under given initial and boundary conditions.

To investigate the existence and uniqueness of solutions to linear and semilinear partial differential equations, several theoretical frameworks have been developed. Notable approaches include the Carathéodory approximation, the Picard approximation, the Faedo–Galerkin method and Rothe's method. In particular, the time discretization of such problems and the proof of existence and uniqueness of solutions are of both theoretical and practical importance. The idea of solving problems in this manner was first proposed by Eberhard Rothe [9] and later developed by Olga Ladyzhenskaya [10], Karel Rektorys [11] and others. A detailed description of this method can also be found in reference [12].

Using this approach, parabolic differential equations have been studied in [13] and [14]. For instance, in [13], the authors investigated an inverse problem for identifying the source function in a semilinear time-fractional diffusion equation posed on a bounded domain. Using the Rothe method, they proved the existence and uniqueness of a weak solution, and in the numerical experiments they demonstrated the convergence of the numerical solution to the exact solution. In [14], an inverse source problem for a non-autonomous time-fractional diffusion equation of order $0 < \beta < 1$ posed on a bounded domain was studied. The existence and uniqueness of the solution were established by means of the Rothe method.

Moreover, the Rothe method has also been applied to pseudoparabolic integro-differential equations [15–19]. Similarly, Guezane-Lakoud and Belakroum [15] demonstrated the existence and uniqueness of weak solutions for an integrodifferential Sobolev-type equation with integral conditions. Maqbul and Raheem [16] investigated the existence and uniqueness of solutions to a semilinear pseudoparabolic equation with purely integral conditions by means of the Rothe method. In references [17–19], the Rothe–Legendre pseudospectral method has been investigated, where the elliptic problem is fully discretized and exact and approximate solutions of the resulting system are compared.

In addition to the introduction, the paper is organized into three main sections. Section 2 is devoted to the statement of the problem and the presentation of the required preliminary concepts. The problem is then reformulated in a suitable weak sense, after which a backward finite-difference time discretization scheme is applied in Section 3 to approximate the resulting formulation by a sequence of classical elliptic

boundary value problems. Within the same section, several stability properties are derived, and the convergence of the time semi-discrete solutions toward the unique solution of the original problem is subsequently established. Section 4 briefly outlines the implementation of the Legendre pseudospectral method for spatial discretization. Finally, the last section presents numerical experiments that demonstrate the accuracy and efficiency of the proposed numerical approach.

Now we present the formulation of the problem and the main concepts and formulas necessary to prove the existence and uniqueness of its solution.

2 Statement of the problem and preliminaries

Let Ω be a bounded subdomain of $\mathbb{R}^d, (d \geq 1)$ with a Lipschitz continuous boundary $\partial\Omega$. In this work, we study the following semilinear pseudoparabolic equation:

$$\partial_t u - \partial_t \Delta u - \Delta u + u = q(t)f(x) + \int_0^t F(x, s, u(x, s))ds, \quad (x, t) \in D_T, \tag{1}$$

where $\partial_t = \frac{\partial}{\partial t}$, $D_T = \Omega \times (0, T)$, Δ is Laplace operator, $q(t)$ is unknown source function, the function $f(x)$ represents the spatial location of the source. and F is assumed to be Lipschitz continuous with respect to all its variables.

The problem (1) is considered together with the following initial and boundary conditions

$$u(x, 0) = \phi(x), \quad x \in \Omega, \tag{2}$$

$$\mathbf{B}u(x, t) = g(x, t), \quad (x, t) \in \partial\Omega \times [0, T], \tag{3}$$

where the boundary operator $\mathbf{B}$ is defined by

$$\mathbf{B}u = \nabla u \cdot \nu.$$

The symbol ν denotes the outward unit normal vector to $\partial\Omega$.

The inverse source problem studied in this paper consists in determining the pair functions $(u(x, t), q(t))$ satisfying (1)–(3) together with the integral condition

$$\int_\Omega u(x, t)\, dx = h(t), \quad t \in [0, T]. \tag{4}$$

Notations. Denote by $(., .)$ the standard inner product of $L^2(\Omega)$ and $\| \cdot \|$ its introduced norm. When working on the boundary $\partial\Omega$ we use a similar notation, namely $(., .)_{\partial\Omega}$. If X is a Banach space with the norm $\| \cdot \|_X$, then by $C([0, T], X)$ we denote the set of abstract functions $\omega : [0, T] \to X$ endowed with the usual norm $\max_{[0,T]} \| \cdot \|_X$.

Integrating equation (1) over the domain Ω, applying Green's theorem, and taking into account condition (4), we obtain the following result

$$h'(t) - (g'(t), 1)_{\partial\Omega} - (g(t), 1)_{\partial\Omega} + h(t) = q(t)(f, 1) + \int_0^t (F(s, u(s)), 1)ds. \quad (5)$$

Assume that $(f, 1) \neq 0$, we have

$$q(t) = \frac{h'(t) + h(t) - (g'(t), 1)_{\partial\Omega} - (g(t), 1)_{\partial\Omega} - \int_0^t (F(s, u(s)), 1)ds}{(f, 1)}. \quad (6)$$

Definition 1 The function u is said to be a weak solution of the problem (1) - (3) if the followings hold:

1. $u \in C([0, T], H^1(\Omega))$;
2. $\partial_t u \in L^2((0, T), H^1(\Omega))$;
3. the function u satisfies the following weak form:

$$(\partial_t u, \varphi) + (\partial_t \nabla u, \nabla\varphi) + (\nabla u, \nabla\varphi) + (u, \varphi) =$$

$$(g'(t), \varphi)_{\partial\Omega} + (g(t), \varphi)_{\partial\Omega} + q(t)(f, \varphi) + \Big(\int_0^t F(s, u(s))ds, \varphi\Big), \quad (7)$$

for any $\varphi \in H^1(\Omega)$ and for all $t \in [0, T]$.

To establish our main result, we impose the following assumptions:

(A1) $\phi \in L^2(\Omega)$,
(A2) $f \in L^2(\Omega)$, $\quad (f, 1) \neq 0$,,
(A3) $g \in C^1(\partial\Omega \times [0, T])$,
(A4) F is a globally Lipschitz continuous function in all variables,
(A5) $h(t) \in C^2[0, T]$.

Preliminaries.

In this work, the analysis is carried out in a variational setting. For this purpose, we begin by introducing the required function spaces together with the corresponding definitions. Throughout the paper, the symbols C (or C_j), ε, and $C(\varepsilon)$ denote generic positive constants depending only on the given data, where ε stands for a sufficiently small parameter.

The notation $\theta(t)$ will be used to represent the function $\theta(x, t)$ when it is viewed as an abstract function of the time variable $t \in [0, T]$ taking values in a suitable functional space over Ω. Strong and weak convergences are denoted by $\to$ and $\rightharpoonup$, respectively.

Cauchy's inequality:(see [20], pp. 706)

$$\beta\gamma \leq \frac{\epsilon}{2}\beta^2 + \frac{1}{2\epsilon}\gamma^2, \quad \forall\beta, \gamma > 0, \quad \forall\epsilon > 0.$$

Young's inequality:(see [20], pp. 706) Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \epsilon a^p + C(\epsilon)b^q, \quad \forall a, b > 0, \quad \forall\epsilon > 0,$$

for $C(\epsilon) = (\epsilon p)^{\frac{-q}{p}} q^{-1}$.

Trace inequality:(see [21], pp. 135-136) If $\omega \in H^1(\Omega)$, then

$$\|\omega\|_{L^2(\partial\Omega)} \leq \epsilon\|\nabla\omega\|_{L^2(\Omega)} + l(\epsilon)\|\omega\|_{L^2(\Omega)},$$

where $l(\varepsilon)$ is a positive constant that depends solely on ε and the domain Ω.

Holder's inequality:(see [22], pp. 63-64) Assume $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(\Omega), v \in L^q(\Omega)$, we have

$$\int_\Omega |uv|dx \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

Gronwall lemma: (see [20], pp. 708-709)

(**i**) Let $x(t) \geq 0, m(t), y(t)$ be real integrable functions on the interval $[a, b]$. If

$$y(t) \leq m(t) + \int_0^t x(\tau)y(\tau)d\tau, \forall t \in [a, b],$$

then

$$y(t) \leq m(t) + \int_0^t m(\tau)x(\tau)exp\Big(\int_\tau^t x(s)ds\Big)d\tau, \forall t \in [a, b].$$

In particular, if $x(t) \equiv C$ is a constant and $m(\tau)$ is nondecreasing, then

$$y(t) \leq m(t)e^{C(t-a)}, \forall t \in [a, b].$$

(**ii**) Let $\{a_i\}$ be a sequence of real nonnegative numbers satisfying

$$a_1 \leq a,$$

$$a_i \leq a + bm\sum_{k=1}^{i-1} a_k, \quad \forall i = 2, ...,$$

where a, b and m are positive constants. Then

$$a_i \leq ae^{b(i-1)m}, \quad \forall i = 1, 2, ...$$

Lax-Milgram theorem.(see [20], pp. 316-320) Assume that

$$A : H \times H \to \mathbb{R}$$

is bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

(**i**) **(continuous)** $|A[u, v]| \leq \alpha\|u\| \cdot \|v\|, \quad (u, v \in H)$, and

(**ii**) **(coercive)** $\beta\|u\|^2 \leq A[u, u], \quad u \in H.$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

$$A[u, v] = (f, v)$$

for all $v \in H$.

Lebesgue dominated theorem. (see [21], pp. 53) Let us assume that the sequence of functions $\{Y_n\}$ converges pointwise to a function Y, that is,

$$\lim_{n\to\infty} Y_n(x) = Y(x)$$

for every $x \in \Omega$. Furthermore, suppose that $\{Y_n\}$ is dominated by some integrable function k, in the sense that

$$|Y_n(x)| \le k(x),$$

for all $x \in \Omega$. Then both Y_n and Y are integrable in the Lebesgue sense, and

$$\lim_{n\to\infty} \int_\Omega Y_n(x)\,dx = \int_\Omega \lim_{n\to\infty} Y_n(x)\,dx = \int_\Omega Y(x)\,dx.$$

Moreover, we have

$$\lim_{n\to\infty} \int_\Omega \Big(Y_n(x) - Y(x)\Big)\,dx = 0.$$

In this section, we discretize the problem (5), (7) with respect to time and prove the existence and uniqueness of the solution at each time level based on the Lax–Milgram theorem.

3 Time discretization

Through Rothe's method, the problem is approximated by a sequence of elliptic problems by discretization of time and solved successively at each time step. The time interval $[0, T]$ is divided into equal subintervals $n \in \mathbb{N}$ $[t_{i-1}, t_i]$, each of length $\tau = \frac{T}{n}$. The value of a function u at time $t = t_i$, for $1 \le i \le n$, is denoted by z_i. This notation is similarly used for any other given function. In addition, the time derivative at t_i is approximated by the backward Euler finite difference, that is,

$$z(t_i) = z_i, \quad z_t(t_i) \approx \delta z_i = \frac{z_i - z_{i-1}}{\tau}, \quad 1 \le i \le n.$$

We approximate problem (5) and (7) at time t_i as follows:

$$\delta h_i - (\delta g_i, 1)_{\partial\Omega} - (g_i, 1)_{\partial\Omega} + h_i = q_i(f, 1) + \Big(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, 1\Big), \tag{8}$$

$$(\delta u_i, \varphi) + (\delta \nabla u_i, \nabla\varphi) + (\nabla u_i, \nabla\varphi) + (u_i, \varphi) =$$

$$(\delta g_i, \varphi)_{\partial\Omega} + (g_i, \varphi)_{\partial\Omega} + q_i(f, \varphi) + \Big(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, \varphi\Big). \tag{9}$$

Equations (9) and (8) are linear with respect to u_i and q_i, respectively, and both equations are decoupled for each index i. Consequently, we first compute q_1 from (8), and then determine u_1 from (9) by using the values $u_0 = \phi$ and $u_{-1} = u(-\tau) = 0$.

This procedure is then repeated for each time step $i = 2, 3, \ldots, n$. Finally, over the entire time interval, we construct the sequences $\{q_i\}_{i=1}^n$ and $\{u_i\}_{i=1}^n$.

Lemma 1 Assume that conditions $(A1) - (A5)$ are satisfied. Then, for each $i \in \{1, 2, \ldots, n\}$, there exists a unique pair (u_i, q_i) where $u_i \in H^1(\Omega)$ and $q_i \in \mathbb{R}$, that solves (9) and (8).

Proof By rewriting (8) to isolate q_i, we get:

$$q_i = \frac{\delta h_i - (\delta g_i, 1)_{\partial\Omega} - (g_i, 1)_{\partial\Omega} + h_i - \left(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, 1\right)}{(f, 1)}. \tag{10}$$

Equation (10) provides a recurrence formula for q_i, $i \geq 1$. Hence, q_i can be determined for $i = 1, \ldots, n$. In addition, for $i = 0$,

$$q_0 = \frac{\delta h_0 - (\delta g_0, 1)_{\partial\Omega} - (g_0, 1)_{\partial\Omega} + h_0}{(f, 1)} \in \mathbb{R}. \tag{11}$$

The identity (9) can be reformulated as:

$$\begin{aligned} &(u_i, \varphi) + (\nabla u_i, \nabla\varphi) + \tau(\nabla u_i, \nabla\varphi) + \tau(u_i, \varphi) \\ &= (g_i, \varphi)_{\partial\Omega} - (g_{i-1}, \varphi)_{\partial\Omega} + \tau(g_i, \varphi)_{\partial\Omega} + \tau q_i(f, \varphi) \\ &+\tau\Big(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, \varphi\Big) + (u_{i-1}, \varphi) + (\nabla u_{i-1}, \nabla\varphi), \quad \varphi \in H^1(\Omega). \end{aligned} \tag{12}$$

The left-hand sides of equations (10) and (12) define a continuous and coercive bilinear form on $H^1(\Omega) \times H^1(\Omega)$, as derived from (8) and (9). Assuming that $u_0, u_1, \ldots, u_{i-1} \in L^2(\Omega)$, the right-hand sides correspond to linear and bounded functionals on $H^1(\Omega)$. By invoking the Lax-Milgram theorem from the theory of linear elliptic partial differential equations, we deduce the existence and uniqueness of u_i in $H^1(\Omega)$. □

In this subsection, we derive energy estimates for the unknown functions involved in the equation.

3.1 Some energy estimates

The next two lemmas are needed to guarantee the existence of a solution to (5)-(7) and to establish the convergence of the approximate solutions toward this solution.

Lemma 2 Assume that the conditions $(A1) - (A5)$ are satisfied. Then, there exist positive constants C and for each $i = 1, 2, \ldots, n$, the following relations hold:

(**i**)

$$\begin{aligned} &\max_{1\leq i\leq n} \|u_i\|^2 + \max_{1\leq i\leq n} \|\nabla u_i\|^2 \\ &+\sum_{i=1}^{n} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{n} \|\nabla u_i - \nabla u_{i-1}\|^2 + \sum_{i=1}^{n} \|u_i\|^2\tau + \sum_{i=1}^{n} \|\nabla u_i\|^2\tau \leq C, \end{aligned}$$

(**ii**) $\max_{1\leq i\leq n} |q_i| \leq C$

Proof Starting from (10) we see that

$$|q_i| \leq C + C\sum_{k=1}^{i} \|F(t_k, u_{k-1})\|\tau \leq C + C\sum_{k=0}^{i-1} \|u_k\|\tau. \tag{13}$$

Take $\varphi = u_i\tau$ in (9) and sum the resulting expression for $i = 1, \ldots, j$ to obtain

$$\sum_{i=1}^{j}(\delta u_i, u_i)\tau + \sum_{i=1}^{j}(\delta\nabla u_i, \nabla u_i)\tau + \sum_{i=1}^{j}(\nabla u_i, \nabla u_i)\tau + \sum_{i=1}^{j}(u_i, u_i)\tau = \sum_{i=1}^{j}(\delta g_i, u_i)_{\partial\Omega}\tau$$
$$+ \sum_{i=1}^{j}(g_i, u_i)_{\partial\Omega}\tau + \sum_{i=1}^{j} q_i(f, u_i)\tau + \sum_{i=1}^{j}\Big(\sum_{k=1}^{i} F(t_k, u_{k-1})\tau, u_i\Big)\tau. \tag{14}$$

The first term on the left hand side (LHS) of (14) can be rewritten as:

$$\sum_{i=1}^{j}(\delta u_i, u_i)\tau = \sum_{i=1}^{j}\Big(\frac{u_i - u_{i-1}}{\tau}, u_i\Big)\tau = \sum_{i=1}^{j}(u_i - u_{i-1}, u_i) = \sum_{i=1}^{j}\Big(\|u_i\|^2 - (u_{i-1}, u_i)\Big). \tag{15}$$

We employ the following identity:

$$\|u_i - u_{i-1}\|^2 = \|u_i\|^2 + \|u_{i-1}\|^2 - 2(u_i, u_{i-1}),$$

hence we obtain

$$(u_i, u_{i-1}) = \frac{1}{2}\Big(\|u_i\|^2 + \|u_{i-1}\|^2 - \|u_i - u_{i-1}\|^2\Big),$$

substitute into the sum:

$$\sum_{i=1}^{j}(u_i, u_{i-1}) = \sum_{i=1}^{j}\frac{1}{2}\Big(\|u_i\|^2 + \|u_{i-1}\|^2 - \|u_i - u_{i-1}\|^2\Big)$$
$$= \frac{1}{2}\sum_{i=1}^{j}(\|u_i\|^2 + \|u_{i-1}\|^2) - \frac{1}{2}\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2.$$

Substituting the above identity into (15), we obtain:

$$\sum_{i=1}^{j}(u_i - u_{i-1}, u_i) = \sum_{i=1}^{j}\|u_i\|^2 - \sum_{i=1}^{j}(u_{i-1}, u_i)$$
$$= \sum_{i=1}^{j}\|u_i\|^2 - \frac{1}{2}\Big[\sum_{i=1}^{j}(\|u_i\|^2 + \|u_{i-1}\|^2) - \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2\Big]$$
$$= \frac{1}{2}\sum_{i=1}^{j}(\|u_i\|^2 - \|u_{i-1}\|^2) + \frac{1}{2}\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2.$$

In the above equality, the first term simplifies as the intermediate terms cancel, leaving only the first and last terms:

$$\sum_{i=1}^{j}(\|u_i\|^2 - \|u_{i-1}\|^2) = \|u_j\|^2 - \|u_0\|^2 = \|u_j\|^2 - \|\phi\|^2.$$

Therefore, LHS of (14) yields the following result:

$$\sum_{i=1}^{j}(\delta u_i, u_i)\tau = \frac{1}{2}\Big[\|u_j\|^2 - \|\phi\|^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2\Big].$$

The second term of equation (14) is rewritten to resemble the first term:

$$\sum_{i=1}^{j}(\delta\nabla u_i, \nabla u_i)\tau = \frac{1}{2}\Big[\|\nabla u_j\|^2 - \|\ \nabla u_0\|^2 + \sum_{i=1}^{j}\|\nabla u_i - \nabla u_{i-1}\|^2\Big].$$

For the other terms on the LHS of equation (14), the following equalities hold:

$$\sum_{i=1}^{j}(\nabla u_i, \nabla u_i)\tau = \sum_{i=1}^{j}\|\nabla u_i\|^2\tau,$$
$$\sum_{i=1}^{j}(u_i, u_i)\tau = \sum_{i=1}^{j}\|u_i\|^2\tau.$$

By using the Cauchy, Young, and Trace inequalities, the initial two terms on the right hand side (RHS) of (14) can be estimated:

$$\sum_{i=1}^{j}(\delta g_i, u_i)_{\partial\Omega}\tau$$
$$= \sum_{i=1}^{j}(\frac{g_i - g_{i-1}}{\tau}, u_i)_{\partial\Omega}\tau = \sum_{i=1}^{j}(g_i - g_{i-1}, u_i)_{\partial\Omega} = \sum_{i=1}^{j}(g_i, u_i)_{\partial\Omega} - \sum_{i=1}^{j}(g_{i-1}, u_i)_{\partial\Omega}$$
$$\leq \Big|\sum_{i=1}^{j}(g_i, u_i)_{\partial\Omega} - \sum_{i=1}^{j}(g_{i-1}, u_i)_{\partial\Omega}\Big| \leq \Big|\sum_{i=1}^{j}(g_i, u_i)_{\partial\Omega}\Big| + \Big|\sum_{i=1}^{j}(g_{i-1}, u_i)_{\partial\Omega}\Big|$$
$$\leq \epsilon\sum_{i=1}^{j}\|u_i\|^2_{\partial\Omega} + C(\epsilon) \leq C_1(\epsilon)\sum_{i=1}^{j}\|u_i\|^2 + C_1(\epsilon)\sum_{i=1}^{j}\|\nabla u_i\|^2 + C(\epsilon),$$
$$\Big|\sum_{i=1}^{j}(g_i, u_i)_{\partial\Omega}\tau\Big| \leq \epsilon\sum_{i=1}^{j}\|u_i\|^2\tau_{\partial\Omega} + C(\epsilon)$$
$$\leq C_2(\epsilon)\sum_{i=1}^{j}\|u_i\|^2\tau + C_2(\epsilon)\sum_{i=1}^{j}\|\nabla u_i\|^2\tau + C(\epsilon).$$

For the third term on the RHS of (14), using Cauchy and Young's inequalities can be readily estimated and taking into account (13)

$$\Big|\sum_{i=1}^{j}q_i(f, u_i)\tau\Big|$$
$$\leq \epsilon\sum_{i=1}^{j}\|u_i\|^2\tau + C(\epsilon)\sum_{i=1}^{j}q_i^2\tau \leq \epsilon\sum_{i=1}^{j}\|u_i\|^2\tau + C(\epsilon) + C(\epsilon)\sum_{i=1}^{j}\sum_{k=0}^{i-1}\|u_k\|^2\tau^2.$$

The last term of (14) we again apply the Cauchy and Young's inequalities to get

$$\Big|\sum_{i=1}^{j}\Big(\sum_{k=1}^{i}F(t_k, u_{k-1})\tau, u_i\Big)\tau\Big| \leq \epsilon\sum_{i=1}^{j}\|u_i\|^2\tau + C(\epsilon) + C(\epsilon)\sum_{i=1}^{j}\sum_{k=0}^{i-1}\|u_k\|^2\tau^2.$$

By gathering all the estimates, we arrive at:

$$\frac{1}{2}\|u_j\|^2+\frac{1}{2}\|\nabla u_j\|^2+\frac{1}{2}\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2+\frac{1}{2}\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2+(1-\epsilon)\sum_{i=1}^{j}\|u_i\|^2\tau$$
$$+(C_0-\epsilon)\sum_{i=1}^{j}\|\nabla u_i\|^2\tau\leq C(\epsilon)+C(\epsilon)\sum_{i=1}^{j}\sum_{l=0}^{i-1}\|u_l\|^2\tau^2,$$

therefore, using the discrete Gronwall lemma with a fixed small $0<\varepsilon<1$:

$$\|u_j\|^2+\|\nabla u_j\|^2+\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2+\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2+\sum_{i=1}^{j}\|u_i\|^2\tau+\sum_{i=1}^{j}\|\nabla u_i\|^2\tau\leq C.$$

This together with (13) imply $|q_i|\leq C$. □

Lemma 3 Assume that the conditions $(A1)-(A5)$ are satisfied. Then, there exist positive constants C and for each $i=1,2,\ldots,n$, the following relations hold:

(**i**)

$$\max_{1\leq i\leq n}\|\delta u_i\|^2+\max_{1\leq i\leq n}\|\delta\nabla u_i\|^2+\sum_{i=1}^{n}\|\delta u_i\|^2\tau+\sum_{i=1}^{n}\|\delta\nabla u_i\|^2\tau+\sum_{i=1}^{n}\|\delta u_i-\delta u_{i-1}\|^2$$
$$+\sum_{i=1}^{n}\|\delta\nabla u_i-\delta\nabla u_{i-1}\|^2\leq C,$$

(**ii**) $|\delta q_i|\leq C$.

Proof Subtract (11) from (10) for $i=1$, the result is divided by τ to get

$$\delta q_1=\frac{q_1-q_0}{\tau}=\frac{\delta^2 h_1-(\delta^2 g_1,1)_{\partial\Omega}-(\delta g_1,1)_{\partial\Omega}+\delta h_1-(F(\tau,u_0),1)}{(f,1)}.$$

The estimate of δq_1

$$|\delta q_1|\leq\frac{|\delta^2 h_1|+|\delta h_1|+\|\delta^2 g_1\|_{L^2(\Omega)}+\|\delta g_1\|_{L^2(\Omega)}+\tau\|u_0\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}},$$

The functions on the right-hand side of the inequality are given functions. If conditions (A1)–(A5) are satisfied, then we obtain $|\delta q_1|\leq C$.

Now taking a difference of (11) for $i\geq 2$ we deduce that

$$\delta q_i=\frac{\delta^2 h_i-(\delta^2 g_i,1)_{\partial\Omega}-(\delta g_i,1)_{\partial\Omega}+\delta h_i-(F(t_i,u_{i-1}),1)}{(f,1)}.$$

That is why the following relationship holds true

$$|\delta q_i|\leq C+C\|u_{i-1}\|,\quad\forall i\geq 1. \tag{16}$$

Taking a difference of (9), we obtain:

$$(\delta(\delta u_i),\varphi)+(\delta(\delta\nabla u_i),\nabla\varphi)+(\delta\nabla u_i,\nabla\varphi)+(\delta u_i,\varphi)$$
$$=(\delta(\delta g_i),\varphi)_{\partial\Omega}+(\delta g_i,\varphi)_{\partial\Omega}+\delta q_i(f,\varphi)+(F(t_i,u_{i-1}),\varphi). \tag{17}$$

For $i \geq 2$, this difference may be used. Setting $\varphi = \delta u_i \tau$ in (17) and summing over $i = 1, \ldots, j$ $(j \leq n)$ yields

$$\sum_{i=1}^{j}(\delta(\delta u_i), \delta u_i)\tau + \sum_{i=1}^{j}(\delta(\delta\nabla u_i), \nabla\delta u_i)\tau + \sum_{i=1}^{j}(\delta\nabla u_i, \delta\nabla u_i)\tau + \sum_{i=1}^{j}(\delta u_i, \delta u_i)\tau$$

$$= \sum_{i=1}^{j}(\delta(\delta g_i), \delta u_i)\tau_{\partial\Omega} + \sum_{i=1}^{j}(\delta g_i, \delta u_i)\tau_{\partial\Omega} + \sum_{i=1}^{j}\delta q_i(f, \delta u_i)\tau + \sum_{i=1}^{j}(F(t_i, u_{i-1}), \delta u_i)\tau. \quad (18)$$

According to the summation by parts formula, we have

$$\sum_{i=1}^{j}(\delta(\delta u_i), \delta u_i)\tau = \frac{1}{2}\Big[\|\delta u_j\|^2 - \|\delta u_0\|^2 + \sum_{i=1}^{j}\|\delta u_i - \delta u_{i-1}\|^2\Big],$$

$$\sum_{i=1}^{j}(\delta(\delta\nabla u_i), \nabla\delta u_i)\tau = \frac{1}{2}\Big[\|\delta\nabla u_j\|^2 - \|\ \delta\nabla u_0\|^2 + \sum_{i=1}^{j}\|\delta\nabla u_i - \delta\nabla u_{i-1}\|^2\Big],$$

$$\sum_{i=1}^{j}(\delta\nabla u_i, \nabla\delta u_i)\tau = \sum_{i=1}^{j}\|\delta\nabla u_i\|^2\tau,$$

$$\sum_{i=1}^{j}(\delta u_i, \delta u_i)\tau = \sum_{i=1}^{j}\|\delta u_i\|^2\tau,$$

on the RHS of (18):

$$\Big|\sum_{i=1}^{j}(\delta(\delta g_i), \delta u_i)\tau_{\partial\Omega}\Big| \leq C_2(\epsilon)\sum_{i=1}^{j}\|\delta u_i\|^2 + C_2(\epsilon)\sum_{i=1}^{j}\|\delta\nabla u_i\|^2 + C(\epsilon),$$

$$\Big|\sum_{i=1}^{j}(\delta g_i, \delta u_i)\tau_{\partial\Omega}\Big| \leq C(\epsilon) + C(\epsilon)\sum_{i=1}^{j}\|\delta u_i\|^2_{\partial\Omega}\tau \leq C(\epsilon) + C(\epsilon)\sum_{i=1}^{j}\|\delta u_i\|^2_{H^1(\Omega)}\tau,$$

$$\Big|\sum_{i=1}^{j}\delta q_i(f, \delta u_i)\tau\Big| \leq C\sum_{i=1}^{j}|\delta q_i|\|\delta u_i\|\tau \leq C(\epsilon) + C(\epsilon)\sum_{i=1}^{j}\|\delta u_i\|^2\tau,$$

and

$$\Big|\sum_{i=1}^{j}(F(t_i, u_{i-1}), \delta u_i)\tau\Big| \leq \epsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau + C(\epsilon).$$

By combining all the above estimates, we obtain:

$$\frac{1}{2}\|\delta u_j\|^2 + \frac{1}{2}\|\delta\nabla u_j\|^2 + \frac{1}{2}\sum_{i=1}^{j}\|\delta u_i - \delta u_{i-1}\|^2 + \frac{1}{2}\sum_{i=1}^{j}\|\delta\nabla u_i - \delta\nabla u_{i-1}\|^2$$

$$+(C_1 - \epsilon)\sum_{i=1}^{j}\|\delta u_i\|^2\tau + (C_0 - \epsilon)\sum_{i=1}^{j}\|\delta\nabla u_i\|^2\tau \leq C(\epsilon).$$

By choosing a sufficiently small $0 < \epsilon < 1$, we obtain:

$$\max_{1\leq j\leq n}\|\delta u_j\|^2 + \max_{1\leq j\leq n}\|\delta\nabla u_j\|^2 + \sum_{i=1}^{j}\|\delta u_i\|^2\tau + \sum_{i=1}^{j}\|\delta\nabla u_i\|^2\tau + \sum_{i=1}^{j}\|\delta u_i - \delta u_{i-1}\|^2$$

$$+\sum_{i=1}^{j}\|\delta\nabla u_i-\delta\nabla u_{i-1}\|^2\le C.$$

This together with (16) implies that

$$|\delta q_i|\le C+C\|u_{i-1}\|\le C.$$

□

In this subsection, we use the obtained energy estimates to prove the existence of a weak solution to (5), (7).

3.2 Existence of a solution

We now define the Rothe function $u_n:[0,T]\to H^1(\Omega)$ by setting

$$u_n: t\longmapsto\begin{cases}u_0, & t=0,\\ u_{i-1}+(t-t_{i-1})\,\delta u_i, & t\in(t_{i-1},t_i],\quad 1\le i\le n,\end{cases}$$

we define the following auxiliary piecewise-constant function as

$$\overline{u}_n: t\longmapsto\begin{cases}u_0, & t=0,\\ u_i, & t\in(t_{i-1},t_i],\quad 1\le i\le n,\end{cases}$$

and

$$\widetilde{u}_n: t\longmapsto\begin{cases}u_0, & t\in[0,\tau]\\ \overline{u}_n(t-\tau), & t\in(t_{i-1},t_i],\quad 2\le i\le n.\end{cases}$$

We expect that the limit $u:=\lim_{n\to\infty}u_n$ exists in an appropriate sense and represents the weak solution desired to our problem (1) - (3). Similarly, we define the functions $q_n,\overline{q}_n,\overline{F}_n,\overline{h'}_n,\ \overline{g'}_n,\overline{g}_n$. Using these Rothe functions, the variational system (7) and (5) can be reformulated throughout the time interval (for $t\in(t_{i-1},t_i]$)

$$(\partial_t u_n,\varphi)+(\partial_t\nabla u_n,\nabla\varphi)+(\nabla\overline{u}_n,\nabla\varphi)+(\overline{u}_n,\varphi)=$$

$$(\overline{g'}_n,\varphi)_{\partial\Omega}+(\overline{g}_n,\varphi)_{\partial\Omega}+\overline{q}_n(f,\varphi)+\Big(\int_0^{t_i}\overline{F}_n(\tau,\widetilde{u}_n(\tau))d\tau,\varphi\Big),\tag{19}$$

$$\overline{h'}_n-(\overline{g'}_n,1)_{\partial\Omega}-(\overline{g}_n,1)_{\partial\Omega}+\overline{h}_n=\overline{q}_n(f,1)+\Big(\int_0^{t_i}\overline{F}_n(\tau,\widetilde{u}_n(\tau))d\tau,1\Big).\tag{20}$$

We are now ready to prove the existence of a variational solution to (5) and (7).

Theorem 1 *(Existence of a solution). Assume that* $(A1)-(A5)$ *and* (11) *are satisfied. Then, there exists a weak solution* (u,q) *to* (5) *and* (7)*, where* $u\in C([0,T],H^1(\Omega))$ *with* $\partial_t u\in L^2((0,T),H^1(\Omega))$*,* $q\in C([0,T])$*.*

Proof The estimate from Lemma 3 (**ii**) implies for $t \in [t_{i-1}, t_i]$ that

$$|q_n'(t)| = |\delta q_i| \leq C.$$

Using the identity $q(t) - q(s) = \int_s^t \partial_t q(z) dz$, therefore

$$|q_n(t+\epsilon) - q_n(t)| = \Big| \int_t^{t+\epsilon} q_n'(x) dx \Big| \leq C\epsilon, \quad t+\epsilon, t \in [t_{i-1}, t_i],$$

which yields the equi-continuity of q_n. Lemma 2 guarantees the equi-boundedness of q_n. By means of the Arzelà-Ascoli theorem ([22], pp. 245-246) we get compactness of $q_n \in C([0,T])$.

From Lemmas 2 and 3, we obtain $\|u_j\| \leq C$ and $\sum_{i=1}^n \|\delta u_i\|^2 \tau \leq C$, which implies that for all $n > 0$,

$$\|\overline{u}_n(t)\|_{H^1(\Omega)} + \int_0^T \|\partial_t u_n\|_{H^1(\Omega)}^2 \, dt \leq C.$$

Due to the compact embedding ensured by the Rellich–Kondrachov theorem ([20], pp. 286-289), we have

$$H^1(\Omega) \hookrightarrow L^2(\Omega),$$

applying the generalized Aubin–Lions lemma ([23], Lemma 7.7), there exist a function $u \in C([0,T], L^2(\Omega)) \cap L^\infty((0,T), H^1(\Omega))$ and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\begin{cases} u_n \to u \quad in \quad C([0,T], L^2(\Omega)), \\ u_n(t) \rightharpoonup u(t) \quad in \quad H^1(\Omega), \quad \forall t \in [0,T], \\ \overline{u}_n \rightharpoonup u(t) \quad in \quad H^1(\Omega), \quad \forall t \in [0,T], \\ \partial_t u_n \rightharpoonup \partial_t u \quad in \quad L^2((0,T), L^2(\Omega)). \end{cases} \tag{21}$$

Using Lemma 3 together with the identity $u(t) - u(s) = \int_s^t \partial_t u(z) dz$, we obtain

$$\|u(t) - u(s)\|_{H^1(\Omega)} \leq \sqrt{t-s} \sqrt{\int_0^T \|\partial_t u(z)\|_{H^1(\Omega)}^2 dz}. \tag{22}$$

Furthermore, since $\|\delta u_j\| \leq C$, it follows that $\partial_t u \in L^\infty((0,T), L^2(\Omega))$. Consequently, the mapping $u : [0,T] \to L^2(\Omega)$ is Lipschitz continuous, that is

$$\|u(t) - u(t')\| \leq C|t - t'| \quad \text{for all } t, t' \in [0,T].$$

From the boundedness of

$$\sum_{i=1}^n \|u_i\|^2 \tau + \sum_{i=1}^n \|\nabla u_i\|^2 \tau$$

and the fact that $u \in L^\infty((0,T), H^1(\Omega))$, we conclude that

$$\int_0^T \|\overline{u}_n - u\|_{\partial\Omega}^2 d\xi \leq \epsilon \int_0^T \|\nabla(\overline{u} - u)\|^2 d\xi + C(\epsilon) \int_0^T \|\overline{u} - u\|^2 d\xi \leq C(\epsilon). \tag{23}$$

Taking the limit and using (21), we obtain that

$$\lim_{n\to+\infty} \int_0^T \|\overline{u}_n - u\|_{\partial\Omega}^2 d\xi \leq \epsilon, \quad \overline{u} \to u, \quad a.e \quad in \quad (0,T) \times \partial\Omega.$$

By the Trace theorem together with Lemmas 2 and 3, we obtain

$$\|u_n(t+\epsilon) - u_n(t)\|_{\partial\Omega} = \Big\| \int_t^{t+\epsilon} \partial_t u_n(s) ds \Big\|_{\partial\Omega} \leq \int_t^{t+\epsilon} \|\partial_t u_n(s)\|_{\partial\Omega} ds$$

$$\leq \sqrt{\epsilon}\sqrt{\int_t^{t+\epsilon} \|\partial_t u_n(s)\|_{\partial\Omega} ds} \leq \sqrt{\epsilon}\sqrt{\int_0^T \|\partial_t u_n(s)\|_{\partial\Omega} ds}$$
$$\leq \sqrt{\epsilon}\sqrt{\int_0^T \|\partial_t u_n(s)\|_{H^1(\Omega)} ds} \leq C\sqrt{\epsilon}.$$

Combining this result with (24), we obtain that $u_n \to u$ in $C([0,T], L^2(\Omega))$. Moreover, the condition $g \in C^1([0,T], L^2(\partial\Omega))$ implies that $\overline{g}_n \to g$ in $C^1([0,T], L^2(\partial\Omega))$. Consequently, the sequence $\{u_n\}$ is Cauchy in $C([0,T], H^1(\Omega))$.

Together with

$$\|u_n(t) - \overline{u}_n(t)\|_{H^1(\Omega)} \leq C\sqrt{\tau}\sqrt{\int_t^{t+\tau} \|\partial_t u_n(s)\|_{H^1(\Omega)} ds} \leq C\sqrt{\tau} \tag{24}$$

in view of (21), we obtain that

$$\overline{u}_n \to u, \quad u_n \to u \quad in \quad C([0,T], H^1(\Omega)).$$

□

In this subsection, we prove the uniqueness of the solution to problem (5), (7).

3.3 Uniqueness

Let (u_1, q_1) and (u_2, q_2) satisfy (7) and (5). Subtracting the corresponding variational forms gives

$$(\partial_t(u_1 - u_2), \varphi) + (\partial_t(\nabla u_1 - \nabla u_2), \nabla\varphi) + (\nabla u_1 - \nabla u_2, \nabla\varphi) + (u_1 - u_2, \varphi) =$$

$$(q_1 - q_2)(f, \varphi) + \Big(\int_0^t [F(s, u_1(s)) - F(s, u_2(s))] ds, \varphi \Big), \tag{25}$$

$$(q_1 - q_2)(f, 1) + \Big(\int_0^t [F(s, u_1(s)) - F(s, u_2(s))] ds, 1 \Big) = 0. \tag{26}$$

We reformulate the above using $e_u = u_1 - u_2$ and $e_q = q_1 - q_2$

$$(\partial_t e_u, \varphi) + (\partial_t \nabla e_u, \nabla\varphi) + (\nabla e_u, \nabla\varphi) + (e_u, \varphi) =$$

$$e_q(f, \varphi) + \Big(\int_0^t [F(s, u_1(s)) - F(s, u_2(s))] ds, \varphi \Big), \tag{27}$$

$$e_q(t)(f, 1) + \Big(\int_0^t [F(s, u_1(s)) - F(s, u_2(s))] ds, 1 \Big) = 0. \tag{28}$$

Theorem 2 *If problem (5), (7) has a solution, then problem (5), (7) has at most one solution* $(u, q) \in C([0,T], H^1(\Omega)) \times C[0,T]$.

Proof F be a Lipschitz continuous function in all variables, the condition $(f,1) \neq 0$, Gronwall's lemma and (28), it follows that

$$|e_q(t)| \leq \Big|\Big(\int_0^t [F(s,u_1(s)) - F(s,u_2(s))]ds, 1\Big)\Big| \leq L\|u_1 - u_2\|. \tag{29}$$

Taking $\varphi = e_u$ in (27) and integrating in time over $(0,\xi)$, we obtain

$$\int_0^\xi (\partial_t e_u, e_u)dt + \int_0^\xi (\partial_t \nabla e_u, \nabla e_u)dt + \int_0^\xi (\nabla e_u, \nabla e_u)dt + \int_0^\xi (e_u, e_u)dt$$

$$= \int_0^\xi e_q(f, e_u)dt + \int_0^\xi \Big(\int_0^t [F(s,u_1(s)) - F(s,u_2(s))]ds, e_u\Big)dt, \tag{30}$$

with the help of Cauchy's inequality, the bounds are obtained successively

$$\int_0^\xi (\partial_t e_u, e_u)dt = \frac{1}{2}\int_0^\xi \frac{d}{dt}\int_\Omega e_u^2 dxdt = \frac{1}{2}\int_0^\xi \frac{d}{dt}\|e_u\|^2_{L^2(\Omega)}dt = \frac{1}{2}\|e_u\|^2,$$

$$\int_0^\xi (\partial_t \nabla e_u, \nabla e_u)dt = \frac{1}{2}\|\nabla e_u\|^2,$$

$$\int_0^\xi (\nabla e_u, \nabla e_u)dt \geq C_0 \int_0^\xi \|\nabla e_u\|^2 dt,$$

$$\int_0^\xi (e_u, e_u)dt \geq C \int_0^\xi \|e_u\|^2 dt,$$

$$\int_0^\xi e_q(f, e_u)dt \leq \epsilon \int_0^\xi e_q^2 dt + C(\epsilon)\int_0^\xi \|e_u\|^2 dt \leq C\int_0^\xi \|e_u\|^2 dt,$$

$$\int_0^\xi \Big(\int_0^t [F(s,u_1(s)) - F(s,u_2(s))]ds, e_u\Big)dt \leq C(\epsilon)\int_0^\xi \int_0^t \|e_u\|^2 dsdt.$$

Combining these estimates, we arrive at

$$\frac{1}{2}\|e_u\|^2 + \frac{1}{2}\|\nabla e_u\|^2 + (C-\epsilon)\int_0^\xi \|e_u\|^2 dt + C_0\int_0^\xi \|\nabla e_u\|^2 dt \leq \int_0^\xi \int_0^t \|e_u\|^2 dsdt.$$

Fixing a small positive ϵ and using Gronwall's lemma, it follows that $e_u = 0$ a.e. in $\Omega \times (0,T)$. Finally, relation (29) implies that $e_q = 0$ a.e. in $(0,T)$. □

In this section, we fully discretize the problem (9), which has been discretized in time, using Legendre polynomials, and obtain a symmetric linear algebraic system.

4 Full discretization

This section deals with the spatial discretization of (9), completing the complete discretization of the problem (1)–(3). For this purpose, we employ the Legendre pseudospectral method.

For numerical purposes, we consider only the one-dimensional scenario to keep the procedure simple. To this end, we choose the space domain Ω to be the finite interval $(-1,1)$. Let $L_k(x)$ denote the Legendre polynomial of degree k.

Let us denote $L_k(x)$ the kth degree Legendre polynomial, which is defined by the following three-term recurrence formula:

$$L_0(x) = 1, \quad L_1(x) = x,$$

$$L_{k+1}(x) = \frac{(2k+1)x}{k+1} L_k(x) - \frac{k}{k+1} L_{k-1}(x), \quad k \geq 1.$$

Recall that the $L_k(x)$ obey the orthogonality relation

$$(L_k(x), L_j(x)) = \frac{2}{2k+1}\delta_{ij}, \quad \forall k, j \geq 0,$$

together with the recurrence relation

$$(2k+1)L_k(x) = L'_{k+1}(x) - L'_{k-1}(x).$$

Let k be positive integer, we define

$$\psi_k(x) = L_k(x) - L_{k+2}(x), \quad 0 \leq k \leq N-2,$$

$$\psi_{N-1}(x) = \frac{1}{2}(L_0(x) + L_1(x)),$$
$$\psi_N(x) = \frac{1}{2}(L_0(x) - L_1(x)).$$

It is easy to verify using linear algebra arguments, that the set $(\psi_k)_{k=0}^N$ forms a basis for the space $P_N(\Omega)$ ($P_N(\Omega)$ is all algebraic polynomials of degree at most N on Ω). The efficiency of our algorithms relies crucially on the following lemma.

Lemma 4 (see [17], [18], [19].) Let two integer k, j, let us denote,

$$m_{k,j} = m_{j,k} = (\psi_k, \psi_j) = \int_{-1}^{1} \psi_k(x)\psi_j(x)\, dx,$$
$$p_{k,j} = p_{k,j} = (\psi'_k, \psi'_j) = \int_{-1}^{1} \psi'_k(x)\, \psi'_j(x)\, dx,$$

then, for $0 \leq k, j \leq N-2$:

$$m_{k,j} = m_{j,k} = \begin{cases} \frac{2}{2k+1} + \frac{2}{2k+5}, & j = k \\ -\frac{2}{2k+5}, & j = k \pm 2 \\ 0, & otherwise \end{cases}$$

and

$$p_{k,j} = p_{j,k} = \delta_{kj} = \begin{cases} 4k+6, & j = k, \\ 0, & k \neq j. \end{cases}$$

Consequently, the standard Legendre–Chebyshev approximation of (9) seeks, for every $i = 1, \ldots, n$, a function $u_i^N \in P_N(\Omega)$ such that

$$(\delta u_i^N, v) + (\delta \partial_x u_i^N, \partial_x v) + (\partial_x u_i^N, \partial v) + (u_i^N, v)$$
$$= (\delta g_i, v)_{\partial\Omega} + (g_i, v)_{\partial\Omega} + q_i(f, v) + \Big(\sum_{k=1}^{i} F(t_k, u_{k-1}^N)\tau, v\Big), \quad v \in P_N(\Omega). \tag{31}$$

Now, for all $t \in (0, T]$ we seek an approximate solution $u_i^N(t)$ for variational problem (31) in $P_N(\Omega)$, and since $\{\psi\}_{k=0}^N$ forms a basis of the admissible space, thus, $u_i^N(t)$ can be written as a linear combination of ψ_k with time-dependent coefficients, namely,

$$u_i^N = \sum_{k=0}^{N} \alpha_k \psi_k(x). \tag{32}$$

Substituting (32) into identity (31) and setting $v = \psi_j, j = 0, \ldots, N$, we obtain the following algebraic linear system:

$$(1+\tau)(\mathbf{M}+\mathbf{P})\mathbf{U}^i = (\mathbf{M}+\mathbf{P})\mathbf{U}^{i-1} + \tau\mathbf{B}, \tag{33}$$

where

$$\mathbf{U}^i = (\alpha_0^i, \alpha_1^i, ..., \alpha_N^i)^T, \quad U^0 = (\alpha_0^0, \alpha_1^0, ..., \alpha_N^0)^T, \quad \alpha_k^0 = (u_0, \psi_k),$$

$$\mathbf{b}_j^i = (g_i - g_{i-1}, \psi_j)_{\partial\Omega} + (g_i, \psi_j)_{\partial\Omega} + q_i(f, \psi_j) + \Big(\sum_{k=1}^{i} F(t_k, u_{k-1}^N)\tau, \psi_j\Big),$$

$$i, j = 0, 1, ...N,$$

$$\mathbf{B}^i = (b_0^i, b_1^i, b_2^i, ..., b_N^i)^T, \quad \mathbf{M} = [m_{kj}]_{0\le j,k\le N}, \quad \mathbf{P} = [p_{kj}]_{0\le j,k\le N}.$$

Since the matrices $\mathbf{M}$ and $\mathbf{P}$ are banded, the linear system (33) can be easily solved using either a direct or an iterative method.

5 Numerical experiment

In this section we test the above-mentioned numerical scheme for approximate solution of (5)-(7), which is based on (8) and (33). Numerical results are presented and discussed.

We consider problem (5)-(7) for $\Omega = (-1, 1)$, with

$$f(x) = 4\cos^2\frac{\pi x}{2}(1 + \sin \pi x),$$

$$F(x, t, u) = -u^2\Big(\cos\frac{\pi x}{2} - \sin\frac{\pi x}{2}\Big)(3t^2 - 2t + 5)e^{2t},$$

along with the initial and boundary conditions

$$\phi(x) = 1 + \sin \pi x + \cos \pi x,$$

$$g(-1, t) = \pi e^{-t},$$
$$g(1, t) = -\pi e^{-t}.$$

Where the time-dependent measurement is

$$h(t) = 2e^{-t}.$$

One can easily verify that functions

$u(x,t) = (1 + \sin \pi x + \cos \pi x)e^{-t}$ and $q(t) = t^3 - t^2 + 5t$

solve the given problem.

In order to obtain approximate solution for numerical solution, we take $N = 6$ polynomial degree besides $T = 5$. Profiles of exact and approximate solutions are plotted in Figure 1 and Figure 2.

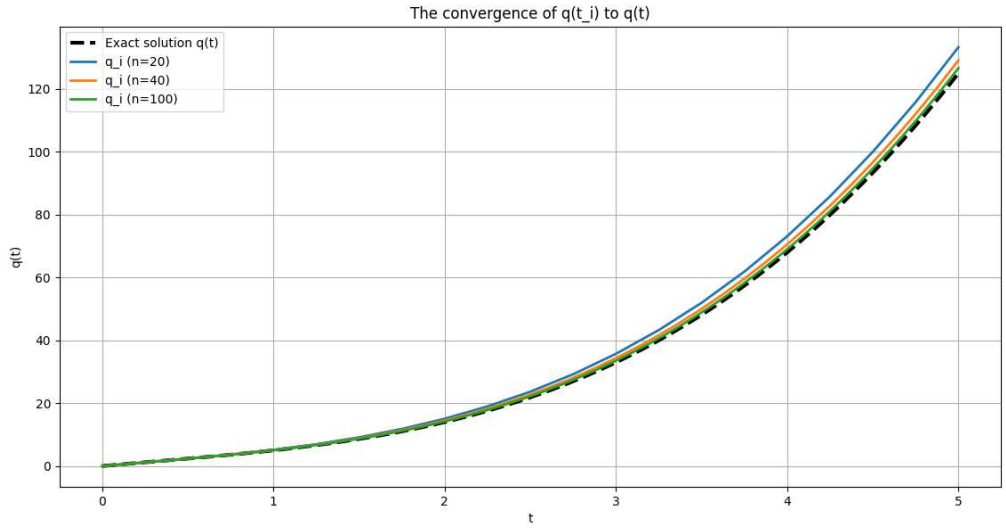


Fig. 1 Numerical and exact solutions for q_i

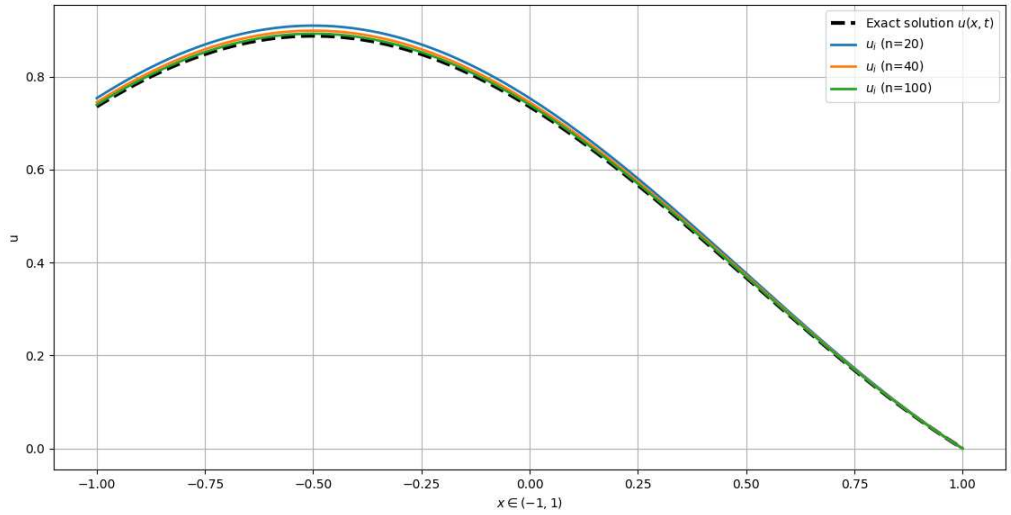


Fig. 2 Numerical and exact solutions for u_i (where $t = T$).

The relative error is defined as

$$\mathrm{RE}_q = \frac{\|q_{\mathrm{num}} - q_{\mathrm{exact}}\|_{L^2(0,T)}}{\|q_{\mathrm{exact}}\|_{L^2(0,T)}}, \quad \mathrm{RE}_q^{(\infty)} = \frac{\max_{t\in[0,T]} |q_{\mathrm{num}} - q_{\mathrm{exact}}|}{\max_{t\in[0,T]} |q_{\mathrm{exact}}|}.$$

In addition, the numerical algorithm was tested for different values of τ using the relative error.

Table 1 Relative errors for q_i for different values of τ.

n	$\tau = T/n$	L^2 Relative Error	L^∞ Relative Error
20	0.25	0.0713839227102934	0.06625
40	0.125	0.035478670049067476	0.0328125
80	0.0625	0.017684865520570553	0.016328125
100	0.05	0.01413909461120035	0.013050000000000068

Table 2 Relative errors for u_i, different values of τ.

n	$\tau = T/n$	L^2 Relative Error	L^∞ Relative Error
20	0.05	0.02449184626154373	0.024508115000112088
40	0.025	0.01237142122410143	0.012387593906366188
80	0.0125	0.00621769612191973	.00623405531075545
100	0.01	0.00497932728187385	0.004995820154653544

The results show that as n increases and τ decreases, the numerical solution converges to the exact solution.

6 Conclusion

We studied a semilinear integro-differential pseudoparabolic equation with initial and boundary conditions. In the inverse problem, the unknown source function was determined using an additional condition. The problem was discretized in time, and the uniqueness of the solution at each time level was proved. Then, Rothe's method was applied to establish the existence of a weak solution over the entire time interval. The uniqueness of the solution was also demonstrated. Using Legendre polynomials, the semi-discretized problem was fully discretized and reduced to a symmetric linear algebraic system. The effectiveness of the constructed algorithm was verified using a numerical method. This study can be further extended by considering the problem with nonclassical boundary conditions and proving the existence and uniqueness of its solution.

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Not applicable.

Declarations

Conflict of interest. This work does not have any conflict of interest.

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