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On the Discrete Past Geometric Vitality Function and Its Properties

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Abstract

Measures based on past lifetime information play an important role in reliability and survival analysis. In this paper, we introduce the *Discrete Past Geometric Vitality Function (PGVF)*, which serves as a multiplicative analogue of the classical past vitality function for discrete lifetime distributions. The proposed measure provides a geometric summary of truncated lifetimes and captures proportional ageing behaviour. Several structural properties of the PGVF are established, including its relationship with the reversed hazard rate, monotonicity behaviour, stochastic ordering, and transformation properties. It is further shown that the PGVF uniquely determines the underlying distribution, leading to characterization results and an information-theoretic interpretation. A nonparametric estimator based on discrete kernel methods with boundary correction is proposed, and its asymptotic properties are derived. Simulation studies and a real-data application illustrate the usefulness and practical applicability of the proposed measure.

Keywords: Discrete lifetime models; geometric vitality function; reversed hazard rate; stochastic ordering; nonparametric estimation; ageing properties.

MSC 2020: Primary 62N05; Secondary 62G05, 62E10, 62B10.

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1 Introduction

Ageing analysis is a central topic in reliability theory, providing quantitative insight into how systems and components deteriorate over time. In many applied fields such as reliability engineering, medical follow-up studies, actuarial science, and warranty analysis, interest extends beyond future lifetime behaviour to the characteristics of systems that have already failed before a specified time. Measures based on *past lifetime* information are therefore essential for retrospective ageing analysis and reliability assessment.

An early contribution in this direction is the *vitality function* introduced by Kupka and Loo [1989]. For a non-negative continuous random variable X with cumulative distribution function (CDF) F and probability density function f , the vitality function is defined as

$$v(t) = \mathbb{E}[X \mid X > t], \quad (1.1)$$

which represents the expected remaining lifetime given survival beyond time t . Its dual concept, known as the *past vitality function* (PVF), is defined as

$$v^*(t) = \mathbb{E}[X \mid X \leq t], \quad (1.2)$$

which summarizes truncated lifetimes through arithmetic averaging and reflects ageing behaviour on an additive scale.

These concepts naturally extend to discrete lifetime models. Let X be a discrete random variable taking values in $\mathbb{N}$ with probability mass function (PMF) $p(x)$ and cumulative distribution function $F(t) = P(X \leq t)$. The survival function is defined as $\bar{F}(t) = P(X > t)$. The discrete vitality and past vitality functions are respectively given by

$$v(t) = E[X \mid X > t] = \frac{\sum_{x>t} x p(x)}{\bar{F}(t)}, \quad (1.3)$$

and

$$v^*(t) = E[X \mid X \leq t] = \frac{\sum_{x\leq t} x p(x)}{F(t)}. \quad (1.4)$$

Closely related retrospective measures include the *mean past lifetime* (MPL), introduced by Asadi and Berred [2007] for continuous distributions

and later extended to discrete models by Goliforushani and Asadi [2008]. For a continuous random variable X , the MPL is defined as

$$K(t) = \mathbb{E}[t - X \mid X < t] = \frac{\int_0^t F(x) dx}{F(t)}, \quad (1.5)$$

while the discrete analogue is given by

$$K(t) = t - \frac{\sum_{x=0}^{t-1} x p(x)}{F(t-1)} = \frac{\sum_{x=1}^t F(x-1)}{F(t-1)}, \quad t \in \mathbb{N}, \quad (1.6)$$

with $K(0) = 0$. These measures are closely related to the reversed hazard rate, which plays an important role in backward-time reliability analysis.

Despite their wide applicability, mean-based vitality and past lifetime measures are inherently *additive* and may not adequately capture proportional or multiplicative ageing effects that arise naturally in many reliability and biological systems. This limitation has motivated the development of geometric-type vitality measures.

In this context, Nair and Rajesh [2000] introduced the *geometric vitality function* (GVF) for continuous lifetime distributions, defined as

$$\log G(t) = \mathbb{E}[\log X \mid X > t] = \frac{1}{\bar{F}(t)} \int_t^\infty \log x f(x) dx. \quad (1.7)$$

For discrete random variables, the corresponding form is

$$\log G(t) = \mathbb{E}[\log X \mid X > t] = \frac{1}{\bar{F}(t)} \sum_{x=t+1}^\infty p(x) \log x, \quad t = 0, 1, 2, \dots \quad (1.8)$$

Extending this idea to past lifetimes, Abdul Sathar and Gayathri [2021] introduced the *past geometric vitality function* (PGVF) for continuous distributions as

$$\log \bar{G}(t) = \mathbb{E}[\log X \mid X \leq t] = \frac{1}{F(t)} \int_0^t \log x f(x) dx. \quad (1.9)$$

However, most existing developments of geometric vitality measures are restricted to continuous lifetime models. This is restrictive since many practical reliability datasets arise in discrete or count-based settings where continuous formulations are not directly applicable. Although discrete analogues

of mean residual life and mean past lifetime have been widely studied, a geometric vitality measure based on *past lifetimes* for discrete distributions has not yet been formally developed.

Motivated by this gap, the present paper introduces the *Discrete Past Geometric Vitality Function (PGVF)*, which extends the geometric vitality framework to discrete lifetime distributions while preserving its multiplicative interpretation. The proposed measure provides a geometric summary of truncated lifetimes and offers a complementary perspective for analysing ageing behaviour.

The main contributions of this paper can be summarized as follows. First, we introduce the *Discrete Past Geometric Vitality Function (PGVF)* for discrete lifetime distributions, providing a multiplicative counterpart to the classical past vitality function. Second, we develop several fundamental structural properties of the PGVF, including its relationship with the reversed hazard rate. Third, monotonicity behaviour, stochastic ordering results, and transformation properties of the proposed measure are established. Furthermore, it is shown that the PGVF uniquely determines the underlying distribution, leading to characterization results. In addition, a nonparametric estimation framework based on discrete kernel methods with boundary correction is proposed. Finally, the usefulness and practical applicability of the proposed measure are illustrated through simulation studies and a real data application.

The remainder of the paper is organized as follows. Section 2 introduces the Discrete PGVF and develops its fundamental properties. Section 3 presents numerical illustrations. Section 4 develops a nonparametric estimation framework based on discrete kernel methods with boundary correction and establishes its asymptotic properties. Section 5 reports a simulation study, and Section 6 presents a real-data application. Finally, Section 7 concludes the paper.

2 Definition and Properties

In this section, we introduce the *Discrete Past Geometric Vitality Function* (Discrete PGVF) and examine its fundamental properties. We also establish its connection with the reversed hazard rate and show that it uniquely determines the underlying distribution.

Let X be a discrete random variable supported on $\mathbb{N}$ with probability

mass function (PMF) $p(x)$ and cumulative distribution function (CDF) $F(x)$. Assume that $\mathbb{E}[\log X] < \infty$. For $t \in \mathbb{N}$, the *Discrete PGVF*, denoted by $\overline{G}(t)$, is defined as

$$\log \overline{G}(t) = \mathbb{E}[\log X \mid X \leq t] \quad (2.1)$$

$$= \frac{1}{F(t)} \sum_{x=1}^t (\log x) p(x), \quad (2.2)$$

with the convention $\overline{G}(0) = 0$.

An alternative representation of $\log \overline{G}(t)$ can be obtained using the identity $p(x) = F(x) - F(x-1)$. Applying discrete summation by parts yields

$$\log \overline{G}(t) = \log(t+1) + \frac{1}{F(t)} \sum_{x=1}^t F(x) \log\left(\frac{x}{x+1}\right). \quad (2.3)$$

Equivalently,

$$\log\left(\frac{\overline{G}(t)}{t+1}\right) = \frac{1}{F(t)} \sum_{x=1}^t F(x) \log\left(\frac{x}{x+1}\right).$$

This representation highlights the intrinsic dependence of the Discrete PGVF on the distribution function $F(\cdot)$ and facilitates its use in structural and characterization results.

To illustrate the interpretation of the proposed measure, consider a sample of observed failure cycles

$$X = \{2, 3, 3, 4, 5, 5, 8, 10\},$$

representing the number of operating cycles completed before failure. For $t = 5$, the Discrete PGVF is computed as $\overline{G}(5) \approx 3.48$, which corresponds to the geometric mean of failure cycles among units failing at or before cycle 5. This value indicates that the truncated lifetime distribution is multiplicatively concentrated around lower cycle counts, reflecting the influence of early failures more strongly than the corresponding arithmetic mean. Thus, the Discrete PGVF provides a sensitive measure of early-life failure clustering in discrete reliability data.

We next examine the relationship between the Discrete PGVF and the reversed hazard rate. Define the backward difference operator by

$$\nabla \log \overline{G}(t) = \log \overline{G}(t) - \log \overline{G}(t-1).$$

Using (2.1), straightforward algebra yields

$$\delta_X(t) = \frac{\nabla \log \overline{G}(t)}{\log \left(\frac{t}{\overline{G}(t-1)} \right)}, \quad (2.4)$$

where $\delta_X(t) = \frac{p(t)}{F(t)}$ denotes the reversed hazard rate.

Equation (2.4) establishes a direct functional relationship between the Discrete PGVF and the reversed hazard rate. Since the reversed hazard rate uniquely determines the distribution of X , it follows that the Discrete PGVF also uniquely characterizes the underlying distribution.

Example 2.1. Let X be a discrete random variable following a uniform distribution over $\{1, 2, \dots, b\}$, where $b \in \mathbb{N}$. The PMF and CDF are

$$p(x) = \frac{1}{b}, \quad F(x) = \frac{x}{b}, \quad x = 1, 2, \dots, b.$$

The reversed hazard rate is therefore

$$\delta_X(t) = \frac{p(t)}{F(t)} = \frac{1/b}{t/b} = \frac{1}{t}, \quad t = 1, 2, \dots, b.$$

The Discrete PGVF becomes

$$\log \overline{G}(t) = \frac{1}{F(t)} \sum_{x=1}^t (\log x) p(x) = \frac{b}{t} \sum_{x=1}^t \frac{\log x}{b} = \frac{1}{t} \sum_{x=1}^t \log x.$$

Hence

$$\nabla \log \overline{G}(t) = \log \overline{G}(t) - \log \overline{G}(t-1) = \frac{1}{t} \log t - \frac{1}{t(t-1)} \sum_{x=1}^{t-1} \log x.$$

Using the identity in (2.4), we obtain

$$\delta_X(t) = \frac{\nabla \log \overline{G}(t)}{\log \left(\frac{t}{\overline{G}(t-1)} \right)} = \frac{1}{t},$$

which verifies (2.4) for the discrete uniform distribution.

The following properties follow directly from the definition of the Discrete PGVF.

Proposition 2.1. *The function $\log \overline{G}(t)$ is increasing in t if $\overline{G}(t-1) < t$ and decreasing in t if $\overline{G}(t-1) > t$.*

Proof. From (2.4), we have

$$\nabla \log \overline{G}(t) = \delta_X(t) \log \left(\frac{t}{\overline{G}(t-1)} \right).$$

Since $\delta_X(t) \geq 0$, the sign of $\nabla \log \overline{G}(t)$ is determined by the sign of $\log \left(\frac{t}{\overline{G}(t-1)} \right)$. Thus $\nabla \log \overline{G}(t) > 0$ if $\overline{G}(t-1) < t$ and $\nabla \log \overline{G}(t) < 0$ if $\overline{G}(t-1) > t$, which proves the result.

Proposition 2.2. *If $\mathbb{E}[\log X] < \infty$, then*

$$\lim_{t \rightarrow \infty} \log \overline{G}(t) = \mathbb{E}[\log X].$$

Proof. From (2.1),

$$\log \overline{G}(t) = \frac{1}{F(t)} \sum_{x=1}^t (\log x) p(x).$$

As $t \rightarrow \infty$, we have $F(t) \rightarrow 1$, and the numerator converges to

$$\sum_{x=1}^{\infty} (\log x) p(x) = \mathbb{E}[\log X].$$

Since $\mathbb{E}[\log X] < \infty$, the desired limit follows.

This result shows that the Discrete PGVF provides a consistent approximation to the logarithmic moment of the distribution as the truncation point increases.

The following definition is useful for studying monotonicity properties of the Discrete PGVF.

Definition 2.1. A discrete random variable X is said to have a *decreasing Past Geometric Vitality Function* (DPGVF) if $\log \overline{G}(t)$ is monotone non-increasing in t , and an *increasing Past Geometric Vitality Function* (IPGVF) if $\log \overline{G}(t)$ is monotone non-decreasing in t , for $t \in \mathbb{N}$.

The following theorem provides a characterization of these ageing classes.

Theorem 2.1. *Let X be a non-negative, non-degenerate discrete random variable. Then X has a DPGVF (respectively, IPGVF) if and only if*

$$\overline{G}(t-1) \geq t \quad (\text{respectively, } \overline{G}(t-1) \leq t),$$

for all $t \geq 1$.

Proof. From Proposition 2.1, we have

$$\nabla \log \overline{G}(t) = \delta_X(t) \log \left(\frac{t}{\overline{G}(t-1)} \right).$$

Since $\delta_X(t) \geq 0$, the sign of $\nabla \log \overline{G}(t)$ is determined by the sign of $\log \left(\frac{t}{\overline{G}(t-1)} \right)$. Thus,

$$\nabla \log \overline{G}(t) \leq 0 \iff \overline{G}(t-1) \geq t,$$

and

$$\nabla \log \overline{G}(t) \geq 0 \iff \overline{G}(t-1) \leq t.$$

Therefore, $\log \overline{G}(t)$ is non-increasing (respectively, non-decreasing) if and only if $\overline{G}(t-1) \geq t$ (respectively, $\overline{G}(t-1) \leq t$) for all $t \geq 1$. This completes the proof.

Example 2.2. To illustrate Theorem 2.1, consider the distribution with CDF

$$F(x) = \frac{x(x+1)}{b(b+1)}, \quad x = 1, 2, \dots, b,$$

and PMF

$$p(x) = \frac{2x}{b(b+1)},$$

which corresponds to the length-biased version of the discrete uniform distribution on $\{1, 2, \dots, b\}$.

The reversed hazard rate is

$$\delta_X(t) = \frac{p(t)}{F(t)} = \frac{2t}{t(t+1)} = \frac{2}{t+1}.$$

The Discrete PGVF is therefore

$$\log \bar{G}(t) = \frac{1}{F(t)} \sum_{x=1}^t (\log x) p(x) = \frac{2}{t(t+1)} \sum_{x=1}^t x \log x.$$

Since the weighting factor x assigns greater importance to larger observations, the geometric average of the truncated lifetimes shifts toward higher values. Consequently, $\bar{G}(t-1)$ tends to remain smaller than t for moderate values of t , implying that $\log \bar{G}(t)$ increases with t . Hence this distribution provides an example of an IPGVF.

Thus, the theorem provides a simple criterion for identifying ageing behaviour in terms of the relative position of $\bar{G}(t-1)$ and t . The following result provides a characterization of a discrete distribution in terms of the Discrete PGVF and the reversed lack-of-memory property.

Theorem 2.2. *Let Z be a non-negative, non-degenerate discrete random variable with support $\{0, 1, 2, \dots, b\}$, where $b < \infty$. Then the following statements are equivalent:*

(a) *The ratio*

$$\frac{\nabla \log \bar{G}(t)}{\log \left(\frac{t}{\bar{G}(t-1)} \right)} = k, \quad \forall t \geq 1,$$

for some constant $0 < k < 1$.

(b) *Z has a constant reversed hazard rate, i.e.,*

$$\delta_Z(t) = k, \quad \forall t \geq 1.$$

(c) *Z satisfies the reversed lack-of-memory property.*

(d) *The distribution function of Z is*

$$F(z) = (1 + c)^{z-b}, \quad z = 0, 1, 2, \dots, b,$$

for some constant $c > 0$.

Proof. From (2.4), we have the identity

$$\delta_Z(t) = \frac{\nabla \log \bar{G}(t)}{\log\left(\frac{t}{\bar{G}(t-1)}\right)}, \quad t \geq 1.$$

Hence, condition (a) holds if and only if $\delta_Z(t) = k$ for all $t \geq 1$, establishing the equivalence between (a) and (b).

The equivalence of (b), (c), and (d) follows directly from the characterization results of Nair and Sankaran [2013], which show that a discrete distribution has a constant reversed hazard rate if and only if it satisfies the reversed lack-of-memory property, and this is further equivalent to the distribution function given in (d). Therefore, all statements (a)–(d) are equivalent. This result shows that the Discrete PGVF provides a natural framework for characterizing distributions with reversed memory properties through a simple functional condition.

In reliability theory and stochastic comparison, it is often useful to compare random variables based on their geometric vitality characteristics. The following definition introduces a partial ordering of discrete random variables based on the PGVF.

Definition 2.2. Let Z and X be two non-negative discrete random variables. We say that Z is larger than X in the PGVF order, denoted by $Z \geq_{\text{PGVF}} X$, if

$$\log \bar{G}_Z(t) \geq \log \bar{G}_X(t), \quad \forall t \in \mathbb{N}.$$

This ordering reflects the idea that larger values of a random variable tend to produce larger truncated logarithmic means. Unlike classical stochastic orders, however, the PGVF order is generally not implied by the usual stochastic order and must therefore be studied independently. In particular, the PGVF order captures multiplicative ageing behaviour rather than tail dominance.

The following lemma describes the behaviour of the Discrete PGVF under positive scaling.

Lemma 2.1. Let Z be a non-negative discrete random variable and let $a > 0$. Define $V = Z/a$. Then, for any $t > 0$,

$$\log \bar{G}_V(t) = \log \bar{G}_Z(\lfloor at \rfloor) - \log a.$$

Proof. By definition,

$$\log \overline{G}_V(t) = \mathbb{E}[\log V \mid V \leq t] = \mathbb{E}[\log(Z/a) \mid Z \leq at].$$

Since Z is integer-valued, the condition $Z \leq at$ is equivalent to $Z \leq \lfloor at \rfloor$. Hence,

$$\log \overline{G}_V(t) = \mathbb{E}[\log Z \mid Z \leq \lfloor at \rfloor] - \log a = \log \overline{G}_Z(\lfloor at \rfloor) - \log a,$$

which proves the result.

The following result establishes preservation of the PGVF order under positive scaling.

Theorem 2.3. *Let Z and X be two non-negative discrete random variables, and define $U_1 = Z/a_1$ and $U_2 = X/a_2$, where $a_1, a_2 > 0$. Suppose that*

- (i) $Z \geq_{\text{PGVF}} X$,
- (ii) $a_1 \geq a_2$,
- (iii) *either $\log \overline{G}_Z(t)$ or $\log \overline{G}_X(t)$ is non-decreasing in t .*

Then $U_1 \geq_{\text{PGVF}} U_2$.

Proof. Assume that $\log \overline{G}_Z(t)$ is non-decreasing. Since $a_1 \geq a_2$, we have

$$\lfloor a_1 t \rfloor \geq \lfloor a_2 t \rfloor \quad \forall t > 0.$$

Therefore,

$$\log \overline{G}_Z(\lfloor a_1 t \rfloor) \geq \log \overline{G}_Z(\lfloor a_2 t \rfloor).$$

From assumption (i),

$$\log \overline{G}_Z(\lfloor a_2 t \rfloor) \geq \log \overline{G}_X(\lfloor a_2 t \rfloor).$$

Combining the above inequalities gives

$$\log \overline{G}_Z(\lfloor a_1 t \rfloor) \geq \log \overline{G}_X(\lfloor a_2 t \rfloor).$$

Using Lemma 2.1, we obtain

$$\log \overline{G}_{U_1}(t) = \log \overline{G}_Z(\lfloor a_1 t \rfloor) - \log a_1, \quad \log \overline{G}_{U_2}(t) = \log \overline{G}_X(\lfloor a_2 t \rfloor) - \log a_2.$$

Hence,

$$\log \overline{G}_{U_1}(t) - \log \overline{G}_{U_2}(t) = [\log \overline{G}_Z(\lfloor a_1 t \rfloor) - \log \overline{G}_X(\lfloor a_2 t \rfloor)] - (\log a_1 - \log a_2).$$

Since $\log a_1 \geq \log a_2$ and the first term is non-negative, it follows that

$$\log \overline{G}_{U_1}(t) \geq \log \overline{G}_{U_2}(t), \quad \forall t > 0.$$

Thus $U_1 \geq_{\text{PGVF}} U_2$.

The case where $\log \overline{G}_X(t)$ is non-decreasing follows by a similar argument.

Example 2.3. Let X follow the arithmetic (length-biased uniform) distribution with

$$F(x) = \frac{x(x+1)}{b(b+1)}, \quad x = 1, 2, \dots, b.$$

For this distribution, the function $\log \overline{G}_X(t)$ is non-decreasing in t . Hence, for any $a_1 \geq a_2 > 0$, the PGVF order is preserved under the scaling transformation described in Theorem 2.3.

Corollary 2.1. *Let $V_1 = Z/a$ and $V_2 = X/a$, where $a > 0$ is a constant, and let Z and X be non-negative discrete random variables such that $Z \geq_{\text{PGVF}} X$. If either $\log \overline{G}_Z(t)$ or $\log \overline{G}_X(t)$ is non-decreasing in t , then*

$$V_1 \geq_{\text{PGVF}} V_2.$$

The above corollary shows that the PGVF order is preserved under common scaling when an appropriate monotonicity condition holds. The following theorem strengthens this observation by showing that the PGVF order remains preserved under common scaling even without the monotonicity assumption.

Theorem 2.4. *Let Z and X be non-negative discrete random variables such that $Z \geq_{\text{PGVF}} X$. For a fixed constant $a > 0$, define $V_1 = Z/a$ and $V_2 = X/a$. Then*

$$V_1 \geq_{\text{PGVF}} V_2.$$

Proof. Since $Z \geq_{\text{PGVF}} X$, by Definition 2.2 we have

$$\log \overline{G}_Z(t) \geq \log \overline{G}_X(t), \quad \forall t \in \mathbb{N}.$$

Using Lemma 2.1, we obtain

$$\log \overline{G}_{V_1}(t) = \log \overline{G}_Z(\lfloor at \rfloor) - \log a, \quad \log \overline{G}_{V_2}(t) = \log \overline{G}_X(\lfloor at \rfloor) - \log a.$$

Since $\log \overline{G}_Z(\lfloor at \rfloor) \geq \log \overline{G}_X(\lfloor at \rfloor)$ for all $t > 0$, it follows that

$$\log \overline{G}_{V_1}(t) \geq \log \overline{G}_{V_2}(t), \quad \forall t > 0.$$

Therefore, by Definition 2.2, we conclude that $V_1 \geq_{\text{PGVF}} V_2$.

This result shows that the PGVF order is invariant under common scaling transformations, without requiring additional regularity conditions. Furthermore, the relationship between the Discrete PGVF and right-truncated reciprocal moments provides a useful framework for characterizing discrete distributions. The following theorem establishes such a characterization.

Theorem 2.5. *Let Z be a non-negative discrete random variable. Suppose that for all $t \geq 1$,*

$$\log \left(\frac{\overline{G}(t)}{t+1} \right) = \sum_{r=1}^{\infty} (\alpha \mu_r(t) + \beta \mu_{r-1}(t)), \quad (2.5)$$

where

$$\mu_r(t) = \mathbb{E}[Z^{-r} \mid Z \leq t].$$

Then, if $\beta = 0$, Z follows a reversed geometric-type distribution with

$$F(z) = (1+c)^{z-b}, \quad z = 0, 1, \dots, b,$$

for some $c > 0$ and $b \in \mathbb{N}$.

Proof. From (2.3), we have

$$\log \left(\frac{\overline{G}(t)}{t+1} \right) = \frac{1}{F(t)} \sum_{z=1}^t F(z) \log \left(\frac{z}{z+1} \right). \quad (2.6)$$

Using the series expansion

$$\log \left(\frac{z}{z+1} \right) = -\log \left(1 + \frac{1}{z} \right) = -\sum_{r=1}^{\infty} \frac{1}{rz^r}, \quad z \geq 1,$$

and substituting into (2.6), we obtain

$$\begin{aligned}\log\left(\frac{\overline{G}(t)}{t+1}\right) &= -\frac{1}{F(t)} \sum_{z=1}^t F(z) \sum_{r=1}^{\infty} \frac{1}{rz^r} \\ &= -\sum_{r=1}^{\infty} \frac{1}{r} \left(\frac{1}{F(t)} \sum_{z=1}^t \frac{F(z)}{z^r} \right).\end{aligned}\quad (2.7)$$

On the other hand, the truncated reciprocal moments satisfy

$$\mu_r(t) = \mathbb{E}[Z^{-r} \mid Z \leq t] = \frac{1}{F(t)} \sum_{z=1}^t \frac{f(z)}{z^r}.$$

Suppose that identity (2.5) holds with $\beta = 0$. Then comparing (2.7) with (2.5) implies that

$$\frac{1}{F(t)} \sum_{z=1}^t \frac{F(z)}{z^r} \propto \frac{1}{F(t)} \sum_{z=1}^t \frac{f(z)}{z^r},$$

for all $r \geq 1$. This proportionality can hold only if $F(z)$ and $f(z)$ are linearly related, that is,

$$F(z) = k f(z)$$

for some constant $k > 0$.

Consequently,

$$\delta_Z(z) = \frac{f(z)}{F(z)} = \frac{1}{k},$$

which shows that the reversed hazard rate is constant.

A discrete distribution with constant reversed hazard rate satisfies

$$F(z) = (1+c)^{z-b}, \quad z = 0, 1, \dots, b,$$

for some $c > 0$. Hence, Z follows a reversed geometric-type distribution.

Conversely, if Z follows the reversed geometric-type distribution, then the reversed hazard rate is constant. Substituting this form of $F(z)$ into (2.7) shows that representation (2.5) holds with $\beta = 0$, completing the proof.

Remark 2.1. The above characterization relies on the series representation of the Discrete PGVF and the proportional relationship between cumulative probabilities and point probabilities. This proportionality induces a constant reversed hazard rate, which characterizes the reversed geometric-type distribution.

3 Structural Comparison of the PVF and PGVF

In this section, we examine the structural differences between the classical Discrete Past Vitality Function (PVF) and the proposed Discrete Past Geometric Vitality Function (PGVF). The aim is to demonstrate, both numerically and graphically, that additive and multiplicative summaries of truncated lifetimes can exhibit substantially different behaviour, particularly under right-skewed discrete lifetime distributions.

The Discrete PVF, defined in (1.4), summarizes past lifetimes through arithmetic averaging. In contrast, the Discrete PGVF, defined in (2.1), represents the geometric mean of truncated lifetimes and captures multiplicative characteristics of the distribution. Owing to its logarithmic formulation, the PGVF reduces the influence of large upper-tail observations and is therefore less sensitive to extreme values.

To illustrate these differences, we consider a Negative Binomial distribution with parameters $\text{size} = 2$ and $\text{prob} = 0.25$, shifted to the support $\{1, 2, \dots\}$. This model exhibits pronounced right-skewness, making it suitable for assessing the impact of tail behaviour on both measures. Table 1 reports the values of the PVF and PGVF for selected truncation levels t .

Table 1: Comparison of the Discrete PVF and PGVF under a right-skewed model

t	$v^*(t)$	$\overline{G}(t)$	$v^*(t) - \overline{G}(t)$
1	1.0000	1.0000	0.0000
5	3.1812	2.8383	0.3430
10	5.0712	4.2819	0.7893
20	6.6606	5.2819	1.3787
30	6.9584	5.4220	1.5364
40	6.9959	5.4359	1.5599

Note. Although the underlying lifetime variable is discrete with support $\mathbb{N}$, both PVF and PGVF are defined only at integer values of t . For visualization purposes, the plotted curves are obtained by joining these points using line segments. This interpolation is used solely for graphical clarity and does not imply a continuous domain. The same convention is adopted for all other figures in this paper.

It is observed that $v^*(t) \geq \overline{G}(t)$ for all $t > 1$, with the difference in-

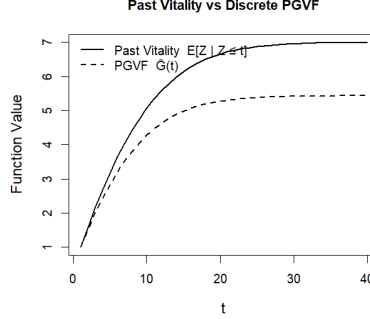


Figure 1: Comparison of PVF (solid line) and PGVF (dashed line) under a right-skewed Negative Binomial model

creasing initially and stabilizing for larger values of t . For example, the gap increases from 0.7893 at $t = 10$ to 1.5599 at $t = 40$. This pattern reflects the greater sensitivity of the arithmetic mean to large observations, whereas the geometric mean remains comparatively stable.

Figure 1 provides a graphical comparison of the two measures. The figure shows that the PVF increases more rapidly than the PGVF, reflecting the inflation effect caused by right-skewness under arithmetic aggregation. Although both measures are derived from the same truncated distribution, their growth patterns differ substantially, highlighting the impact of multiplicative versus additive summarization.

From an applied perspective, the PGVF is particularly appropriate in settings where lifetimes evolve multiplicatively or exhibit proportional growth. Such situations arise in reliability systems with cumulative wear, biological processes governed by replication, actuarial models involving compounding, and highly skewed count data. In these contexts, the PGVF provides a more representative measure of central tendency, whereas the PVF may be distorted by extreme observations.

Overall, this analysis demonstrates that the Discrete PGVF is not merely a transformation of the PVF but a structurally distinct measure capable of capturing multiplicative ageing behaviour that may not be evident under additive summaries.

To further illustrate this distinction, we construct two discrete distributions with identical PVF values but different PGVF values.

Let Z be supported on $\{1, 2, 3, 4\}$, and consider two distributions with

probability mass functions

$$f_A = (0.2, 0.3, 0.2, 0.3), \quad f_B = (0.3, 0.1, 0.3, 0.3).$$

For $t = 3$, both distributions satisfy $F_A(3) = F_B(3) = 0.7$ and

$$\sum_{z=1}^3 z f_A(z) = \sum_{z=1}^3 z f_B(z) = 1.4,$$

which yields

$$v_A^*(3) = v_B^*(3) = 2.$$

However, the logarithmic truncated sums differ:

$$\begin{aligned} \sum_{z=1}^3 \log z f_A(z) &= 0.3 \log 2 + 0.2 \log 3, \\ \sum_{z=1}^3 \log z f_B(z) &= 0.1 \log 2 + 0.3 \log 3. \end{aligned}$$

Consequently,

$$\overline{G}_A(3) \neq \overline{G}_B(3).$$

This example shows that equality of the Discrete PVF does not imply equality of the Discrete PGVF. While the PVF captures additive structure, the PGVF reflects multiplicative characteristics and can therefore distinguish between distributions that appear identical under arithmetic averaging.

4 Nonparametric Estimation

In this section, we develop a nonparametric framework for estimating the Discrete Past Geometric Vitality Function (PGVF). The proposed methodology is based on discrete kernel smoothing with an appropriate boundary correction to respect the support of the distribution. We first study the estimation of the probability mass function (PMF) and establish its asymptotic properties. These results will subsequently be used to construct estimators of the PGVF through a ratio-type representation.

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (i.i.d.) discrete random variables with probability mass function

$$f(x) = \Pr(X = x), \quad x \in \mathbb{N}, \quad (4.1)$$

where $f(x) > 0$ and $\sum_{x=1}^{\infty} f(x) = 1$.

To derive the asymptotic properties of the proposed estimators, we impose the following regularity conditions.

Assumption 4.1. The PMF f is locally smooth in the sense that

$$f(x+1) - f(x) = O(h), \quad x \in \mathbb{N},$$

and admits a first-order discrete expansion in a neighborhood of each point x .

Assumption 4.2. Let $K_{x,h}(y)$ be a discrete kernel defined on $\mathbb{Z}$ with bandwidth $h > 0$ satisfying the following conditions:

1. $K_{x,h}(y) \geq 0$ and $\sum_{y \in \mathbb{Z}} K_{x,h}(y) = 1$,
2. $\sum_{y \in \mathbb{Z}} (y - x) K_{x,h}(y) = 0$,
3. the kernel has a finite second moment,
4. $h \rightarrow 0$ as $n \rightarrow \infty$.

Assumption 4.3. The bandwidth h satisfies

$$h \rightarrow 0, \quad nh \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

Under Assumptions 4.1–4.3, standard discrete kernel smoothing arguments (see [Kokonendji and Zocchi, 2010](#)) imply that, provided the kernel satisfies the required moment conditions and the PMF admits a local linear expansion, the following local approximation holds:

$$\sum_{y \in \mathbb{Z}} K_{x,h}(y) f(y) = f(x) + C_x h + o(h), \quad (4.2)$$

where C_x is a finite constant depending on the local behaviour of f in a neighborhood of x .

We now introduce kernel-based estimators for the PMF. A natural estimator is the classical discrete kernel estimator defined by

$$\hat{f}_{K,h}(x) = \frac{1}{n} \sum_{i=1}^n K_{x,h}(X_i), \quad (4.3)$$

where $K_{x,h}(\cdot)$ denotes a discrete kernel function. A commonly used choice in practice is the discrete triangular kernel (see [Kokonendji et al., 2007](#)).

Since the support of X is $\mathbb{N}$, the estimator $\hat{f}_{K,h}(x)$ may suffer from boundary bias near $x = 1$, as the kernel assigns probability mass to points outside the support. To address this issue, we define the lost mass

$$\alpha_x(h) = \sum_{z < 1} K_{x,h}(z), \quad (4.4)$$

and construct the truncated and renormalized kernel

$$K_{x,h}^+(y) = \begin{cases} \frac{K_{x,h}(y)}{1 - \alpha_x(h)}, & y \geq 1, \\ 0, & y < 1. \end{cases} \quad (4.5)$$

This modification ensures that the kernel remains a valid probability mass function on the support $\mathbb{N}$. The corresponding boundary-corrected estimator of the PMF is therefore given by

$$\hat{f}_{K,h}^+(x) = \frac{1}{n} \sum_{i=1}^n K_{x,h}^+(X_i). \quad (4.6)$$

The asymptotic properties of the boundary-corrected estimator $\hat{f}_{K,h}^+(x)$ are summarized below.

Proposition 4.1. *Under Assumptions [4.1–4.3](#),*

$$E[\hat{f}_{K,h}^+(x)] = f(x) + C_x h + o(h), \quad (4.7)$$

$$\text{Var}[\hat{f}_{K,h}^+(x)] = O\left(\frac{1}{nh}\right), \quad (4.8)$$

so that $\text{Bias}[\hat{f}_{K,h}^+(x)] = O(h)$.

Proof. By the linearity of expectation,

$$E[\hat{f}_{K,h}^+(x)] = \sum_{y \geq 1} K_{x,h}^+(y) f(y).$$

Since the boundary-corrected kernel $K_{x,h}^+(y)$ is a renormalized version of $K_{x,h}(y)$, the expansion in [\(4.2\)](#) continues to hold up to the first order, yielding

$$E[\hat{f}_{K,h}^+(x)] = f(x) + C_x h + o(h).$$

For the variance, note that $\hat{f}_{K,h}^+(x)$ is the sample mean of i.i.d. random variables:

$$\text{Var}[\hat{f}_{K,h}^+(x)] = \frac{1}{n} \text{Var}(K_{x,h}^+(X_1)).$$

Under the kernel assumptions, the effective support of $K_{x,h}^+$ has width of order h , which implies

$$\text{Var}(K_{x,h}^+(X_1)) = O\left(\frac{1}{h}\right).$$

Consequently,

$$\text{Var}[\hat{f}_{K,h}^+(x)] = O\left(\frac{1}{nh}\right).$$

Proposition 4.2. *If $h \rightarrow 0$ and $nh \rightarrow \infty$, then*

$$\hat{f}_{K,h}^+(x) \xrightarrow{P} f(x).$$

Moreover, if $\sqrt{nh} \rightarrow 0$, then

$$\sqrt{n} \left(\hat{f}_{K,h}^+(x) - f(x) \right) \xrightarrow{d} N(0, \sigma_x^2),$$

where $\sigma_x^2 = \text{Var}(K_{x,h}^+(X_1))$.

Proof. Consistency follows from Chebyshev's inequality using the bias and variance orders established in Proposition 4.1. In particular, the bias tends to zero as $h \rightarrow 0$, while the variance converges to zero since $nh \rightarrow \infty$.

For asymptotic normality, observe that $\hat{f}_{K,h}^+(x)$ is a normalized sum of i.i.d. random variables. The condition $\sqrt{nh} \rightarrow 0$ ensures that the bias term is negligible compared with the stochastic fluctuation. The result then follows from the Lindeberg–Feller central limit theorem.

Proposition 4.3. *The mean integrated squared error satisfies*

$$\text{MISE}(\hat{f}_{K,h}^+) = O(h^2) + O\left(\frac{1}{nh}\right),$$

and is minimized for a bandwidth of order

$$h_{\text{opt}} \asymp n^{-1/3}.$$

Proof. The MISE admits the decomposition

$$\text{MISE} = \sum_{x \in \mathbb{N}} \text{Bias}^2\left(\hat{f}_{K,h}^+(x)\right) + \sum_{x \in \mathbb{N}} \text{Var}\left(\hat{f}_{K,h}^+(x)\right).$$

Using the bias and variance orders from Proposition 4.1, we obtain

$$\text{MISE} = O(h^2) + O\left(\frac{1}{nh}\right).$$

Minimizing this expression with respect to h yields

$$h_{\text{opt}} \asymp n^{-1/3}.$$

We now extend the kernel-based framework to estimate the Discrete Past Geometric Vitality Function (PGVF). Recall that

$$\theta(t) = \frac{A(t)}{B(t)}, \quad A(t) = \sum_{x=1}^t \log(x) f(x), \quad B(t) = \sum_{x=1}^t f(x), \quad B(t) > 0. \quad (4.9)$$

A natural estimator of $\theta(t)$ is obtained by replacing the unknown PMF $f(x)$ with its kernel estimator. Define

$$\psi_t(X_i) = \sum_{x=1}^t \log(x) K_{x,h}^+(X_i), \quad (4.10)$$

$$\phi_t(X_i) = \sum_{x=1}^t K_{x,h}^+(X_i), \quad (4.11)$$

and the corresponding estimators

$$\hat{A}(t) = \frac{1}{n} \sum_{i=1}^n \psi_t(X_i), \quad (4.12)$$

$$\hat{B}(t) = \frac{1}{n} \sum_{i=1}^n \phi_t(X_i), \quad (4.13)$$

$$\hat{\theta}(t) = \frac{\hat{A}(t)}{\hat{B}(t)}. \quad (4.14)$$

Theorem 4.1. *Under Assumptions 4.1–4.3, with $h \rightarrow 0$ as $n \rightarrow \infty$,*

$$\hat{\theta}(t) \xrightarrow{P} \theta(t), \quad n \rightarrow \infty.$$

Proof. Observe that $\hat{A}(t)$ and $\hat{B}(t)$ can be written as sample averages of i.i.d. random variables:

$$\hat{A}(t) = \frac{1}{n} \sum_{i=1}^n \psi_t(X_i), \quad \hat{B}(t) = \frac{1}{n} \sum_{i=1}^n \phi_t(X_i).$$

From the definitions of $\psi_t(\cdot)$ and $\phi_t(\cdot)$ and the kernel expansion in (4.2), we obtain

$$E[\psi_t(X_1)] = \sum_{x=1}^t \log(x) f(x) + O(h) = A(t) + O(h),$$

and

$$E[\phi_t(X_1)] = \sum_{x=1}^t f(x) + O(h) = B(t) + O(h).$$

Hence, by the weak law of large numbers,

$$\hat{A}(t) \xrightarrow{P} A(t), \quad \hat{B}(t) \xrightarrow{P} B(t),$$

as $n \rightarrow \infty$ and $h \rightarrow 0$.

Since $B(t) > 0$, Slutsky's theorem implies

$$\hat{\theta}(t) = \frac{\hat{A}(t)}{\hat{B}(t)} \xrightarrow{P} \frac{A(t)}{B(t)} = \theta(t).$$

Theorem 4.2. *Let $\mathbf{Z}_i = (\psi_t(X_i), \phi_t(X_i))^\top$ and denote by $\Sigma = \text{Cov}(\mathbf{Z}_1)$ the covariance matrix of $\mathbf{Z}_1$. If $\sqrt{nh} \rightarrow 0$, then*

$$\sqrt{n}(\hat{\theta}(t) - \theta(t)) \xrightarrow{d} N(0, \tau^2),$$

where

$$\tau^2 = \nabla g^\top \Sigma \nabla g, \quad g(a, b) = \frac{a}{b}, \quad \nabla g = \begin{pmatrix} \frac{1}{B(t)} \\ \theta(t) \\ -\frac{1}{B(t)} \end{pmatrix}.$$

Proof. Recall that

$$\hat{A}(t) = \frac{1}{n} \sum_{i=1}^n \psi_t(X_i), \quad \hat{B}(t) = \frac{1}{n} \sum_{i=1}^n \phi_t(X_i),$$

so that $(\hat{A}(t), \hat{B}(t))^\top$ is a vector of sample means of i.i.d. random variables.

From the kernel expansion,

$$E[\psi_t(X_1)] = A(t) + O(h), \quad E[\phi_t(X_1)] = B(t) + O(h).$$

Thus, the bias of $(\hat{A}(t), \hat{B}(t))$ is of order $O(h)$. Under the condition $\sqrt{nh} \rightarrow 0$, this bias becomes negligible relative to the stochastic variation.

By the multivariate central limit theorem,

$$\sqrt{n} \left((\hat{A}(t), \hat{B}(t))^\top - (A(t), B(t))^\top \right) \xrightarrow{d} N(\mathbf{0}, \Sigma).$$

Since $\hat{\theta}(t) = g(\hat{A}(t), \hat{B}(t))$ with $g(a, b) = a/b$, and $B(t) > 0$, the function g is differentiable in a neighborhood of $(A(t), B(t))$. Applying the delta method yields

$$\sqrt{n}(\hat{\theta}(t) - \theta(t)) \xrightarrow{d} N(0, \nabla g^\top \Sigma \nabla g),$$

which completes the proof.

Corollary 4.1. *The mean squared error of $\hat{\theta}(t)$ satisfies*

$$\text{MSE}(\hat{\theta}(t)) = O(h^2) + O\left(\frac{1}{n}\right).$$

Consequently, balancing the squared bias and variance terms leads to an optimal bandwidth of order

$$h_{\text{opt}} \asymp n^{-1/3}.$$

5 Simulation Study

In this section, we investigate the finite-sample performance of three estimators of the Discrete Past Geometric Vitality Function (PGVF): the empirical estimator, the asymmetric discrete triangular kernel estimator [Kokonendji and Zocchi, 2010, Kiessé, 2017], and the proposed boundary-corrected kernel

estimator. The comparison focuses on bias, variance, and mean squared error (MSE).

Let $X_1, X_2, \dots, X_n$ be a random sample drawn from a zero-truncated Poisson distribution with parameter $\lambda = 3$, defined on the positive integers $\{1, 2, \dots\}$.

The PGVF is defined as

$$\overline{G}(t) = \exp \left(\frac{\sum_{x=1}^t \log(x) f(x)}{\sum_{x=1}^t f(x)} \right), \quad t \geq 1,$$

which corresponds to the exponential of the conditional expectation $E[\log X \mid X \leq t]$.

The PGVF was evaluated at truncation levels $t = 1, \dots, 15$. For each configuration, Monte Carlo simulations were conducted using 1000 replications for sample sizes $n = 50$ and $n = 200$. The smoothing parameter in the kernel estimators was selected using the triangular kernel support parameter $a = 2$, corresponding to a bandwidth of order $h \asymp n^{-1/3}$, which follows from the mean integrated squared error expansion

$$\text{MISE} = O(h^2) + O\left(\frac{1}{nh}\right).$$

Tables 2 and 3 report the empirical mean, bias, variance, and MSE of the estimators for $n = 50$. The empirical estimator is nearly unbiased across all truncation levels but exhibits relatively large variance, particularly for larger values of t , which results in moderate MSE values. The asymmetric triangular kernel estimator reduces variance compared to the empirical estimator but introduces noticeable bias, especially for larger truncation levels, which increases its overall MSE.

The proposed boundary-corrected kernel estimator achieves a more balanced bias–variance trade-off. Although a small negative bias appears for smaller values of t , this bias decreases as t increases. The reduction in variance relative to the empirical estimator leads to improved MSE performance for moderate and large truncation levels.

Tables 4 and 5 present the corresponding results for $n = 200$. As expected, increasing the sample size substantially reduces the variance of all estimators. The empirical estimator remains essentially unbiased and performs well in terms of MSE for small truncation levels. The asymmetric triangular kernel estimator continues to exhibit noticeable bias, particularly in the upper range of t , which limits its efficiency despite variance reduction.

Table 2: PGVF Monte Carlo results: mean and bias ($n = 50$).

t	True	Empirical	Asym. Kernel	Proposed	Emp. Bias	Asym. Bias	Prop. Bias
2	1.516	1.519	1.490	1.430	0.00343	-0.02609	-0.08578
4	2.290	2.291	2.351	2.163	0.00064	0.06093	-0.12708
6	2.631	2.629	2.847	2.592	-0.00268	0.21592	-0.03968
8	2.717	2.715	2.974	2.720	-0.00128	0.25765	0.00317
10	2.729	2.727	2.984	2.730	-0.00159	0.25477	0.00151
12	2.730	2.728	2.984	2.731	-0.00157	0.25424	0.00111
15	2.730	2.729	2.984	2.731	-0.00156	0.25420	0.00108

Table 3: PGVF Monte Carlo results: variance and MSE ($n = 50$).

t	Emp. Var	Asym. Var	Prop. Var	Emp. MSE	Asym. MSE	Prop. MSE
2	0.01439	0.00767	0.00048	0.01440	0.00836	0.00784
4	0.03107	0.02826	0.00771	0.03107	0.03197	0.02386
6	0.04398	0.05067	0.02303	0.04398	0.09729	0.02460
8	0.04876	0.06349	0.03162	0.04877	0.12987	0.03163
10	0.05003	0.06574	0.03290	0.05003	0.13065	0.03290
12	0.05013	0.06584	0.03292	0.05013	0.13048	0.03292
15	0.05005	0.06581	0.03289	0.05005	0.13043	0.03289

Table 4: PGVF Monte Carlo results: mean and bias ($n = 200$).

t	True	Empirical	Asym. Kernel	Proposed	Emp. Bias	Asym. Bias	Prop. Bias
2	1.516	1.518	1.483	1.429	0.00179	-0.03314	-0.08622
4	2.290	2.292	2.344	2.163	0.00137	0.05371	-0.12692
6	2.631	2.632	2.846	2.572	0.00012	0.21421	-0.05907
8	2.717	2.718	3.011	2.719	0.00119	0.29399	0.00188
10	2.729	2.730	3.028	2.737	0.00062	0.29893	0.00769
12	2.730	2.731	3.028	2.737	0.00056	0.29843	0.00729
15	2.730	2.731	3.028	2.737	0.00054	0.29838	0.00725

Table 5: PGVF Monte Carlo results: variance and MSE ($n = 200$).

t	Emp. Var	Asym. Var	Prop. Var	Emp. MSE	Asym. MSE	Prop. MSE
2	0.00336	0.00166	0.00012	0.00337	0.00276	0.00755
4	0.00771	0.00693	0.00186	0.00771	0.00981	0.01797
6	0.01089	0.01235	0.00526	0.01089	0.05823	0.00874
8	0.01208	0.01532	0.00770	0.01208	0.10176	0.00770
10	0.01223	0.01610	0.00802	0.01223	0.10546	0.00808
12	0.01229	0.01616	0.00804	0.01229	0.10522	0.00810
15	0.01229	0.01617	0.00804	0.01230	0.10519	0.00809

The proposed boundary-corrected estimator maintains relatively small bias while achieving variance reduction compared with the empirical estimator for moderate and large truncation levels. Consequently, it generally yields smaller MSE values in this region and exhibits stable finite-sample behavior.

Figure 2 provides a visual comparison of the three estimators for $n = 50$ and $n = 200$. The proposed estimator closely follows the true PGVF while exhibiting smoother behavior than the empirical estimator. The asymmetric triangular kernel estimator smooths the empirical estimator but introduces visible bias for larger t values.

Overall, the simulation results indicate that the proposed boundary-corrected kernel estimator provides a competitive and stable approach for estimating the discrete PGVF, particularly for moderate sample sizes and truncation levels where boundary effects may influence traditional kernel estimators.

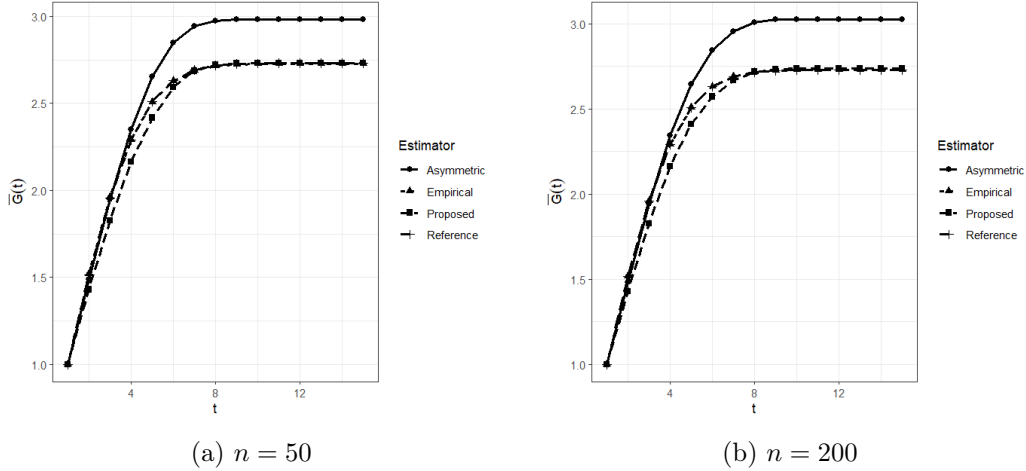


Figure 2: Comparison of PGVF estimators for two sample sizes. The proposed estimator reduces variance and MSE while closely following the true PGVF.

6 Real-Data Illustration: Insurance Claims Data

To illustrate the practical applicability of the proposed Discrete Past Geometric Vitality Function (PGVF) and to assess the performance of different

nonparametric estimators, we consider a real-world dataset on insurance policy durations. Such data naturally arise in actuarial science and reliability analysis, where the duration of customer engagement can be modeled as a discrete lifetime variable.

The dataset is obtained from [Aqqad \[2023\]](#), and the variable `months_as_customer` is used as the lifetime variable. Let $Z_1, Z_2, \dots, Z_n$ denote the observed durations (in months), which are treated as realizations of a discrete random variable supported on $\mathbb{N} = \{1, 2, \dots\}$. Missing and non-positive observations are excluded prior to analysis.

The empirical probability mass function (PMF) is defined as

$$\hat{f}_{\text{emp}}(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(Z_i = x), \quad x \in \mathbb{N}.$$

To smooth the empirical distribution, two kernel-based estimators are employed: the asymmetric discrete triangular kernel [[Kokonendji and Zocchi, 2010](#), [Kiessé, 2017](#)] and a symmetric triangular kernel (referred to as the proposed estimator). For a fixed bandwidth parameter $a = 2$, the smoothed PMF is given by

$$\hat{f}(x) = \sum_{z \in \mathbb{N}} K_a(x, z) \hat{f}_{\text{emp}}(z),$$

where $K_a(x, z)$ denotes the corresponding kernel function.

For any PMF estimator $\hat{f}(x)$, the Discrete PGVF is estimated by

$$\hat{G}(t) = \exp \left(\frac{\sum_{x=1}^t \log(x) \hat{f}(x)}{\sum_{x=1}^t \hat{f}(x)} \right), \quad t = 1, 2, \dots \quad (6.1)$$

Since the true underlying distribution is unknown, a parametric reference model is constructed by fitting a Negative Binomial distribution via maximum likelihood estimation. The fitted PMF, denoted by $\hat{f}_{\text{ref}}(x)$, is used to compute the reference PGVF, $\bar{G}_{\text{ref}}(t)$.

The performance of the estimators is evaluated using bias and mean squared error (MSE). As only a single dataset is available, the MSE is approximated by the squared bias:

$$\text{MSE}(t) \approx \text{Bias}(t)^2, \quad \text{Bias}(t) = \hat{G}(t) - \bar{G}_{\text{ref}}(t).$$

Table 6 reports the estimated PGVF values, bias, and MSE for ten representative values of t spanning the support of the data.

Table 6: Comparison of PGVF estimators for selected values of t (insurance data)

t	Reference	Empirical	Asym.	Prop.	Bias _{Emp}	Bias _{Asym}	Bias _{Prop}	MSE _{Emp}	MSE _{Asym}	MSE _{Prop}
54	31.9948	18.9688	18.7957	19.1565	-13.0260	-13.1991	-12.8383	169.6754	174.2153	164.8227
107	59.4958	45.2495	44.5112	45.3049	-14.2462	-14.9846	-14.1909	202.9548	224.5378	201.3815
160	82.8386	71.7680	71.2980	71.7889	-11.0706	-11.5406	-11.0497	122.5588	133.1860	122.0963
213	102.0737	92.2504	91.9635	92.4939	-9.8233	-10.1101	-9.5798	96.4966	102.2143	91.7722
267	117.7061	117.4892	117.0004	117.4556	-0.2170	-0.7057	-0.2505	0.0471	0.4981	0.0627
320	129.5721	134.5026	134.3127	134.6626	4.9305	4.7406	5.0905	24.3099	22.4733	25.9136
373	138.5042	142.6077	142.4083	142.7023	4.1035	3.9042	4.1981	16.8385	15.2425	17.6238
426	145.0444	148.9928	148.7814	149.1757	3.9484	3.7370	4.1313	15.5898	13.9653	17.0677
479	149.7128	158.3478	158.0070	158.4638	8.6350	8.2942	8.7510	74.5631	68.7942	76.5799

Figure 3 displays the estimated PGVF curves.

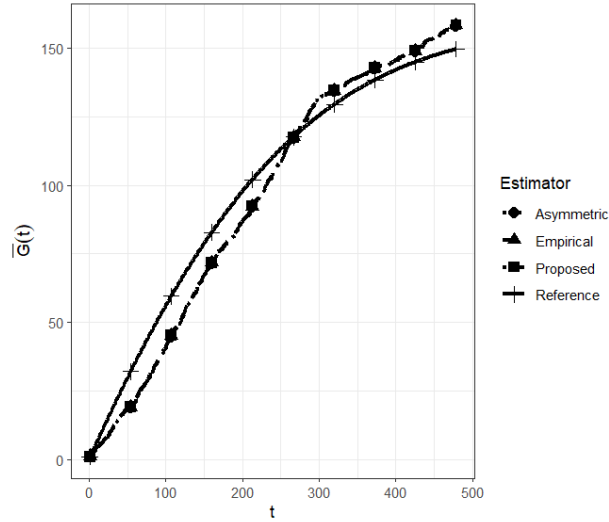


Figure 3: Estimated PGVF for insurance data using empirical, asymmetric kernel, and symmetric kernel estimators, along with the fitted Negative Binomial reference curve.

The results reveal a systematic pattern across truncation levels. For smaller values of t , all estimators substantially underestimate the reference PGVF, indicating a notable discrepancy between the empirical distribution and the fitted Negative Binomial model. As t increases, this discrepancy

gradually decreases, reflecting the cumulative nature of the PGVF, which stabilizes estimation at higher truncation levels.

A comparison of the estimators shows that none of the methods uniformly dominates the others. For moderate values of t (e.g., $t = 54$ to $t = 213$), the proposed estimator exhibits slightly smaller bias and MSE compared to the asymmetric estimator. However, for larger values of t (e.g., $t \geq 320$), the asymmetric estimator tends to achieve marginally lower MSE. The empirical estimator performs comparably to both kernel estimators across most of the range.

Overall, the differences among the three estimators are relatively small, suggesting that kernel smoothing has only a limited impact for this dataset. This behaviour can be attributed to the wide support and dense distribution of the data, where local smoothing does not substantially modify the empirical PMF. The relatively large bias observed across all estimators is primarily due to model misspecification in the Negative Binomial reference, rather than deficiencies in the nonparametric estimation procedures.

7 Conclusion

This paper introduced the *Discrete Past Geometric Vitality Function (PGVF)* as a multiplicative measure for analysing past lifetime behaviour in discrete reliability models. In contrast to classical past vitality measures based on arithmetic averaging, the proposed PGVF summarizes truncated lifetimes through a geometric framework, thereby capturing proportional ageing characteristics that may not be reflected by additive summaries.

We established several fundamental properties of the Discrete PGVF, including its relationship with the reversed hazard rate, monotonicity behaviour, stochastic ordering, and transformation properties. A key result shows that the PGVF uniquely determines the underlying distribution, leading to characterization results and a reconstruction perspective. Further insights were obtained through representations involving right-truncated reciprocal moments and an information-theoretic connection with the Shannon entropy of the truncated distribution.

To support practical implementation, a nonparametric estimation procedure based on discrete kernel methods with boundary correction was developed. Theoretical results on consistency, bias, and asymptotic normality were derived. Simulation studies demonstrated that the proposed estima-

tor achieves a favourable bias–variance trade-off compared to existing approaches, while the real data application illustrated its practical relevance.

Overall, the Discrete PGVF provides a flexible and informative framework for analysing past lifetime characteristics in discrete settings and contributes to the broader development of geometric approaches in reliability theory. Future work may focus on extensions to censored data, adaptive bandwidth selection, and applications to more complex count data models.

Data Availability Statement

The empirical dataset used in this study was obtained from a previously published source cited in the manuscript. The simulated datasets generated for illustration and performance evaluation are available from the corresponding author upon reasonable request.

Declarations

The authors declare no conflict of interest and no funding was received for this study.

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