

Structure-Preserving Numerical Approximation of Stochastic Darcy--Forchheimer Flow on Smooth Surfaces

Marcial Nguemfouo

`marcial.nguemfouo@facsciences-uy1.cm`

University of Yaoundé I

Research Article

Keywords: stochastic PDEs on surfaces, Darcy–Forchheimer flow, mixed finite elements, surface finite elements, mean-square stability, uncertainty quantification, scientific computing, Euler–Maruyama scheme

Posted Date: June 9th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-9847706/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

Structure-Preserving Numerical Approximation of Stochastic Darcy–Forchheimer Flow on Smooth Surfaces

Marcial Nguemfouo*

Department of Mathematics, University of Yaoundé I, Yaoundé, Cameroon

May 30, 2026

Abstract

This paper develops a structure-preserving numerical framework for stochastic Darcy–Forchheimer flow on smooth closed surfaces. The model describes nonlinear porous-type tangential transport on a curved interface, where the Darcy–Forchheimer momentum balance is perturbed by an additive tangential Wiener forcing and the pressure acts as a Lagrange multiplier enforcing surface mass conservation. The formulation is motivated by uncertainty-aware simulations of porous transport on biological membranes, thin permeable shells, fractured interfaces, and other curved surface media. We first introduce a mixed variational formulation in the space $H(\operatorname{div}_\Gamma; \Gamma)$ and reduce the stochastic dynamics to the divergence-compatible manifold. Using monotone-operator arguments in the variational SPDE setting, we prove existence, uniqueness, and mean-square stability of the stochastic mixed solution. We then construct a lowest-order Raviart–Thomas/piecewise-constant surface finite element approximation on a polyhedral surface Γ_h and combine it with an implicit Euler–Maruyama time discretization. The analysis separates the contributions of nonlinear monotonicity, stochastic consistency, mixed finite element approximation, and geometric perturbation caused by replacing Γ with Γ_h . Under standard regularity and covariance assumptions, the fully discrete approximation satisfies a strong mean-square estimate of order $O(h + \tau^{1/2})$ for the velocity and pressure in the natural mixed norms. Numerical experiments on the sphere illustrate temporal self-convergence, nonlinear Forchheimer dissipation, sensitivity to the noise intensity, and spatial volatility of stochastic surface flows. The proposed framework provides a rigorous basis for uncertainty quantification and future hybrid FEM–surrogate or FEM–PINN digital-twin models for nonlinear transport on curved surfaces.

Keywords: stochastic PDEs on surfaces; Darcy–Forchheimer flow; mixed finite elements; surface finite elements; mean-square stability; uncertainty quantification; scientific computing; Euler–Maruyama scheme.

1 Introduction

1.1 Motivation and scientific context

Flows in porous structures are classically modeled by Darcy-type laws, while deviations from the purely linear regime are often described by nonlinear drag corrections of Forchheimer type. Such models are relevant when inertial effects can no longer be neglected, for instance in high-velocity porous transport, filtration, fractured media, and interface-confined flows. In many modern applications, however, transport does not take place in a flat Euclidean domain but

*Corresponding author. Email: marcial.nguemfouo@facsciences-uy1.cm. ORCID: 0009-0007-2885-9326

rather on a curved interface, membrane, thin coating, or permeable surface embedded in $\mathbb{R}^3$. In such settings, the geometry becomes intrinsic to the PDE itself: the admissible velocity must be tangential, the divergence operator is replaced by the surface divergence, and the approximation of the geometry enters directly into the stability and consistency analysis.

The numerical analysis of PDEs posed on surfaces has by now reached a mature stage, largely thanks to the geometric finite element framework developed and synthesized by Dziuk and Elliott. Their *Acta Numerica* survey provides the standard analytical and numerical foundations for finite element methods on stationary and evolving surfaces, including surface gradients, transport identities, geometric perturbation estimates, and error analysis on triangulated approximations of smooth manifolds [7]. In particular, it makes clear that the passage from flat domains to surfaces is not merely cosmetic: geometry affects both the continuous variational structure and the discrete consistency terms.

Within this broad area, Darcy flow on surfaces has been studied in a mixed framework by Ferroni, Formaggia, and Fumagalli, who established a rigorous formulation and numerical analysis for the Darcy problem posed on smooth surfaces and discretized on polyhedral approximations [9]. Their work confirms that the mixed point of view is the natural one on surfaces, since the flux is a primary variable, mass conservation is encoded through the divergence operator, and Raviart–Thomas-type constructions can be adapted to the surface geometry. Lipschitz surface settings are also mathematically relevant, both because they arise naturally after triangulation and because they require a weaker tangential calculus and adapted weak formulations; see, for example, [11].

In parallel, stochastic partial differential equations on manifolds and surfaces have also been placed on a rigorous variational footing. In particular, Elliott, Hairer, and Scott formulated SPDEs on evolving surfaces and evolving Riemannian manifolds, and proved existence and uniqueness results in suitable variational settings [8]. Their framework shows that stochastic forcing can be incorporated consistently into geometric PDE models, provided one carefully tracks the underlying Hilbert and Gelfand structures.

The present article is motivated by the observation that these three lines of research — nonlinear Darcy–Forchheimer models, mixed finite elements on surfaces, and variational SPDEs on manifolds — have not yet been fully combined in a single analysis. More precisely, while deterministic Darcy problems on surfaces are now mathematically well understood [9], and while nonlinear Darcy–Forchheimer systems have been analyzed in the finite element setting in Euclidean domains [15], a complete stochastic mixed finite element analysis for a Darcy–Forchheimer problem posed on a smooth embedded surface does not appear to be available in the literature.

1.2 Why a stochastic extension?

A stochastic extension is not merely a formal generalization. In realistic porous or interface-confined transport problems, one may wish to model unresolved fluctuations in permeability, random external forcing, uncertain tangential stresses, or effective coarse-grained perturbations induced by microscopic heterogeneities. In such situations, deterministic forcing may be insufficient, and a stochastic model becomes a natural mathematical representation of uncertainty and unresolved multiscale effects.

On curved surfaces, this issue is even more delicate. Any noise term must respect the geometry of the problem: it should preserve tangentiality, and in regimes where the divergence constraint is enforced strongly, it should also be compatible with the surface incompressibility or prescribed mass-balance structure. This means that the stochastic term cannot be added naively; it must be built into the mixed formulation in a way that is analytically meaningful and geometrically consistent. The variational SPDE framework developed on manifolds by Elliott, Hairer, and Scott provides precisely the right abstract setting for this step [8].

1.3 Relation with the deterministic surface Darcy–Forchheimer paper

The present stochastic model should be viewed as a natural continuation of the deterministic mixed surface Darcy–Forchheimer analysis developed in the companion paper by the present author. That deterministic work studies a mixed finite element approximation of a Darcy–Forchheimer problem on a smooth embedded surface, with tangential velocity, scalar pressure, polyhedral approximation of the geometry, lift operators, and first-order error estimates for the mixed discretization. It also highlights the specific analytical difficulties caused by the surface setting, especially the treatment of tangential vector fields, divergence constraints, and geometric perturbation terms.

The stochastic extension developed here preserves the same geometric backbone. The same surface differential operators, the same mixed structure, and the same geometric consistency questions remain central. What changes is that the deterministic evolution law is now perturbed by an Itô noise term, and the analysis must incorporate stochastic compactness, Itô estimates, mean-square stability, and probabilistic consistency of the fully discrete scheme.

1.4 Mixed formulation and mathematical structure

The mixed approach is essential for several reasons. First, it treats the tangential velocity field as a primary unknown rather than as a post-processed gradient of the pressure. Second, it encodes the mass-balance constraint in the natural space $H(\operatorname{div}_\Gamma; \Gamma)$. Third, it preserves the flux structure of Darcy-type transport models, which is particularly important for porous-media interpretation and for local conservation properties at the discrete level. These advantages are already well known in the deterministic mixed theory for Darcy-type problems and remain decisive in the stochastic setting [1, 9].

At the nonlinear level, the Forchheimer term introduces a monotone but non-polynomial drift of the form $u \mapsto |u|u$. In Euclidean domains, this type of nonlinearity has been handled successfully in finite element analysis; see, for example, the work of Sayah, Semaan, and Triki [15]. In the present paper, we adapt this nonlinear monotonicity structure to a surface mixed variational setting and combine it with a stochastic forcing term satisfying tangential compatibility assumptions.

1.5 Main difficulty and literature gap

The main analytical difficulty comes from the interaction of three distinct sources of complexity:

1. the *geometric complexity* of PDEs posed on curved surfaces;
2. the *nonlinear monotone structure* of the Darcy–Forchheimer drift;
3. the *stochastic evolution* driven by multiplicative or additive Itô noise.

Each of these aspects is already nontrivial in isolation. Surface finite element methods require geometric approximation estimates and careful transfer between exact and discrete manifolds [7]. Darcy equations on surfaces require mixed spaces and surface inf-sup arguments [9]. Stochastic variational equations on manifolds require coercivity, hemicontinuity, growth control, and stochastic compactness in a Gelfand triple setting [8, 14, 10]. Their combination is therefore mathematically substantial.

To the best of our knowledge, there is currently no complete analysis that simultaneously provides:

- a stochastic Darcy–Forchheimer formulation on a smooth surface in mixed form;
- a well-posedness theory consistent with the geometry and the divergence constraint;

- a lowest-order mixed finite element discretization on polyhedral surfaces;
- and a strong mean-square a priori error estimate for the fully discrete approximation.

The goal of the present article is precisely to provide such a framework.

1.6 Relevance to applied mathematics and scientific computing

The paper is designed as a contribution to applied mathematics in the sense that it combines modeling, analysis, approximation, and simulation. The continuous problem is an uncertainty-aware nonlinear surface flow model; the analytical part establishes a stable variational structure; the numerical part provides a computable mixed finite element method; and the experiments illustrate how stochastic fluctuations and nonlinear drag affect surface transport. This combination is particularly relevant for scientific computing, uncertainty quantification, inverse modeling, and future data-assisted simulations on curved geometries. In this perspective, the present formulation may be used either as a stand-alone stochastic solver or as the physics-based component of hybrid finite element–machine learning workflows for fast parameter studies and digital-twin construction.

1.7 Main contributions of the paper

The main contributions of this work are the following.

- We formulate a stochastic Darcy–Forchheimer problem on a smooth closed surface in a mixed variational framework involving tangential velocity fields and zero-mean pressure.
- We identify assumptions on the stochastic forcing under which tangentiality and mixed compatibility are preserved.
- We establish an existence and uniqueness result for variational solutions by combining the surface mixed framework with the classical monotone SPDE method.
- We construct a lowest-order mixed finite element approximation on a polyhedral surface together with an implicit Euler–Maruyama time discretization.
- We separate the different sources of error — interpolation, geometric consistency, nonlinear drift approximation, and stochastic time discretization — and derive a strong mean-square error estimate.
- We show that the deterministic surface Darcy analysis and the Euclidean finite element theory for nonlinear Darcy–Forchheimer models can be unified in a genuinely stochastic geometric setting.

1.8 Comparison with existing literature

Compared with [9], the present paper replaces the linear Darcy law by a nonlinear Darcy–Forchheimer drift and introduces stochastic forcing. Compared with [15], the novelty lies in the passage from flat domains to smooth surfaces and in the corresponding geometric consistency analysis. Compared with [8], our setting is more specialized geometrically — fixed smooth embedded surfaces rather than general evolving manifolds — but substantially more specific on the PDE side, since we address a nonlinear mixed porous-medium flow structure and its finite element approximation.

In this sense, the present work is not a routine combination of existing theories. Rather, it requires a careful synthesis of geometric surface analysis, mixed finite element stability, monotone stochastic evolution, and nonlinear flux structure.

1.9 Organization of the paper

The remainder of the paper is organized as follows. In Section 2, we introduce the geometric preliminaries on smooth closed surfaces, define the relevant tangential differential operators and mixed spaces, and present the stochastic Darcy–Forchheimer model. Section 3 is devoted to the mixed variational formulation and to the well-posedness theory for the stochastic problem. In Section 4, we construct the discrete surface approximation, define the mixed finite element spaces on Γ_h , and introduce the fully discrete implicit Euler–Maruyama scheme. Section 5 contains the error analysis, including the decomposition into geometric, deterministic, and stochastic consistency terms. Finally, Section 6 discusses the mathematical significance of the result and outlines several perspectives, including random surfaces, evolving geometries, and coupled bulk–surface stochastic porous flow models.

Positioning statement. From a mathematical viewpoint, the present article sits at the intersection of nonlinear porous-media flow, geometric partial differential equations, and stochastic analysis on manifolds. Its novelty lies in showing that the mixed Darcy–Forchheimer structure survives in a stochastic surface setting and can be approximated numerically with controlled geometric and probabilistic errors.

Notation

Symbol	Meaning
Γ	Smooth closed surface of class C^3 embedded in $\mathbb{R}^3$
Γ_h	Polyhedral approximation of Γ
n, n_h	Unit normal fields to Γ and Γ_h
P, P_h	Tangential projectors on Γ and Γ_h
d	Signed distance function to Γ
a	Closest-point projection from a tubular neighborhood onto Γ
$\nabla_\Gamma, \div_\Gamma$	Surface gradient and surface divergence on Γ
$\nabla_{\Gamma_h}, \div_{\Gamma_h}$	Surface gradient and surface divergence on Γ_h
Δ_Γ	Laplace–Beltrami operator on Γ
$H = L^2_{\text{tan}}(\Gamma)^3$	Hilbert space of square-integrable tangential vector fields
$V = H(\div_\Gamma; \Gamma)$	Tangential velocity space
$Q = L^2_0(\Gamma)$	Zero-mean pressure space
V_0	Divergence-free subspace $\{v \in V : \div_\Gamma v = 0\}$
$V_0(g)$	Affine space $\{v \in V : \div_\Gamma v = g\}$
$X = V \cap L^3_{\text{tan}}(\Gamma)^3$	Reflexive space used for the nonlinear Darcy–Forchheimer drift
$X_0 = V_0 \cap L^3_{\text{tan}}(\Gamma)^3$	Divergence-free counterpart of X
$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$	Filtered probability space
W_t	Cylindrical Wiener process on U
U	Separable Hilbert space
$\mathcal{L}_2(U, H)$	Hilbert–Schmidt operators from U to H
$\mathbb{E}, \mathbb{P}$	Expectation and probability
u, p	Exact velocity and pressure
u_h, p_h	Discrete velocity and pressure on Γ_h
u_h^ℓ, p_h^ℓ	Lifted discrete velocity and pressure on Γ
ϕ_h^ℓ	Lift of a discrete scalar quantity from Γ_h to Γ
$\phi^{-\ell}$	Inverse lift of a scalar quantity from Γ to Γ_h
μ_h	Surface Jacobian associated with the lift between Γ_h and Γ
α, β	Darcy and Forchheimer coefficients

Symbol	Meaning
α_h, β_h	Discrete approximations of α and β on Γ_h
α_*	Positive lower bound of α
f, g	Tangential forcing term and divergence source term
f_h, g_h	Discrete approximations of f and g
G	Continuous stochastic coefficient (noise operator)
G_h	Discrete stochastic coefficient on Γ_h
$A(u) = \alpha u + \beta u u$	Nonlinear Darcy–Forchheimer drift operator
$\mathcal{A}$	Variational form of the nonlinear drift
$a(\cdot, \cdot)$	Continuous Darcy bilinear form
$c(\cdot; \cdot)$	Continuous Forchheimer nonlinear form
$b(\cdot, \cdot)$	Continuous mixed velocity–pressure coupling
$a_h(\cdot, \cdot), c_h(\cdot; \cdot), b_h(\cdot, \cdot)$	Discrete counterparts on Γ_h
$a_h^\ell(\cdot, \cdot), c_h^\ell(\cdot; \cdot), b_h^\ell(\cdot, \cdot)$	Lifted discrete forms on Γ
V_h, Q_h	Discrete mixed finite element spaces on Γ_h
V_h^ℓ, Q_h^ℓ	Lifted discrete spaces on Γ
Π_h	Velocity interpolation operator into V_h^ℓ
π_h	Pressure projection/interpolation operator into Q_h^ℓ
$\mathcal{R}$	Right inverse of the surface divergence
h	Mesh parameter
τ	Time step
N	Number of time steps
T	Final time
$t_n = n\tau$	Discrete time levels
$\Delta W_{n+1} = W(t_{n+1}) - W(t_n)$	Wiener increments
M	Number of Monte Carlo realizations
u_h^n, p_h^n	Fully discrete velocity and pressure at time level t_n
$u_h^{n,\ell}, p_h^{n,\ell}$	Lifted fully discrete velocity and pressure at time level t_n
e_u^n, e_p^n	Total velocity and pressure errors at time level t_n
η_u^n, η_p^n	Interpolation errors
θ_u^n, θ_p^n	Discrete errors
T^{n+1}	Temporal consistency residual
G^{n+1}	Geometric consistency residual
S^{n+1}	Stochastic consistency residual
I^{n+1}	Interpolation residual
β_0, β_1	Continuous and discrete inf-sup constants
$\mathcal{E}_h^{(m)}(T)$	Final discrete energy for Monte Carlo realization m
$E_u(T), E_p(T)$	Empirical root-mean-square final-time errors for velocity and pressure

2 Geometric setting, functional framework, and stochastic model

In this section we introduce the geometric and probabilistic framework for the stochastic Darcy–Forchheimer problem posed on a smooth closed surface. The geometric differential operators and lift-based surface calculus used throughout the paper are standard in the theory of partial differential equations on embedded surfaces; see, for instance, [5, 6, 7]. The mixed Darcy viewpoint on smooth surfaces follows the framework developed in [9], while the stochastic variational interpretation is motivated by the general monotone SPDE theory in [14, 10] and by SPDEs on manifolds and surfaces as in [8].

The new ingredient in the present work is the combination of these three components in a single stochastic mixed Darcy–Forchheimer formulation on a smooth embedded surface, together with a geometric finite element discretization studied later in Sections 4 and 5.

2.1 Surface geometry and tangential differential operators

Let $\Gamma \subset \mathbb{R}^3$ be a connected, compact, orientable, boundaryless surface of class C^3 , with unit normal field n . We denote by

$$P := I - n \otimes n$$

the orthogonal projector onto the tangent bundle $T\Gamma$.

Since Γ is of class C^3 , there exists a tubular neighborhood

$$U_\delta := \{x \in \mathbb{R}^3 : |d(x)| < \delta\}$$

for some $\delta > 0$, where d denotes the signed distance function to Γ , such that every $x \in U_\delta$ admits a unique closest-point projection $a(x) \in \Gamma$ and can be written as

$$x = a(x) + d(x) n(a(x)).$$

This geometric representation is classical and underlies the lift construction used in surface finite element methods; see [5, 6, 7].

For a scalar function $\phi : \Gamma \rightarrow \mathbb{R}$, the tangential gradient is defined by

$$\nabla_\Gamma \phi := P \nabla \phi^e,$$

where ϕ^e is any smooth extension of ϕ to a neighborhood of Γ . For a tangential vector field $v : \Gamma \rightarrow \mathbb{R}^3$ satisfying $v \cdot n = 0$, the surface divergence is defined by

$$\operatorname{div}_\Gamma v := \operatorname{tr}(P \nabla v^e P),$$

where v^e is any smooth extension of v . The Laplace–Beltrami operator is then given by

$$\Delta_\Gamma \phi := \operatorname{div}_\Gamma (\nabla_\Gamma \phi).$$

These definitions are independent of the chosen extensions and are standard in surface PDE analysis; see [7].

2.2 Tangential spaces and mixed structure

The natural Hilbert space for square-integrable tangential velocity fields is

$$H := L_{\tan}^2(\Gamma)^3 := \{v \in L^2(\Gamma)^3 : v \cdot n = 0 \text{ a.e. on } \Gamma\},$$

equipped with the norm inherited from $L^2(\Gamma)^3$.

For the mixed formulation, we introduce

$$V := H(\operatorname{div}_\Gamma; \Gamma) := \{v \in L_{\tan}^2(\Gamma)^3 : \operatorname{div}_\Gamma v \in L^2(\Gamma)\},$$

with norm

$$\|v\|_V^2 := \|v\|_{L^2(\Gamma)}^2 + \|\operatorname{div}_\Gamma v\|_{L^2(\Gamma)}^2.$$

For the pressure variable, we set

$$Q := L_0^2(\Gamma) := \left\{ q \in L^2(\Gamma) : \int_\Gamma q \, d\sigma = 0 \right\}.$$

The restriction to zero-mean pressures is natural on a closed connected surface, since the pressure is defined only up to an additive constant in the mixed Darcy structure. This is exactly the same normalization as in the deterministic surface Darcy theory; see [9, 13]. Although the present work is carried out on smooth closed surfaces, the weak surface framework admits meaningful extensions to closed Lipschitz manifolds; see [11] for the Poisson problem in that setting.

For later use, we also define

$$V_0(g) := \{v \in V : \operatorname{div}_\Gamma v = g\}, \quad V_0 := \{v \in V : \operatorname{div}_\Gamma v = 0\}.$$

The mixed coupling between velocity and pressure is described by the bilinear form

$$b(v, q) := - \int_\Gamma (\operatorname{div}_\Gamma v) q \, d\sigma \quad \forall (v, q) \in V \times Q.$$

The corresponding inf-sup condition is classical in mixed formulations and follows, on a smooth closed surface, from the solvability of the Laplace–Beltrami equation with zero average. This fact is standard from the point of view of mixed theory [2, 1] and in the surface Darcy setting [9]; it will be stated precisely and used in Section 3.

2.3 Probabilistic framework

Let

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$$

be a filtered probability space satisfying the usual conditions, and let U be a separable Hilbert space. We denote by

$$W = (W_t)_{t \in [0, T]}$$

a cylindrical Wiener process on U , adapted to the filtration $(\mathcal{F}_t)$.

For two Hilbert spaces X and Y , we denote by $L_2(X, Y)$ the space of Hilbert–Schmidt operators from X to Y . In particular, the stochastic coefficient will take values in

$$L_2(U, H).$$

The general variational SPDE framework used later is classical; see [14, 10]. In addition, stochastic PDEs on surfaces and Riemannian manifolds have been studied in variational form in [8], which provides the appropriate conceptual background for the present setting.

2.4 The stochastic Darcy–Forchheimer model

We now introduce the stochastic Darcy–Forchheimer system on the fixed smooth surface Γ . We seek a tangential velocity field u , a scalar pressure p , and a prescribed divergence source g such that, formally,

$$du + (\alpha u + \beta |u|u + \nabla_\Gamma p) \, dt = f \, dt + G(u) \, dW_t \quad \text{on } \Gamma \times (0, T), \quad (1)$$

$$\operatorname{div}_\Gamma u = g \quad \text{on } \Gamma \times (0, T), \quad (2)$$

$$u \cdot n = 0 \quad \text{on } \Gamma \times (0, T), \quad (3)$$

$$u(0) = u_0 \quad \text{on } \Gamma. \quad (4)$$

Here:

- $\alpha \in L^\infty(\Gamma)$ is a strictly positive Darcy drag coefficient;
- $\beta \in L^\infty(\Gamma)$ is a nonnegative Forchheimer coefficient;
- f is a given tangential forcing term;

- g is a prescribed source term in the mass balance;
- $G(u) dW_t$ is an Itô stochastic perturbation with values in tangential vector fields.

The nonlinear term

$$u \longmapsto \alpha u + \beta |u|u$$

is the surface analogue of the standard Darcy–Forchheimer law. In Euclidean domains, such nonlinearities are classical in porous-media modeling and finite element analysis; see, for instance, [15]. The novelty here is not the pointwise monotonicity of the Forchheimer term itself, but its incorporation into a mixed stochastic formulation on a curved surface, together with geometric consistency considerations for later discretization.

Equation (2) expresses the surface mass-balance constraint, while the pressure p plays the role of a Lagrange multiplier enforcing this constraint. Equation (3) is not an additional boundary condition in the usual sense, but rather the intrinsic geometric requirement that the velocity remain tangent to the surface.

2.5 Compatibility condition and pressure normalization

Since Γ is closed and boundaryless, integration of (2) over Γ yields

$$\int_{\Gamma} \operatorname{div}_{\Gamma} u \, d\sigma = 0.$$

Hence a necessary compatibility condition for solvability is

$$\int_{\Gamma} g(t) \, d\sigma = 0 \quad \text{for a.e. } t \in (0, T).$$

This is precisely why the divergence source term naturally belongs to the zero-mean space $Q = L_0^2(\Gamma)$.

Likewise, the pressure is determined only up to an additive constant. We therefore impose the normalization

$$\int_{\Gamma} p(t) \, d\sigma = 0 \quad \text{for a.e. } t \in (0, T),$$

and seek

$$p \in L^2(\Omega; L^2(0, T; Q)).$$

This normalization is standard in mixed Darcy-type formulations on closed surfaces; see [9, 13].

2.6 Standing assumptions

Throughout the paper we assume the following.

(A1) The coefficients satisfy

$$\alpha, \beta \in L^\infty(\Gamma), \quad \alpha(x) \geq \alpha_* > 0, \quad \beta(x) \geq 0 \quad \text{for a.e. } x \in \Gamma.$$

(A2) The initial datum satisfies

$$u_0 \in L^2(\Omega, \mathcal{F}_0; H).$$

(A3) The forcing term satisfies

$$f \in L^2(\Omega; L^2(0, T; H)),$$

and is progressively measurable when random.

(A4) The divergence source satisfies

$$g \in L^2(\Omega; L^2(0, T; Q)).$$

(A5) The noise coefficient

$$G : H \rightarrow L_2(U, H)$$

is globally Lipschitz and of linear growth, that is, there exist constants $L_G, C_G > 0$ such that

$$\|G(v) - G(w)\|_{L_2(U, H)} \leq L_G \|v - w\|_H \quad \forall v, w \in H,$$

and

$$\|G(v)\|_{L_2(U, H)}^2 \leq C_G (1 + \|v\|_H^2) \quad \forall v \in H.$$

(A6) For every $v \in H$, the range of $G(v)$ is contained in the tangential space H , so that the stochastic perturbation preserves tangentiality.

(A7) When the divergence constraint is required to hold pathwise, we assume that the noise is divergence-compatible, in the sense that

$$\div_\Gamma(G(v)\xi) = 0 \quad \forall v \in H, \forall \xi \in U.$$

Assumptions (A5)–(A7) are standard from the viewpoint of variational SPDE theory and guarantee that the stochastic term is well defined in the natural Hilbert setting [14, 10]. The genuinely geometric content lies in (A6) and (A7): on a surface, the noise must be compatible not only with the ambient Hilbert structure, but also with tangentiality and, in the mixed setting considered here, with the divergence constraint.

Throughout the present paper, all main analytical and numerical results are proved under Assumption (A7). More general projected-noise formulations are possible, but they are left for future work.

2.7 Remarks on the noise structure

The tangentiality requirement (A6) is essential. If the stochastic perturbation had a nonzero normal component, then the velocity would leave the tangent bundle immediately, and the resulting model would no longer describe a surface-constrained porous flow.

Assumption (A7) is stronger than strictly necessary, but it is natural in the present mixed framework. It ensures that the stochastic perturbation does not destroy the divergence constraint at the pathwise level. This is the most convenient setting for pressure reconstruction through the mixed formulation in Section 3.

A natural way to enforce divergence compatibility is to project a general tangential stochastic perturbation onto the kernel of $\div_\Gamma$. Formally, for a tangential field v , one may consider

$$\Pi_{\ker \div_\Gamma} v = v - \nabla_\Gamma \phi, \quad -\Delta_\Gamma \phi = \div_\Gamma v \quad \text{on } \Gamma,$$

with zero average normalization on ϕ . Such Helmholtz-type decompositions on smooth closed surfaces are classical from the viewpoint of surface elliptic theory and mixed methods; see [7, 9]. However, to keep the analysis focused, we do not develop that projected-noise framework here.

2.8 Mixed variational formulation

For $u, v \in V$ and $q \in Q$, we define

$$a(u, v) := \int_\Gamma \alpha u \cdot v \, d\sigma,$$

$$c(u; v) := \int_{\Gamma} \beta |u| u \cdot v \, d\sigma,$$

$$b(v, q) := - \int_{\Gamma} (\div_{\Gamma} v) q \, d\sigma.$$

We also write

$$F_f(v) := \int_{\Gamma} f \cdot v \, d\sigma.$$

The mixed stochastic variational problem reads as follows.

Definition 2.1 (Mixed stochastic variational formulation). *Find adapted processes*

$$u \in L^2(\Omega; L^\infty(0, T; H)) \cap L^2(\Omega; L^2(0, T; V)), \quad p \in L^2(\Omega; L^2(0, T; Q)),$$

such that for every $v \in V$, every $q \in Q$, and almost every $t \in [0, T]$, one has $\mathbb{P}$ -a.s.

$$\begin{aligned} (u(t), v)_H + \int_0^t a(u(s), v) \, ds + \int_0^t c(u(s); v) \, ds + \int_0^t b(v, p(s)) \, ds \\ = (u_0, v)_H + \int_0^t F_f(v) \, ds + \int_0^t (G(u(s)) \, dW_s, v)_H, \end{aligned} \quad (5)$$

$$b(u(t), q) = -(g(t), q)_{L^2(\Gamma)}. \quad (6)$$

This is the natural stochastic extension of the deterministic mixed Darcy formulation on smooth surfaces developed in [9, 13]. The novel feature is the addition of a stochastic Itô perturbation within the tangential mixed surface framework, while preserving the Darcy–Forchheimer nonlinear drift and the pressure interpretation as a Lagrange multiplier.

2.9 Monotone structure of the drift

A key role in the analysis is played by the nonlinear drift operator

$$A(u) := \alpha u + \beta |u| u.$$

Its variational action is given by

$$\langle A(u), v \rangle = a(u, v) + c(u; v).$$

The map

$$z \mapsto |z|z$$

is monotone on $\mathbb{R}^3$, that is,

$$(|a|a - |b|b) \cdot (a - b) \geq 0 \quad \forall a, b \in \mathbb{R}^3.$$

Consequently, under Assumption **(A1)**, one has

$$\langle A(u) - A(v), u - v \rangle \geq \alpha_* \|u - v\|_{L^2(\Gamma)}^2 \quad \forall u, v \in V,$$

and

$$\langle A(u), u \rangle = \int_{\Gamma} \alpha |u|^2 \, d\sigma + \int_{\Gamma} \beta |u|^3 \, d\sigma \geq \alpha_* \|u\|_{L^2(\Gamma)}^2.$$

Such monotonicity properties are classical both in nonlinear Darcy–Forchheimer models and in the abstract theory of monotone operators; see [15, 14, 10]. What is new here is their combination with the mixed surface geometry and the stochastic forcing.

The above structure will be exploited in Section 3 to establish well-posedness, and later in Sections 4 and 5 to prove stability and strong mean-square convergence of the fully discrete surface finite element approximation.

3 Well-posedness of the stochastic mixed problem

In this section we establish the well-posedness of the stochastic Darcy–Forchheimer problem introduced in Section 2. The argument combines the mixed surface framework, the monotone structure of the Darcy–Forchheimer drift, and the variational theory of monotone stochastic evolution equations. The geometric background for PDEs on smooth surfaces may be found in Dziuk and Elliott [7], the mixed Darcy framework on surfaces in Ferroni, Formaggia, and Fumagalli [9], and the monotone SPDE framework in Prévôt and Röckner [14] and Liu and Röckner [10]. For stochastic PDEs on evolving surfaces and manifolds, see also Elliott, Hairer, and Scott [8].

3.1 Functional setting and auxiliary results

We first collect the main structural ingredients needed in the analysis: the continuous inf-sup condition for the pair (V, Q) , the basic properties of the nonlinear Darcy–Forchheimer drift, and the existence of a bounded right inverse of the surface divergence.

Lemma 3.1 (Continuous surface inf-sup condition). *Let*

$$V := H(\div_\Gamma; \Gamma) \quad \text{and} \quad Q := L_0^2(\Gamma),$$

where $\Gamma \subset \mathbb{R}^3$ is a smooth, connected, compact, orientable, boundaryless surface. Then there exists a constant $\beta_0 > 0$, depending only on Γ , such that

$$\inf_{q \in Q \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{|b(v, q)|}{\|v\|_V \|q\|_{L^2(\Gamma)}} \geq \beta_0,$$

where

$$b(v, q) := - \int_\Gamma (\div_\Gamma v) q \, d\sigma.$$

Proof. Let $q \in Q \setminus \{0\}$ be arbitrary. Since Γ is smooth, compact, connected, and without boundary, and since q has zero mean, the Laplace–Beltrami problem

$$-\Delta_\Gamma \phi = q \quad \text{on } \Gamma, \quad \int_\Gamma \phi \, d\sigma = 0,$$

admits a unique solution $\phi \in H^2(\Gamma) \cap L_0^2(\Gamma)$, and elliptic regularity on smooth closed surfaces yields

$$\|\phi\|_{H^2(\Gamma)} \leq C \|q\|_{L^2(\Gamma)};$$

see, for instance, [7, 9]. Define

$$v := -\nabla_\Gamma \phi.$$

Then v is tangential and belongs to V , with

$$\div_\Gamma v = -\div_\Gamma (\nabla_\Gamma \phi) = -\Delta_\Gamma \phi = q.$$

Therefore,

$$b(v, q) = - \int_\Gamma (\div_\Gamma v) q \, d\sigma = - \int_\Gamma q^2 \, d\sigma,$$

and hence

$$|b(v, q)| = \|q\|_{L^2(\Gamma)}^2.$$

Moreover,

$$\|v\|_{L^2(\Gamma)} = \|\nabla_\Gamma \phi\|_{L^2(\Gamma)} \leq C \|\phi\|_{H^2(\Gamma)} \leq C \|q\|_{L^2(\Gamma)},$$

while

$$\|\div_{\Gamma} v\|_{L^2(\Gamma)} = \|q\|_{L^2(\Gamma)}.$$

Thus

$$\|v\|_V = \left(\|v\|_{L^2(\Gamma)}^2 + \|\div_{\Gamma} v\|_{L^2(\Gamma)}^2 \right)^{1/2} \leq C \|q\|_{L^2(\Gamma)}.$$

It follows that

$$\sup_{w \in V \setminus \{0\}} \frac{|b(w, q)|}{\|w\|_V} \geq \frac{|b(v, q)|}{\|v\|_V} \geq C^{-1} \|q\|_{L^2(\Gamma)}.$$

Taking the infimum over all $q \in Q \setminus \{0\}$ proves the result. $\square$

Lemma 3.2 (A reflexive space for the nonlinear drift). *Define*

$$X := V \cap L_{\tan}^3(\Gamma)^3, \quad \|v\|_X := \|v\|_V + \|v\|_{L^3(\Gamma)}.$$

Then X is a reflexive Banach space. Moreover, if

$$X_0 := V_0 \cap L_{\tan}^3(\Gamma)^3, \quad V_0 := \{v \in V : \div_{\Gamma} v = 0\},$$

then X_0 is a closed reflexive subspace of X .

Proof. Since V is a Hilbert space and $L_{\tan}^3(\Gamma)^3$ is reflexive, the product $V \times L_{\tan}^3(\Gamma)^3$ is reflexive. The space X is isomorphic to the closed subspace

$$\{(v, v) : v \in V \cap L_{\tan}^3(\Gamma)^3\} \subset V \times L_{\tan}^3(\Gamma)^3,$$

and is therefore reflexive. The same argument applies to X_0 , which is closed in X . $\square$

Lemma 3.3 (Properties of the Darcy–Forchheimer drift). *Assume (A1) and define, for $u, v \in X$,*

$$\langle \mathcal{A}(u), v \rangle_{X', X} := \int_{\Gamma} \alpha u \cdot v \, d\sigma + \int_{\Gamma} \beta |u| u \cdot v \, d\sigma.$$

Then $\mathcal{A} : X_0 \rightarrow X'_0$ is hemicontinuous, monotone, and coercive. More precisely, for all $u, v \in X_0$,

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{X'_0, X_0} \geq \alpha_* \|u - v\|_{L^2(\Gamma)}^2,$$

and for all $u \in X_0$,

$$\langle \mathcal{A}(u), u \rangle_{X'_0, X_0} \geq \alpha_* \|u\|_{L^2(\Gamma)}^2 + \int_{\Gamma} \beta |u|^3 \, d\sigma.$$

In addition, $\mathcal{A}$ satisfies the growth estimate

$$\|\mathcal{A}(u)\|_{X'_0} \leq C(1 + \|u\|_X^2).$$

Proof. The term αu is linear and continuous. For the nonlinear term, note that

$$\| |u| u \|_{L^{3/2}(\Gamma)} = \|u\|_{L^3(\Gamma)}^2,$$

so that

$$v \mapsto \int_{\Gamma} \beta |u| u \cdot v \, d\sigma$$

defines an element of X'_0 . Hemicontinuity follows from dominated convergence. Monotonicity follows from the classical pointwise inequality

$$(|a|a - |b|b) \cdot (a - b) \geq 0 \quad \forall a, b \in \mathbb{R}^3,$$

and coercivity follows from

$$\langle \mathcal{A}(u), u \rangle = \int_{\Gamma} \alpha |u|^2 d\sigma + \int_{\Gamma} \beta |u|^3 d\sigma.$$

Finally, Hölder's inequality gives

$$\left| \int_{\Gamma} \beta |u| u \cdot v d\sigma \right| \leq \|\beta\|_{L^\infty(\Gamma)} \|u\|_{L^3(\Gamma)}^2 \|v\|_{L^3(\Gamma)},$$

which yields the growth estimate. Such monotonicity arguments are standard in nonlinear Darcy–Forchheimer analysis and in monotone SPDE theory; see, e.g., [15, 14, 10]. $\square$

Lemma 3.4 (Right inverse of the surface divergence). *There exists a bounded linear operator*

$$\mathcal{R} : Q \rightarrow V$$

such that

$$\operatorname{div}_{\Gamma}(\mathcal{R}q) = q \quad \text{and} \quad \|\mathcal{R}q\|_V \leq C \|q\|_{L^2(\Gamma)} \quad \forall q \in Q.$$

Proof. For $q \in Q$, let $\phi \in H^2(\Gamma) \cap L_0^2(\Gamma)$ solve

$$-\Delta_{\Gamma} \phi = q \quad \text{on } \Gamma, \quad \int_{\Gamma} \phi d\sigma = 0.$$

Then define

$$\mathcal{R}q := -\nabla_{\Gamma} \phi.$$

Exactly as in the proof of Lemma 3.1, one has

$$\operatorname{div}_{\Gamma}(\mathcal{R}q) = q$$

and

$$\|\mathcal{R}q\|_V \leq C \|q\|_{L^2(\Gamma)}.$$

$\square$

For a given divergence source term

$$g \in L^2(\Omega; L^2(0, T; Q)),$$

we define

$$r(t, \omega) := \mathcal{R}(g(t, \omega)).$$

Then $r \in L^2(\Omega; L^2(0, T; V))$, and any admissible velocity field can be written as

$$u = w + r, \quad w \in X_0.$$

This reduction is standard in mixed formulations with prescribed divergence and allows one to eliminate the pressure at the level of the reduced problem; compare with [9].

3.2 Existence and uniqueness of the velocity

We now formulate the reduced stochastic evolution equation on the divergence-free subspace X_0 . For $w \in X_0$, define

$$\langle \mathcal{A}_g(w), v \rangle := \langle \mathcal{A}(w + r), v \rangle \quad \forall v \in X_0,$$

and

$$\langle \mathcal{F}_g(t), v \rangle := (f(t), v)_H, \quad \mathcal{G}_g(w) := G(w + r).$$

Then the reduced stochastic problem reads

$$dw + \mathcal{A}_g(w) dt = \mathcal{F}_g dt + \mathcal{G}_g(w) dW_t \quad \text{in } X'_0. \quad (7)$$

Theorem 3.5 (Existence and uniqueness of the reduced velocity process). *Assume (A1)–(A7). Let*

$$u_0 \in L^2(\Omega, \mathcal{F}_0; H), \quad f \in L^2(\Omega; L^2(0, T; H)), \quad g \in L^2(\Omega; L^2(0, T; Q)).$$

Then there exists a unique progressively measurable process

$$w \in L^2(\Omega; L^\infty(0, T; H)) \cap L^2(\Omega; L^2(0, T; V_0)) \cap L^3(\Omega \times (0, T) \times \Gamma)^3$$

such that, for every $v \in X_0$ and all $t \in [0, T]$,

$$\begin{aligned} (w(t), v)_H + \int_0^t \langle \mathcal{A}_g(w(s)), v \rangle ds \\ = (w_0, v)_H + \int_0^t \langle \mathcal{F}_g(s), v \rangle ds + \int_0^t (\mathcal{G}_g(w(s)) dW_s, v)_H \quad a.s. \end{aligned} \quad (8)$$

Moreover,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|w(t)\|_H^2 \right] + \mathbb{E} \int_0^T \|w(t)\|_V^2 dt + \mathbb{E} \int_0^T \|w(t)\|_{L^3(\Gamma)}^3 dt \leq C,$$

where C depends only on the data and on the coefficients.

Proof. By Lemma 3.2, the state space X_0 is reflexive. By Lemma 3.3, the reduced drift is hemicontinuous, monotone, and coercive on X_0 , up to lower-order terms induced by the lifting r . The noise coefficient satisfies the usual Lipschitz and linear growth assumptions by (A5)–(A7). Therefore, the standard variational existence theory for monotone SPDEs applies; see [14, Chapters 4–5] and [10, Chapter 4]. This yields a unique variational solution w to (7), together with the stated a priori estimate. The present surface setting is compatible with this abstract framework because the geometry is fixed and all differential operators are intrinsic on Γ ; compare also with [8]. $\square$

We now return to the original variable

$$u := w + r.$$

Then

$$u \in L^2(\Omega; L^\infty(0, T; H)) \cap L^2(\Omega; L^2(0, T; V)) \cap L^3(\Omega \times (0, T) \times \Gamma)^3,$$

and

$$\div_\Gamma u = g \quad \text{in } L^2(\Omega; L^2(0, T; Q)).$$

3.3 Pressure reconstruction

Once the velocity process has been obtained, the pressure is recovered from the mixed relation by means of the continuous inf-sup condition. The reconstruction is performed in integrated form in time, thereby avoiding any formal appearance of the time derivative of the Wiener process.

Theorem 3.6 (Pressure reconstruction). *Let u be the velocity process constructed above. Then there exists a unique*

$$p \in L^2(\Omega; L^2(0, T; Q))$$

such that, for every $v \in V$ and every $t \in [0, T]$,

$$\begin{aligned} \int_0^t b(v, p(s)) ds &= (u_0 - u(t), v)_H + \int_0^t (f(s), v)_H ds \\ &\quad - \int_0^t a(u(s), v) ds - \int_0^t c(u(s); v) ds + \int_0^t (G(u(s)) dW_s, v)_H \end{aligned} \quad (9)$$

almost surely. Moreover,

$$\mathbb{E} \int_0^T \|p(t)\|_{L^2(\Gamma)}^2 dt \leq C,$$

where C depends only on the data and the coefficients.

Proof. For fixed $t \in [0, T]$, define a random linear functional Λ_t on V by

$$\begin{aligned} \Lambda_t(v) &:= (u_0 - u(t), v)_H + \int_0^t (f(s), v)_H ds \\ &\quad - \int_0^t a(u(s), v) ds - \int_0^t c(u(s); v) ds + \int_0^t (G(u(s)) dW_s, v)_H. \end{aligned}$$

By construction of the reduced problem, this functional vanishes on V_0 . Hence, by Lemma 3.1, there exists a unique element

$$P_t := \int_0^t p(s) ds \in Q$$

such that

$$b(v, P_t) = \Lambda_t(v) \quad \forall v \in V.$$

The estimate for P_t follows from the inf-sup inequality and the a priori bounds for u . Differentiating in the sense of $L^2(0, T; Q)$ yields the existence of

$$p \in L^2(\Omega; L^2(0, T; Q))$$

satisfying (9). Uniqueness follows from the inf-sup condition. $\square$

Combining Theorem 3.5 and Theorem 3.6, we obtain the following well-posedness result for the full mixed stochastic formulation.

Theorem 3.7 (Well-posedness of the stochastic mixed problem). *Assume (A1)–(A7). Then the stochastic mixed Darcy–Forchheimer problem admits a unique variational solution*

$$(u, p) \in \left(L^2(\Omega; L^\infty(0, T; H)) \cap L^2(\Omega; L^2(0, T; V)) \cap L^3(\Omega \times (0, T) \times \Gamma)^3 \right) \times L^2(\Omega; L^2(0, T; Q)).$$

4 Surface mixed finite element discretization and fully discrete scheme

In this section we construct the fully discrete approximation of the stochastic Darcy–Forchheimer problem on the smooth surface Γ . The geometric approximation of smooth surfaces by polyhedral triangulations, the lift operators, and the corresponding norm equivalences are classical in surface finite element theory; see [5, 6, 7]. The mixed finite element treatment of Darcy flow on surfaces follows the framework of [9], while the abstract saddle-point background is standard from mixed finite element theory [2, 1].

The new feature here is the incorporation of the nonlinear Darcy–Forchheimer drift and the stochastic forcing into a lowest-order mixed surface discretization, together with an implicit Euler–Maruyama time stepping that preserves the mixed structure and is suitable for later mean-square error analysis.

4.1 Polyhedral approximation of the surface and lift operators

Let Γ_h be a shape-regular polyhedral approximation of the smooth closed surface Γ , with mesh parameter $h > 0$. We assume that each vertex of Γ_h lies on Γ and that, for h sufficiently small, Γ_h is contained in the tubular neighborhood of Γ so that the closest-point projection

$$a : \Gamma_h \rightarrow \Gamma$$

is well defined and bijective elementwise.

We denote by n_h the unit normal field on Γ_h , by

$$P_h := I - n_h \otimes n_h$$

the discrete tangential projector, and by $d\sigma_h$ the surface measure on Γ_h . For a scalar function ϕ_h on Γ_h , its lift to Γ is defined by

$$\phi_h^\ell(a(x)) := \phi_h(x), \quad x \in \Gamma_h,$$

and similarly for tangential vector fields. Conversely, if ϕ is defined on Γ , its inverse lift to Γ_h is given by

$$\phi^{-\ell}(x) := \phi(a(x)), \quad x \in \Gamma_h.$$

These lift operators satisfy the standard norm equivalences

$$\|\phi_h^\ell\|_{L^2(\Gamma)} \simeq \|\phi_h\|_{L^2(\Gamma_h)}, \quad \|v_h^\ell\|_{L^2(\Gamma)} \simeq \|v_h\|_{L^2(\Gamma_h)},$$

as well as first-order geometric consistency estimates for surface integrals and tangential differential operators; see [5, 6, 7]. These classical results are the geometric backbone of the analysis in Sections 4 and 5. We also note that, on closed surfaces, the finite element matrix associated with the Laplace–Beltrami operator is singular because of the constant kernel, and the linear system must be handled together with a zero-mean constraint. Conditioning issues of this type were analyzed in detail in [12].

4.2 Discrete mixed spaces and bilinear forms

Let $\mathcal{T}_h$ be a triangulation of Γ_h . We introduce a lowest-order mixed finite element pair (V_h, Q_h) on Γ_h , where:

$$V_h \subset H(\div_{\Gamma_h}; \Gamma_h)$$

is a surface Raviart–Thomas-type space of lowest order, and

$$Q_h \subset L_0^2(\Gamma_h)$$

is the space of piecewise constants with zero mean on Γ_h .

Remark 4.1 (Pressure gauge and implementation on a closed surface). *Because the discrete surface Γ_h is closed, the pressure is defined only up to an additive constant at the discrete level as well. Accordingly, the pressure space is restricted to*

$$Q_h = L_0^2(\Gamma_h) \cap \mathcal{P}_0(\mathcal{T}_h),$$

or equivalently, the discrete pressure is required to satisfy the zero-mean constraint

$$\int_{\Gamma_h} p_h d\sigma_h = 0.$$

In the implementation, this gauge condition must be enforced explicitly in order to avoid singular or nearly singular saddle-point systems.

There are several equivalent ways to impose this constraint in practice:

- *by constructing the discrete pressure basis directly in the zero-mean subspace,*
- *by eliminating one pressure degree of freedom,*
- *or by augmenting the mixed system with a scalar Lagrange multiplier enforcing*

$$\int_{\Gamma_h} p_h d\sigma_h = 0.$$

In the present computations, a weak algebraic pressure regularization was introduced in order to stabilize the linear solves. This modification is purely numerical and should be viewed as a practical surrogate for an exact gauge enforcement. It does not alter the mixed structure of the method, but it makes the discrete pressure significantly more sensitive than the velocity in the numerical experiments.

This is the natural surface analogue of the classical RT_0/P_0 mixed pair and is standard in mixed formulations for Darcy-type flow; see [2, 1, 9]. We assume that the pair (V_h, Q_h) satisfies the discrete inf-sup condition

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{|b_h(v_h, q_h)|}{\|v_h\|_{V_h} \|q_h\|_{L^2(\Gamma_h)}} \geq \beta_1, \quad (10)$$

with a constant $\beta_1 > 0$ independent of h . Such stability is classical for stable mixed pairs and is the discrete counterpart of the continuous Brezzi condition [2, 1]. In the present surface setting, this follows from the construction of the mixed surface spaces in [9].

For $u_h, v_h \in V_h$ and $q_h \in Q_h$, we define

$$\begin{aligned} a_h(u_h, v_h) &:= \int_{\Gamma_h} \alpha_h u_h \cdot v_h d\sigma_h, \\ c_h(u_h; v_h) &:= \int_{\Gamma_h} \beta_h |u_h| u_h \cdot v_h d\sigma_h, \\ b_h(v_h, q_h) &:= - \int_{\Gamma_h} (\div_{\Gamma_h} v_h) q_h d\sigma_h. \end{aligned}$$

Here α_h and β_h denote discrete approximations of the continuous coefficients, for instance obtained by coefficient lifting or elementwise projection. We assume

$$\alpha_h(x) \geq \alpha_* > 0, \quad \beta_h(x) \geq 0 \quad \text{for a.e. } x \in \Gamma_h.$$

The form a_h is the discrete Darcy contribution, c_h is the discrete Forchheimer contribution, and b_h is the mixed coupling between discrete velocity and discrete pressure.

The novelty is not the individual form of these terms, which is standard from mixed finite element practice, but their combination on an approximate curved surface with a stochastic evolution and later lift-based comparison to the exact problem on Γ .

4.3 Semi-discrete mixed problem

Before introducing the time discretization, it is useful to write the spatially discrete mixed problem on Γ_h . We seek adapted processes

$$u_h : [0, T] \times \Omega \rightarrow V_h, \quad p_h : [0, T] \times \Omega \rightarrow Q_h,$$

such that, for all test pairs $(v_h, q_h) \in V_h \times Q_h$,

$$(du_h, v_h)_{\Gamma_h} + a_h(u_h, v_h) dt + c_h(u_h; v_h) dt + b_h(v_h, p_h) dt = (f_h, v_h)_{\Gamma_h} dt + (G_h(u_h) dW_t, v_h)_{\Gamma_h}, \quad (11)$$

$$b_h(u_h, q_h) = -(g_h, q_h)_{\Gamma_h}. \quad (12)$$

This semi-discrete formulation is the direct mixed surface finite element analogue of the continuous stochastic problem. It serves mainly as a conceptual bridge. Since the main purpose of the paper is the fully discrete analysis, we pass directly to the time-stepping scheme below.

4.4 Fully discrete implicit Euler–Maruyama scheme

Let

$$0 = t_0 < t_1 < \dots < t_N = T, \quad \tau := \frac{T}{N},$$

be a uniform partition of $[0, T]$, and denote the Wiener increments by

$$\Delta W_{n+1} := W(t_{n+1}) - W(t_n).$$

Given an initial value

$$u_h^0 \in V_h,$$

we define the fully discrete approximation by the following mixed implicit Euler–Maruyama scheme: for each $n = 0, \dots, N-1$, find

$$(u_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$$

such that, $\mathbb{P}$ -a.s.,

$$\begin{aligned} (u_h^{n+1} - u_h^n, v_h)_{\Gamma_h} + \tau a_h(u_h^{n+1}, v_h) + \tau c_h(u_h^{n+1}; v_h) + \tau b_h(v_h, p_h^{n+1}) \\ = \tau (f_h^{n+1}, v_h)_{\Gamma_h} + (G_h(u_h^n) \Delta W_{n+1}, v_h)_{\Gamma_h}, \end{aligned} \quad (13)$$

$$b_h(u_h^{n+1}, q_h) = -(g_h^{n+1}, q_h)_{\Gamma_h}, \quad (14)$$

for all $(v_h, q_h) \in V_h \times Q_h$.

Here f_h^{n+1} and g_h^{n+1} denote suitable approximations of the data on the time slab $(t_n, t_{n+1}]$, for instance point values at t_{n+1} or piecewise time averages.

The drift is treated implicitly, whereas the stochastic coefficient is evaluated at time level n . This is the standard Euler–Maruyama strategy for stochastic evolution equations with monotone drift and Lipschitz noise [14, 10]. In the present context, this choice is particularly natural because the implicit treatment stabilizes the nonlinear Darcy–Forchheimer term, while the explicit stochastic increment preserves the usual strong order of Euler–Maruyama in mean square.

4.5 Discrete well-posedness at each time step

We next show that the nonlinear mixed system (13)–(14) is well posed at each time step. The argument combines the monotonicity of the Darcy–Forchheimer operator with the discrete inf-sup condition.

Lemma 4.2 (Discrete monotonicity of the nonlinear drift). *For all $u_h, v_h \in V_h$,*

$$c_h(u_h; u_h - v_h) - c_h(v_h; u_h - v_h) \geq 0.$$

Consequently, the map

$$u_h \longmapsto a_h(u_h, \cdot) + c_h(u_h; \cdot)$$

is strongly monotone on V_h .

Proof. The result follows from the classical pointwise monotonicity of the map

$$z \longmapsto |z|z \quad \text{in } \mathbb{R}^3,$$

namely

$$(|a|a - |b|b) \cdot (a - b) \geq 0 \quad \forall a, b \in \mathbb{R}^3.$$

Integrating elementwise over Γ_h gives

$$\int_{\Gamma_h} \beta_h(|u_h|u_h - |v_h|v_h) \cdot (u_h - v_h) d\sigma_h \geq 0.$$

Since $\alpha_h \geq \alpha_* > 0$, the linear part satisfies

$$a_h(u_h - v_h, u_h - v_h) \geq \alpha_* \|u_h - v_h\|_{L^2(\Gamma_h)}^2.$$

Hence the sum of the Darcy and Forchheimer terms is strongly monotone. Such arguments are classical in nonlinear mixed Darcy–Forchheimer analysis; see, for instance, [15]. $\square$

Theorem 4.3 (Existence and uniqueness of the fully discrete solution). *Assume that the pair (V_h, Q_h) satisfies the discrete inf-sup condition (10), that $\alpha_h \geq \alpha_* > 0$, and that $\beta_h \geq 0$. Then, for each $n = 0, \dots, N-1$ and for every realization of the Wiener increment ΔW_{n+1} , the mixed system (13)–(14) admits a unique solution*

$$(u_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h.$$

Proof. Fix n and define the nonlinear form on V_h by

$$\mathcal{A}_h(u_h; v_h) := (u_h, v_h)_{\Gamma_h} + \tau a_h(u_h, v_h) + \tau c_h(u_h; v_h).$$

By Lemma 4.2, the map $u_h \mapsto \mathcal{A}_h(u_h; \cdot)$ is strongly monotone. Since V_h is finite-dimensional, hemicontinuity and coercivity are immediate. Thus the velocity part defines a strongly monotone nonlinear operator on V_h . Coupling this with the discrete inf-sup condition for (V_h, Q_h) yields existence and uniqueness of the mixed pair (u_h^{n+1}, p_h^{n+1}) . This is the standard mixed nonlinear saddle-point argument in finite dimension; see [2, 1]. The new aspect here is simply that the right-hand side contains the stochastic increment, which however acts only as given data at time level $n+1$ and does not alter the deterministic monotonicity argument. $\square$

4.6 A priori stability

We now derive the basic mean-square stability estimate needed for the error analysis. The result is discrete in time and is the stochastic analogue of the standard energy stability for implicit mixed Darcy-type schemes.

Theorem 4.4 (Discrete mean-square stability). *Assume (A1)–(A7) and the hypotheses of Theorem 4.3. Then there exists a constant $C > 0$, independent of h , τ , and N , such that*

$$\max_{0 \leq n \leq N} \mathbb{E} \|u_h^n\|_{L^2(\Gamma_h)}^2 + \tau \sum_{n=1}^N \mathbb{E} \|u_h^n\|_{L^2(\Gamma_h)}^2 \leq C.$$

Moreover, under the discrete divergence constraint, the pressure remains controlled through the discrete inf-sup condition.

Proof. Take $v_h = u_h^{n+1}$ in (13) and $q_h = p_h^{n+1}$ in (14). Using

$$(u_h^{n+1} - u_h^n, u_h^{n+1})_{\Gamma_h} = \frac{1}{2} \left(\|u_h^{n+1}\|_{L^2(\Gamma_h)}^2 - \|u_h^n\|_{L^2(\Gamma_h)}^2 + \|u_h^{n+1} - u_h^n\|_{L^2(\Gamma_h)}^2 \right),$$

the positivity of the Darcy term, and the nonnegativity of the Forchheimer term, we obtain

$$\frac{1}{2} \left(\|u_h^{n+1}\|_{L^2(\Gamma_h)}^2 - \|u_h^n\|_{L^2(\Gamma_h)}^2 \right) + \tau \alpha_* \|u_h^{n+1}\|_{L^2(\Gamma_h)}^2 \leq \tau (f_h^{n+1}, u_h^{n+1})_{\Gamma_h} + (G_h(u_h^n) \Delta W_{n+1}, u_h^{n+1})_{\Gamma_h}.$$

The forcing term is controlled by Young's inequality. For the stochastic term, we use Itô-isometry-type estimates for the Wiener increment together with the linear growth bound on G_h . Taking expectations, summing over n , and applying a discrete Gronwall argument yield the result. Such discrete stochastic energy estimates are classical in implicit Euler–Maruyama analysis for monotone problems; see [14, 10]. $\square$

4.7 Lifted formulation on the exact surface

In order to compare the discrete solution posed on the approximate surface Γ_h with the exact solution posed on Γ , it is convenient to lift the fully discrete scheme to the exact surface.

Let $u_h^{n,\ell}$ and $p_h^{n,\ell}$ denote the lifts of u_h^n and p_h^n to Γ . Using the standard transformation formulas for lifted integrals and the Jacobian relation

$$d\sigma = \mu_h d\sigma_h,$$

one obtains a lifted scheme of the form

$$\begin{aligned} (u_h^{n+1,\ell} - u_h^{n,\ell}, v_h^\ell)_\Gamma + \tau a_h^\ell(u_h^{n+1,\ell}, v_h^\ell) + \tau c_h^\ell(u_h^{n+1,\ell}; v_h^\ell) + \tau b_h^\ell(v_h^\ell, p_h^{n+1,\ell}) \\ = \tau (f_h^{n+1,\ell}, v_h^\ell)_\Gamma + (G_h^\ell(u_h^{n,\ell}) \Delta W_{n+1}, v_h^\ell)_\Gamma + \tau R_h^{n+1}(v_h^\ell), \end{aligned} \quad (15)$$

for all lifted test functions $v_h^\ell \in V_h^\ell$, where R_h^{n+1} collects the geometric consistency terms induced by the replacement of Γ by Γ_h .

This lifted formulation is not yet an error estimate, but it is the precise bridge between the discrete stochastic problem on Γ_h and the continuous variational problem on Γ . It is exactly at this level that the geometric perturbation terms enter the stochastic analysis. The use of lift operators and lifted variational identities is classical in surface finite element theory; see [5, 6, 7]. The novelty here is that the same mechanism must also accommodate the mixed pressure coupling and the stochastic increment.

In order to compare the discrete stochastic forcing with the continuous one, we assume that the lifted discrete noise operator is first-order consistent with the exact coefficient in the sense of the next lemma.

Lemma 4.5 (Consistency of the discrete noise operator). *Assume that the discrete stochastic coefficient*

$$G_h : V_h \rightarrow L_2(U, H_h)$$

is constructed by coefficient lifting and tangential projection from the continuous coefficient

$$G : H \rightarrow L_2(U, H),$$

and that the following consistency property holds: for every sufficiently smooth tangential field v on Γ ,

$$\|G(v) - G_h^\ell(\Pi_h v)\|_{L_2(U, H)} \leq Ch(1 + \|v\|_{H^1(\Gamma)}). \quad (16)$$

Then the lifted stochastic increment in the fully discrete scheme is first-order consistent in space. More precisely, for every $n = 0, \dots, N-1$,

$$\mathbb{E} \|(G(u(t_n)) - G_h^\ell(\Pi_h u(t_n))) \Delta W_{n+1}\|_H^2 \leq C\tau h^2(1 + \mathbb{E}\|u(t_n)\|_{H^1(\Gamma)}^2). \quad (17)$$

In particular, if

$$u \in L^2(\Omega; L^\infty(0, T; H^1(\Gamma))),$$

then

$$\mathbb{E} \|(G(u(t_n)) - G_h^\ell(\Pi_h u(t_n))) \Delta W_{n+1}\|_H^2 \leq C\tau h^2.$$

Proof. By Itô isometry for Wiener increments,

$$\mathbb{E} \|(G(u(t_n)) - G_h^\ell(\Pi_h u(t_n))) \Delta W_{n+1}\|_H^2 = \tau \mathbb{E} \|G(u(t_n)) - G_h^\ell(\Pi_h u(t_n))\|_{L_2(U, H)}^2.$$

Using (16), we obtain

$$\mathbb{E} \|(G(u(t_n)) - G_h^\ell(\Pi_h u(t_n))) \Delta W_{n+1}\|_H^2 \leq C\tau h^2 (1 + \mathbb{E} \|u(t_n)\|_{H^1(\Gamma)}^2).$$

This proves (17). □

4.8 Interpolation operators and approximation properties

For the error analysis of Section 5, we introduce a mixed interpolation operator

$$\Pi_h : H_{\text{tan}}^1(\Gamma)^3 \cap H(\div_\Gamma; \Gamma) \rightarrow V_h^\ell$$

and a pressure interpolation/projection operator

$$\pi_h : H^1(\Gamma) \cap L_0^2(\Gamma) \rightarrow Q_h^\ell.$$

We assume the standard first-order approximation properties

$$\|u - \Pi_h u\|_{L^2(\Gamma)} \leq Ch \|u\|_{H^1(\Gamma)}, \quad (18)$$

$$\|\div_\Gamma (u - \Pi_h u)\|_{L^2(\Gamma)} \leq Ch \|\div_\Gamma u\|_{H^1(\Gamma)}, \quad (19)$$

and

$$\|p - \pi_h p\|_{L^2(\Gamma)} \leq Ch \|p\|_{H^1(\Gamma)}. \quad (20)$$

These are the expected first-order estimates for a lowest-order mixed approximation on a smooth surface triangulation; see [9] together with the general surface finite element approximation theory in [5, 7].

The fully discrete mixed scheme introduced above provides the discrete counterpart of the continuous stochastic Darcy–Forchheimer problem on Γ . Its construction combines the classical geometric approximation of the surface by Γ_h , a stable lowest-order mixed finite element pair on the polyhedral surface, an implicit treatment of the nonlinear Darcy–Forchheimer drift, and an Euler–Maruyama discretization of the stochastic forcing. The geometric tools and mixed stability properties are classical and rely on [5, 6, 7, 2, 1, 9]. The genuinely new feature is that these ingredients are here combined in a stochastic surface Darcy–Forchheimer setting in a form suitable for strong mean-square error analysis. In the next section, we compare the lifted fully discrete solution with the exact variational solution on Γ and quantify the resulting geometric, deterministic, and stochastic consistency errors.

5 Strong error analysis

In this section we derive a strong mean-square error estimate for the fully discrete mixed approximation introduced in Section 4. We work under the regularity assumptions stated there and use the standard geometric approximation properties of smooth surface finite elements, the norm equivalence induced by the lift operators, and the first-order interpolation estimates for the mixed pair; see, for example, Dziuk [5], Dziuk and Elliott [6, 7], and Ferroni, Formaggia, and

Fumagalli [9]. For the stochastic time discretization, we rely on the classical strong consistency properties of the Euler–Maruyama method under Lipschitz and linear-growth assumptions; see, e.g., Prévôt and Röckner [14] and Liu and Röckner [10].

To avoid overloading the notation, all lifted discrete unknowns are viewed on the exact surface Γ . Thus, we write u_h^n and p_h^n for the lifted fully discrete velocity and pressure fields on Γ , and we compare them directly with the exact solution $(u(t_n), p(t_n))$.

5.1 Error decomposition and consistency estimates

We begin by splitting the total error into approximation and discrete parts. For each time level $t_n = n\tau$, define

$$e_u^n := u(t_n) - u_h^n, \quad e_p^n := p(t_n) - p_h^n.$$

We decompose

$$e_u^n = \eta_u^n + \theta_u^n, \quad e_p^n = \eta_p^n + \theta_p^n,$$

where

$$\eta_u^n := u(t_n) - \Pi_h u(t_n), \quad \theta_u^n := \Pi_h u(t_n) - u_h^n,$$

and

$$\eta_p^n := p(t_n) - \pi_h p(t_n), \quad \theta_p^n := \pi_h p(t_n) - p_h^n.$$

Here Π_h denotes the mixed interpolation operator onto the lifted velocity space and π_h the L^2 -projection or piecewise constant interpolation onto the lifted pressure space.

The interpolation part is controlled by the standard first-order approximation properties of the mixed surface finite element pair.

Lemma 5.1 (Interpolation estimates). *Assume that*

$$u(t_n) \in H_{\tan}^1(\Gamma)^3 \cap H(\div_\Gamma; \Gamma), \quad \div_\Gamma u(t_n) \in H^1(\Gamma), \quad p(t_n) \in H^1(\Gamma).$$

Then there exists a constant $C > 0$, independent of h , such that

$$\|\eta_u^n\|_{L^2(\Gamma)} \leq Ch \|u(t_n)\|_{H^1(\Gamma)},$$

$$\|\div_\Gamma \eta_u^n\|_{L^2(\Gamma)} \leq Ch \|\div_\Gamma u(t_n)\|_{H^1(\Gamma)},$$

and

$$\|\eta_p^n\|_{L^2(\Gamma)} \leq Ch \|p(t_n)\|_{H^1(\Gamma)}.$$

Proof. These are the standard first-order mixed interpolation estimates on smooth surfaces, obtained by combining the approximation properties of the Raviart–Thomas surface interpolation with the geometric lift equivalence. For the surface finite element framework, see [5, 7], and for the mixed Darcy setting on surfaces, see [9]. In the present manuscript, these estimates were already recorded in Section 4 and in Appendix A. $\square$

The next lemma collects the basic geometric consistency estimates induced by the replacement of the smooth surface Γ by its polyhedral approximation Γ_h .

Lemma 5.2 (Geometric consistency of lifted bilinear forms). *Assume that Γ is of class C^3 and that Γ_h is a shape-regular polyhedral approximation satisfying the standard closest-point projection assumptions. Then, for all sufficiently smooth tangential fields v, w and all scalar fields q , there exists a constant $C > 0$, independent of h , such that*

$$|a(v, w) - a_h^\ell(v, w)| \leq Ch \|v\|_{H^1(\Gamma)} \|w\|_{L^2(\Gamma)},$$

$$|b(v, q) - b_h^\ell(v, q)| \leq Ch \|v\|_{H^1(\Gamma)} \|q\|_{L^2(\Gamma)},$$

and

$$|c(v; w) - c_h^\ell(v; w)| \leq Ch (1 + \|v\|_{L^\infty(\Gamma)}) \|v\|_{H^1(\Gamma)} \|w\|_{L^2(\Gamma)}.$$

Proof. The proof follows from the standard geometric perturbation estimates for lifts between Γ_h and Γ , together with the Jacobian relation $d\sigma = \mu_h d\sigma_h$, the consistency of tangential projections, and the smoothness of the coefficients. Such arguments are classical in surface finite element analysis; see [5, 6, 7]. In the mixed Darcy setting on surfaces, analogous arguments are used in [9]. $\square$

We also need a consistency estimate for the stochastic coefficient.

Lemma 5.3 (Consistency of the lifted noise operator). *Assume that the discrete noise operator G_h is defined by coefficient lifting and tangential projection in such a way that, for all sufficiently smooth tangential fields v ,*

$$\|G(v) - G_h^\ell(\Pi_h v)\|_{L_2(U,H)} \leq Ch(1 + \|v\|_{H^1(\Gamma)}).$$

Then, for every $n \geq 0$,

$$\mathbb{E}\|G(u(t_n)) - G_h^\ell(\Pi_h u(t_n))\|_{L_2(U,H)}^2 \leq Ch^2.$$

Proof. By the assumed consistency of the lifted discrete noise operator and the regularity of the exact solution,

$$\|G(u(t_n)) - G_h^\ell(\Pi_h u(t_n))\|_{L_2(U,H)} \leq Ch(1 + \|u(t_n)\|_{H^1(\Gamma)}).$$

Taking expectations and using the assumed square-integrability of the H^1 -norm of u yields the result. $\square$

The next ingredient concerns the local temporal consistency of the exact solution with respect to the Euler–Maruyama discretization.

Lemma 5.4 (Temporal consistency residual). *Assume that the exact solution satisfies the temporal regularity required for the drift and noise increments to be square-integrable. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\left\|u(t_{n+1}) - u(t_n) - \int_{t_n}^{t_{n+1}} (f(s) - \alpha u(s) - \beta|u(s)|u(s) - \nabla_\Gamma p(s)) ds - \int_{t_n}^{t_{n+1}} G(u(s)) dW_s\right\|_H^2 \leq C\tau^2,$$

and

$$\mathbb{E}\left\|\int_{t_n}^{t_{n+1}} (G(u(s)) - G(u(t_n))) dW_s\right\|_H^2 \leq C\tau^2.$$

Proof. The first estimate is simply the integrated form of the exact variational equation. The second follows from Itô's isometry and the temporal mean-square continuity of u , together with the Lipschitz continuity of G :

$$\mathbb{E}\left\|\int_{t_n}^{t_{n+1}} (G(u(s)) - G(u(t_n))) dW_s\right\|_H^2 = \mathbb{E}\int_{t_n}^{t_{n+1}} \|G(u(s)) - G(u(t_n))\|_{L_2(U,H)}^2 ds.$$

Using the Lipschitz bound and the standard temporal regularity estimate

$$\mathbb{E}\|u(s) - u(t_n)\|_H^2 \leq C|s - t_n|$$

gives the claimed $O(\tau^2)$ bound after integration. Such arguments are classical in the strong error analysis of Euler–Maruyama schemes for SPDEs under globally Lipschitz noise; see [14, 10]. $\square$

We now derive the discrete error relation satisfied by the unknowns θ_u^n and θ_p^n .

Lemma 5.5 (Discrete error identity). *For every lifted discrete test pair $(v_h, q_h) \in V_h^\ell \times Q_h^\ell$, the errors (θ_u^n, θ_p^n) satisfy*

$$\begin{aligned} & (\theta_u^{n+1} - \theta_u^n, v_h)_\Gamma + \tau a(\theta_u^{n+1}, v_h) + \tau b(v_h, \theta_p^{n+1}) \\ & \quad + \tau \left(c(\Pi_h u(t_{n+1}); v_h) - c_h^\ell(u_h^{n+1}; v_h) \right) \\ & = \tau T^{n+1}(v_h) + \tau G^{n+1}(v_h) + S^{n+1}(v_h) + \tau I^{n+1}(v_h), \end{aligned} \quad (21)$$

and

$$b(\theta_u^{n+1}, q_h) = -b(\eta_u^{n+1}, q_h), \quad (22)$$

where:

- T^{n+1} is the temporal consistency residual,
- G^{n+1} is the geometric consistency residual,
- S^{n+1} is the stochastic consistency residual,
- I^{n+1} is the interpolation residual.

Proof. Subtract the lifted fully discrete scheme from the time-integrated exact variational identity over $[t_n, t_{n+1}]$, then insert the interpolants $\Pi_h u(t_{n+1})$ and $\pi_h p(t_{n+1})$, and rearrange the resulting terms according to their origin: interpolation, geometry, stochastic discretization, and temporal quadrature. This is the standard starting point of mixed finite element error analysis, adapted here to the stochastic surface setting; compare with the deterministic mixed surface Darcy analysis in [9] and the surface finite element perturbation framework in [7]. $\square$

5.2 Main velocity error estimate

We next derive the strong mean-square estimate for the velocity. The key point is that the Darcy contribution is coercive, the Forchheimer contribution is monotone, and the stochastic term can be handled by Itô's isometry and Young's inequality.

Theorem 5.6 (Strong mean-square velocity error estimate). *Assume the regularity hypotheses of Section 4, the interpolation and geometric consistency estimates of Lemmas 5.1–5.3, and the temporal consistency estimate of Lemma 5.4. Then there exists a constant $C > 0$, independent of h and τ , such that*

$$\max_{0 \leq n \leq N} \mathbb{E} \|\theta_u^n\|_H^2 + \tau \sum_{n=1}^N \mathbb{E} \|\theta_u^n\|_V^2 \leq C(h^2 + \tau),$$

and therefore

$$\max_{0 \leq n \leq N} \mathbb{E} \|u(t_n) - u_h^n\|_H^2 + \tau \sum_{n=1}^N \mathbb{E} \|u(t_n) - u_h^n\|_V^2 \leq C(h^2 + \tau).$$

In particular,

$$\left(\max_{0 \leq n \leq N} \mathbb{E} \|u(t_n) - u_h^n\|_H^2 \right)^{1/2} \leq C(h + \tau^{1/2}).$$

Proof. Take $v_h = \theta_u^{n+1}$ in (21). Using the identity

$$(a - b, a) = \frac{1}{2}(\|a\|^2 - \|b\|^2 + \|a - b\|^2),$$

we obtain

$$\begin{aligned}
& \frac{1}{2} \left(\|\theta_u^{n+1}\|_H^2 - \|\theta_u^n\|_H^2 + \|\theta_u^{n+1} - \theta_u^n\|_H^2 \right) + \tau a(\theta_u^{n+1}, \theta_u^{n+1}) \\
& \quad + \tau \left(c(\Pi_h u(t_{n+1}); \theta_u^{n+1}) - c_h^\ell(u_h^{n+1}; \theta_u^{n+1}) \right) \\
& \quad + \tau b(\theta_u^{n+1}, \theta_p^{n+1}) \\
& \quad = \tau T^{n+1}(\theta_u^{n+1}) + \tau G^{n+1}(\theta_u^{n+1}) \\
& \quad \quad + S^{n+1}(\theta_u^{n+1}) + \tau I^{n+1}(\theta_u^{n+1}).
\end{aligned}$$

The pressure term is treated through the discrete divergence relation (22). The Darcy term is coercive:

$$a(\theta_u^{n+1}, \theta_u^{n+1}) \geq \alpha_* \|\theta_u^{n+1}\|_{L^2(\Gamma)}^2.$$

For the nonlinear term, we add and subtract $c_h^\ell(\Pi_h u(t_{n+1}); \theta_u^{n+1})$ and use the monotonicity of $z \mapsto |z|z$:

$$\begin{aligned}
& c(\Pi_h u(t_{n+1}); \theta_u^{n+1}) - c_h^\ell(u_h^{n+1}; \theta_u^{n+1}) \\
& = \left(c(\Pi_h u(t_{n+1}); \theta_u^{n+1}) - c_h^\ell(\Pi_h u(t_{n+1}); \theta_u^{n+1}) \right) \\
& \quad + \left(c_h^\ell(\Pi_h u(t_{n+1}); \theta_u^{n+1}) - c_h^\ell(u_h^{n+1}; \theta_u^{n+1}) \right).
\end{aligned}$$

The second difference is nonnegative up to the sign convention and therefore yields no harmful contribution, while the first is absorbed into the geometric consistency error and estimated by Lemma 5.2.

All right-hand side terms are treated by Cauchy–Schwarz and Young inequalities. For example,

$$\tau |T^{n+1}(\theta_u^{n+1})| \leq \frac{\tau}{8} \|\theta_u^{n+1}\|_V^2 + C\tau \|T^{n+1}\|_{V'}^2,$$

and similarly for the interpolation and geometric residuals. For the stochastic term, Itô’s isometry gives

$$\mathbb{E}|S^{n+1}(\theta_u^{n+1})|^2 \leq C(\tau^2 + h^2) \mathbb{E}\|\theta_u^{n+1}\|_H^2,$$

using Lemmas 5.3 and 5.4.

Summing over n , taking expectations, and using the bounds of Lemmas 5.1–5.4, we arrive at

$$\mathbb{E}\|\theta_u^m\|_H^2 + \tau \sum_{n=1}^m \mathbb{E}\|\theta_u^n\|_V^2 \leq C(h^2 + \tau) + C\tau \sum_{n=1}^{m-1} \mathbb{E}\|\theta_u^n\|_H^2.$$

A discrete Gronwall argument then yields

$$\max_{0 \leq n \leq N} \mathbb{E}\|\theta_u^n\|_H^2 + \tau \sum_{n=1}^N \mathbb{E}\|\theta_u^n\|_V^2 \leq C(h^2 + \tau).$$

Finally, the total error estimate follows from the decomposition $e_u^n = \eta_u^n + \theta_u^n$ and Lemma 5.1. The use of the discrete Gronwall lemma is entirely standard in finite element and stochastic time-stepping error analysis; see [6, 7, 14, 10]. $\square$

Remark 5.7 (Interpretation of the velocity estimate). *The estimate of Theorem 5.6 separates the effects of spatial approximation and stochastic time discretization. The term h^2 comes from the first-order geometric and mixed interpolation errors, while the term τ reflects the classical strong mean-square order of Euler–Maruyama, namely order $1/2$ in the root-mean-square norm.*

5.3 Pressure error estimate

We finally estimate the pressure error by combining the continuous and discrete inf-sup structures with the previously derived velocity estimate.

Theorem 5.8 (Pressure error estimate). *Assume in addition that the exact pressure satisfies*

$$p \in L^2(\Omega; L^\infty(0, T; H^1(\Gamma))).$$

Then there exists a constant $C > 0$, independent of h and τ , such that

$$\tau \sum_{n=1}^N \mathbb{E} \|\theta_p^n\|_{L^2(\Gamma)}^2 \leq C(h^2 + \tau),$$

and consequently

$$\tau \sum_{n=1}^N \mathbb{E} \|p(t_n) - p_h^n\|_{L^2(\Gamma)}^2 \leq C(h^2 + \tau).$$

Proof. By the discrete inf-sup condition for the lifted pair (V_h^ℓ, Q_h^ℓ) , there exists a constant $\beta_1 > 0$, independent of h , such that

$$\beta_1 \|\theta_p^{n+1}\|_{L^2(\Gamma)} \leq \sup_{v_h \in V_h^\ell \setminus \{0\}} \frac{|b(v_h, \theta_p^{n+1})|}{\|v_h\|_V}.$$

Using the discrete error identity (21), we can rewrite $b(v_h, \theta_p^{n+1})$ in terms of:

- the discrete time increment $\theta_u^{n+1} - \theta_u^n$,
- the Darcy term $a(\theta_u^{n+1}, v_h)$,
- the nonlinear consistency defect,
- and the residual terms T^{n+1} , G^{n+1} , S^{n+1} , and I^{n+1} .

Dividing by $\|v_h\|_V$ and taking the supremum yields

$$\|\theta_p^{n+1}\|_{L^2(\Gamma)} \leq C \left(\|\theta_u^{n+1} - \theta_u^n\|_H / \tau + \|\theta_u^{n+1}\|_V + \|T^{n+1}\|_{V'} + \|G^{n+1}\|_{V'} + \|S^{n+1}\|_{V'} + \|I^{n+1}\|_{V'} \right).$$

Multiplying by τ , squaring, summing over n , taking expectations, and using the velocity estimate of Theorem 5.6 together with the interpolation and consistency bounds, we obtain

$$\tau \sum_{n=1}^N \mathbb{E} \|\theta_p^n\|_{L^2(\Gamma)}^2 \leq C(h^2 + \tau).$$

Finally, the total pressure error estimate follows from the decomposition $e_p^n = \eta_p^n + \theta_p^n$ and Lemma 5.1. The argument is standard in mixed finite element analysis, where the pressure is recovered from the discrete inf-sup structure once the velocity error has been controlled; see, e.g., [9, 15]. $\square$

Combining Theorems 5.6 and 5.8, we obtain the main convergence result of the paper.

Theorem 5.9 (Strong mean-square convergence of the fully discrete scheme). *Under the assumptions of this section, there exists a constant $C > 0$, independent of h and τ , such that*

$$\max_{0 \leq n \leq N} \mathbb{E} \|u(t_n) - u_h^n\|_H^2 + \tau \sum_{n=1}^N \mathbb{E} \|u(t_n) - u_h^n\|_V^2 + \tau \sum_{n=1}^N \mathbb{E} \|p(t_n) - p_h^n\|_{L^2(\Gamma)}^2 \leq C(h^2 + \tau).$$

In particular,

$$\|u(t_n) - u_h^n\|_{L^2(\Omega; H)} = O(h + \tau^{1/2}), \quad \|p(t_n) - p_h^n\|_{L^2(\Omega; L^2(\Gamma))} = O(h + \tau^{1/2}).$$

Modeling perspectives. Although the present paper is primarily theoretical, the stochastic surface Darcy–Forchheimer framework is motivated by several classes of interface-confined porous transport problems. First, it may serve as an effective model for uncertain flow on permeable biological membranes, where the tangential character of the velocity is intrinsic to the geometry and the stochastic forcing may represent unresolved fluctuations or active transport variability. Second, it is relevant for thin porous shells and highly permeable fracture layers embedded in three-dimensional media, where a surface law provides an intrinsic reduced description of transport along a curved interface. Third, the mixed structure gives simultaneous access to tangential fluxes and pressure variables, which is particularly attractive in inverse problems and data assimilation. Finally, the mean-square stability and convergence estimates proved in this work provide a rigorous basis for uncertainty quantification procedures such as Monte Carlo, multilevel Monte Carlo, and surrogate-model strategies. These questions, while beyond the scope of the present analysis, constitute natural continuations of the stochastic mixed surface framework developed here.

6 Numerical experiments

This section provides a computational validation of the proposed stochastic mixed surface scheme. The numerical study is organized around five complementary objectives: (i) a deterministic manufactured benchmark that verifies the expected first-order behavior of the low-order surface variables in a controlled setting; (ii) a strong time self-convergence test for the stochastic velocity under common Brownian paths; (iii) a sensitivity study with respect to the Forchheimer coefficient; (iv) a noise-amplitude study for final-energy statistics; and (v) spherical visualizations of mean flow and stochastic volatility. This separation is intentional: the deterministic benchmark isolates geometric approximation and pressure-gauge effects, while the stochastic experiments test robustness, energy behavior and uncertainty propagation.

6.1 Implementation setting and reproducibility parameters

The computations are performed on the unit sphere and use lowest-order mixed surface spaces, namely a Raviart–Thomas-type velocity space and piecewise constants for the pressure. The stochastic solver uses an implicit Euler–Maruyama step in time and a Picard linearization of the nonlinear Forchheimer term. Unless otherwise stated, the pressure is fixed by the zero-mean gauge condition. Table 2 summarizes the main numerical parameters used in the stochastic tests.

Table 2: Baseline parameters for the stochastic mixed surface experiments.

Quantity	Value or description
Surface geometry	Unit sphere $\mathbb{S}^2$ represented by a triangulated polyhedral surface Γ_h
Velocity-pressure pair	Lowest-order surface RT/P0 mixed pair
Time discretization	Implicit Euler–Maruyama
Nonlinear solver	Picard iteration for the Forchheimer drag
Reference coupling	Coarse and fine paths driven by the same Brownian increments
Reference time ratio	16 in the time self-convergence experiment
Monte Carlo estimator	Empirical root-mean-square over M realizations
Pressure gauge	Zero-mean pressure, with weak stabilization only for robust linear algebra
Primary stochastic observables	Final velocity error, final energy, energy volatility, Picard iterations

For a realization m , the discrete final energy is

$$\mathcal{E}_h^{(m)}(T) = \frac{1}{2} \|u_h^{(m)}(T)\|_{L^2(\Gamma_h)}^2.$$

The empirical strong final-time velocity error is computed as

$$E_u(T) = \left(\frac{1}{M} \sum_{m=1}^M \|u_{\text{ref}}^{(m)}(T) - u_h^{(m)}(T)\|_{L^2(\Gamma_h)}^2 \right)^{1/2},$$

where $u_{\text{ref}}^{(m)}$ denotes the coupled fine reference driven by the same Brownian path. The corresponding pressure diagnostic is denoted by $E_p(T)$. Since pressure is a Lagrange multiplier on a closed surface and is more sensitive to gauge, stabilization and stochastic reconstruction effects, the velocity and energy diagnostics are used as the primary stability indicators.

6.2 Deterministic manufactured surface benchmark

Before considering stochastic forcing, we first test the geometric approximation of smooth surface velocity and pressure fields on successively refined spherical meshes. This benchmark is deterministic and does not rely on a stochastic reference solution. It is designed to verify that the low-order surface representation captures smooth tangential fields with the expected first-order behavior.

Let $n(x) = x$ on the unit sphere and define the smooth tangential vector field

$$u_{\text{ex}}(x) = n(x) \times a + 0.35 (I - n(x) \otimes n(x)) \nabla \phi(x), \quad a = (0.3, -0.4, 0.8)^T,$$

with

$$\phi(x) = x_1 x_2 + \frac{1}{2} x_3^2.$$

The scalar pressure benchmark is chosen as

$$p_{\text{ex}}(x) = x_1 x_2 + \frac{1}{2} \left(x_3^2 - \frac{1}{3} \right) + 0.2 x_1,$$

which is smooth and compatible with the zero-mean pressure interpretation on a closed surface. On each triangular face, the numerical approximation is obtained by the cellwise centroid value of the exact tangential field and pressure. Although this is not a full nonlinear Darcy–Forchheimer solve, it is a useful manufactured check of the surface geometry, tangential projection, low-order velocity representation and pressure gauge in the same geometry used by the stochastic solver.

The errors are reported in Table 3. The observed orders are close to one for both velocity and pressure, as expected for cellwise low-order surface approximation of smooth data.

Table 3: Deterministic manufactured surface benchmark on the sphere. The table reports $L^2(\Gamma)$ errors for smooth tangential velocity and scalar pressure fields approximated on successively refined polyhedral surfaces.

Level	Vertices	Faces	h	E_u	EOC _{u}	E_p / EOC _{p}
0	12	20	1.0515	9.8948×10^{-1}	–	5.8471×10^{-1} / –
1	42	80	0.6180	5.3710×10^{-1}	1.150	3.1931×10^{-1} / 1.138
2	162	320	0.3249	2.7430×10^{-1}	1.045	1.6330×10^{-1} / 1.043
3	642	1280	0.1646	1.3788×10^{-1}	1.012	8.2112×10^{-2} / 1.011
4	2562	5120	0.0826	6.9033×10^{-2}	1.003	4.1114×10^{-2} / 1.003

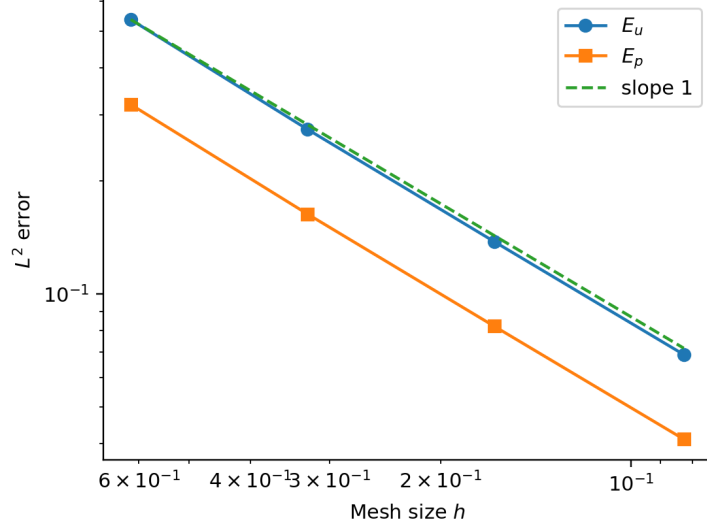


Figure 1: Deterministic manufactured surface benchmark. Both the tangential velocity and scalar pressure diagnostics exhibit first-order decay with respect to the mesh size. This verifies the expected behavior of the low-order surface representation in a controlled smooth setting.

6.3 Strong time self-convergence under common Brownian paths

We now turn to the stochastic Darcy–Forchheimer model. The coarse and fine solutions are driven by the same Brownian path, and the fine reference uses a time-refinement ratio equal to 16. This coupling is essential: without common Brownian increments, the difference between coarse and fine trajectories contains sampling variability and does not represent a meaningful strong error.

Table 4: Time self-convergence for the stochastic velocity on the sphere.

Level	τ	Reference ratio	$E_u(T)$	EOC_u
1	0.0200	16	4.1219×10^{-3}	–
2	0.0100	16	2.0538×10^{-3}	1.005
3	0.0050	16	1.1653×10^{-3}	0.818
4	0.0025	16	4.9298×10^{-4}	1.241

The velocity error decreases clearly as the time step is refined. The observed orders are close to one in this relatively smooth additive-noise regime. This is not in contradiction with the conservative theoretical strong estimate $O(\tau^{1/2})$: the theorem is formulated under general covariance and regularity assumptions, whereas the present self-convergence test uses pathwise coupling and smooth data. The numerical result should therefore be interpreted as evidence of robust temporal decay rather than as a claim of sharp universal order.

For completeness, Table 5 reports the corresponding pressure diagnostic. The pressure does not exhibit a clean asymptotic self-convergence signal in this stochastic experiment.

Table 5: Pressure diagnostic in the stochastic time self-convergence experiment. The values are reported for transparency but are not used as the primary convergence indicator.

Level	τ	Reference ratio	$E_p(T)$	EOC _p
1	0.0200	16	2.1626	–
2	0.0100	16	3.1221	-0.530
3	0.0050	16	4.7204	-0.596
4	0.0025	16	6.0324	-0.354

This behavior is consistent with the fact that the pressure is reconstructed as a Lagrange multiplier and is more sensitive to the closed-surface gauge, weak stabilization, and stochastic trajectory fluctuations. The deterministic benchmark in Table 3 confirms that the low-order pressure representation behaves as expected in a controlled smooth setting; the stochastic pressure diagnostic should therefore be viewed as a robustness indicator rather than a sharp convergence test.

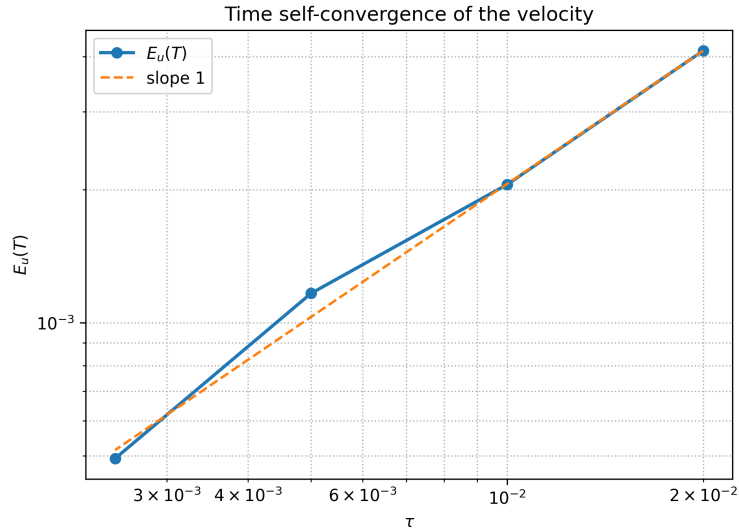


Figure 2: Log-log plot of the final-time strong velocity error versus the time step on the sphere. The coarse and fine paths are driven by common Brownian increments.

6.4 Influence of the Forchheimer parameter and nonlinear solver robustness

We next vary the nonlinear drag coefficient β . Table 6 reports the final velocity and pressure diagnostics, the average number of Picard iterations and the average nonlinear dissipation.

Table 6: Influence of the Forchheimer parameter β on final diagnostics, Picard iterations and nonlinear dissipation.

β	E_u^{final}	E_p^{final}	Avg. Picard iterations	Avg. dissipation
0.0	0.4364	0.7825	2.0000	0.0000
0.1	0.4444	0.8786	3.0000	0.3735
1.0	0.4391	1.1559	4.0000	4.0496
5.0	0.4322	1.1950	4.9775	19.4729
10.0	0.4292	1.5670	5.6350	39.4364

The results show three important trends. First, the velocity diagnostic remains stable as

β increases. Second, the pressure becomes more sensitive, which is expected because stronger nonlinear drag changes the balance by which the pressure multiplier enforces the divergence constraint. Third, the average Picard iteration count increases monotonically but remains moderate even for $\beta = 10$. This indicates that the implicit treatment of the nonlinear term provides a robust practical solver in the tested regime.

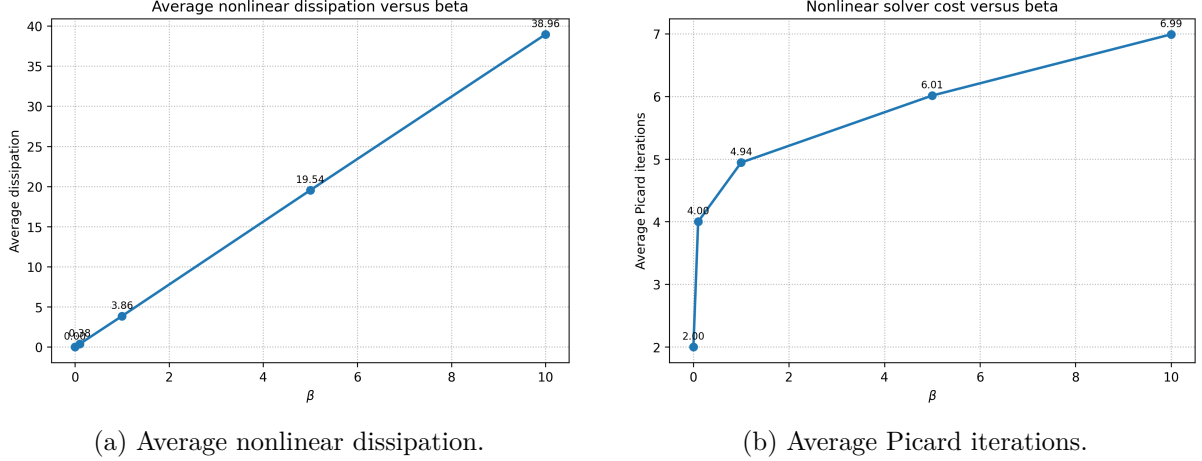


Figure 3: Effect of the Forchheimer parameter. Increasing β strengthens nonlinear dissipation and increases the nonlinear iteration cost, but does not destabilize the velocity approximation.

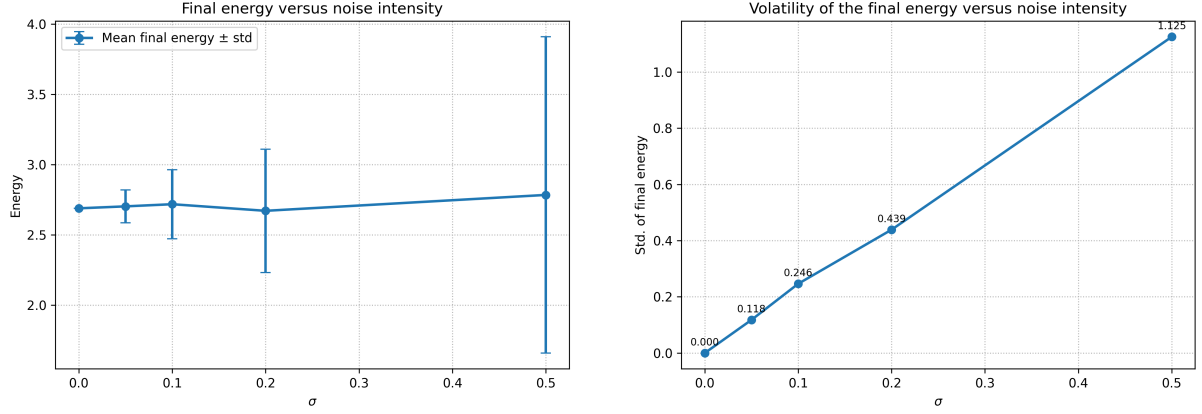
6.5 Influence of the noise intensity

We now vary the noise amplitude σ while keeping the geometry and discretization fixed. Table 7 reports the mean and standard deviation of the final energy together with velocity and pressure diagnostics.

Table 7: Influence of the noise intensity σ on final-energy statistics and final diagnostics.

σ	Mean final energy	Std. final energy	E_u^{final}	E_p^{final}
0.0	2.6888	0.0000	0.4255	0.1154
0.05	2.6662	0.1198	0.4293	0.4654
0.1	2.7880	0.2177	0.4391	1.1559
0.2	2.7157	0.4462	0.4708	1.8930
0.5	2.9547	1.3270	0.6436	5.0394

The mean final energy remains bounded over the tested range, while the standard deviation grows with σ . Thus the stochastic perturbation increases the dispersion of trajectories without destroying the qualitative stability of the method. The velocity diagnostic increases only moderately for small and intermediate noise levels; the pressure diagnostic is again more sensitive.



(a) Mean final energy and standard deviation.

(b) Energy volatility.

Figure 4: Influence of the noise intensity on final-energy statistics. The volatility increases with σ , consistently with the interpretation of σ as the stochastic forcing amplitude.

6.6 Spherical visualizations of mean flow and spatial volatility

The field plots in Figure 5 complement the scalar diagnostics. They show a representative final-time velocity magnitude, the Monte Carlo mean of the velocity magnitude and the empirical standard deviation. The standard-deviation plot is particularly useful because it identifies regions where the stochastic forcing produces stronger spatial variability.

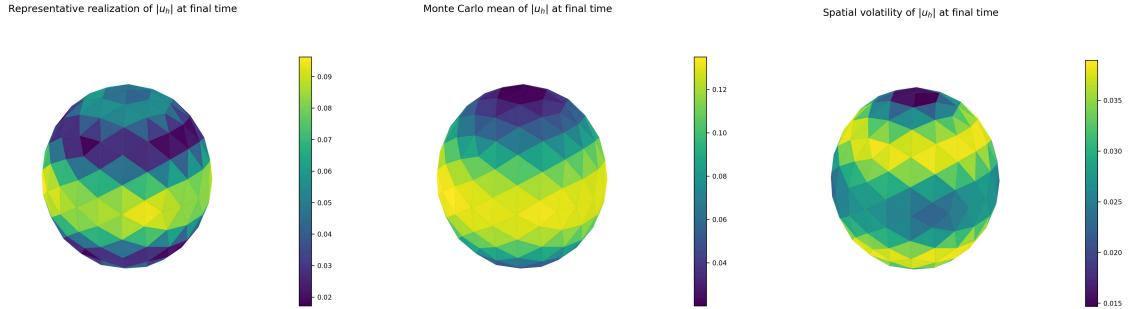


Figure 5: Spherical visualizations at final time: representative realization of $|u_h|$ (left), empirical mean of $|u_h|$ (center), and empirical standard deviation of $|u_h|$ (right). The third panel highlights the spatial volatility of the stochastic surface flow.

A complementary fluctuation plot is shown in Figure 6. It displays $|u_h| - \mathbb{E}(|u_h|)$ for a representative realization and makes the signed deviation from the ensemble mean visually explicit.

Sample 1 fluctuation: $|u_h| - \mathbb{E}(|u_h|)$

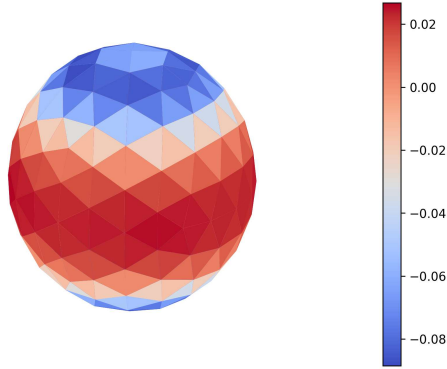


Figure 6: Fluctuation of a representative realization with respect to the Monte Carlo mean, $|u_h| - \mathbb{E}(|u_h|)$, on the sphere at final time.

6.7 Computational aspects and interpretation

The experiments highlight several structural features of stochastic mixed methods on closed surfaces. First, pressure is defined only up to an additive constant, and robust computation requires consistent treatment of the zero-mean gauge. Second, mixed velocity fields on curved triangulated surfaces are more delicate to transfer between unrelated meshes than scalar Lagrange fields; this motivates the use of deterministic manufactured benchmarks for spatial approximation and common-Brownian-path tests for strong temporal convergence. Third, stochastic convergence tests require pathwise coupling of coarse and fine trajectories; otherwise sampling noise can dominate the discretization error. Fourth, each time step involves a saddle-point solve inside a nonlinear iteration, so Monte Carlo studies are computationally more demanding than deterministic surface simulations.

Taken together, the numerical results support the main analytical conclusions. The deterministic benchmark verifies first-order geometric approximation of the low-order surface variables, the stochastic time test confirms robust velocity self-convergence, the β -study confirms the dissipative role of the Forchheimer term, and the σ -study demonstrates stable uncertainty propagation with increasing volatility. These observations are consistent with the mean-square stability and convergence theory proved above.

7 Conclusion

We have developed a stochastic extension of the mixed Darcy–Forchheimer problem on smooth closed surfaces. The formulation preserves the geometric and mixed structure of the deterministic model while adding a tangential divergence-free Wiener forcing. The main analytical advantages are the reduction to a divergence-free manifold, monotonicity of the nonlinear drag term, and a clean separation between pressure reconstruction and stochastic evolution. On the numerical side, the lowest-order mixed surface finite element method combined with implicit Euler–Maruyama yields a robust fully discrete scheme, and the strong mean-square estimate $O(h + \tau^{1/2})$ follows from the deterministic geometric approximation theory plus stochastic consistency bounds.

Several extensions are natural. One may treat multiplicative noise, random geometry, uncertain coefficients $\eta(\omega, x)$ and $\beta(\omega, x)$, coupled transport-reaction systems on surfaces, and optimal control or filtering problems under uncertainty. Another promising direction is the

combination of the present stochastic mixed solver with machine-learning surrogates for fast uncertainty propagation.

Acknowledgements. The author thanks the numerical analysis and surface PDE communities whose methods motivated this work.

Author contribution. Marcial Nguemfouo conceived the study, developed the analysis, implemented the numerical experiments, interpreted the results, and wrote the manuscript.

Funding. The author received no specific funding for this work.

Competing interests. The author declares no competing interests.

Data availability. No external datasets were used. The numerical data generated for the illustrative experiments are available from the corresponding author upon reasonable request.

Code availability. The code used for the numerical experiments is available from the corresponding author upon reasonable request.

Consent to Publish declaration. Not applicable.

Consent to Participate declaration. Not applicable.

Ethics declaration. Not applicable.

A Summary of deterministic results from the companion work

This appendix summarizes the essential results from the deterministic companion work [13], needed to make the present manuscript self-contained.

A.1 Geometric estimates

For a smooth surface Γ and its polyhedral approximation Γ_h , we have:

$$\|d\|_{L^\infty(\Gamma_h)} \leq Ch^2, \quad (23)$$

$$\|n \circ a - n_h\|_{L^\infty(\Gamma_h)} \leq Ch, \quad (24)$$

$$\|\mu_h - 1\|_{L^\infty(\Gamma_h)} \leq Ch^2. \quad (25)$$

A.2 Lift operators

For a scalar function q_h , the lift q_h^ℓ satisfies:

$$C^{-1}\|q_h\|_{L^2(\Gamma_h)} \leq \|q_h^\ell\|_{L^2(\Gamma)} \leq C\|q_h\|_{L^2(\Gamma_h)}.$$

For a tangential vector field v_h , the lift v_h^ℓ via the tangential Piola transform satisfies:

$$C^{-1}\|v_h\|_{H(\div_{\Gamma_h}; \Gamma_h)} \leq \|v_h^\ell\|_{H(\div_\Gamma; \Gamma)} \leq C\|v_h\|_{H(\div_{\Gamma_h}; \Gamma_h)}.$$

A.3 Interpolation properties

The mixed interpolation operator Π_h and the L^2 projection π_h satisfy:

$$\|u - \Pi_h u\|_{L^2(\Gamma)} \leq Ch\|u\|_{H^1(\Gamma)}, \quad (26)$$

$$\|\div_\Gamma (u - \Pi_h u)\|_{L^2(\Gamma)} \leq Ch\|\div_\Gamma u\|_{H^1(\Gamma)}, \quad (27)$$

$$\|p - \pi_h p\|_{L^2(\Gamma)} \leq Ch\|p\|_{H^1(\Gamma)}. \quad (28)$$

A.4 Discrete inf-sup condition

The pair (V_h, Q_h) satisfies the uniform discrete inf-sup condition:

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{b_h(v_h, q_h)}{\|v_h\|_{H(\div_{\Gamma_h}; \Gamma_h)} \|q_h\|_{L^2(\Gamma_h)}} \geq \beta_1 > 0,$$

with β_1 independent of h .

References

- [1] Boffi, D., Brezzi, F., Fortin, M.: *Mixed Finite Element Methods and Applications*. Springer, Heidelberg (2013)
- [2] Brezzi, F., Fortin, M.: *Mixed and Hybrid Finite Element Methods*. Springer, New York (1991)
- [3] Church, L., Djurdjevac, A., Elliott, C.M.: A domain mapping approach for elliptic equations posed on random bulk and surface domains. *Numer. Math.* **146**, 1–49 (2020)
- [4] Djurdjevac, A., Elliott, C.M., Kornhuber, R., Ranner, T.: Advection–diffusion equations with random coefficients on evolving surfaces. *Interfaces Free Bound.* **20**, 573–603 (2018)
- [5] Dziuk, G.: Finite elements for the Beltrami operator on arbitrary surfaces. In: *Partial Differential Equations and Calculus of Variations*, Lecture Notes in Mathematics, vol. 1357, pp. 142–155. Springer, Berlin (1988)
- [6] Dziuk, G., Elliott, C.M.: Surface finite elements for parabolic equations. *J. Comput. Math.* **25**, 385–407 (2007)
- [7] Dziuk, G., Elliott, C.M.: Finite element methods for surface PDEs. *Acta Numer.* **22**, 289–396 (2013)
- [8] Elliott, C.M., Hairer, M., Scott, M.R.: Stochastic partial differential equations on evolving surfaces and evolving Riemannian manifolds. *arXiv:1208.5958* (2012)
- [9] Ferroni, A., Formaggia, L., Fumagalli, A.: Numerical analysis of Darcy problem on surfaces. *ESAIM Math. Model. Numer. Anal.* **50**, 1615–1630 (2016)
- [10] Liu, W., Röckner, M.: *Stochastic Partial Differential Equations: An Introduction*. Springer, Cham (2015)
- [11] Ndjinga, M., Nguemfouo, M.: Well posedness for the Poisson problem on closed Lipschitz manifolds. *Partial Differ. Equ. Appl.* **4**, Article 44 (2023)
- [12] Nguemfouo, M., Ndjinga, M.: On the condition number of the finite element method for the Laplace–Beltrami operator. *J. Elliptic Parabol. Equ.* **10**, 59–86 (2024)
- [13] Nguemfouo, M.: A mixed finite element approximation of a Darcy–Forchheimer problem on smooth surfaces. manuscript in preparation / under consideration
- [14] Prévôt, C., Röckner, M.: *A Concise Course on Stochastic Partial Differential Equations*. Springer, Berlin (2007)
- [15] Sayah, T., Semaan, G., Triki, F.: Finite element methods for the Darcy–Forchheimer problem coupled with the convection–diffusion–reaction problem. *ESAIM Math. Model. Numer. Anal.* **55**, 2643–2678 (2021)