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A Variable-Step IMEX-BDF2 Finite Element Method for the Navier–Stokes Equations: Optimal L^2 Error Estimates for Velocity and Pressure

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Abstract

This paper presents a rigorous error analysis of a variable time-step, second-order implicit-explicit backward differentiation formula finite element method (IMEX-BDF2-FEM) for solving the nonstationary incompressible Navier–Stokes equations. The scheme combines a skew-symmetric form of the convection term with variable time steps, enabling efficient simulation of multiscale flow phenomena. The analysis establishes fully optimal a priori error estimates in both space and time for the velocity in the L^2 norm, as well as for the pressure—a result not previously available in the variable-step setting. A key technical contribution is the use of the discrete orthogonal convolution (DOC) kernel to handle time-step variations, coupled with a negative-norm argument to upgrade the H^1 estimate to an L^2 estimate without sacrificing optimality. The initial step is computed using a Crank–Nicolson method to preserve second-order accuracy. Numerical experiments confirm the predicted convergence rates and illustrate the effectiveness of the proposed approach. This work closes a notable gap in the theory of variable-step BDF2 methods for the Navier–Stokes equations, providing a complete and optimal error portrait for the scheme.

Keywords: Navier–Stokes equations, variable time steps, optimal L^2 -norm error estimate, negative-norm technique, IMEX scheme

1. Introduction

This paper presents an analysis of a variable time-step, fully discrete scheme for solving the non-stationary incompressible Navier-Stokes (NS) equations in primitive variables:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, & \text{in } (0, T] \times \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \text{in } (0, T] \times \Omega, \end{cases} \quad (1.1)$$

subject to Dirichlet boundary conditions $\mathbf{u}(t, \mathbf{x}) = \mathbf{u}_D(t, \mathbf{x})$ on $(0, T] \times \partial\Omega$ and initial condition $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x})$ in Ω . Here, $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is an open bounded domain, $T > 0$ is the terminal time, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ and $p = p(t, \mathbf{x})$ denote the fluid velocity and pressure, respectively, and $\nu = 1/\text{Re}$ is the kinematic viscosity for Reynolds number Re . The term $\mathbf{f}$ represents external body forces. The development of efficient, accurate, and long-term stable numerical methods for this system remains a central challenge in scientific and engineering applications.

The NS equations exhibit intrinsic multiscale behavior, posing significant challenges for numerical simulations, particularly at high Reynolds numbers. Kolmogorov’s pioneering work [26] characterized this phenomenon, distinguishing between deterministic large-scale eddies and stochastic

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small-scale structures. The latter, governed by viscous dissipation, decay rapidly below the Kolmogorov microscale $\eta = O(\text{Re}^{-3/4})$. Common strategies to address this multiscale interplay include Large Eddy Simulation (LES) [11, 32], Variational Multiscale Methods (VMS) [19, 23], and Time Relaxation Models (TRM) [34, 34]. Specifically, LES filters subgrid scales and VMS reintroduces them via resolved scales, TRM employs convolution-based filtering, each offering distinct trade-offs between accuracy and computational cost. An alternative, complementary approach is adaptive time-stepping, a vital tool for balancing accuracy and efficiency by naturally refining the temporal resolution during dynamic flow transitions. John and Rang [24] underscored its utility in CFD, noting that “adaptive time stepping is an important tool for controlling accuracy and enhancing efficiency.” Among variable-step methods, the second-order backward differentiation formula (BDF2) is particularly attractive for stiff problems like the incompressible NS equations due to its strong stability and robustness in handling differential-algebraic constraints [9, 15, 36].

Recent work on variable-step BDF2 schemes for NS equations reveals a specific and important gap in the theory. Frutos et al. [5] analyzed an IMEX variable-step method but established stability and convergence only for a first-order scheme. García-Archilla and Novo [10] performed a rigorous error analysis for a variable-step BDF2 scheme applied to the NS equations; however, while obtaining an optimal temporal rate in L^2 for constant steps, their variable-step analysis provided estimates only in a G-functional norm. Ji and Zhao [21] later provided an L^2 error estimate for a variable-step BDF2 scheme using the discrete orthogonal convolution (DOC) kernel technique [28–30], but their result remained suboptimal in space. Notably, a concurrent, optimal error estimate for the pressure field in the variable-step setting is absent from these studies.

Achieving a fully optimal, simultaneous L^2 error estimate in both time and space typically requires analytical tools beyond an optimal H^1 estimate. While the duality (Aubin-Nitsche) argument is a standard approach [2], its application to nonlinear problems is nontrivial. The negative-norm technique [16] offers an effective alternative pathway to this goal.

In this work, we close this theoretical gap by establishing fully optimal L^2 error estimates in space and time for a variable-step IMEX-BDF2 finite element discretization of the incompressible NS equations. To ensure exact second-order accuracy from the initial step, we generate the first step via a Crank-Nicolson scheme. Leveraging the DOC framework, we precisely quantify the influence of this initial step in both L^2 and H^{-1} norms. We then prove an optimal H^1 error estimate via mathematical induction, which yields explicit constants. Finally, we employ a negative-norm argument to lift this result to an optimal L^2 error estimate for the velocity and, concurrently, derive an optimal estimate for the pressure. This analysis provides a complete error portrait, solidifying the theoretical foundation for this widely used variable-step scheme.

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries and introduces the notation used throughout the analysis. In Section 3, we formulate the fully discrete scheme for variable time steps, which employs the IMEX-BDF2 method combined with a skew-symmetric finite element discretization. Subsequently, Section 4 establishes an optimal-order error estimate for the velocity in the H^1 norm via a mathematical induction argument. This result is complemented in Section 5 by deriving an optimal error estimate on the pressure and Section 6 by deriving an optimal error estimate in the L^2 norm via the negative norm technique. Section 7 provides two numerical experiments that illustrate and confirm the theoretical convergence rates. Finally, concluding remarks and directions for future work are summarized in Section 8.

2. Notations and preliminaries

To simplify the presentation, we first introduce some standard notations. Let $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and the corresponding $L^2(\Omega)$ norm, respectively. Sobolev spaces are denoted as usual: for $k \geq 0$ and $1 \leq p \leq \infty$, the $L^p(\Omega)$ norm and the $W^{k,p}(\Omega)$ norm are written as $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$, respectively. For integer $k \geq 0$ and $1 \leq p \leq \infty$, we denote the usual $L^p(\Omega)$ and Sobolev $W^{k,p}(\Omega)$ norms by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$. When $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$, with $\|\cdot\|_k$ and $|\cdot|_k$ representing its norm and seminorm, respectively.

Let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ consisting of functions with zero trace on $\partial\Omega$, and denote by $\|\cdot\|_{-1}$ the norm of its dual space $H^{-1}(\Omega)$. For time-dependent functions on $(0, T) \times \Omega$, we

shall also use the Bochner norms $\|\cdot\|_{L^m(0,T;W^{k,p}(\Omega))}$ for $1 \leq m \leq \infty$. Unless otherwise stated, we assume throughout the paper a homogeneous Dirichlet boundary condition, i.e., $\mathbf{u}_D(t, \mathbf{x}) = 0$ for all $(t, \mathbf{x}) \in (0, T] \times \partial\Omega$.

We now define the continuous velocity–pressure spaces $\mathbf{X} \times M$ as

$$\mathbf{X} = (H_0^1(\Omega))^d := \{\mathbf{v} \in (H^1(\Omega))^d \mid \mathbf{v}|_{\partial\Omega} = 0\}, \quad M = L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_{\Omega} q = 0\}.$$

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with the mesh size $h = \max\{\text{diam}(K) \mid K \in \mathcal{T}_h\}$. For spatial discretization, we employ the standard Taylor-Hood mixed finite element pair $\mathbf{X}_h \times M_h$ [35], defined as

$$\mathbf{X}_h := \{\mathbf{v}_h \in \mathbf{X} \mid \mathbf{u}_h|_K \in (P^k(K))^d, \forall K \in \mathcal{T}_h\}, \quad M_h := \{q_h \in M \mid q_h|_K \in P^{k-1}(K), \forall K \in \mathcal{T}_h\},$$

where $P^k(K)$ denotes the set of polynomials of total degree at most k on the element K . In this work, we restrict ourselves to the classical case $k = 2$. The following approximation properties [4] hold:

$$\begin{cases} \inf_{\mathbf{v} \in \mathbf{X}_h} \|\mathbf{w} - \mathbf{v}\|_k \leq C_{\text{itp}} h^{3-k} \|\mathbf{w}\|_3, & \forall \mathbf{w} \in (H^3(\Omega))^d, k = 0, 1, \\ \inf_{q \in M_h} \|r - q\| \leq C_{\text{itp}} h^2 \|r\|_2, & \forall r \in H^2(\Omega), \end{cases} \quad (2.1)$$

where C_{itp} is a positive constant depending only on the domain Ω . It is well-known that the Taylor–Hood element satisfies the discrete inf-sup condition, i.e. there exists a constant $\gamma > 0$ such that

$$\inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|q_h\| \| \mathbf{v}_h \|_1} \geq \gamma. \quad (2.2)$$

Furthermore, let us define $\mathbf{V}$ as the divergence-free subspace of $\mathbf{X}$ and $\mathbf{V}_h$ as the weakly divergence-free subspace of $\mathbf{X}_h$, i.e.,

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} \mid \nabla \cdot \mathbf{v} = 0, \text{ a.e. in } \Omega\}, \quad \mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{X}_h \mid (q_h, \nabla \cdot \mathbf{v}_h) = 0, \forall q_h \in M_h\}.$$

It is noted that $\mathbf{V}_h \not\subseteq \mathbf{V}$. For any given $(\mathbf{w}, q) \in \mathbf{X} \times M$, we introduce the Stokes projection $(P_{\text{st}}\mathbf{w}, Q_{\text{st}}q) \in \mathbf{X}_h \times M_h$ such that

$$\nu(\nabla(\mathbf{w} - P_{\text{st}}\mathbf{w}), \nabla \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, q - Q_{\text{st}}q) + (\nabla \cdot (\mathbf{w} - P_{\text{st}}\mathbf{w}), q_h) = 0, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h. \quad (2.3)$$

Regarding the Stokes projection (2.3), the following conclusions hold.

Lemma 2.1 ([6, 13, 14]). *For any fixed $r \in [2, \infty]$, there exists a constant $C_{\text{st}} > 0$ depending only on Ω such that*

$$\|\nabla P_{\text{st}}\mathbf{w}\|_{L^r} \leq C_{\text{st}} \|\nabla \mathbf{w}\|_{L^r}, \quad \forall \mathbf{w} \in \mathbf{V}.$$

In addition, the following error estimates hold:

$$\|\mathbf{w} - P_{\text{st}}\mathbf{w}\| + h \|\mathbf{w} - P_{\text{st}}\mathbf{w}\|_1 \leq C_{\text{st}} h^3 \|\mathbf{w}\|_3, \quad \forall \mathbf{w} \in \mathbf{V} \cap (H^3(\Omega))^d.$$

We shall also use the following inverse inequality [4]:

Lemma 2.2 ([4]). *For any fixed $0 \leq n \leq m \leq 1$, $1 \leq q \leq p \leq \infty$, there exists a constant $C_{\text{inv}} > 0$, depending only on Ω , such that*

$$\|\mathbf{w}_h\|_{W^{m,p}} \leq C_{\text{inv}} h^{n-m-d(\frac{1}{q}-\frac{1}{p})} \|\mathbf{w}_h\|_{W^{n,q}}, \quad \forall \mathbf{w}_h \in \mathbf{X}_h.$$

In particular, we will frequently use the following specific inverse estimates:

$$\|\mathbf{w}_h\|_{L^\infty} \leq C_{\text{inv}} h^{-\frac{d}{2}} \|\mathbf{w}_h\|, \quad \|\mathbf{w}_h\|_{H^1} \leq C_{\text{inv}} h^{-1} \|\mathbf{w}_h\|, \quad \|\mathbf{w}_h\|_{L^{\frac{2d}{d-1}}} \leq C_{\text{inv}} h^{-1/2} \|\mathbf{w}_h\|.$$

Next, let us also introduce some operators which are needed when estimating the pressure. Let us first define the linear operator $A_h : \mathbf{V}_h \rightarrow \mathbf{V}_h$ by

$$(A_h \mathbf{w}_h, \mathbf{v}_h) = (\nabla \mathbf{w}_h, \nabla \mathbf{v}_h), \quad \forall \mathbf{w}_h, \mathbf{v}_h \in \mathbf{V}_h,$$

and it holds that $\|A_h^{1/2} \mathbf{w}_h\| = \|\nabla \mathbf{w}_h\|$ and $\|\nabla A_h^{-1/2} \mathbf{w}_h\| = \|\mathbf{w}_h\|$. The linear operator $D_h : L^2(\Omega) \rightarrow \mathbf{V}_h$ is defined by

$$(D_h q, \mathbf{v}_h) = (\nabla \cdot \mathbf{v}_h, q), \quad \forall q \in L^2(\Omega), \mathbf{v}_h \in \mathbf{V}_h.$$

The discrete Leray projection $\Pi_h^{\text{div}} : (L^2(\Omega))^d \rightarrow \mathbf{V}_h$ is defined by

$$(\Pi_h^{\text{div}} \mathbf{w}, \mathbf{v}_h) = (\mathbf{w}, \mathbf{v}_h), \quad \forall \mathbf{w} \in (L^2(\Omega))^d, \mathbf{v}_h \in \mathbf{V}_h.$$

There also exist constants $C_{\text{neg}}^{(1)} > 0$ and $C_{\text{neg}}^{(2)} > 0$ depending only on Ω such that

$$\|\mathbf{w}\|_{-1} \leq C_{\text{neg}}^{(1)} \|\mathbf{w}\|, \quad \forall \mathbf{w} \in (L^2(\Omega))^d, \quad (2.4)$$

and the following inverse inequality holds [6]

$$\|\mathbf{w}_h\|_{-1} \leq C_{\text{neg}}^{(2)} \left(h \|\mathbf{w}_h\| + \|A_h^{-1/2} \mathbf{w}_h\| \right), \quad \forall \mathbf{w}_h \in \mathbf{V}_h. \quad (2.5)$$

We set $C_{\text{neg}} = \max\{C_{\text{neg}}^{(1)}, C_{\text{neg}}^{(2)}\}$ for simplicity.

We conclude this section with some auxiliary results for the nonlinear term. Define the skew-symmetric trilinear form $c(\cdot, \cdot, \cdot) : \mathbf{X} \times \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$, that is

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w}) - \frac{1}{2}((\mathbf{u} \cdot \nabla) \mathbf{w}, \mathbf{v}),$$

such that $c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$, and there exists a positive constant $C_{\text{tr}} > 0$ depending on Ω satisfying

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C_{\text{tr}} \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\|.$$

Furthermore, by the Sobolev embedding theorems [1], there also exists a positive constant $C_{\text{pi}} > 0$ depending only on Ω such that

$$\|\mathbf{w}\|_{L^p} \leq C_{\text{pi}} \|\mathbf{w}\|_1, \quad 1 \leq p \leq 4, \text{ if } d = 2; \quad 1 \leq p \leq 6, \text{ if } d = 3.$$

3. Variable-step IMEX-BDF2-FEM scheme

For the temporal discretization, let us consider a nonuniform partition of the time interval $[0, T]$: $0 = t_0 < t_1 < \dots < t_N = T$ with time step-size $\tau_k = t_k - t_{k-1}$ for $1 \leq k \leq N$. Denote the maximum step-size $\tau = \max_{1 \leq k \leq N} \tau_k$ and the adjacent time-step ratio $r_k := \tau_k / \tau_{k-1}$ for $2 \leq k \leq N$. Besides, we make the following assumption:

$$\mathbf{A0.} \quad 0 < r_k \leq r_{\max} \text{ for } 2 \leq k \leq N, \text{ where } 1 < r_{\max} < 4.864.$$

For a generic variable φ , let $\varphi^n = \varphi(t_n)$, and define $\nabla_\tau \varphi^n = \varphi^n - \varphi^{n-1}$ and $\partial_\tau \varphi^n = \nabla_\tau \varphi^n / \tau_n$. Then, the variable-step BDF2 discretization is given as

$$D_2 \varphi^n := \frac{1 + 2r_n}{\tau_n (1 + r_n)} \nabla_\tau \varphi^n - \frac{r_n^2}{\tau_n (1 + r_n)} \nabla_\tau \varphi^{n-1} = \sum_{j=1}^n b_{n-j}^{(n)} \nabla_\tau \varphi^j, \quad n \geq 2,$$

where $\{b_{n-j}^{(n)}\}_{j=1}^n$ for $n \geq 2$ are the discrete convolution kernels [21, 29] and defined as

$$b_0^{(n)} := \frac{1 + 2r_n}{\tau_n (1 + r_n)}, \quad b_1^{(n)} := -\frac{r_n^2}{\tau_n (1 + r_n)}, \quad \text{and} \quad b_j^{(n)} := 0 \text{ for } 2 \leq j \leq n-1.$$

As its counterpart, we introduce the discrete orthogonal convolution (DOC) kernels $\{\beta_{n-k}^{(n)}\}_{k=2}^n$ as

$$\beta_0^{(n)} := \frac{1}{b_0^{(n)}} \quad \text{and} \quad \beta_{n-k}^{(n)} := -\frac{1}{b_0^{(k)}} \sum_{j=k+1}^n \beta_{n-j}^{(n)} b_{j-k}^{(j)}, \quad \text{for } 2 \leq k \leq n-1,$$

so that the following orthogonality identity holds

$$\sum_{j=k}^n \beta_{n-j}^{(n)} b_{j-k}^{(j)} \equiv \delta_{n,k}, \quad \text{for } 2 \leq k \leq n, \quad 2 \leq n \leq N, \quad (3.1)$$

where $\delta_{n,k}$ is the Kronecker delta symbol. By exchanging the summation order and using the definition (3.1), it is easy to check that

$$\begin{aligned} \sum_{m=2}^n \beta_{n-m}^{(n)} D_2 \varphi^m &= \sum_{m=2}^n \beta_{n-m}^{(n)} b_{m-1}^{(m)} \nabla_\tau \varphi^1 + \sum_{m=2}^n \beta_{n-m}^{(n)} \sum_{j=2}^m b_{m-j}^{(m)} \nabla_\tau \varphi^j \\ &= \nabla_\tau \varphi^n + \beta_{n-2}^{(n)} b_1^{(2)} \nabla_\tau \varphi^1, \quad 2 \leq n \leq N. \end{aligned} \quad (3.2)$$

Several useful lemmas are presented for the subsequent theoretical analysis.

Lemma 3.1 ([8, 28, 30]). *If A0 holds, then the DOC kernels $\{\beta_{n-k}^{(n)}\}$ defined in (3.1) are positive semi-definite, i.e., for any real sequence $\{w^k\}_{k=2}^n$, it holds that*

$$\sum_{k=2}^n w^k \sum_{j=2}^k \beta_{k-j}^{(k)} w^j \geq 0 \quad \text{for } n \geq 2.$$

Moreover, the following properties hold

$$\beta_{n-k}^{(n)} > 0 \quad \text{for } 2 \leq k \leq n; \quad \sum_{k=2}^n \beta_{n-k}^{(n)} \leq \tau_n, \quad \sum_{k=2}^n \sum_{j=2}^k \beta_{k-j}^{(k)} \leq t_n \quad \text{for } n \geq 2.$$

Lemma 3.2 ([28]). *If A0 holds, there exists three positive constants $\mathbf{m}_i$ ($i = 1, 2, 3$) such that they are independent of the time t_n , time-step sizes τ_n and the step-size ratios r_n . Then for any real vector sequences $\{\mathbf{v}^k\}_{k=2}^n$ and $\{\mathbf{w}^k\}_{k=2}^n$, there holds that*

$$\frac{\mathbf{m}_1}{2\mathbf{m}_2} \sum_{k=2}^n (\mathbf{v}^k)^\top \mathbf{v}^k \leq \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\mathbf{v}^k)^\top \mathbf{v}^j,$$

and for $\varepsilon > 0$

$$\begin{aligned} \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\mathbf{w}^k)^\top \mathbf{v}^j &\leq \varepsilon \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\mathbf{v}^k)^\top \mathbf{v}^j + \frac{1}{2\mathbf{m}_1 \varepsilon} \sum_{k=2}^n \tau_k (\mathbf{w}^k)^\top \mathbf{w}^k, \\ \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\mathbf{w}^k)^\top \mathbf{v}^j &\leq \varepsilon \sum_{k=2}^n \tau_k (\mathbf{v}^k)^\top \mathbf{v}^k + \frac{\mathbf{m}_3}{4\mathbf{m}_1 \varepsilon} \sum_{k=2}^n \tau_k (\mathbf{w}^k)^\top \mathbf{w}^k, \end{aligned}$$

where the symbol $\sum_{k,j}^{n,k} := \sum_{k=2}^n \sum_{j=2}^k$.

We will also make use of the following discrete Gronwall's inequality in our error analysis.

Lemma 3.3 ([30]). *Let $\lambda \geq 0$ and the sequences $\{\xi^k\}_{k=0}^N$ and $\{V^k\}_{k=1}^N$ be nonnegative. If*

$$V_n \leq \lambda \sum_{j=1}^{n-1} \tau_j V^j + \sum_{j=0}^n \xi^j \quad \text{for } 1 \leq n \leq N,$$

then it holds that

$$V^n \leq \exp(\lambda t_{n-1}) \sum_{j=0}^n \xi^j \quad \text{for } 1 \leq n \leq N.$$

The following temporal truncation error result is used for the first time level error analysis.

Lemma 3.4 ([37]). *There exists a positive constant $C_t > 0$ depending only on Ω such that*

(i) *If $f''(t) \in L^2([t^{n-1}, t^n])$, then we have*

$$f(t^{n-1/2}) = \frac{1}{2} (f(t^n) + f(t^{n-1})) + \Upsilon_2, \quad |\Upsilon_2| \leq C_t \|f''\|_{L^2([t^{n-1}, t^n])} \tau^{3/2}.$$

(ii) *If $f'''(t) \in L^2([t^{n-1}, t^n])$, then we have*

$$f'(t^{n-1/2}) = \frac{f(t^n) - f(t^{n-1})}{\tau} + \Theta_2, \quad |\Theta_2| \leq C_t \|f'''\|_{L^2([t^{n-1}, t^n])} \tau^{3/2}.$$

The weak form for the NS equations (1.1) is given as follows: find $(\mathbf{u}(t), p(t)) \in \mathbf{X} \times M$ for $t \in (0, T]$ such that

$$\begin{cases} (\partial_t \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{X}, \\ (\nabla \cdot \mathbf{u}, q) = 0, & \forall q \in M, \end{cases} \quad (3.3)$$

where we denote $c(\mathbf{u}, \mathbf{u}, \mathbf{v}) := (\mathcal{N}(\mathbf{u}, \mathbf{u}), \mathbf{v})$ is the skew-symmetric form of the nonlinear term. Next, we discretize the weak form (3.3) using the Taylor–Hood finite element method in space and the second-order implicit-explicit backward differentiation formula in time, and propose the following variable-step IMEX-BDF2 scheme.

Variable-step IMEX-BDF2-FEM. Given $(\mathbf{u}_h^\ell, p_h^\ell)$ for $\ell = n-2, n-1$ and $n \geq 2$, find $(\mathbf{u}_h^n, p_h^n) \in \mathbf{X}_h \times M_h$ such that

$$\begin{cases} (D_2 \mathbf{u}_h^n, \mathbf{v}_h) + c(\tilde{\mathbf{u}}_h^n, \mathbf{u}_h^n, \mathbf{v}_h) - (p_h^n, \nabla \cdot \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^n, \nabla \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{X}_h, \\ (\nabla \cdot \mathbf{u}_h^n, q_h) = 0, & \forall q_h \in M_h, \end{cases} \quad (3.4)$$

where $\tilde{\mathbf{u}}_h^n := (1 + r_n) \mathbf{u}_h^{n-1} - r_n \mathbf{u}_h^{n-2}$ denotes the second-order extrapolation at $t = t_n$. For the first step $t = t_1$, we approximate (3.3) using the implicit Crank-Nicolson scheme, that is, given $\mathbf{u}_h^0 = \mathbf{P}_{st} \mathbf{u}_0$, find $(\mathbf{u}_h^1, p_h^1) \in \mathbf{X}_h \times M_h$ such that

$$\begin{cases} (\partial_\tau \mathbf{u}_h^1, \mathbf{v}_h) + c(\mathbf{u}_h^{1/2}, \mathbf{u}_h^{1/2}, \mathbf{v}_h) - (p_h^1, \nabla \cdot \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{1/2}, \nabla \mathbf{v}_h) = (\mathbf{f}^{1/2}, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{X}_h, \\ (\nabla \cdot \mathbf{u}_h^1, q_h) = 0, & \forall q_h \in M_h, \end{cases} \quad (3.5)$$

where $\mathbf{u}_h^{1/2} = \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}$.

4. Optimal H^1 error estimate on velocity

Let $(\mathbf{u}, p)$ be the exact solutions of the NS system (3.3) and $\{(\mathbf{u}_h^n, p_h^n)\}_{n=1}^N$ be the numerical solutions produced by the variable-step FEM (3.4)–(3.5). Moreover, we assume that the exact solutions $(\mathbf{u}, p)$ satisfy the following regularity requirement:

$$\begin{cases} \mathbf{u} \in L^\infty(0, T; (H_0^1(\Omega))^d \cap (H^3(\Omega))^d), \quad \partial_t \mathbf{u} \in L^\infty(0, T; (H^3(\Omega))^d), \\ \partial_{tt} \mathbf{u} \in L^\infty(0, T; (H^1(\Omega))^d), \quad \partial_{ttt} \mathbf{u} \in L^\infty(0, T; (H^1(\Omega))^d), \\ p \in L^\infty(0, T; H^2(\Omega)). \end{cases} \quad (4.1)$$

In this section, we will carefully analyze the convergence behavior in time and space of the proposed variable-step IMEX-BDF2-FEM scheme.

Define $\eta(t) := \mathbf{u}(t) - \mathbf{P}_{st} \mathbf{u}(t)$, $e_{\mathbf{u}}^n := \mathbf{P}_{st} \mathbf{u}(t_n) - \mathbf{u}_h^n$, then $\mathbf{u}(t_n) - \mathbf{u}_h^n = \eta(t_n) + e_{\mathbf{u}}^n$. Since the first step is the standard Crank-Nicolson finite element scheme, it has been fully analyzed, see [3, 18, 22]. Hence, we will directly present its error estimate and omit its proof here.

Lemma 4.1. Assume that (4.1) holds. There exists a positive constant $\mathcal{B}_1$ such that

$$\|e_{\mathbf{u}}^1\|^2 + \tau_1 \nu h^2 \|\nabla e_{\mathbf{u}}^1\|^2 \leq \mathcal{B}_1(\tau^4 + h^6). \quad (4.2)$$

Furthermore, there exists a positive constant $0 < h_0 \leq 1$ such that if $h \leq h_0$, then

$$\|e_{\mathbf{u}}^1\|^2 + \tau_1 \nu \|\nabla e_{\mathbf{u}}^1\|^2 \leq h^{-1/2}(\tau^4 + h^4). \quad (4.3)$$

Next, we give an estimate for $\partial_\tau e_{\mathbf{u}}^1$ with optimal temporal convergent rate.

Lemma 4.2. Assume that (4.1) holds. If the first time step-size $\tau_1 \leq \tau^* := \frac{1}{2} \left(\frac{246C_{\text{st}}^2}{\nu} \|\mathbf{u}(t)\|_{L^\infty(0,T;L^\infty)}^2 \right)^{-1}$, then there exist positive constants $\mathcal{B}_2^1$ and $\mathcal{B}_2^2$, such that

$$\tau \|\partial_\tau e_{\mathbf{u}}^1\|^2 \leq \mathcal{B}_2^1(\tau_1^4 + h^4) \quad \text{and} \quad \tau \|\partial_\tau e_{\mathbf{u}}^1\|_{-1}^2 \leq \mathcal{B}_2^2(\tau_1^4 + h^6). \quad (4.4)$$

Proof. The proof is divided into two parts.

Part I: the first conclusion of (4.4). Let $\mathbf{v} = \mathbf{v}_h$ in (3.3) and $t = t_{1/2}$, and then by subtracting (3.5) from the resulting equations, it yields

$$\begin{cases} (\partial_t \mathbf{u}(t_{1/2}) - \partial_\tau \mathbf{u}_h^1, \mathbf{v}_h) + c(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), \mathbf{v}_h) - c(\mathbf{u}_h^{1/2}, \mathbf{u}_h^{1/2}, \mathbf{v}_h) \\ \quad - (p(t_1) - p_h^1, \nabla \cdot \mathbf{v}_h) + \nu (\nabla(\mathbf{u}(t_{1/2}) - \mathbf{u}_h^{1/2}), \nabla \mathbf{v}_h) = 0, \end{cases} \quad (4.5a)$$

$$(\nabla \cdot (\mathbf{u}(t_1) - \mathbf{u}_h^1), q_h) = 0. \quad (4.5b)$$

Take $\mathbf{v}_h = \nabla_\tau e_{\mathbf{u}}^1 = e_{\mathbf{u}}^1 - e_{\mathbf{u}}^0 = e_{\mathbf{u}}^1$, then (4.5a) turns to be

$$\begin{aligned} \tau_1 \|\partial_\tau e_{\mathbf{u}}^1\|^2 + \frac{\nu}{2} \|\nabla e_{\mathbf{u}}^1\|^2 &= -(\partial_t \mathbf{u}(t_{1/2}) - \mathbf{P}_{\text{st}} \partial_\tau \mathbf{u}(t_1), e_{\mathbf{u}}^1) \\ &\quad - \nu \left(\nabla \mathbf{P}_{\text{st}} \left(\mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right), \nabla e_{\mathbf{u}}^1 \right) \\ &\quad - c(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), e_{\mathbf{u}}^1) + c \left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, e_{\mathbf{u}}^1 \right) =: \sum_{k=1}^4 \mathbb{J}_k, \end{aligned} \quad (4.6)$$

where each term $\mathbb{J}_k$ ($k = 1, \dots, 4$) can be estimated as follows.

By Taylor's expansion and Lemma 2.1, we first have

$$\begin{aligned} \mathbb{J}_1 &= -(\partial_t \mathbf{u}(t_{1/2}) - \partial_\tau \mathbf{u}(t_1), e_{\mathbf{u}}^1) - (\partial_\tau \mathbf{u}(t_1) - \mathbf{P}_{\text{st}} \partial_\tau \mathbf{u}(t_1), e_{\mathbf{u}}^1) \\ &\leq \frac{\nu}{12C_{\text{pi}}^2} \|e_{\mathbf{u}}^1\|^2 + \frac{6C_{\text{pi}}^2}{\nu} \|\partial_t \mathbf{u}(t_{1/2}) - \partial_\tau \mathbf{u}(t_1)\|^2 + \frac{6C_{\text{pi}}^2}{\nu} \|\partial_\tau \mathbf{u}(t_1) - \mathbf{P}_{\text{st}} \partial_\tau \mathbf{u}(t_1)\|^2 \\ &\leq \frac{\nu}{12} \|\nabla e_{\mathbf{u}}^1\|^2 + \frac{3C_{\text{pi}}^2 \tau_1^3}{640\nu} \|\partial_{ttt} \mathbf{u}\|_{L^2(t_0, t_1; L^2)}^2 + \frac{6C_{\text{pi}}^2 C_{\text{st}}^2 h^6}{\nu} \|\partial_t \mathbf{u}\|_{L^2(t_0, t_1; H^3)}^2. \end{aligned}$$

Via Young's inequality and Lemma 3.4, it yields

$$\mathbb{J}_2 \leq \frac{\nu}{24} \|\nabla e_{\mathbf{u}}^1\|^2 + 6\nu \left\| \nabla \mathbf{P}_{\text{st}} \left(\mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right) \right\|^2 \leq \frac{\nu}{24} \|\nabla e_{\mathbf{u}}^1\|^2 + \frac{\nu C_{\text{st}}^2 \tau_1^3}{8} \|\partial_{ttt} \mathbf{u}\|_{L^2(t_0, t_1; H^1)}^2.$$

$$\begin{aligned} \mathbb{J}_3 + \mathbb{J}_4 &= -c \left(\mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, \mathbf{u}(t_{1/2}), e_{\mathbf{u}}^1 \right) - c \left(\frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, \mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, e_{\mathbf{u}}^1 \right) \\ &\quad - c \left(\frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} - \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, e_{\mathbf{u}}^1 \right) \\ &\quad - c \left(\mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} - \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, e_{\mathbf{u}}^1 \right) \\ &\quad - c \left(\frac{e_{\mathbf{u}}^1}{2}, \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2}, e_{\mathbf{u}}^1 \right) - c \left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \frac{e_{\mathbf{u}}^1}{2}, e_{\mathbf{u}}^1 \right) := \sum_{i=1}^6 \mathbb{K}_i, \end{aligned}$$

where

$$\begin{aligned}\mathbb{K}_1 &\leq C_{\text{tr}} \left\| \nabla \left(\mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right) \right\| \left\| \nabla \mathbf{u}(t_{1/2}) \right\| \left\| \nabla e_{\mathbf{u}}^1 \right\| \\ &\leq \frac{\nu}{48} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 + \frac{C_{\text{tr}}^2 \tau_1^3}{4\nu} \left\| \partial_{tt} \mathbf{u} \right\|_{L^2(t_0, t_1; H^1)}^2 \left\| \nabla \mathbf{u}(t_{1/2}) \right\|^2.\end{aligned}$$

Similarly,

$$\mathbb{K}_2 \leq \frac{\nu}{48} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 + \frac{C_{\text{tr}}^2 \tau_1^3}{4\nu} \left\| \partial_{tt} \mathbf{u} \right\|_{L^2(t_0, t_1; H^1)}^2 \left\| \nabla \mathbf{u}(t_{1/2}) \right\|^2.$$

In addition,

$$\begin{aligned}\mathbb{K}_3 &\leq C_{\text{tr}} \left\| \nabla \left(\frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} - \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right) \right\| \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\| \left\| \nabla e_{\mathbf{u}}^1 \right\| \\ &\leq \frac{\nu}{48} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 + \frac{12C_{\text{tr}}^2 C_{\text{st}}^2 h^4}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2.\end{aligned}$$

As for $\mathbb{K}_4$, it holds

$$\mathbb{K}_4 \leq \frac{\nu}{48} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 + \frac{12C_{\text{tr}}^2 C_{\text{st}}^4 h^4}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2.$$

Next,

$$\mathbb{K}_5 \leq \frac{\nu}{48} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 + \frac{3C_{\text{st}}^2 \tau_1^2}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_{L^\infty}^2 \left\| \partial_\tau e_{\mathbf{u}}^1 \right\|^2.$$

Lastly, by integration-by-part, we have

$$\begin{aligned}\mathbb{K}_6 &= \frac{3}{2} \left((e_{\mathbf{u}}^1 \cdot \nabla) e_{\mathbf{u}}^1, \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right) + \frac{3}{2} \left((\nabla \cdot e_{\mathbf{u}}^1) e_{\mathbf{u}}^1, \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right) \\ &\quad + \frac{3}{2} \left(\left(\mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \cdot \nabla \right) e_{\mathbf{u}}^1, e_{\mathbf{u}}^1 \right) \\ &\leq \frac{\nu}{48} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 + \frac{243C_{\text{st}}^2 \tau_1^2}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_{L^\infty}^2 \left\| \partial_\tau e_{\mathbf{u}}^1 \right\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{J}_3 + \mathbb{J}_4 &\leq \frac{\nu}{8} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 + \frac{246C_{\text{st}}^2 \tau_1^2}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_{L^\infty}^2 \left\| \partial_\tau e_{\mathbf{u}}^1 \right\|^2 + \frac{C_{\text{tr}}^2 \tau_1^3}{\nu} \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(t_0, t_1; H^1)}^2 \left\| \nabla \mathbf{u}(t_{1/2}) \right\|^2 \\ &\quad + \frac{12C_{\text{tr}}^2 C_{\text{st}}^2 h^4}{\nu} (1 + C_{\text{st}}^2) \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2.\end{aligned}$$

Collecting all the estimates $\mathbb{J}_1 \sim \mathbb{J}_4$ into (4.6), if $\tau_1 \leq \frac{1}{2} \left(\frac{246C_{\text{st}}^2}{\nu} \left\| \mathbf{u}(t) \right\|_{L^\infty(0, T; L^\infty)}^2 \right)^{-1}$, then we have

$$\begin{aligned}\tau_1 \left\| \partial_\tau e_{\mathbf{u}}^1 \right\|^2 + \frac{\nu}{2} \left\| \nabla e_{\mathbf{u}}^1 \right\|^2 &\leq \frac{3C_{\text{pi}}^2 \tau_1^3}{160\nu} \left\| \partial_{tt} \mathbf{u} \right\|_{L^2(t_0, t_1; L^2)}^2 + \frac{24C_{\text{pi}}^2 C_{\text{st}}^2 h^6}{\nu} \left\| \partial_t \mathbf{u} \right\|_{L^2(t_0, t_1; H^3)}^2 \\ &\quad + \frac{\nu C_{\text{st}}^2 \tau_1^3}{2} \left\| \partial_{tt} \mathbf{u} \right\|_{L^2(t_0, t_1; H^1)}^2 + \frac{4C_{\text{tr}}^2 \tau_1^3}{\nu} \left\| \partial_{tt} \mathbf{u} \right\|_{L^2(t_0, t_1; H^1)}^2 \left\| \nabla \mathbf{u}(t_{1/2}) \right\|^2 \\ &\quad + \frac{12C_{\text{tr}}^2 C_{\text{st}}^2 h^4}{\nu} (1 + C_{\text{st}}^2) \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2 \\ &\leq \mathcal{B}_2^1 (\tau_1^4 + h^4),\end{aligned}$$

where

$$\begin{aligned}\mathcal{B}_2^1 &:= \frac{3C_{\text{pi}}^2}{160\nu} \left\| \partial_{ttt} \mathbf{u} \right\|_{L^\infty(0, T; L^2)}^2 + \frac{24C_{\text{pi}}^2 C_{\text{st}}^2 h^2}{\nu} \left\| \partial_t \mathbf{u} \right\|_{L^\infty(0, T; H^3)}^2 + \frac{\nu C_{\text{st}}^2 \tau_1^3}{2} \left\| \partial_{ttt} \mathbf{u} \right\|_{L^\infty(0, T; H^1)}^2 \\ &\quad + \frac{4C_{\text{tr}}^2}{\nu} \left\| \partial_{tt} \mathbf{u} \right\|_{L^\infty(0, T; H^1)}^2 \left\| \nabla \mathbf{u}(t_{1/2}) \right\|^2 + \frac{12C_{\text{tr}}^2 C_{\text{st}}^2}{\nu} (1 + C_{\text{st}}^2) \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2.\end{aligned}$$

Part II: the second conclusion of (4.4). Let $\mathbf{v}_h = A_h^{-1} \tau_1 \partial_\tau e_{\mathbf{u}}^1 = A_h^{-1} e_{\mathbf{u}}^1$, then (4.5a) turns to be

$$\begin{aligned}\tau_1 \left\| \partial_\tau e_{\mathbf{u}}^1 \right\|_{-1}^2 + \frac{\nu}{2} \left\| e_{\mathbf{u}}^1 \right\|^2 &= - \left(\partial_t \mathbf{u}(t_{1/2}) - \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) - \mathbf{u}(t_0)}{\tau_1}, A_h^{-1} e_{\mathbf{u}}^1 \right) - \nu \left(\mathbf{P}_{\text{st}} \left(\mathbf{u}(t_{1/2}) - \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right), e_{\mathbf{u}}^1 \right) \\ &\quad - c(\mathbf{u}(t_{1/2}), \mathbf{u}(t_{1/2}), A_h^{-1} e_{\mathbf{u}}^1) + c \left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, A_h^{-1} e_{\mathbf{u}}^1 \right) := \sum_{1 \leq k \leq 4} \widetilde{\mathbb{J}}_k.\end{aligned}$$

Adopting the arguments in Part I with (2.4), we can directly have the following estimates:

$$\begin{aligned}\tilde{\mathbb{J}}_1 &\leq \frac{\nu}{12} \|e_{\mathbf{u}}^1\|^2 + \frac{3C_{\text{neg}}^2 C_{\text{pi}}^2 \tau_1^3}{640\nu} \|\partial_{tt}\mathbf{u}\|_{L^2(t_0, t_1; L^2)}^2 + \frac{6C_{\text{neg}}^2 C_{\text{pi}}^2 C_{\text{st}}^2 h^6}{\nu} \|\partial_t\mathbf{u}\|_{L^2(t_0, t_1; H^3)}^2, \\ \tilde{\mathbb{J}}_2 &\leq \frac{\nu}{24} \|e_{\mathbf{u}}^1\|^2 + \frac{\nu C_{\text{neg}}^2 C_{\text{st}}^2 \tau_1^3}{8} \|\partial_{tt}\mathbf{u}\|_{L^2(t_0, t_1; H^1)}^2.\end{aligned}$$

In addition, $\tilde{\mathbb{J}}_3 + \tilde{\mathbb{J}}_4 = \sum_{i=1}^6 \tilde{\mathbb{K}}_i$, where $\tilde{\mathbb{K}}_i$ corresponds to $\mathbb{K}_i$ with the modified test function $A_h^{-1} e_{\mathbf{u}}^1$. Then, we can also follow the arguments in Part I to obtain

$$\begin{cases} \tilde{\mathbb{K}}_1 \leq \frac{\nu}{48} \|e_{\mathbf{u}}^1\|^2 + \frac{C_{\text{tr}}^2 \tau_1^3}{4\nu} \|\partial_{tt}\mathbf{u}\|_{L^2(t_0, t_1; L^2)}^2 \|\nabla \mathbf{u}(t_{1/2})\|^2, \\ \tilde{\mathbb{K}}_2 \leq \frac{\nu}{48} \|e_{\mathbf{u}}^1\|^2 + \frac{C_{\text{tr}}^2 \tau_1^3}{4\nu} \|\partial_{tt}\mathbf{u}\|_{L^2(t_0, t_1; L^2)}^2 \|\nabla \mathbf{u}(t_{1/2})\|^2, \\ \tilde{\mathbb{K}}_5 \leq \frac{\nu}{48} \|e_{\mathbf{u}}^1\|^2 + \frac{3C_{\text{st}}^2 \tau_1^2}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_{L^\infty}^2 \|\partial_\tau e_{\mathbf{u}}^1\|_{-1}^2, \\ \tilde{\mathbb{K}}_6 \leq \frac{\nu}{48} \|e_{\mathbf{u}}^1\|^2 + \frac{243C_{\text{st}}^2 \tau_1^2}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_{L^\infty}^2 \|\partial_\tau e_{\mathbf{u}}^1\|_{-1}^2. \end{cases}$$

Now, let us focus on $\tilde{\mathbb{K}}_3$ and $\tilde{\mathbb{K}}_4$, that is,

$$\begin{aligned}\tilde{\mathbb{K}}_3 &\leq C_{\text{tr}} \left\| \left(\frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} - \mathbf{P}_{\text{st}} \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right) \right\| \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\| \|e_{\mathbf{u}}^1\| \\ &\leq \frac{\nu}{48} \|e_{\mathbf{u}}^1\|^2 + \frac{12C_{\text{tr}}^2 C_{\text{st}}^2 h^6}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2.\end{aligned}$$

Similarly, we have

$$\tilde{\mathbb{K}}_4 \leq \frac{\nu}{48} \|e_{\mathbf{u}}^1\|^2 + \frac{12C_{\text{tr}}^2 C_{\text{st}}^4 h^6}{\nu} \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2.$$

Combining all the above estimate and assuming $\tau_1 \leq 1/2 \left(\frac{246C_{\text{st}}^2}{\nu} \|\mathbf{u}\|_{L^\infty(0, T; L^\infty)}^2 \right)^{-1}$, we have

$$\tau_1 \|\partial_\tau e_{\mathbf{u}}^1\|_{-1}^2 + \frac{\nu}{2} \|e_{\mathbf{u}}^1\|^2 \leq \mathcal{B}_2^2(\tau_1^4 + h^6),$$

where

$$\begin{aligned}\mathcal{B}_2^2 &:= \frac{3C_{\text{neg}}^2 C_{\text{pi}}^2}{160\nu} \|\partial_{tt}\mathbf{u}\|_{L^2(0, T; L^2)}^2 + \frac{24C_{\text{neg}}^2 C_{\text{pi}}^2 C_{\text{st}}^2 h^6}{\nu} \|\partial_t\mathbf{u}\|_{L^2(t_0, t_1; H^3)}^2 + \frac{\nu C_{\text{st}}^2 \tau_1^3}{2} \|\partial_{tt}\mathbf{u}\|_{L^2(t_0, t_1; H^1)}^2 \\ &\quad + \frac{4C_{\text{tr}}^2}{\nu} \|\partial_{tt}\mathbf{u}\|_{L^2(t_0, t_1; H^1)}^2 \|\nabla \mathbf{u}(t_{1/2})\|^2 + \frac{12C_{\text{tr}}^2 C_{\text{st}}^2 h^4}{\nu} (1 + C_{\text{st}}^2) \left\| \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|_3^2 \left\| \nabla \frac{\mathbf{u}(t_1) + \mathbf{u}(t_0)}{2} \right\|^2.\end{aligned}$$

The proof is completed. $\square$

Next, we will present an updated optimal H^1 -norm error estimate for (3.4). Similar theoretical results are present in [21]. However, we update their results from $O(\tau_1^{\frac{3}{2}} + \tau^2 + h^2)$ to $O(\tau^2 + h^2)$, which is strictly second-order accurate in time due to the Crank-Nicolson scheme for the first step. In addition, we adopt the DOC from [28] with index starting from $n = 2$, which is more reasonable since the first time level does not fit the discrete convolution kernel. In addition, we perform the error estimate to have an explicit upper bound on the mesh size h with an improved mesh ratio.

First, let us denote

$$\begin{aligned}\mathcal{C}_1 &:= \frac{256\mathbf{m}_2\mathbf{m}_3 C_{\text{tr}}^2 C_{\text{st}}^2 (1 + r_{\text{max}})^4 r_{\text{max}}}{\nu \mathbf{m}_1^2} \|\nabla \mathbf{u}\|_{L^\infty(0, t_n; L^3)}^2 + \frac{56\mathbf{m}_2\mathbf{m}_3 C_{\text{tr}}^2 C_{\text{inv}}^2 (1 + r_{\text{max}})^4 r_{\text{max}}}{\nu \mathbf{m}_1^2}, \\ \mathcal{B}_3(t_n) &:= \frac{8\tau}{3\tau_1} \mathcal{B}_2^1 + \frac{32\mathbf{m}_3(1+r_{\text{max}})^2}{\mathbf{m}_1} t_n \|\partial_{ttt}\mathbf{u}\|_{L^\infty(0, t_n; L^2)} + \frac{72C_{\text{st}}^2 \mathbf{m}_3}{\mathbf{m}_1} t_n \|\partial_t\mathbf{u}\|_{L^\infty(0, t_n; H^3)} \\ &\quad + \frac{32\mathbf{m}_3 C_{\text{tr}}^2 (1+5r_{\text{max}})^2}{\mathbf{m}_1} t_n (1 + r_{\text{max}})^2 \|\partial_{tt}\mathbf{u}\|_{L^\infty(0, t_n; H^1)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0, t_n; L^3)}^2 \\ &\quad + \frac{1024\mathbf{m}_3 C_{\text{tr}}^2 (1+C_{\text{tr}}^2)(1+r_{\text{max}})^4}{\mathbf{m}_1} t_n \|\mathbf{u}\|_{L^\infty(0, t_n; H^3)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0, t_n; L^3)}^2.\end{aligned}$$

Theorem 4.3. Assume that (4.1) holds. There exists a positive constant $0 < h_1 \leq 1$ such that if $h \leq h_1$ and $\tau \leq \min \left\{ h^{\frac{d}{12} + \frac{5}{8}}, \tau^*, 1 \right\}$, then

$$\|e_{\mathbf{u}}^n\|^2 + \frac{\nu \mathfrak{m}_1}{\mathfrak{m}_2} \sum_{k=1}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 \leq 2 \exp((2\mathcal{C}_1 + 1)t_{n-1}) (\mathcal{B}_1 + 2\mathcal{B}_3(t_n)) (h^4 + \tau^4). \quad (4.7)$$

Proof. In order to show (4.7), we will first estimate the following sub-optimal estimate

$$\|e_{\mathbf{u}}^n\|^2 + \frac{\nu \mathfrak{m}_1}{\mathfrak{m}_2} \sum_{k=1}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 \leq h^{-1/2} (h^4 + \tau^4), \quad (4.8)$$

which is devoted to showing the boundness of the discrete velocity, then the estimate (4.7) comes directly by repeating the following arguments.

Thanks to Lemma 4.1, we have

$$\begin{aligned} \|e_{\mathbf{u}}^1\|_{L^\infty}^2 &\leq C_{\text{inv}}^2 h^{-d} \|e_{\mathbf{u}}^1\|^2 \leq C_{\text{inv}}^2 h^{-d-1/2} (\tau^4 + h^4) \leq 2C_{\text{inv}}^2, \\ \|\nabla e_{\mathbf{u}}^1\|_{L^3}^2 &\leq C_{\text{inv}}^2 h^{-2-\frac{d}{3}} \|e_{\mathbf{u}}^1\|^2 \leq C_{\text{inv}}^2 h^{-\frac{d}{3}-\frac{5}{2}} (\tau^4 + h^4) \leq 2C_{\text{inv}}^2. \end{aligned}$$

Now, we assume (4.8) is true for $i < n$, so that

$$\begin{aligned} \|e_{\mathbf{u}}^i\|_{L^\infty}^2 &\leq C_{\text{inv}}^2 h^{-d} \|e_{\mathbf{u}}^i\|^2 \leq C_{\text{inv}}^2 h^{-d-1/2} (\tau^4 + h^4) \leq 2C_{\text{inv}}^2, \\ \|\nabla e_{\mathbf{u}}^i\|_{L^3}^2 &\leq C_{\text{inv}}^2 h^{-2-\frac{d}{3}} \|e_{\mathbf{u}}^i\|^2 \leq C_{\text{inv}}^2 h^{-\frac{d}{3}-\frac{5}{2}} (\tau^4 + h^4) \leq 2C_{\text{inv}}^2, \end{aligned}$$

which implies that $\|e_{\mathbf{u}}^i\|_{L^\infty}$ and $\|\nabla e_{\mathbf{u}}^i\|_{L^3}$ are bounded. When $t = t_n$, the error equation is

$$\begin{aligned} (D_2 e_{\mathbf{u}}^n, \mathbf{v}_h) + \nu (\nabla e_{\mathbf{u}}^n, \nabla \mathbf{v}_h) &= -(\partial_t \mathbf{u}(t_n) - D_2 \mathbf{u}(t_n), \mathbf{v}_h) - (D_2 \eta(t_n), \mathbf{v}_h) \\ &\quad - c(\mathbf{u}(t_n), \mathbf{u}(t_n), \mathbf{v}_h) + c(\tilde{\mathbf{u}}_h^n, \mathbf{u}_h^n, \mathbf{v}_h) + (\mathbf{Q}_{\text{stp}}(t_n) - p_h^n, \nabla \cdot \mathbf{v}_h). \end{aligned} \quad (4.9)$$

Changing the index n to j , multiplying (4.9) by $\beta_{k-j}^{(k)}$, summing up j from 2 to k , and letting $\mathbf{v}_h = e_{\mathbf{u}}^k$, it yields

$$\begin{aligned} (\nabla_\tau e_{\mathbf{u}}^k, e_{\mathbf{u}}^k) + \nu \sum_{j=2}^k \beta_{k-j}^{(k)} (\nabla e_{\mathbf{u}}^j, \nabla e_{\mathbf{u}}^k) &= - \sum_{j=2}^k \beta_{k-j}^{(k)} (\partial_t \mathbf{u}(t_j) - D_2 \mathbf{u}(t_j), e_{\mathbf{u}}^k) \\ &\quad - \sum_{j=2}^k \beta_{k-j}^{(k)} (D_2 \eta(t_j), e_{\mathbf{u}}^k) - \sum_{j=2}^k \beta_{k-j}^{(k)} \left(c(\mathbf{u}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, e_{\mathbf{u}}^k) \right) \\ &\quad - \left(\beta_{k-2}^{(k)} b_1^{(2)} \nabla_\tau e_{\mathbf{u}}^1, e_{\mathbf{u}}^k \right). \end{aligned}$$

Summing up k from 2 to n with (3.2), we have

$$\begin{aligned} \|e_{\mathbf{u}}^n\|^2 - \|e_{\mathbf{u}}^1\|^2 + \sum_{k=2}^n \|e_{\mathbf{u}}^k - e_{\mathbf{u}}^{k-1}\|^2 + 2\nu \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\nabla e_{\mathbf{u}}^j, \nabla e_{\mathbf{u}}^k) \\ = -2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\partial_t \mathbf{u}(t_j) - D_2 \mathbf{u}(t_j), e_{\mathbf{u}}^k) - 2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (D_2 \eta(t_j), e_{\mathbf{u}}^k) \\ - 2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} \left(c(\mathbf{u}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, e_{\mathbf{u}}^k) \right) - 2 \sum_{k=2}^n \left(\beta_{k-2}^{(k)} b_1^{(2)} \nabla_\tau e_{\mathbf{u}}^1, e_{\mathbf{u}}^k \right). \end{aligned} \quad (4.10)$$

By Lemma 3.2, the left-hand side could be estimated as

$$\|e_{\mathbf{u}}^n\|^2 - \|e_{\mathbf{u}}^1\|^2 + \sum_{k=2}^n \|e_{\mathbf{u}}^k - e_{\mathbf{u}}^{k-1}\|^2 + 2\nu \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\nabla e_{\mathbf{u}}^j, \nabla e_{\mathbf{u}}^k) \geq \|e_{\mathbf{u}}^n\|^2 - \|e_{\mathbf{u}}^1\|^2 + \frac{\nu \mathfrak{m}_1}{\mathfrak{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2. \quad (4.11)$$

Compared to [21], $-\left(\beta_{k-2}^{(k)} b_1^{(2)} \nabla_\tau e_{\mathbf{u}}^1, \mathbf{v}_h\right)$ is the extra term. This term can be estimated as

$$\begin{aligned} -2 \sum_{k=2}^n \left(\beta_{k-2}^{(k)} b_1^{(2)} \nabla_\tau e_{\mathbf{u}}^1, e_{\mathbf{u}}^k \right) &\leq 2 \left\| \partial_\tau e_{\mathbf{u}}^1 \right\| \sum_{k=2}^n \frac{\tau_k}{2^{k-1}} \left\| e_{\mathbf{u}}^k \right\| \\ &\leq \sum_{k=2}^n \frac{\tau_k}{2^{k-2}} \left\| \partial_\tau e_{\mathbf{u}}^1 \right\| \left\| e_{\mathbf{u}}^k \right\| \\ &\leq \frac{1}{8} \sum_{k=2}^n \tau_k \left\| e_{\mathbf{u}}^k \right\|^2 + \frac{8}{3} \tau \left\| \partial_\tau e_{\mathbf{u}}^1 \right\|^2. \end{aligned} \quad (4.12)$$

In addition,

$$\begin{aligned} &-2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} \left(\partial_t \mathbf{u}(t_j) - D_2 \mathbf{u}(t_j), e_{\mathbf{u}}^k \right) \\ &\leq 4 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (1 + r_{\max}) \tau^2 \left\| \partial_{ttt} \mathbf{u} \right\|_{L^\infty(t_{j-1}, t_j; L^2)} \left\| e_{\mathbf{u}}^k \right\| \\ &\leq \frac{1}{8} \sum_{k=2}^n \tau_k \left\| e_{\mathbf{u}}^k \right\|^2 + \frac{32 \mathbf{m}_3 (1 + r_{\max})^2}{\mathbf{m}_1} \tau^4 \sum_{k=2}^n \tau_k \left\| \partial_{ttt} \mathbf{u} \right\|_{L^\infty(t_{k-1}, t_k; L^2)}. \end{aligned} \quad (4.13)$$

Since

$$D_2 \eta(t_j) = \frac{1 + 2r_j}{\tau_j (1 + r_j)} \int_{t_{j-1}}^{t_j} \partial_t \eta(t) dt - \frac{r_j^2}{\tau_j (1 + r_j)} \int_{t_{j-2}}^{t_{j-1}} \partial_t \eta(t) dt,$$

then there holds

$$-2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} \left(D_2 \eta(t_j), e_{\mathbf{u}}^k \right) \leq \frac{1}{8} \sum_{k=2}^n \tau_k \left\| e_{\mathbf{u}}^k \right\|^2 + \frac{72 C_{\text{st}}^2 h^6 \mathbf{m}_3}{\mathbf{m}_1} \sum_{k=2}^n \tau_k \left\| \partial_t \mathbf{u} \right\|_{L^\infty(t_{k-2}, t_k; H^3)}^2. \quad (4.14)$$

As for the nonlinear terms, we first have

$$\begin{aligned} c(\mathbf{u}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, e_{\mathbf{u}}^k) &= c(\mathbf{u}(t_j) - \tilde{\mathbf{u}}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) + c(\tilde{\mathbf{u}}(t_j), \mathbf{u}(t_j) - \tilde{\mathbf{u}}(t_j), e_{\mathbf{u}}^k) \\ &\quad + c(\tilde{\eta}(t_j), \tilde{\mathbf{u}}(t_j), e_{\mathbf{u}}^k) + c(\text{P}_{\text{st}} \tilde{\mathbf{u}}(t_j), \tilde{\eta}(t_j), e_{\mathbf{u}}^k) \\ &\quad + c(\tilde{e}_u^j, \text{P}_{\text{st}} \tilde{\mathbf{u}}(t_j), e_{\mathbf{u}}^k) + c(\tilde{\mathbf{u}}_h^j, \tilde{e}_u^j, e_{\mathbf{u}}^k) := \sum_{i=1}^6 \mathbb{L}_i, \end{aligned}$$

where

$$\begin{cases} \mathbb{L}_1 \leq C_{\text{tr}} \left\| \nabla(\mathbf{u}(t_j) - \tilde{\mathbf{u}}(t_j)) \right\| \left\| \nabla \mathbf{u}(t_j) \right\|_{L^3} \left\| e_{\mathbf{u}}^k \right\| \\ \leq C_{\text{tr}} \tau^2 \frac{1 + 5r_{\max}}{2} \left\| \partial_{tt} \mathbf{u} \right\|_{L^\infty(t_{j-2}, t_j; H^1)} \left\| \nabla \mathbf{u}(t_j) \right\|_{L^3} \left\| e_{\mathbf{u}}^k \right\|, \\ \mathbb{L}_2 \leq C_{\text{tr}} \tau^2 \frac{1 + 5r_{\max}}{2} \left\| \partial_{tt} \mathbf{u} \right\|_{L^\infty(t_{j-2}, t_j; H^1)} \left\| \nabla \tilde{\mathbf{u}}(t_j) \right\|_{L^3} \left\| e_{\mathbf{u}}^k \right\|, \\ \mathbb{L}_3 \leq C_{\text{tr}} \left\| \nabla(\tilde{\eta}(t_j)) \right\| \left\| \nabla \tilde{\mathbf{u}}(t_j) \right\|_{L^3} \left\| e_{\mathbf{u}}^k \right\| \\ \leq C_{\text{tr}} C_{\text{st}} h^2 \left\| \tilde{\mathbf{u}}(t_j) \right\|_3 \left\| \nabla \tilde{\mathbf{u}}(t_j) \right\|_{L^3} \left\| e_{\mathbf{u}}^k \right\|, \\ \mathbb{L}_4 \leq C_{\text{tr}} C_{\text{st}}^2 h^2 \left\| \nabla \tilde{\mathbf{u}}(t_j) \right\|_{L^3} \left\| \tilde{\mathbf{u}}(t_j) \right\|_3 \left\| e_{\mathbf{u}}^k \right\|, \\ \mathbb{L}_5 \leq C_{\text{tr}} C_{\text{st}} \left\| \nabla \tilde{\mathbf{u}}(t_j) \right\|_{L^3} \left\| \tilde{e}_u^j \right\| \left\| \nabla e_{\mathbf{u}}^k \right\|, \\ \mathbb{L}_6 \leq C_{\text{tr}} (\left\| \nabla \tilde{e}_u^j \right\|_{L^3} + C_{\text{st}} \left\| \nabla \tilde{\mathbf{u}}^j \right\|_{L^3}) \left\| \tilde{e}_u^j \right\| \left\| \nabla e_{\mathbf{u}}^k \right\|. \end{cases}$$

Therefore, by Lemma 3.2 we have

$$\begin{aligned}
& -2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} \left(c(\mathbf{u}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, e_{\mathbf{u}}^k) \right) \\
& \leq \frac{1}{8} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|^2 + \frac{32\mathbf{m}_3 C_{\text{tr}}^2 (1+5r_{\max})^2 \tau^4}{\mathbf{m}_1} t_n (1+r_{\max})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0,t_n;H^1)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\
& \quad + \frac{1024\mathbf{m}_3 C_{\text{tr}}^2 (1+C_{\text{tr}}^2)(1+r_{\max})^4 h^4}{\mathbf{m}_1} t_n \|\mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\
& \quad + \frac{\nu \mathbf{m}_1}{4\mathbf{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 + \frac{256\mathbf{m}_2 \mathbf{m}_3 C_{\text{tr}}^2 C_{\text{st}}^2 (1+r_{\max})^4 r_{\max}}{\nu \mathbf{m}_1^2} \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \sum_{k=1}^{n-1} \tau_k \|e_{\mathbf{u}}^k\|^2 \\
& \quad + \frac{56\mathbf{m}_2 \mathbf{m}_3 C_{\text{tr}}^2 C_{\text{inv}}^2 (1+r_{\max})^4 r_{\max}}{\nu \mathbf{m}_1^2} \sum_{k=1}^{n-1} \tau_k \|e_{\mathbf{u}}^k\|^2.
\end{aligned} \tag{4.15}$$

Combining all the above estimate from (4.11) to (4.15) and applying Lemma 4.2, it yields

$$\begin{aligned}
& \|e_{\mathbf{u}}^n\|^2 - \|e_{\mathbf{u}}^1\|^2 + \frac{3\nu \mathbf{m}_1}{4\mathbf{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 \\
& \leq \frac{1}{2} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|^2 + \frac{8\tau}{3\tau_1} \mathcal{B}_2(\tau^4 + h^4) + \frac{32\mathbf{m}_3 (1+r_{\max})^2}{\mathbf{m}_1} \tau^4 t_n \|\partial_{ttt} \mathbf{u}\|_{L^\infty(0,t_n;L^2)}^2 \\
& \quad + \frac{72C_{\text{st}}^2 h^6 \mathbf{m}_3}{\mathbf{m}_1} t_n \|\partial_t \mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 + \mathcal{C}_1 \sum_{k=1}^{n-1} \tau_k \|e_{\mathbf{u}}^k\|^2 \\
& \quad + \frac{32\mathbf{m}_3 C_{\text{tr}}^2 (1+5r_{\max})^2 \tau^4}{\mathbf{m}_1} t_n (1+r_{\max})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0,t_n;H^1)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\
& \quad + \frac{1024\mathbf{m}_3 C_{\text{tr}}^2 (1+C_{\text{tr}}^2)(1+r_{\max})^4 h^4}{\mathbf{m}_1} t_n \|\mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2.
\end{aligned}$$

If $\tau < 1$, then we can further have

$$\frac{1}{2} \|e_{\mathbf{u}}^n\|^2 + \frac{3\nu \mathbf{m}_1}{4\mathbf{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 \leq (\mathcal{C}_1 + \frac{1}{2}) \sum_{k=1}^{n-1} \tau_k \|e_{\mathbf{u}}^k\|^2 + \|e_{\mathbf{u}}^1\|^2 + \mathcal{B}_3(t_n)(\tau^4 + h^4).$$

Finally, by Grönwall's inequality (cf. Lemma 3.3), we have

$$\begin{aligned}
\|e_{\mathbf{u}}^n\|^2 + \frac{\nu \mathbf{m}_1}{\mathbf{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 & \leq \exp((2\mathcal{C}_1 + 1)t_{n-1}) (2\|e_{\mathbf{u}}^1\|^2 + 2\mathcal{B}_3(t_n)(\tau^4 + h^4)) \\
& \leq 2 \exp((2\mathcal{C}_1 + 1)t_{n-1}) (\mathcal{B}_1 + \mathcal{B}_3(t_n)) (\tau^4 + h^4).
\end{aligned}$$

By setting $h_1 := \min \left\{ (2 \exp((2\mathcal{C}_1 + 1)T) (\mathcal{B}_1 + 2\mathcal{B}_3(T)))^{-2}, 1 \right\}$, the proof is complete. $\square$

Remark 4.4. We found the explicit formula for the upper bound h_1 of mesh size h . In addition, the chosen h_1 is independent of the index n , which is uniform for each time step.

5. Optimal error estimate on the pressure

In this section, we establish the optimal error estimate for the pressure in the variable time-step setting. This result is a novel contribution, absent from the existing literature. The proof relies crucially on a negative-norm technique, following the arguments in [6, 27].

Theorem 5.1. Assume that (4.1) holds. If $h \leq h_1$ and $\tau \leq \min \left\{ h^{\frac{d}{12} + \frac{5}{8}}, \tau^*, 1 \right\}$, then the following error estimate for the pressure holds

$$\tau \sum_{i=1}^n \|p(t_{i+1}) - p_h^{i+1}\|^2 \leq \mathcal{K}(T, \nu)(h^4 + \tau^4) + Th^4 \|p\|_{L^\infty(0,T;H^2)}^2,$$

for $1 \leq n \leq N-1$, where $\mathcal{K}(T, \nu) > 0$ is a constant depending on T and ν .

Proof. We can obtain the following error equation at $t = t_{i+1}$ ($1 \leq i \leq N-1$):

$$\begin{aligned} & (D_2 e_{\mathbf{u}}^{i+1}, \mathbf{v}_h) + \nu (\nabla e_{\mathbf{u}}^{i+1}, \nabla \mathbf{v}_h) \\ &= -(\partial_t \mathbf{u}(t_{i+1}) - D_2 \mathbf{u}(t_{i+1}), \mathbf{v}_h) - (D_2 \eta^{i+1}, \mathbf{v}_h) - c(\mathbf{u}(t_{i+1}), \mathbf{u}(t_{i+1}), \mathbf{v}_h) + c(\tilde{\mathbf{u}}_h^{i+1}, \tilde{\mathbf{u}}_h^{i+1}, \mathbf{v}_h) \\ & \quad + (p(t_{i+1}) - p_h^{i+1}, \nabla \cdot \mathbf{v}_h). \end{aligned}$$

Projecting the above equation onto $\mathbf{V}_h$ by picking $\mathbf{v}_h \in \mathbf{V}_h$, we then have

$$\begin{aligned} D_2 e_{\mathbf{u}}^{i+1} &= -\nu A_h e_{\mathbf{u}}^{i+1} - \Pi_h^{\text{div}} (\mathcal{N}(\mathbf{u}(t_{i+1}), \mathbf{u}(t_{i+1})) - \mathcal{N}(\tilde{\mathbf{u}}_h^{i+1}, \mathbf{u}_h^{i+1})) \\ & \quad - D_h(p(t_{i+1}) - \Pi_h p(t_{i+1})) - \Pi_h^{\text{div}} (\partial_t \mathbf{u}(t_{i+1}) - D_2 \mathbf{u}(t_{i+1})) - \Pi_h^{\text{div}} D_2 \eta^{i+1}. \end{aligned} \quad (5.1)$$

By using the discrete inf-sup condition (2.2), it holds

$$\begin{aligned} \gamma \|\Pi_h p(t_{i+1}) - p_h^{i+1}\| &\leq \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(\Pi_h p(t_{i+1}) - p_h^{i+1}, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_1} \\ &\leq \|\partial_t \mathbf{u}(t_{i+1}) - D_2 \mathbf{u}(t_{i+1})\|_{-1} + \|D_2 \eta^{i+1}\|_{-1} + \|D_2 e_{\mathbf{u}}^{i+1}\|_{-1} \\ & \quad + \nu \|\nabla e_{\mathbf{u}}^{i+1}\| + \|\mathcal{N}(\mathbf{u}(t_{i+1}), \mathbf{u}(t_{i+1})) - \mathcal{N}(\tilde{\mathbf{u}}_h^{i+1}, \mathbf{u}_h^{i+1})\|_{-1} \\ & \quad + C_{\text{itp}} h^2 \|p(t_{i+1})\|_2 =: \sum_{i=1}^6 \mathbb{K}_i. \end{aligned} \quad (5.2)$$

For the first two terms on the right-hand side of (5.2), we have

$$\mathbb{K}_1 \leq C_{\text{neg}} \|\partial_t \mathbf{u}(t_{i+1}) - D_2 \mathbf{u}(t_{i+1})\| \leq C_{\text{neg}} (1 + r_{\max})^{\frac{3}{2}} \tau^{\frac{3}{2}} \|\partial_{ttt} \mathbf{u}\|_{L^2(t_{i-1}, t_{i+1}; L^2)},$$

and

$$\mathbb{K}_2 \leq C_{\text{st}} C_{\text{neg}} (1 + r_{\max}) h^3 \|\partial_t \mathbf{u}\|_{L^\infty(t_{i-1}, t_{i+1}; H^3)}.$$

For the fifth term on the right-hand side of (5.2), we have

$$\begin{aligned} \mathbb{K}_5 &= \sup_{\|\phi\|_1=1} |c(\mathbf{u}(t_{i+1}), \mathbf{u}(t_{i+1}), \phi) - c(\tilde{\mathbf{u}}_h^{i+1}, \mathbf{u}_h^{i+1}, \phi)| \\ &\leq \sup_{\|\phi\|_1=1} \left(|c(\mathbf{u}(t_{i+1}) - \tilde{\mathbf{u}}(t_{i+1}), \mathbf{u}(t_{i+1}), \phi)| + |c(\tilde{\mathbf{u}}(t_{i+1}), \mathbf{u}(t_{i+1}), \phi) - c(\tilde{\mathbf{u}}_h^{i+1}, \mathbf{u}_h^{i+1}, \phi)| \right), \end{aligned}$$

where

$$|c(\mathbf{u}(t_{i+1}) - \tilde{\mathbf{u}}(t_{i+1}), \mathbf{u}(t_{i+1}), \phi)| \leq \frac{1 + 5r_{\max}}{2} \tau^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0, T; L^2)} \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})},$$

and

$$\begin{aligned} & |c(\tilde{\mathbf{u}}(t_{i+1}), \mathbf{u}(t_{i+1}), \phi) - c(\tilde{\mathbf{u}}_h^{i+1}, \mathbf{u}_h^{i+1}, \phi)| \\ &= |c(\tilde{\mathbf{u}}(t_{i+1}) - \tilde{\mathbf{u}}_h^{i+1}, \mathbf{u}(t_{i+1}), \phi) - c(\mathbf{P}_{st} \mathbf{u}(t_{i+1}), \mathbf{u}(t_{i+1}) - \mathbf{u}_h^{i+1}, \phi) + c(\tilde{\mathbf{e}}_u^{i+1}, \mathbf{u}(t_{i+1}) - \mathbf{u}_h^{i+1}, \phi)| \\ &\leq \|\mathbf{u}(t_{i+1})\|_{W^{1, \infty}} \|\tilde{\mathbf{u}}(t_{i+1}) - \tilde{\mathbf{u}}_h^{i+1}\| + C_{\text{st}} \|\mathbf{u}(t_{i+1})\|_{W^{1, \infty}} \|\mathbf{u}(t_{i+1}) - \mathbf{u}_h^{i+1}\| \\ & \quad + C_{\text{tr}} \|\tilde{\mathbf{e}}_u^{i+1}\|_1 \|\mathbf{u}(t_{i+1}) - \mathbf{u}_h^{i+1}\|_1 \\ &\leq (1 + r_{\max}) \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} (\|\eta^i\| + \|\eta^{i-1}\| + \|e_{\mathbf{u}}^i\| + \|e_{\mathbf{u}}^{i-1}\|) + C_{\text{st}} \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} (\|\eta^{i+1}\| + \|e_{\mathbf{u}}^{i+1}\|) \\ & \quad + C_{\text{tr}} C_{\text{inv}} h^{-1} (1 + r_{\max}) (\|e_{\mathbf{u}}^i\| + \|e_{\mathbf{u}}^{i-1}\|) (\|\eta^{i+1}\|_1 + C_{\text{inv}} h^{-1} \|e_{\mathbf{u}}^{i+1}\|). \end{aligned}$$

Denote $\mathcal{C}^p := \sqrt{2 \exp((2\mathcal{C}_1 + 1)T) (\mathcal{B}_1 + 2\mathcal{B}_3(T))}$. Thus by Theorem 4.3, we get

$$\begin{aligned} \mathbb{K}_5 &\leq \tau^2 \frac{1 + 5r_{\max}}{2} \|\partial_{tt} \mathbf{u}\|_{L^\infty(0, T; L^2)} \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} \\ & \quad + h^3 (2(1 + r_{\max}) + C_{\text{st}}) C_{\text{st}} \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} \|\mathbf{u}\|_{L^\infty(0, T; H^3)} \\ & \quad + 2C_{\text{tr}} C_{\text{inv}} C_{\text{st}} (1 + r_{\max}) (h^3 + h\tau^2 s) \mathcal{C}^p \|\mathbf{u}\|_{L^\infty(0, T; H^3)} \\ & \quad + (2(1 + r_{\max}) + C_{\text{st}}) \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} \mathcal{C}^p (h^2 + \tau^2) + 4(1 + r_{\max}) C_{\text{tr}} C_{\text{inv}}^2 (\mathcal{C}^p)^2 (h^2 + \tau^2). \end{aligned}$$

Now, we are turning to analyze $\mathbb{K}_3$ on the right-hand side of (5.2). According to [6], we have by (2.5)

$$\mathbb{K}_3 \leq C_{\text{neg}} (C_{\text{inv}} + 1) \|A_h^{-1/2} D_2 e_{\mathbf{u}}^{i+1}\|,$$

by (5.1)

$$\begin{aligned} \left\| A_h^{-1/2} D_2 e_{\mathbf{u}}^{i+1} \right\| &\leq \nu \|\nabla e_{\mathbf{u}}^{i+1}\| + \|\mathcal{N}(\mathbf{u}(t_{i+1}), \mathbf{u}(t_{i+1})) - \mathcal{N}(\tilde{\mathbf{u}}_h^{i+1}, \mathbf{u}_h^{i+1})\|_{-1} \\ &\quad + C_{\text{itp}} h^2 \|p(t_{i+1})\|_2 + \|\partial_t \mathbf{u}(t_{i+1}) - D_2 \mathbf{u}(t_{i+1})\|_{-1} + \|D_2 \eta^{i+1}\|_{-1}. \end{aligned}$$

Combination of all the above estimates gives us

$$\begin{aligned} &\gamma \|p(t_{i+1}) - p_h^{i+1}\| \\ &\leq (C_{\text{neg}}(C_{\text{inv}} + 1) + 1) \left\{ C_{\text{neg}}(1 + r_{\text{max}})^{\frac{3}{2}} \tau^{\frac{3}{2}} \|\partial_{ttt} \mathbf{u}\|_{L^2(t_{i-1}, t_{i+1}; L^2)} \right. \\ &\quad + C_{\text{st}} C_{\text{neg}}(1 + r_{\text{max}}) h^3 \|\partial_t \mathbf{u}\|_{L^\infty(t_{i-1}, t_{i+1}; H^3)} + C_{\text{itp}} h^2 \|p(t_{i+1})\|_2 \\ &\quad + \nu \|\nabla e_{\mathbf{u}}^{i+1}\| + \tau^2 \frac{1+5r_{\text{max}}}{2} \|\partial_{tt} \mathbf{u}\|_{L^\infty(0, T; L^2)} \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} \\ &\quad + h^3 (2(1 + r_{\text{max}}) + C_{\text{st}}) C_{\text{st}} \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} \|\mathbf{u}\|_{L^\infty(0, T; H^3)} \\ &\quad + 2C_{\text{tr}} C_{\text{inv}} C_{\text{st}} (1 + r_{\text{max}}) (h^3 + h\tau^2) \mathcal{C}^p \|\mathbf{u}\|_{L^\infty(0, T; H^3)} \\ &\quad \left. + (2(1 + r_{\text{max}}) + C_{\text{st}}) \mathcal{C}^p \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})} (h^2 + \tau^2) + 4(1 + r_{\text{max}}) C_{\text{tr}} C_{\text{inv}}^2 (\mathcal{C}^p)^2 (h^2 + \tau^2) \right\}. \end{aligned}$$

Taking square on the both sides, multiplying by τ , summing up from $i = 1$ to n , and applying the Cauchy-Schwarz inequality and Theorem 4.3, we finally obtain

$$\tau \sum_{i=1}^n \|p(t_{i+1}) - p_h^{i+1}\|^2 \leq \mathcal{K}(T, \nu) (h^4 + \tau^4),$$

where

$$\begin{aligned} \mathcal{K}(T, \nu) &= \frac{9(C_{\text{neg}}(C_{\text{inv}}+1)+1)^2}{\gamma^2} \left(C_{\text{neg}}^2 (1 + r_{\text{max}})^3 \|\partial_{ttt} \mathbf{u}\|_{L^2(0, T; L^2)}^2 + C_{\text{st}}^2 C_{\text{neg}}^2 (1 + r_{\text{max}})^2 T \|\partial_t \mathbf{u}\|_{L^\infty(0, T; H^3)}^2 \right. \\ &\quad + C_{\text{itp}}^2 T \|p\|_{L^\infty(0, T; H^2)}^2 + \frac{2\mathbf{m}_2}{\mathbf{m}_1} \exp((2\mathcal{C}_1 + 1)T) (\mathcal{B}_1 + 2\mathcal{B}_3(T)) \\ &\quad + \frac{(1+5r_{\text{max}})^2}{4} T \|\partial_{tt} \mathbf{u}\|_{L^\infty(0, T; L^2)}^2 \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})}^2 \\ &\quad + (2(1 + r_{\text{max}}) + C_{\text{st}})^2 C_{\text{st}}^2 T \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})}^2 \|\mathbf{u}\|_{L^\infty(0, T; H^3)}^2 \\ &\quad + 4C_{\text{tr}}^2 C_{\text{inv}}^2 C_{\text{st}}^2 (1 + r_{\text{max}})^2 (\mathcal{C}^p)^2 \|\mathbf{u}\|_{L^\infty(0, T; H^3)}^2 \\ &\quad \left. + (2(1 + r_{\text{max}}) + C_{\text{st}})^2 (\mathcal{C}^p)^2 \|\mathbf{u}\|_{L^\infty(0, T; W^{1, \infty})}^2 + 16(1 + r_{\text{max}})^2 C_{\text{tr}}^2 C_{\text{inv}}^4 (\mathcal{C}^p)^4 \right), \end{aligned}$$

which completes the proof. $\square$

Remark 5.2. Regarding the initial step of the CN scheme, the earlier pressure error estimate given in [18] is suboptimal. Recently, [31, 33] have provided an optimal error estimate for the pressure. Consequently, the CN initial step can indeed yield optimal pressure accuracy—a point that is not further addressed in this work.

6. Optimal L^2 error estimate on velocity

In this section, we are going to show the optimal $l^\infty(0, T; L^2(\Omega))$ error estimate on the velocity via the negative norm technique [16]. First of all, we will show the optimal $l^2(0, T; L^2(\Omega))$ error estimate. Denote

$$\begin{aligned} \mathcal{B}_4(t_n) &:= \frac{8\tau}{3\tau_1} \mathcal{B}_2^2 + \frac{32\mathbf{m}_3(1+r_{\text{max}})^2 C_{\text{neg}}^2}{\mathbf{m}_1} t_n \|\partial_{ttt} \mathbf{u}\|_{L^\infty(0, t_n; L^2)}^2 \\ &\quad + \frac{72C_{\text{st}}^2 C_{\text{neg}}^2 \mathbf{m}_3}{\mathbf{m}_1} t_n \|\partial_t \mathbf{u}\|_{L^\infty(0, t_n; H^3)}^2 \\ &\quad + \frac{32\mathbf{m}_3 C_{\text{tr}}^2 (1+5r_{\text{max}})^2}{\mathbf{m}_1} t_n (1 + r_{\text{max}})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0, t_n; L^2)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0, t_n; L^3)}^2 \\ &\quad + \frac{1024\mathbf{m}_3 C_{\text{tr}}^2 (1+C_{\text{tr}}^2)(1+r_{\text{max}})^4}{\mathbf{m}_1} t_n \|\mathbf{u}\|_{L^\infty(0, t_n; H^3)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0, t_n; L^3)}^2. \end{aligned}$$

Lemma 6.1. Assume that (4.1) holds. If $h \leq h_1$ and $\tau \leq \min \left\{ h^{\frac{d}{12} + \frac{5}{8}}, \tau^*, 1 \right\}$, then

$$\begin{aligned} & \|e_{\mathbf{u}}^n\|_{-1}^2 + \frac{\nu \mathbf{m}_1}{\mathbf{m}_2} \sum_{k=1}^n \tau_k \|\mathbf{u}(t_k) - \mathbf{u}_h^k\|^2 \\ & \leq 2 \exp((2\mathcal{C}_1 + 1)t_{n-1}) (C_{neg}^2 \mathcal{B}_1 + \mathcal{B}_4(t_n)) (\tau^4 + h^6) + \frac{\nu \mathbf{m}_1 C_{st}^2 t_n}{\mathbf{m}_2} h^6 \|\mathbf{u}\|_{L^\infty(0, t_n; H^3)}^2. \end{aligned} \quad (6.1)$$

Proof. Setting $\mathbf{v}_h = A_h^{-1} e_{\mathbf{u}}^k$ in (4.9), similar to the derivation of (4.10) we have

$$\begin{aligned} & \|e_{\mathbf{u}}^n\|_{-1}^2 - \|e_{\mathbf{u}}^1\|_{-1}^2 + \sum_{k=2}^n \|e_{\mathbf{u}}^k - e_{\mathbf{u}}^{k-1}\|_{-1}^2 + 2\nu \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (e_{\mathbf{u}}^j, e_{\mathbf{u}}^k) \\ & = -2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\partial_t \mathbf{u}(t_j) - D_2 \mathbf{u}(t_j), A_h^{-1} e_{\mathbf{u}}^k) - 2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (D_2 \eta(t_j), A_h^{-1} e_{\mathbf{u}}^k) \\ & \quad - 2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} \left(c(\mathbf{u}(t_j), \mathbf{u}(t_j), A_h^{-1} e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, A_h^{-1} e_{\mathbf{u}}^k) \right) \\ & \quad - 2 \sum_{k=2}^n \left(\beta_{k-2}^{(k)} b_1^{(2)} \nabla_\tau e_{\mathbf{u}}^1, A_h^{-1} e_{\mathbf{u}}^k \right). \end{aligned} \quad (6.2)$$

Observed from the proof of Theorem 4.3, we should deal with $\mathbb{L}_3$ and $\mathbb{L}_4$ to obtain the higher spatial accuracy. The other terms can be estimated similarly. First of all, the left hand side of (6.2) can be estimated as

$$\|e_{\mathbf{u}}^n\|_{-1}^2 - \|e_{\mathbf{u}}^1\|_{-1}^2 + \sum_{k=2}^n \|e_{\mathbf{u}}^k - e_{\mathbf{u}}^{k-1}\|_{-1}^2 + 2\nu \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (e_{\mathbf{u}}^j, e_{\mathbf{u}}^k) \geq \|e_{\mathbf{u}}^n\|_{-1}^2 - \|e_{\mathbf{u}}^1\|_{-1}^2 + \frac{\nu \mathbf{m}_1}{\mathbf{m}_2} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|^2.$$

Now, let us turn to the right hand side. That is,

$$\begin{aligned} & -2 \sum_{k=2}^n \left(\beta_{k-2}^{(k)} b_1^{(2)} \nabla_\tau e_{\mathbf{u}}^1, A_h^{-1} e_{\mathbf{u}}^k \right) \leq \frac{1}{8} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 + \frac{8}{3} \frac{\tau}{\tau_1} \tau_1 \|\partial_\tau e_{\mathbf{u}}^1\|_{-1}^2, \\ & -2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (\partial_t \mathbf{u}(t_j) - D_2 \mathbf{u}(t_j), A_h^{-1} e_{\mathbf{u}}^k) \\ & \leq 4C_{neg} \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (1 + r_{\max}) \tau^2 \|\partial_{ttt} \mathbf{u}\|_{L^\infty(t_{j-1}, t_j; L^2)} \|e_{\mathbf{u}}^k\|_{-1} \\ & \leq \frac{1}{8} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 + \frac{32\mathbf{m}_3(1+r_{\max})^2 C_{neg}^2}{\mathbf{m}_1} \tau^4 \sum_{k=2}^n \tau_k \|\partial_{ttt} \mathbf{u}\|_{L^\infty(t_{k-1}, t_k; L^2)}^2, \end{aligned}$$

and

$$-2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} (D_2 \eta(t_j), A_h^{-1} e_{\mathbf{u}}^k) \leq \frac{1}{8} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 + \frac{72C_{st}^2 C_{neg}^2 h^6 \mathbf{m}_3}{\mathbf{m}_1} \sum_{k=2}^n \tau_k \|\partial_t \mathbf{u}\|_{L^\infty(t_{k-2}, t_k; H^3)}^2.$$

Next, let us denote $\tilde{\mathbb{L}}_i$ ($i = 1, \dots, 6$) as the corresponding term $\mathbb{L}_i$ with the test function $A_h^{-1} e_{\mathbf{u}}^k$. Then, there hold

$$\begin{cases} \tilde{\mathbb{L}}_1 \leq C_{tr} \tau^2 \frac{1 + 5r_{\max}}{2} \|\partial_{tt} \mathbf{u}\|_{L^\infty(t_{j-2}, t_j; L^2)} \|\nabla \mathbf{u}(t_j)\|_{L^3} \|e_{\mathbf{u}}^k\|_{-1}, \\ \tilde{\mathbb{L}}_2 \leq C_{tr} \tau^2 \frac{1 + 5r_{\max}}{2} \|\partial_{tt} \mathbf{u}\|_{L^\infty(t_{j-2}, t_j; L^2)} \|\nabla \tilde{\mathbf{u}}(t_j)\|_{L^3} \|e_{\mathbf{u}}^k\|_{-1}, \\ \tilde{\mathbb{L}}_3 \leq C_{tr} \|\tilde{\eta}(t_j)\| \|\nabla \tilde{\mathbf{u}}(t_j)\|_{L^3} \|e_{\mathbf{u}}^k\|_{-1} \\ \quad \leq C_{tr} C_{st} h^3 \|\tilde{\mathbf{u}}(t_j)\|_3 \|\nabla \tilde{\mathbf{u}}(t_j)\|_{L^3} \|e_{\mathbf{u}}^k\|_{-1}, \\ \tilde{\mathbb{L}}_4 \leq C_{tr} C_{st}^2 h^3 \|\nabla \tilde{\mathbf{u}}(t_j)\|_{L^3} \|\tilde{\mathbf{u}}(t_j)\|_3 \|e_{\mathbf{u}}^k\|_{-1}, \\ \tilde{\mathbb{L}}_5 \leq C_{tr} C_{st} \|\nabla \tilde{\mathbf{u}}(t_j)\|_{L^3} \|\tilde{e}_{\mathbf{u}}^j\|_{-1} \|e_{\mathbf{u}}^k\|, \\ \tilde{\mathbb{L}}_6 \leq C_{tr} (\|\nabla \tilde{e}_{\mathbf{u}}^j\|_{L^3} + C_{st} \|\nabla \tilde{\mathbf{u}}^j\|_{L^3}) \|\tilde{e}_{\mathbf{u}}^j\|_{-1} \|e_{\mathbf{u}}^k\|. \end{cases}$$

Therefore, we have

$$\begin{aligned}
& -2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} \left(c(\mathbf{u}(t_j), A_h^{-1} \mathbf{u}(t_j), e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, A_h^{-1} e_{\mathbf{u}}^k) \right) \\
& \leq \frac{1}{8} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 + \frac{32\mathbf{m}_3 C_{\text{tr}}^2 (1+5r_{\max})^2 \tau^4}{\mathbf{m}_1} t_n (1+r_{\max})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0,t_n;L^2)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\
& \quad + \frac{1024\mathbf{m}_3 C_{\text{tr}}^2 (1+C_{\text{tr}}^2)(1+r_{\max})^4 h^6}{\mathbf{m}_1} t_n \|\mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\
& \quad + \frac{\nu \mathbf{m}_1}{4\mathbf{m}_2} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 + \frac{256\mathbf{m}_2 \mathbf{m}_3 C_{\text{tr}}^2 C_{\text{st}}^2 (1+r_{\max})^4 r_{\max}}{\nu \mathbf{m}_1^2} \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \sum_{k=1}^{n-1} \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 \\
& \quad + \frac{56\mathbf{m}_2 \mathbf{m}_3 C_{\text{tr}}^2 C_{\text{inv}}^2 (1+r_{\max})^4 r_{\max}}{\nu \mathbf{m}_1^2} \sum_{k=1}^{n-1} \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2.
\end{aligned}$$

Combining all the above estimates and the second conclusion of (4.4), it yields

$$\begin{aligned}
& 1/2 \|e_{\mathbf{u}}^n\|_{-1}^2 - \|e_{\mathbf{u}}^1\|_{-1}^2 + \frac{3\nu \mathbf{m}_1}{4\mathbf{m}_2} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 \\
& \leq (C_1 + 1/2) \sum_{k=2}^{n-1} \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 + \frac{8\tau}{3\tau_1} \mathcal{B}_2(\tau^4 + h^6) + \frac{32\mathbf{m}_3 (1+r_{\max})^2 C_{\text{neg}}^2}{\mathbf{m}_1} \tau^4 t_n \|\partial_{ttt} \mathbf{u}\|_{L^\infty(0,t_n;L^2)}^2 \\
& \quad + \frac{72C_{\text{st}}^2 C_{\text{neg}}^2 h^6 \mathbf{m}_3}{\mathbf{m}_1} t_n \|\partial_t \mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \\
& \quad + \frac{32\mathbf{m}_3 C_{\text{tr}}^2 (1+5r_{\max})^2 \tau^4}{\mathbf{m}_1} t_n (1+r_{\max})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0,t_n;L^2)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\
& \quad + \frac{1024\mathbf{m}_3 C_{\text{tr}}^2 (1+C_{\text{tr}}^2)(1+r_{\max})^4 h^6}{\mathbf{m}_1} t_n \|\mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2.
\end{aligned}$$

Thus, by Grönwall's inequality and (4.2), we have

$$\begin{aligned}
\|e_{\mathbf{u}}^n\|_{-1}^2 + \frac{\nu \mathbf{m}_1}{\mathbf{m}_2} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|_{-1}^2 & \leq \exp((2C_1 + 1)t_{n-1}) (2\|e_{\mathbf{u}}^1\|_{-1}^2 + 2\mathcal{B}_4(t_n)(\tau^4 + h^6)) \\
& \leq 2 \exp((2C_1 + 1)t_{n-1}) (C_{\text{neg}}^2 \mathcal{B}_1 + \mathcal{B}_4(t_n)) (\tau^4 + h^6).
\end{aligned}$$

Therefore, the conclusion (6.1) is proved via the triangle inequality. $\square$

Next, we have the following optimal $\ell^\infty(0, t_n; L^2)$ error estimate on the velocity.

Theorem 6.2. Assume that (4.1) holds. If $h \leq h_1$ and $\tau \leq \min\{h, \tau^*, 1\}$, then

$$\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|^2 \leq 2 \exp(t_n) \mathcal{B}_5(t_n) (\tau^4 + h^6) + C_{\text{st}}^2 h^6 \|\mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2. \quad (6.3)$$

Proof. We will show (6.3) by modifying the proof of Theorem 4.3. Specifically, we just need to modify Eq. (4.12) and the nonlinear part. That is

$$\begin{aligned}
-2 \sum_{k=2}^n \left(\beta_{k-2}^{(k)} b_1^{(2)} \nabla_\tau e_{\mathbf{u}}^1, e_{\mathbf{u}}^k \right) & \leq \sum_{k=2}^n \frac{\tau_k}{2^{k-2}} \|\partial_\tau e_{\mathbf{u}}^1\|_{-1} \|\nabla e_{\mathbf{u}}^k\| \\
& \leq \frac{\nu \mathbf{m}_1}{8\mathbf{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 + \frac{\mathbf{m}_2}{\nu \mathbf{m}_1} \|\partial_\tau e_{\mathbf{u}}^1\|_{-1}^2 \sum_{k=2}^n \frac{2\tau_k}{4^{k-2}} \\
& \leq \frac{\nu \mathbf{m}_1}{8\mathbf{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 + \frac{8\mathbf{m}_2}{3\nu \mathbf{m}_1} \frac{\tau}{\tau_1} \tau_1 \|\partial_\tau e_{\mathbf{u}}^1\|_{-1}^2.
\end{aligned}$$

and

$$\begin{aligned}
c(\mathbf{u}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, e_{\mathbf{u}}^k) & = c(\mathbf{u}(t_j) - \tilde{\mathbf{u}}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) + c(\tilde{\mathbf{u}}(t_j), \mathbf{u}(t_j) - \tilde{\mathbf{u}}(t_j), e_{\mathbf{u}}^k) \\
& \quad + c(\tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j, \tilde{\mathbf{u}}(t_j), e_{\mathbf{u}}^k) + c(\tilde{\mathbf{u}}_h^j - \tilde{\mathbf{u}}(t_j), \tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j, e_{\mathbf{u}}^k) \\
& \quad + c(\tilde{\mathbf{u}}(t_j), \tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j, e_{\mathbf{u}}^k),
\end{aligned}$$

where the first two terms are $\mathbb{L}_1$ and $\mathbb{L}_2$, so we will not estimate them here. As for the other three terms, they can be estimated as

$$\begin{aligned} c\left(\tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j, \tilde{\mathbf{u}}(t_j), e_{\mathbf{u}}^k\right) &\leq C_{\text{tr}} \|\tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j\| \|\tilde{\mathbf{u}}(t_j)\|_2 \|\nabla e_{\mathbf{u}}^k\|, \\ c\left(\tilde{\mathbf{u}}_h^j - \tilde{\mathbf{u}}(t_j), \tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j, e_{\mathbf{u}}^k\right) &\leq C_{\text{tr}} C_{\text{inv}} h^{-1} \|\tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j\| \|\nabla(\tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j)\| \|\nabla e_{\mathbf{u}}^k\|, \\ c\left(\tilde{\mathbf{u}}(t_j), \tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j, e_{\mathbf{u}}^k\right) &\leq C_{\text{tr}} \|\tilde{\mathbf{u}}(t_j) - \tilde{\mathbf{u}}_h^j\| \|\tilde{\mathbf{u}}(t_j)\|_2 \|\nabla e_{\mathbf{u}}^k\|. \end{aligned}$$

Hence, (4.15) can be updated as

$$\begin{aligned} &-2 \sum_{k,j}^{n,k} \beta_{k-j}^{(k)} \left(c(\mathbf{u}(t_j), \mathbf{u}(t_j), e_{\mathbf{u}}^k) - c(\tilde{\mathbf{u}}_h^j, \mathbf{u}_h^j, e_{\mathbf{u}}^k) \right) \\ &\leq \frac{1}{8} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|^2 + \frac{16\mathfrak{m}_3 C_{\text{tr}}^2 (1+5r_{\max})^2 \tau^4}{\mathfrak{m}_1} t_n (1+r_{\max})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0,t_n;H^1)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\ &\quad + \frac{\nu \mathfrak{m}_1}{4\mathfrak{m}_2} \sum_{k=2}^n \tau_k \|\nabla e_{\mathbf{u}}^k\|^2 + \frac{96\mathfrak{m}_2 \mathfrak{m}_3 C_{\text{tr}}^2 r_{\max} (1+r_{\max})^2}{\nu \mathfrak{m}_1^2} \|\mathbf{u}\|_{L^\infty(0,t_n;H^2)}^2 \sum_{k=1}^{n-1} \tau_k \|\mathbf{u}(t_k) - \mathbf{u}_h^k\|^2 \\ &\quad + \frac{48\mathfrak{m}_2 \mathfrak{m}_3 C_{\text{tr}}^2 C_{\text{inv}}^2 (1+r_{\max})^4 r_{\max}}{\nu \mathfrak{m}_1^2} h^{-2} \sum_{k=1}^{n-1} \tau_k \|\mathbf{u}(t_k) - \mathbf{u}_h^k\|^2 \|\nabla(\mathbf{u}(t_k) - \mathbf{u}_h^k)\|^2. \end{aligned}$$

Due to Theorem 4.3, we have

$$\|\mathbf{u}(t_k) - \mathbf{u}_h^k\|^2 \leq 2\|\eta(t_k)\|^2 + 2\|e_{\mathbf{u}}^k\|^2 \leq 2C_{\text{st}}^2 h^6 \|\mathbf{u}(t_k)\|_3^2 + 4 \exp((2\mathcal{C}_1 + 1)t_{n-1}) (\mathcal{B}_1 + 2\mathcal{B}_3(t_n)) (h^4 + \tau^4),$$

then,

$$h^{-2} \|\mathbf{u}(t_k) - \mathbf{u}_h^k\|^2 \leq 2(2C_{\text{st}}^2 \|\mathbf{u}(t)\|_{L^\infty(0,t_n;H^3)}^2 + 4 \exp((2\mathcal{C}_1 + 1)t_{n-1}) (\mathcal{B}_1 + 2\mathcal{B}_3(t_n))) h^2 =: \mathcal{C}_2 h^2.$$

Hence, we have the following estimate

$$\begin{aligned} \|e_{\mathbf{u}}^n\|^2 - \|e_{\mathbf{u}}^1\|^2 &\leq \frac{3}{8} \sum_{k=2}^n \tau_k \|e_{\mathbf{u}}^k\|^2 + \frac{8\mathfrak{m}_2}{3\nu \mathfrak{m}_1} \frac{\tau}{\tau_1} \mathcal{B}_2^2(\tau^4 + h^6) + \frac{32\mathfrak{m}_3 (1+r_{\max})^2}{\mathfrak{m}_1} \tau^4 t_n \|\partial_{ttt} \mathbf{u}\|_{L^\infty(0,t_n;L^2)}^2 \\ &\quad + \frac{72C_{\text{st}}^2 h^6 \mathfrak{m}_3}{\mathfrak{m}_1} t_n \|\partial_t \mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \\ &\quad + \frac{16\mathfrak{m}_3 C_{\text{tr}}^2 (1+5r_{\max})^2 \tau^4}{\mathfrak{m}_1} t_n (1+r_{\max})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0,t_n;H^1)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\ &\quad + \frac{96\mathfrak{m}_2^2 \mathfrak{m}_3 C_{\text{tr}}^2 r_{\max} (1+r_{\max})^2}{\nu^2 \mathfrak{m}_1^3} \|\mathbf{u}\|_{L^\infty(0,t_n;H^2)}^2 \frac{\nu \mathfrak{m}_1}{\mathfrak{m}_2} \sum_{k=1}^{n-1} \tau_k \|\mathbf{u}(t_k) - \mathbf{u}_h^k\|^2 \\ &\quad + \frac{48\mathfrak{m}_2^2 \mathfrak{m}_3 C_{\text{tr}}^2 C_{\text{inv}}^2 (1+r_{\max})^4 r_{\max}}{\nu^2 \mathfrak{m}_1^3} \mathcal{C}_2 h^2 \frac{\nu \mathfrak{m}_1}{\mathfrak{m}_2} \sum_{k=1}^{n-1} \tau_k \|\nabla(\mathbf{u}(t_k) - \mathbf{u}_h^k)\|^2. \end{aligned}$$

Define

$$\begin{aligned} \mathcal{B}_5(t_n) &:= \mathcal{B}_1 + \frac{8\mathfrak{m}_2}{3\nu \mathfrak{m}_1} \frac{\tau}{\tau_1} \mathcal{B}_2^2 + \frac{32\mathfrak{m}_3 (1+r_{\max})^2}{\mathfrak{m}_1} \tau^4 \|\partial_{ttt} \mathbf{u}\|_{L^\infty(0,t_n;L^2)}^2 + \frac{72C_{\text{st}}^2 \mathfrak{m}_3}{\mathfrak{m}_1} t_n \|\partial_t \mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \\ &\quad + \frac{16\mathfrak{m}_3 C_{\text{tr}}^2 (1+5r_{\max})^2}{\mathfrak{m}_1} t_n (1+r_{\max})^2 \|\partial_{tt} \mathbf{u}\|_{L^\infty(0,t_n;H^1)}^2 \|\nabla \mathbf{u}\|_{L^\infty(0,t_n;L^3)}^2 \\ &\quad + \frac{96\mathfrak{m}_2^2 \mathfrak{m}_3 C_{\text{tr}}^2 r_{\max} (1+r_{\max})^2}{\nu^2 \mathfrak{m}_1^3} \|\mathbf{u}\|_{L^\infty(0,t_n;H^2)}^2 2 \exp((2\mathcal{C}_1 + 1)t_{n-1}) (\mathcal{C}_{\text{neg}}^2 \mathcal{B}_1 + \mathcal{B}_4(t_n)) \\ &\quad + \frac{96\mathfrak{m}_2^2 \mathfrak{m}_3 C_{\text{tr}}^2 r_{\max} (1+r_{\max})^2}{\nu^2 \mathfrak{m}_1^3} \|\mathbf{u}\|_{L^\infty(0,t_n;H^2)}^2 \frac{\nu \mathfrak{m}_1 C_{\text{st}}^2 t_n}{\mathfrak{m}_2} \|\mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2 \\ &\quad + \frac{48\mathfrak{m}_2^2 \mathfrak{m}_3 C_{\text{tr}}^2 C_{\text{inv}}^2 (1+r_{\max})^4 r_{\max}}{\nu^2 \mathfrak{m}_1^3} \mathcal{C}_2 2 \exp((2\mathcal{C}_1 + 1)t_{n-1}) (\mathcal{B}_1 + 2\mathcal{B}_3(t_n)) \\ &\quad + \frac{48\mathfrak{m}_2 \mathfrak{m}_3 C_{\text{tr}}^2 C_{\text{inv}}^2 (1+r_{\max})^4 r_{\max}}{\nu \mathfrak{m}_1^2} \mathcal{C}_2 t_n C_{\text{st}}^2 \|\mathbf{u}\|_{L^\infty(0,t_n;H^3)}^2. \end{aligned}$$

By Theorem 4.3, Lemma 6.1, the Gröwall inequality (cf. Lemma 3.3) and triangle inequality, we can eventually obtain the conclusion. $\square$

7. Numerical experiments

In this section, we present several numerical experiments to verify the optimal-order error estimates for the proposed variable-step IMEX-BDF2-FEM scheme (3.4) with variable time steps. We further examine mesh robustness and evaluate the effectiveness of the adaptive time-stepping strategy. All computations are performed with FreeFEM++ [17], using the $P2 - P1$ Taylor–Hood elements (quadratic for velocity, linear for pressure) for spatial discretization.

7.1. Accuracy test

To verify the numerical accuracy of the IMEX-BDF2 scheme (3.4), we solve the two-dimensional lattice-vortex problem [7] with zero body force. The problem admits the following exact solution on the unit square $\Omega = (0, 1)^2$:

$$\begin{aligned} \mathbf{u}(x, y, t) &= \begin{pmatrix} \sin(2\pi x) \sin(2\pi y) \\ \cos(2\pi x) \cos(2\pi y) \end{pmatrix} \exp(-8\nu\pi^2 t), \\ p(x, y, t) &= -\frac{1}{2} \left(\sin^2(2\pi x) + \cos^2(2\pi y) \right) \exp(-16\nu\pi^2 t). \end{aligned}$$

The Dirichlet boundary conditions and the initial data are determined by the exact solution, and the simulation proceeds up to a final time $T = 0.25$.

We investigate spatial and temporal convergence using two distinct classes of time grids, with the kinematic viscosity set to $\nu = 0.1$:

- (i) **Graded grids.** The time nodes are distributed according to $t_k = T(k/N)^\gamma$ with $\gamma \geq 1$. Let $\tau_k = t_k - t_{k-1}$ and $r_k = \tau_k/\tau_{k-1}$. The maximum step ratio then satisfies $\max_k r_k = 2^\gamma - 1$.
- (ii) **Random grids.** The time steps are defined by $\tau_k = \frac{T\epsilon_k}{\sum_{j=1}^N \epsilon_j}$, $1 \leq k \leq N$, where ϵ_k are independent and identically distributed random variables uniformly distributed in $(0, 1)$.

Table 1: Temporal accuracy of the IMEX-BDF2 scheme on graded time meshes with $\gamma = 3$ and $N_x = 160$

N	$\ e_u\ $	Rate	$\ e_u\ _{H^1}$	Rate	$\ e_p\ $	Rate
10	5.38e-03	–	5.67e-02	–	1.13e-02	–
20	1.43e-03	1.92	1.49e-02	1.93	2.83e-03	2.00
40	3.61e-04	1.98	3.77e-03	1.99	6.98e-04	2.02
80	9.04e-05	2.00	9.51e-04	1.99	1.72e-04	2.02

Table 2: Temporal accuracy of the IMEX-BDF2 scheme on random time meshes with $N_x = 160$

N	$\ e_u\ $	Rate	$\ e_u\ _{H^1}$	Rate	$\ e_p\ $	Rate
10	2.37e-03	–	2.32e-02	–	2.23e-03	–
20	4.84e-04	2.29	4.89e-03	2.25	5.06e-04	2.14
40	1.28e-04	1.92	1.32e-03	1.89	1.86e-04	1.44
80	2.84e-05	2.17	3.15e-04	2.06	4.89e-05	1.93

The L^2 - and H^1 -norm errors for velocity, along with the L^2 -norm errors for pressure, and their corresponding convergence rates, are summarized in Tables 1–4. Using a fixed spatial mesh with $N_x = 160$ and $h = 1/N_x$, both the graded grid with $\gamma = 2$ and the random grid yields the expected second-order temporal convergence for both velocity and pressure, which is illustrated in Tables 1–2. Notably, for $\gamma = 3$, the maximum step ratio reaches $\max_k r_k = 7$, which exceeds the theoretical bound of ≈ 4.864 (Condition **S0**). Despite this, the scheme maintains the expected accuracy, suggesting that the **S0** condition is just sufficient. For a fixed number of time steps $N = 1000$, the expected optimal spatial convergence rates are observed for both grid families in Tables 3–4: the velocity exhibits third-order convergence in the L^2 norm and second-order convergence in the H^1 norm, and the pressure exhibits second-order convergence in the L^2 norm.

Table 3: Spatial accuracy of the IMEX-BDF2 scheme on graded time meshes with $\gamma = 2$ and $N = 1000$

N_x	$\ e_u\ $	Rate	$\ e_u\ _{H^1}$	Rate	$\ e_p\ $	Rate
10	5.07e-04	—	3.31e-02	—	6.39e-04	—
20	5.77e-05	3.14	8.43e-03	1.97	1.56e-04	2.04
40	7.04e-06	3.03	2.12e-03	1.99	3.93e-05	1.98
80	9.35e-07	2.91	5.31e-04	2.00	9.82e-06	2.00

Table 4: Spatial accuracy of the IMEX-BDF2 scheme on random time meshes with $N = 1000$

N_x	$\ e_u\ $	Rate	$\ e_u\ _{H^1}$	Rate	$\ e_p\ $	Rate
10	5.07e-04	—	3.31e-02	—	6.39e-04	—
20	5.77e-05	3.14	8.43e-03	1.97	1.56e-04	2.04
40	7.02e-06	3.04	2.12e-03	1.99	3.94e-05	1.98
80	9.00e-07	2.96	5.31e-04	2.00	9.88e-06	2.00

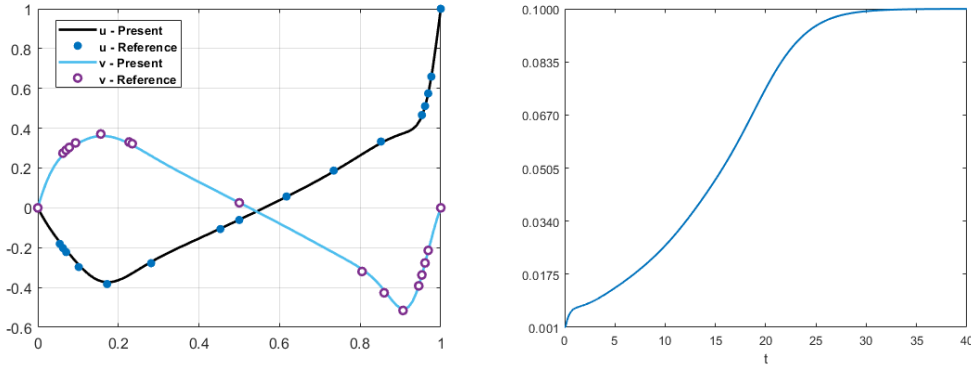
7.2. The lid-driven cavity flow problem

We consider the classical lid-driven cavity flow problem in a unit square domain $\Omega = [0, 1]^2$. The flow is initialized from a state of rest with $\mathbf{u}_0 = \mathbf{0}$. Dirichlet boundary conditions are imposed as $\mathbf{u} = (1, 0)$ on the top boundary ($y = 1$) and $\mathbf{u} = \mathbf{0}$ on the remaining three walls (see, e.g., [12, 21, 25]).

To enhance computational efficiency, we incorporate the adaptive time-stepping strategy proposed in [20] into the IMEX-BDF2 scheme. The subsequent time-step size τ_{n+1} is governed by the following update rule:

$$\tau_{\text{ada}} = \max \left\{ \tau_{\min}, \frac{\tau_{\max}}{\sqrt{1 + \alpha \|\nabla \mathbf{u}_h^n\|^2 \|\partial_t \mathbf{u}_h^n\|^2}} \right\}, \quad \tau_{n+1} = \min \{ \tau_{\text{ada}}, r_{\text{user}} \tau_n \}, \quad (7.1)$$

where $\tau_{\max}$ and $\tau_{\min}$ represent the prescribed maximum and minimum time-step sizes, respectively; r_{user} denotes the maximum allowable step-size ratio; and $\alpha > 0$ is a sensitivity parameter controlling the degree of adaptivity. For the simulations presented herein, we set $\tau_{\min} = 10^{-3}$, $\tau_{\max} = 10^{-1}$, $\alpha = 10^3$, and $r_{\text{user}} = 4$.

Figure 1: Profiles of the u -velocity along $x = 0.5$ and of the v -velocity along $y = 0.5$ (left), and evolution of the adaptive time-step size as a function of time (right), for the lid-driven cavity flow at $Re = 10^3$.

Simulations are performed at a Reynolds number $Re = 10^3$ up to the final time $T = 40$, by which point the flow has reached a steady state. The spatial domain is discretized using a uniform triangular mesh generated by 128 subdivisions along each coordinate direction. The profiles of the u -velocity along $x = 0.5$ and of the v -velocity along $y = 0.5$ are plotted in Fig. 1(a), where the benchmark data of Ghia et al. [12] are included for comparison. The two sets of results agree very

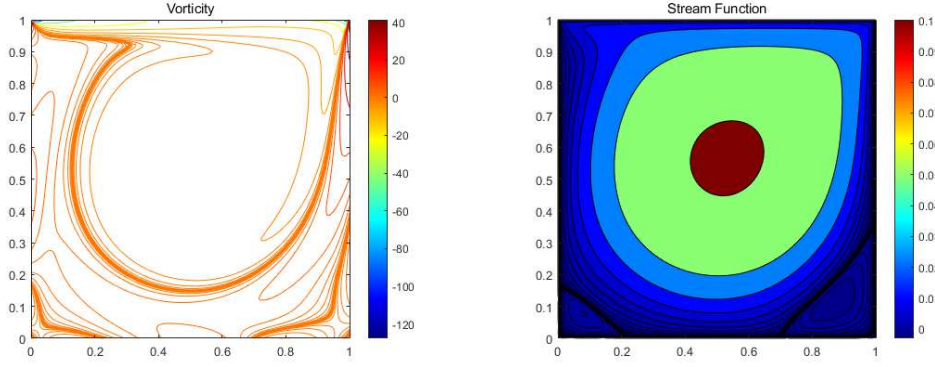


Figure 2: Vorticity and stream function of the lid-driven cavity flow with $Re = 10^3$ at $T = 40$.

well. The evolution of the adaptive time-step size is shown in Fig. 1(b). At early times, relatively small time steps are selected in order to resolve the rapid variation of the velocity field, whereas, as the flow approaches its steady state, the time-step size quickly increases and remains close to $\tau_{\max}$. In contrast to the adaptive strategy of [21], the rule (7.1) updates the time-step size at each time level as a smooth function of the solution norms, so that the sequence $\{\tau_k\}$ varies gradually and the profile of $\tau(t)$ is nearly continuous rather than staircase-like.

The vorticity and stream-function contours of the lid-driven cavity flow are displayed in Fig. 2. The flow exhibits a large primary vortex occupying most of the cavity, together with two secondary vortices located in the lower corners. This vortex structure is consistent with the reference solutions reported in [12, 25].

8. Conclusions

This work has established optimal L^2 error estimates for the velocity and pressure of a variable-step Variable-step IMEX-BDF2-FEM scheme for the incompressible Navier–Stokes equations. By combining the discrete orthogonal convolution (DOC) kernel technique with negative-norm estimates, we have derived error bounds that explicitly capture Reynolds number dependencies. This work also addresses the initialization challenge inherent to two-step BDF schemes by using a first step based on the Crank–Nicolson method; the analysis shows that this initialization preserves an overall second-order temporal accuracy while maintaining variable time steps. Future work will extend this framework in two natural directions: first, applying Aubin–Nitsche duality arguments to establish long-time stability estimates—particularly relevant for high Reynolds number flows—and second, developing analogous results for coupled multiphysics systems where the Navier–Stokes equations interact with other physical processes, such as heat transfer, phase change, or electromagnetism.

The current Reynolds-number-dependent estimates provide a foundation for these planned extensions, while the DOC methodology offers promising adaptability to more complex physical scenarios. Additional developments will examine adaptive strategies combining variable time-stepping with spatial refinement, particularly for multiscale multiphysics applications where different physical processes exhibit competing characteristic time scales. Such extensions would significantly enhance the scheme’s utility in real-world engineering and scientific simulations requiring both accuracy and computational efficiency.

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