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THEORETICAL AND EXPERIMENTAL STUDIES OF THE BUCKLING MODES AND FAILURE OF TEST SPECIMENS MADE OF LAYERED COMPOSITES UNDER THREE-POINT BENDING

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ABSTRACT

Approximate analytical solutions are constructed for linearized problems of transverse-shear buckling modes of fiber reinforced composite test specimens under the three-point bending tests. The corresponding problems are formulated based on three versions of a refined geometrically nonlinear theory of plane bending of composite rods, previously constructed by taking into account transverse shear and compression at different orders of displacement approximations through the rod thickness. The first two versions of the theory are based on approximations of axial displacements by third-degree polynomials through the thickness, with the number of unknown functions reduced by one order in the first one by introducing transverse shear functions according to the Reddy-Nemirovsky model. The third version is based on a linear approximation of axial displacement. In all three versions of the theory, a linear approximation through the thickness is adopted for deflection, with the third version of the theory fully consistent with the well-known shear S.P. Timoshenko's model. The model was constructed taking into account transverse compression. In all of the above equations, geometrically nonlinear terms are introduced that allow the theoretical identification of the investigated buckling modes of composite test specimens of the considered class under the bending loading. It is shown that their failure is entirely due to the implementation of such a shear buckling mode, which is accompanied by delamination in the longitudinal direction of elongated specimens and shear failure in the central section of short specimens. For their theoretical identification, deformation models must be constructed with mandatory consideration of transverse compression, preserving the first-order internal moments of transverse shear stresses in the equations, as well as taking into account the geometrically nonlinear deformation process based on the use of elasticity theory relations in a non-classical (non-standard) geometrically nonlinear approximation. An analysis of the degree of accuracy and meaningfulness of the employed equations is performed. Comparison of the obtained theoretical results in the form of simple analytical formulas with the data of previously conducted experiments on three-point bending of short and elongated test specimens is given.

Keywords: fiber reinforced plastic, test specimen, three-point bending, mechanical properties, geometrical and physical nonlinearity, refined linearized equations, stability, analytical solutions, experimental results.

1. Introduction

A significant number of scientific papers have been devoted to theoretical and experimental studies related to three-point bending tests of special test specimens made of composite materials with a rectangular cross-section. [1–29]. Such tests are performed on short and elongated test specimens of layered fiber composites, which are regulated by the current composite testing standards [30–35]. Their purpose is to determine the strength limits and elastic modulus of the first

kind when bending elongated test specimens and the shear modulus and shear strength when bending short test specimens. A review of the current state of the problem and an analysis of the results obtained in articles [1-29] on the issues discussed in this article were presented in [36]. The main focus was on the formulation and development of numerical methods for solving linearized and geometrically and physically nonlinear problems related to the modeling of three-point bending tests of short and elongated test specimens made of layered fiber composites. It also provides an analysis of the test results for carbon fiber made from HSE 180 REM prepreg based on carbon tape with straight fibers. It was shown that failure of such composites during three-point bending tests occurs not due to compressive stresses reaching the compressive strength limit (for elongated test specimens) and transverse tangential stresses reaching the shear strength limit (for short test specimens), but due to the implementation of a non-classical, almost purely transverse-shear buckling mode of test specimens under transverse bending conditions.

It should be noted that the non-classical buckling mode of composite test specimens under three-point bending conditions was theoretically identified in [36] by using refined geometrically nonlinear equations [37] based on the shear model of S.P. Timoshenko, taking into account transverse compression. When setting up the problem in [36, 37], all geometrically nonlinear terms that appear when reducing three-dimensional geometrically nonlinear equations of elasticity theory, compiled in [38, 39] in a consistent quadratic approximation, to one-dimensional equations of the theory of rods, were preserved. However, as it was established in [36], the preservation of not all geometrically nonlinear terms of the equations [37] is due to the possibility of identifying on their basis the experimentally realized transverse-shear buckling mode. In addition, it should be pointed out that within the framework of using the above-mentioned refined shear model of S.P. Timoshenko in [36, 37], the description of the resulting transverse shear stresses along the thickness is not accurate enough. In this regard, in the works [40, 41], based on known approaches [42–49, etc.] and in the development of previously obtained results, two variants of refined mathematical models were proposed to describe the process of nonlinear cylindrical deformation of plates made of fiber composites. They are mainly designed to identify and study both classical (bending, flexural-shear) and non-classical (transverse-shear) buckling mode of composite plates under various types of loading. The first of them [40] develops models of the Reddy-Nemirovsky type [43, 48, etc.] and is based on the use of approximations in plate thickness by the linear law for deflection and the cubic law for tangential displacements when boundary conditions for tangential forces are satisfied at the points of the boundary (face) planes. In the second variant [41], in contrast to the first, the corresponding expansion coefficients are considered independent in the third-degree polynomial for tangential displacements in the transverse coordinate. Kinematic relations for determining the components of deformations are constructed on the basis of the previously proposed consistent

geometrically nonlinear relations of the theory of elasticity [38,39] while preserving the terms that make it possible to identify the classical and non-classical (in particular, transverse shear under conditions of transverse bending) described above buckling mode of plates. The physically nonlinear behavior of the material is taken into account in the equations for transverse tangential stresses [36,51]. The article discusses the correctness of the equations used to formulate the conditions for fixing and loading the plate's end sections. It is shown that in solutions determined based on the equations of the first variant, when the boundary conditions of the cross-section are satisfied, the solution is transformed to the classical Kirchhoff-Love model. In contrast, solutions based on the equations of the second variant of the theory are converted to solutions based on the S.P. Timoshenko model with a constant transverse tangential stress along the thickness. This conclusion is based on an analysis of the exact analytical solutions of the linear problem of transverse bending of a cantilevered plate in a cylindrical shape.

The purpose of this work is to construct approximate analytical solutions to the problem of transverse-shear buckling mode of composite plates during their bending in a cylindrical shape, with their applications to the problem of modeling the testing of test specimens made of fiber composites in three-point bending, as well as to analyze the accuracy and meaningfulness of the equations constructed in [40,41] in comparison with the equations constructed based on the shear model of S.P. Timoshenko, taking into account transverse compression.

2. Three variants of geometrically nonlinear one-dimensional equations

Let us consider test specimen with rectangular cross-section made of a unidirectional fiber reinforced pastic, which is supported by end cylinders and subjected to three-point bending by loading it through a rigid cylinder in the center of the specimen (Fig. 1).

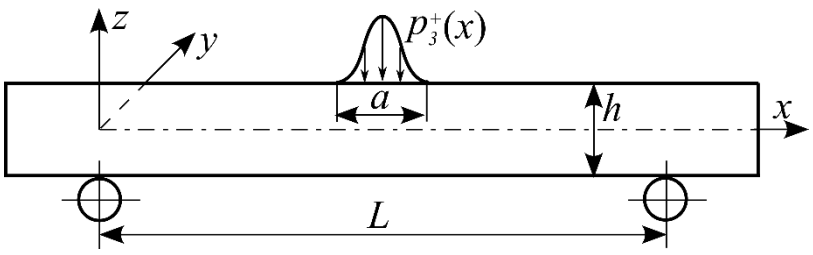


Fig. 1. Coordinate system and loading diagram for the test specimen

With such loading and fixing of the specimen, as shown by the results of the conducted studies [36, 50], for long specimens with their flat transverse bending the dependence between the applied force *P* and the deflection of the specimen is linear and remains almost up to the beginning of the process of its failure. In contrast, for short specimen this dependence is nonlinear. According to the results of the work [36], a correct and sufficiently accurate study of the deformation processes of such

specimens can be carried out in a plane formulation of the problem using the kinematic relations of the nonlinear theory of elasticity, which are based on the approximation

$$\eta_1 = U_{,x} + W_{,x}^2/2,\ \eta_2 = W_{,z} + U_{,z}^2/2, \eta_{13} = W_{,x} + U_{,z} \tag{1}$$

where U, W – components of the displacement vector $\mathbf{U}$ of arbitrary point $M(x,z)$ in the selected coordinate system.

Based on (1), one-dimensional equations for studying the deformation process of a specimen can be constructed with varying degrees of accuracy and content.

2.1. The refined model of S.P. Timoshenko. In studies [36, 37], the description of the deformation process of the specimen was based on the representation of the displacement vector according to the S. P. Timoshenko model (e_1, e_3 – unit vectors)

$$\mathbf{U} = U\mathbf{e}_1 + W\mathbf{e}_3 = (u_1 + z\gamma_1)\mathbf{e}_1 + (w + z\varphi)\mathbf{e}_3, \quad -h/2 \le z \le h/2 \tag{2}$$

In which u_1, w – displacements of the axial line points in the direction of the axes x, z; γ_1 – angle of rotation of the cross-section $x = const$ around the axis y; φ – transverse compression function in the z - direction.

In accordance with (1) and (2), deformation components η_1, η_2, η_{13} will be further determined by one-dimensional relations drawn up in the approximation

$$\eta_1 = \varepsilon_1 + z\text{æ}_1,\ \eta_3 = \varphi + \underline{\gamma_1^2/2},\ \eta_{13} = \gamma_{13} + z\text{æ}_{13} \tag{3}$$

where

$$\varepsilon_1 = u_{,x} + \underline{w_{,x}^2/2},\ \gamma_{13} = w_{,x} + \gamma_1,\ \text{æ}_1 = \gamma_{1,x} + \underline{w_{,x}\varphi_{,x}},\ \text{æ}_{13} = \varphi_{,x} \tag{4}$$

The stresses forming on the $x = const$ and $z = const$ areas in the deformed state of the specimen are denoted by $\sigma_{11}, \sigma_{33}, \sigma_{13}$. In accordance with (3), we introduce into consideration the forces and moments reduced to the middle plane $z = 0$, which are calculated using the formulas

$$\begin{aligned} T_{11} &= \int_{-h/2}^{h/2} \sigma_{11} dz, \quad T_{13} = \int_{-h/2}^{h/2} \sigma_{13} dz, \quad T_{33} = \int_{-h}^{h/2} \sigma_{33} dz \\ M_{11} &= \int_{-h/2}^{h/2} \sigma_{11} z dz, \quad M_{13} = \int_{-h/2}^{h/2} \sigma_{13} z dz \end{aligned} \tag{5}$$

Then, using relations (3) and basing on the Lagrange variational principle, it is possible to obtain a system of four geometrically nonlinear differential equations of equilibrium, formulated in the approximation

$$T_{11,x}=0,\ Q_{1,x}+p_3^-+p_3^+=0,\ M_{11,x}-T_{13}-\underline{T_{33}\gamma_1}=0,$$
$$R_{1,x}-T_{33}+\frac{h}{2}\left(p_3^+-p_3^-\right)=0 \tag{6}$$

where

$$Q_1=T_{13}+\underline{T_{11}w_{,x}}+\underline{M_{11}\varphi_{,x}},\ R_1=M_{13}+\underline{M_{11}w_{,x}}, \tag{7}$$

p_3^+, p_3^- – normal surface forces applied at the points $z=\pm h/2$ of the plate.

2.2. The refined Reddy-Nemirovsky type model (the first version of the refined theory). In [40], a refined Reddy-Nemirovsky type deformation model was proposed [42,43], based on the representation of deflection in the thickness direction z, as in (2), by a linear law

$$W=w+z\varphi, \tag{8}$$

by quadratic approximation of transverse shear stress σ_{13}

$$\sigma_{13}=\left(1-\frac{4z^2}{h^2}\right)\tau_1, \tag{9}$$

and by cubic approximation of displacement U_1

$$U_1=u_1-z\left[\left(1+\varphi\right)w_{,x}-\hat{\tau}_1\right]-\frac{z^2}{2}\left(1+\varphi\right)\varphi_{,x}-\frac{4z^3}{3t^2}\hat{\tau}_1. \tag{10}$$

Note that in (9) τ_1 – transverse shear stress at $z=0$, $\tilde{\tau}_1=\tau_1/G_{13}^0$, $G_{13}^0=G_{13}\left(z=0\right)$, where $G_{13}=G_{13}\left(z\right)$. When using (8) – (10) to determine the components of deformations within the accepted degree of accuracy and content, the following relations take place

$$\eta_1=u_{1,x}+\frac{1}{2}w_{,1}^2-z\left[\left(1+\varphi\right)w_{,xx}-\hat{\tau}_{1,1}\right]-\frac{z^2}{2}\left(1+\varphi\right)\varphi_{,xx}-\frac{4z^3}{3t^2}\hat{\tau}_{1,1},$$
$$\eta_3=\varphi+\frac{1}{2}\underline{\left[\left(1+\varphi\right)w_{,x}-\hat{\tau}_1\right]^2},\ \eta_{13}=\sigma_{13}/G_{13}\left(z\right). \tag{11}$$

In accordance with (11), we introduce the forces and moments reduced to the middle plane $z=0$, calculated using the formulas

$$T_{13}=\int_{-t/2}^{t/2}\sigma_{13}\left(1-\frac{4z^2}{t^2}\right)dz,\quad T_{33}=\int_{-t/2}^{t/2}\sigma_{33}dz,$$
$$T_{11}=\int_{-t/2}^{t/2}\sigma_{11}dz,\quad M_{11}=\int_{-t/2}^{t/2}\sigma_{11}zdz,\quad R_{11}=\frac{1}{2}\int_{-t/2}^{t/2}\sigma_{11}z^2dz, \tag{12}$$
$$N_{11}=\frac{4}{3t^2}\int_{-t/2}^{t/2}\sigma_{11}z^3dz,$$

which allows, based on the Lagrange variational principle, to obtain equilibrium equations

$$T_{11,x}=0,\quad Q_{1,x}+\hat{p}_3^-+\hat{p}_3^+=0,$$
$$R_{1,x}-T_{33}-M_{11,x}w_{,x}-R_{11,x}\varphi_{,x}+\frac{t}{2}\left(\hat{p}_3^+-\hat{p}_3^-\right)=0, \tag{13}$$
$$N_{11,x}-M_{11,x}+T_{13}-T_{33}\gamma_1=0.$$

In them, unlike (7),

$$Q_1=M_{11,x}+\underline{T_{11}w_{,x}}+\underline{M_{11}\varphi_{,x}}+T_{33}\gamma_1,\ R_1=R_{11,x}+\underline{M_{11}w_{,x}}+2R_{11}\varphi_{,x} \tag{14}$$

2.3. The second version of the refined theory. In [41], a second variant of the refined deformation model was also proposed. It is based on the approximation of deflection by the linear law (8), and displacement U_1 – by a polynomial

$$U_1=u_1-z\gamma_1-\frac{z^2}{2}\psi_1-\frac{4z^3}{3t^2}\chi_1, \tag{15}$$

in which the functions $u_1,\gamma_1,\psi_1,\chi_1$, as well as w and φ, are classified as unknowns. At the same time, for the deformation components in the approximation adopted above, we arrive at expressions

$$\varepsilon_1=u_{1,x}+\frac{1}{2}w_{,x}^2-z\left(\gamma_{1,x}-\underline{w_{,x}\varphi_{,x}}\right)-\frac{z^2}{2}\left(\psi_{1,x}-\varphi_{,x}^2\right)-\frac{4z^3}{3t^2}\chi_{1,x},$$
$$2\varepsilon_{13}=w_{,x}-\gamma_1+z\left(\varphi_{,x}-\psi_1\right)-\frac{4z^2}{t^2}\chi_1,\ \varepsilon_{33}=\varphi+\frac{1}{2}\underline{\gamma_1^2}, \tag{16}$$

in which the underlined nonlinear terms have the same significance as in the relationships of refined models of the S.P. Timoshenko type of the first version of the theory.

If, instead of T_{13} according to (12) we introduce into consideration other internal forces and moments

$$T_{13}=\int_{-t/2}^{t/2}\sigma_{13}dz,\ M_{13}=\int_{-t/2}^{t/2}\sigma_{13}zdz,\ R_{13}=\frac{4}{t^2}\int_{-t/2}^{t/2}\sigma_{13}z^2dz, \tag{17}$$

then instead of (13) we can obtain a system of six differential equations of equilibrium

$$T_{11,x}=0,\quad R_{1,x}-T_{33}+\frac{t}{2}\left(p_3^+-p_3^-\right)=0,\ R_{11,x}-M_{13}=0,$$
$$Q_{1,x}+p_3^-+p_3^+=0,\quad f_\gamma=M_{11,x}-T_{13}+T_{33}\gamma_1=0,\ f_\chi=N_{11,x}-R_{13}=0, \tag{18}$$

in which, unlike (14),

$$Q_1=T_{13}+T_{11}w_{,x}+\underline{M_{11}\varphi_{,x}},\quad R_1=M_{13}+\underline{M_{11}w_{,x}}+2R_{11}\varphi_{,x}. \tag{19}$$

Note that in (19) the presence of the underlined terms makes it possible to identify transverse-shear buckling modes under plate bending conditions.

2.4. About the relations of elasticity. Tests of test specimens made of unidirectional fiber reinforced plastic, manufactured in accordance with existing standards, show that the stresses σ_{11} and σ_{33}, together with the corresponding strains η_1 and η_3, can be related up to failure by the relations of the generalized Hooke's law

$$\sigma_{11} = E_1^*\left(\eta_1 + \nu_{31}\eta_3\right),\ \sigma_{33} = E_3^*\left(\eta_3 + \nu_{13}\eta_1\right)$$
$$E_1^* = E_1/\left(1-\nu_{13}\nu_{31}\right),\ E_3^* = E_3/\left(1-\nu_{13}\nu_{31}\right) \tag{20}$$

where $E_1, E_3, \nu_{13}, \nu_{31} = E_3\nu_{13}/E_1$ are the values of the corresponding Young's modulus and Poisson's ratios determined in [50]. To establish the physical relation $\sigma_{13} = \sigma_{13}\left(\eta_{13}\right)$, assuming that for the composite under consideration it practically coincides with the relation $\sigma_{12} = \sigma_{12}\left(\eta_{12}\right)$, dedicated tensile tests are performed on specimens with cross-ply lay-up with a fiber orientation angle of $\alpha = \pm 45^0$ in relation to the tension direction x.

Thus, the following dependence for the tangent transverse shear modulus is obtained:

$$\sigma_{13} = G_0\eta_{13} + \frac{G^{(1)}}{2}\eta_{13}^2 + \frac{G^{(2)}}{3}\eta_{13}^3 \tag{21}$$

The coefficients included in (21) are determined by the formulas given in articles [41,51].

3. Linearized equations of neutral equilibrium and transverse-shear buckling mode of test specimen under bending conditions

Suppose that the composite rod (test specimen) is in transverse bending conditions and has an internal bending moment $M_{11}^0 = M^0$ and a shearing force Q^0, which are predominant in comparison with the others. In particular, such a stress state is formed during testing of composite test specimens for three-point bending, regulated by currently existing composite testing standards. These tests are conducted to determine the ultimate stress and Young's modulus in bending of elongated test specimens and the transverse shear moduli and ultimate shear stress in bending of short test specimens. The results of such tests for HSE 180 REM prepreg carbon fiber reinforced plastic based on unidirectional carbon tape have been reported, in particular, in [36, 49, 51]. Based on experimental and theoretical investigations, it was shown that the failure of such composites under three-point bending does not occur due to reaching the compressive ultimate stress(for elongated specimens) or the transverse shear stress (for short specimens), but rather due to the realization of a nonclassical, nearly shear buckling mode of the specimens under transverse bending conditions. Below, the results of similar studies are presented, carried out with the aim of

qualitatively analyzing the refined geometrically nonlinear models of deformation of composite structural elements developed in [40].

3.1. Construction of an approximate analytical solution of the linearized three-point bending problem based on the equations of the first version of the theory. We assume that in the equilibrium equations constructed in [40] and based on the model (8)–(10), geometrically nonlinear terms introduced into consideration, containing the internal force factors $R_{11}^0, T_{11}^0, N_{11}^0$ are insignificant. Then equations (13), linearized in the vicinity of an initial strain-stress state (equilibrium equations of perturbed state) without taking into account the parameters of the initial deformed state (i.e., functions $w^0, \varphi^0, u_1^0, \hat{\tau}_1^0$) will have the form

$$T_{11,x} = 0,\ Q_{1,x} = 0,\ R_{1,x} - T_{33} - M_{11,x}^0 w_{,x} = 0,\ N_{11,x} - M_{11,x} + T_{13} = 0, \tag{22}$$

where in the case under consideration, instead of (14), we have the relations

$$Q_1 = M_{11,x} + M_{11}^0 \varphi_{,x}, \quad R_1 = R_{11,x} + M_{11}^0 w_{,x}, \tag{23}$$

and expressions

$$\begin{gathered} T_{11} = B_{11}\left(u_{1,x} - \frac{h^2}{24}\varphi_{,xx}\right),\ R_{11} = \frac{D}{2}\left(u_{1,x} - \frac{3h^2}{40}\varphi_{,xx}\right),\ T_{33} = B_{33}\varphi, \\ M_{11} = -D\left(w_{,xx} - \frac{4}{5}\hat{\tau}_{1,x}\right),\ T_{13} = \frac{8h}{15}\tau_1, N_{11} = -D\left(\frac{1}{15}w_{,xx} - \frac{16}{105}\hat{\tau}_{1,x}\right) \end{gathered}. \tag{24}$$

From the first equation of the system (22), taking into account (24), under the condition $T_{11} = 0$ the dependence follows

$$u_{1,x} = \frac{h^2}{24}\varphi_{,xx},$$

therefore

$$R_{11} = -\frac{Dh^2}{60}\varphi_{,xx},\ R_1 = -\frac{Dh^2}{60}\varphi_{,xxx} + M^0 w_{,x}. \tag{25}$$

When using (25), assuming $M^0 = const$, the third, fourth, and second equations of system (22) are transformed to the form

$$\varphi^{IV} - \frac{60M^0}{Dh^2}w'' + \frac{60B_{33}}{Dh^2}\varphi = 0, \tag{26}$$

$$w''' - \frac{16}{21\tilde{G}_{13}}\tau_1'' + \frac{2h}{3D}\tau_1 = 0, \tag{27}$$

$$w^{IV} - \frac{4}{5\tilde{G}_{13}}\tau_1''' - \frac{M^0}{D}\varphi'' = 0. \tag{28}$$

The solutions of the formulated equations (26) – (28) are represented as (λ – parameter to be determined)

$$w = \tilde{w} sin\,\lambda x,\ \varphi = \tilde{\varphi} sin\,\lambda x,\ \tau_1 = \tilde{\tau} cos\,\lambda x,$$

according to which from (27) the dependence follows

$$\tilde{\tau} = \frac{21\lambda^3 \tilde{G}_{13} D}{16\lambda^2 D + 14h\tilde{G}_{13}} \tilde{w}.$$

Using this dependence, equations (26), (28) take the form

$$\begin{gathered} \left(\lambda^4 + \frac{60B_{33}}{Dh^2}\right)\tilde{\varphi} + \frac{60\lambda^2}{h^2}\frac{M^0}{D}\tilde{w} = 0, \\ \frac{M^0}{D}\tilde{\varphi} + \lambda\left[\lambda - \frac{4}{5\tilde{G}_{13}}\left(\frac{21\lambda^3 D}{16\lambda^2 D + 14h\tilde{G}_{13}}\right)\right]\tilde{w} = 0. \end{gathered} \tag{29}$$

From the condition of non-triviality of the solutions of the obtained system of algebraic equations (29) for determining the buckling value of the bending moment, the formula follows

$$\left(\frac{M_1^{cr}}{Dh}\right)^2 = \lambda^4\left(\frac{1}{60} + \frac{E_3}{Dh\lambda^4}\right)\left[1 - \frac{4}{5\tilde{G}_{13}}\left(\frac{21\lambda^2 D}{16\lambda^2 D + 14h\tilde{G}_{13}}\right)\right]. \tag{30}$$

therefore,

$$M_1^{cr} = Dh\lambda_*^2\left(\frac{1}{60} + \frac{B_{33}}{Dh^2\lambda_*^4}\right)^{1/2}\left[1 - \frac{4}{5\tilde{G}_{13}}\left(\frac{21\lambda_*^2 D}{16\lambda_*^2 D + 14h\tilde{G}_{13}}\right)\right]^{1/2}, \tag{31}$$

where λ_* is the value of parameter λ at which the right side of expression (30) reaches its minimum value.

In accordance with (31), for calculating the maximum value of the normal stress σ_{*1}^{cr} arising at the points of the face planes of the plate at the moment of buckling, the formula takes the form

$$\sigma_{*1}^{cr} = \frac{6D\lambda_*^2}{h}\left(\frac{1}{60} + \frac{B_{33}}{Dh^2\lambda_*^4}\right)^{1/2}\left[1 - \frac{4}{5\tilde{G}_{13}}\left(\frac{21\lambda_*^2 D}{16\lambda_*^2 D + 14h\tilde{G}_{13}}\right)\right]^{1/2}, \tag{32}$$

3.2. Construction of an approximate analytical solution of the linearized three-point bending problem based on the equations of the second version of the theory. Assuming that in the equilibrium equations (18), (19), constructed in the work [41] on the basis of the model (8), (15), the parametric terms containing the initial bending moment $M_{11}^0 = M^0$ are predominant, we will compose the linearized equilibrium equations of perturbed state

$$\begin{gathered} T_{11,x} = 0,\ R_{1,x} - T_{33} = 0,\ R_{11,x} - M_{13} = 0, \\ Q_{1,x} = 0,\ M_{11,x} - T_{13} = 0,\ N_{11,x} - R_{13} = 0. \end{gathered} \tag{33}$$

In this case, in contrast to (23)

$$Q_1 = T_{13} + M^0\varphi_{,x},\ R_1 = M_{13} + M^0 w_{,x} = 0, \tag{34}$$

and the elasticity relations will have the form [41]

$$\begin{gathered}
T_{13} = B_{13}\left(w_{,x} - \gamma_1 - \frac{1}{3}\chi_1\right),\ N_{11} = -D\left(\frac{1}{5}\gamma_{1,x} + \frac{1}{21}\chi_{1,x}\right), \\
M_{11} = -D\left(\gamma_{1,x} + \frac{1}{5}\chi_{1,x}\right),\ R_{13} = B_{13}\left[\frac{1}{3}\left(w_{,x} - \gamma_1\right) - \frac{1}{5}\chi_1\right], \\
T_{11} = B_{11}u_{1,x} - \frac{D}{2}\psi_{1,x},\ R_{11} = \frac{D}{2}\left(u_{1,x} - \frac{3h^2}{40}\psi_{1,x}\right), \\
T_{33} = B_{33}\varphi,\ M_{13} = D_{13}\left(\varphi_{,x} - \psi_1\right),
\end{gathered} \tag{35}$$

in which $B_{11} = E_1 h,\ D = B_{11}h^2/12,\ \tilde{B}_{13} = \tilde{G}_{13}h,\ D_{13} = \tilde{B}_{13}h^2/12$. Represent the solutions of equations (33)–(35) in the form

$$w = \tilde{w}\sin\lambda x,\ \varphi = \tilde{\varphi}\sin\lambda x,\ u_1 = \tilde{u}\cos\lambda x,\ \gamma_1 = \tilde{\gamma}\cos\lambda x,$$
$$\psi_1 = \tilde{\psi}\cos\lambda x,\ \chi_1 = \tilde{\chi}\cos\lambda x,$$

in the present case, we obtain a system of homogeneous algebraic equations

$$r_{11}\tilde{w} + M^0\tilde{\varphi} = 0,\ M^0\tilde{w} + r_{22}\tilde{\varphi} = 0, \tag{36}$$

where

$$\begin{gathered}
r_{11} = \tilde{B}_{13}\left[1 + \frac{\tilde{B}_{13}}{Db}\left(\frac{5}{9}b_{11} - b_{22}\right)\right],\ r_{22} = \frac{B_{33}}{2\lambda^2} + D_{13}\left(1 - \frac{D_{13}}{c}\right), \\
c = D_{13} + \frac{Dh^2\lambda^2}{120},\ b = b_{11}b_{22} - 5b_{12}^2,\ b_{11} = \frac{\tilde{B}_{13}}{3D} + \lambda^2, \\
b_{12} = \frac{\tilde{B}_{13}}{3D} + \frac{\lambda^2}{5},\ b_{22} = \frac{\tilde{B}_{13}}{D} + \frac{5\lambda^2}{21}.
\end{gathered}$$

From the condition of nontrivial solutions of the system of equations (36) for determining M_2^{cr} and $\sigma_2^{cr} = 6M_2^{cr}/h^2$, the formulas follow

$$M_2^{cr} = \sqrt{r_{11}r_{22}},\ \sigma_2^{cr} = \frac{6}{h^2}\sqrt{r_{11}r_{22}} \tag{37}$$

3.2.1. Accounting for transverse compression. When taking into account the transverse compression stress σ_{33}^0 formed in the vicinity of the applied localized transverse force P, the equations of system (33), with the exception of the equation

$$f_\gamma = M_{11,x} - T_{13} + T_{33}^0\gamma_1 = 0,\ T_{33}^0 \approx h\sigma_{33}^0, \tag{38}$$

expressions (34) and elasticity relations (35) remain unchanged. In this case, the above solution of the problem also remains unchanged except for the coefficient b_{11}, which is calculated in the considered approximation by the formula

$$b_{11} = \frac{\tilde{B}_{13}}{3D} + \lambda^2 - \frac{T_{33}^0}{D}. \tag{39}$$

For a given value of external load $P_э = \int\limits_{L/2-a/2}^{L/2+a/2} p_3^+(x)dx$ (a is the length of the contact section of the loading roller with the test specimen, Fig. 1), applied in the center of the test specimen with a width B, length L, stress σ_{33}^0, averaged over the length of the contact a, will be equal to $\sigma_{33}^0 = P_э/(aB) = 4M^0/(aL)$. In this case,

$$T_{33}^0 = h\sigma_{33}^0 = æM^0,\ æ = 4h/(aL), \tag{40}$$

and when using (39) and taking into account (40), instead of the first formula in (37), we obtain the characteristic equation of the form

$$b_{22}æ\left(M^0\right)^3 - Db\left(M^0\right)^2 - \left(r_{22}\tilde{B}_{13} + \frac{5\tilde{B}_{13}b_{22}}{9D}\right)æM^0 +$$
$$+r_{22}\tilde{B}_{13}Db + \tilde{B}_{13}\left(\frac{5b_{11}}{9} - b_{22}\right) = 0,$$

where the above formulas apply to the incoming coefficients. The critical value of the bending moment determined by the solution of this equation is denoted by M_3^{cr}, and the corresponding normal stress is denoted by σ_3^{cr}.

3.3. Construction of an approximate analytical solution of the linearized three-point bending problem based on the equations of S.P. Timoshenko type theory. If instead of (8)–(10) (or (8), (15)) we adopt simplified representations $U_3 = w + z\varphi,\ U_1 = u_1 + z\gamma$, corresponding to the well-known shear model of S.P. Timoshenko with account for transverse compression [36, et al.], then, under the adopted geometrically nonlinear approximation, instead of (35) we arrive at a system of linearized equations

$$\begin{gathered} u_{,xx} = 0,\ \tilde{B}_{13}\left(w_{,xx} + \gamma_{,x}\right) + \underline{M^0\varphi_{,xx}} = 0, \\ D\gamma_{,xx} - \tilde{B}_{13}\left(w_{,x} + \gamma\right) = 0,\ \tilde{D}_{13}\varphi_{,xx} - B_{33}\varphi + \underline{M^0 w_{,xx}} = 0, \\ \tilde{B}_{13} = k_c h\tilde{G}_{13},\ \tilde{D}_{13} = \tilde{B}_{13}h^2/12. \end{gathered} \tag{41}$$

where k_c is a correction factor accounting for the nonuniform distribution of stresses σ_{13} along the z direction.

By solving equations (41), we again arrive at formulas of the form (37)

$$M_T^{cr} = \sqrt{r_{11} r_{22}}, \ \sigma_T^{cr} = \frac{6}{h^2} \sqrt{r_{11} r_{22}}, \tag{42}$$

where, in the present approximation,

$$r_{11} = \tilde{B}_{13} \left[1 - \left(1 + \frac{D\lambda^2}{\tilde{B}_{13}} \right)^{-1} \right], \ r_{22} = \frac{\tilde{D}_{13} \lambda^2 + B_{33}}{\lambda^2} \tag{43}$$

The conducted analysis shows that, in accordance with formulas (43), as $\lambda^2 \to \infty$, $r_{11} \to \tilde{B}_{13}$, $r_{22} \to \tilde{D}_{13}$, $\left(M_T^{cr} \right)^2 \to \tilde{B}_{13}^2 h^2 / 12$. Consequently, when $\lambda \to \infty$ and $k_c = 1$, we have the formula

$$M_T^{cr} = \sqrt{\tilde{B}_{13} \tilde{D}_{13}}, \ \sigma_{T*}^{cr} \approx 1,73 \tilde{G}_{13}. \tag{44}$$

If in system (41) the third equation is written as a dependence (here and below $\gamma_1 = \gamma$)

$$w_{,x} = \frac{D}{B_{13}} \gamma_{,xx} - \gamma$$

and entered into the second and fourth equations, we obtain

$$D\gamma_{,xxx} + M^0 \varphi_{,xx} = 0, \tag{45}$$

$$\tilde{D}_{13} \varphi_{,xx} - B_{33} \varphi + M^0 \left(\frac{D}{B_{13}} \gamma_{,xxx} - \gamma_{,x} \right) = 0. \tag{46}$$

From (45), we obtain the relation $\varphi_{,xx} = -\frac{D}{M^0} \gamma_{,xxx}$ and the integrals

$$\varphi_{,x} = c_1 - \frac{D}{M^0} \gamma_{,xx}, \ \varphi = c_1 x + c_2 - \frac{D}{M^0} \gamma_{,x}, \tag{47}$$

where c_1, c_2 are constant integrations.

When using (47), equation (46) is transformed to the form

$$\left(M^0 \frac{D}{\tilde{B}_{13}} - D_{13} \frac{D}{M^0} \right) \gamma_{,xxx} + \left(B_{33} \frac{D}{M^0} - M_0 \right) \gamma_{,x} = B_{33} \left(c_1 x + c_2 \right). \tag{48}$$

The first term of this equation degenerates at

$$M^{02} = \tilde{D}_{13} \tilde{B}_{13}, \text{ i.e. at } M_{cr}^0 = \sqrt{\tilde{D}_{13} \tilde{B}_{13}}, \tag{49}$$

which coincides with solution (42) as $\lambda^2 \to \infty$. In this case, due to (49), equation (48) and its solution can be written in the form

$$\gamma_{,x} = \frac{B_{33}}{B_{33}D - \tilde{D}_{13}\tilde{B}_{13}}\left(c_1 x + c_2\right),\ \gamma = \frac{B_{33}}{B_{33}D - \tilde{D}_{13}\tilde{B}_{13}}\left(c_1 \frac{x^2}{2} + c_2 x\right) + c_3, \quad (50)$$

where c_3 is an integration constant.

Therefore, when the bending moment M^0 at the cross section $x = L/2$ reaches the value (49) (which is the same as (42) as $\lambda \to \infty$), buckling of the test specimen occurs in a purely transverse-shear mode under three-point bending, and the post buckling behavior of the specimen is described by function (50).

In conclusion of this section, we note that under three-point bending of a test specimen having a distance L between the supports, the shear force Q^0 (constant within the half-length of the specimen) and the bending moment M^0 (maximal at the specimen's midpoint) developing at the midpoint of the specimen are related, according to classical beam theory, by the dependence

$$M^0 = Q^0 \frac{L}{2} = \frac{PL}{4},$$

where P is the bending load applied in the center of the specimen. In this case, according to the Zhuravsky formula, the maximum value of the transverse shear stress σ_{13}^0, formed at the points $z = 0$, and the maximum value of the normal stress σ^0, formed at the point $z = t/2$ of the midspan cross section of the specimen, are related by the dependence

$$\sigma_{13}^0 = \frac{3Q^0}{2h} = \frac{3M^0}{Lh} = \frac{\sigma^0 h}{2L}. \quad (51)$$

Based on formula (51), it is possible to refine the above solutions by taking into account the physical nonlinear dependence $\tilde{G}_{13} = \tilde{G}_{13}\left(\gamma_{13}\right)$ through the use of the approximation formula (21) and the results presented in [51].

4. Calculation results and their comparison with experimental data

For the composite made of HSE180 REM prepreg, Table 1 presents the results of three-point bending tests of the test specimens, which were averaged over the data for six specimens in each series [25,26].

Table 1

L, mm	h, mm	$P_э^{cr}$, kN	$\tilde{G}_{13}$, MPa	$\sigma_э^{cr}$, MPa	σ_{*1}^{cr}, MPa	σ_{*2}^{cr}, MPa
170	9,07	2,3	460	801	5320 as $\lambda_* = 262{,}8\,(1/\text{m})$	796,7 as $\lambda \to \infty$
50	8,48	5,4	460	634	5320 as $\lambda_* = 281\,(1/\text{m})$	796,7 as $\lambda \to \infty$

The long specimens had a distance between supports $L = 170$ mm, an average thickness of $h = 9{,}07$ mm, while the short specimens had – $L = 50$ mm, $h = 8{,}48$ mm. In this table, $P_э^{cr}$ denotes the ultimate force P applied at the center of the specimen, corresponding to its failure; $\tilde{G}_{13} = G_* = 460$ MPa is the value of the shear modulus of transverse shear appearing in formula (49); $\sigma_э^{cr}$ is the experimental maximum value of normal stress at the point of application of the external load, while σ_{1*}^{cr} and σ_{2*}^{cr} denote the minimum theoretical values of stresses (32) and (37). The values of σ_{1*}^{cr} were found at $\lambda = \lambda_*$ determined by formula (31), and the values of σ_{2*}^{cr} were found as $\lambda \to \infty$.

It should be noted that the relation $M_1^{cr} = M_1^{cr}(\lambda)$, constructed using Eq. (30) and based on the equations of the first version of the theory, possesses a limit point (31), at which M_1^{cr} attains its minimum value, and according to solution (37), based on the equations of the second version of the theory, M_2^{cr} tends to a finite limiting value as $\lambda \to \infty$.

Such dependences $\sigma^{cr} = \sigma^{cr}(\lambda)$, obtained at $\tilde{G}_{13} = 460$ MPa, are shown in Fig. 2, 3 as an illustration.

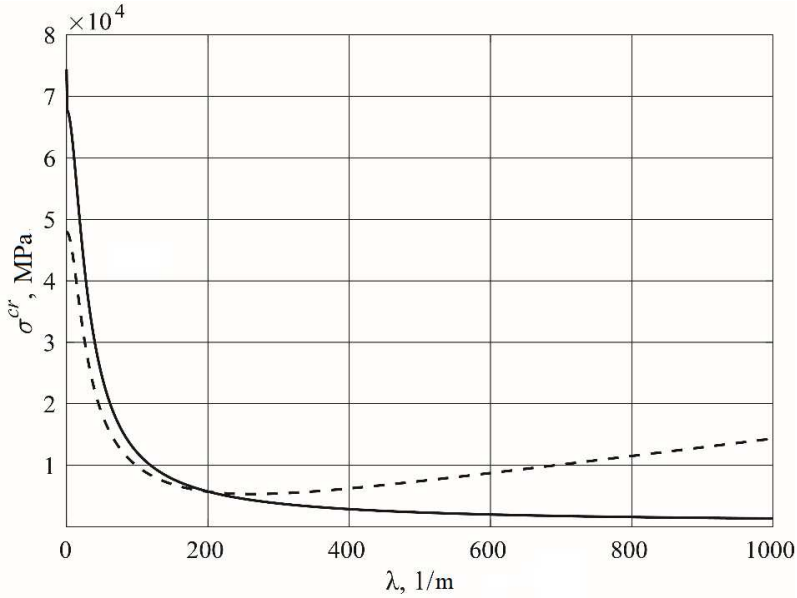


Fig.2. Dependence $\sigma_2^{cr} = \sigma_2^{cr}(\lambda)$, $\sigma_3^{cr} = \sigma_3^{cr}(\lambda)$ at $t = 9{,}07$ mm, $P_э^{cr} = 2{,}3$ kN (σ_2^{cr} – solid line, σ_1^{cr} – dashed line)

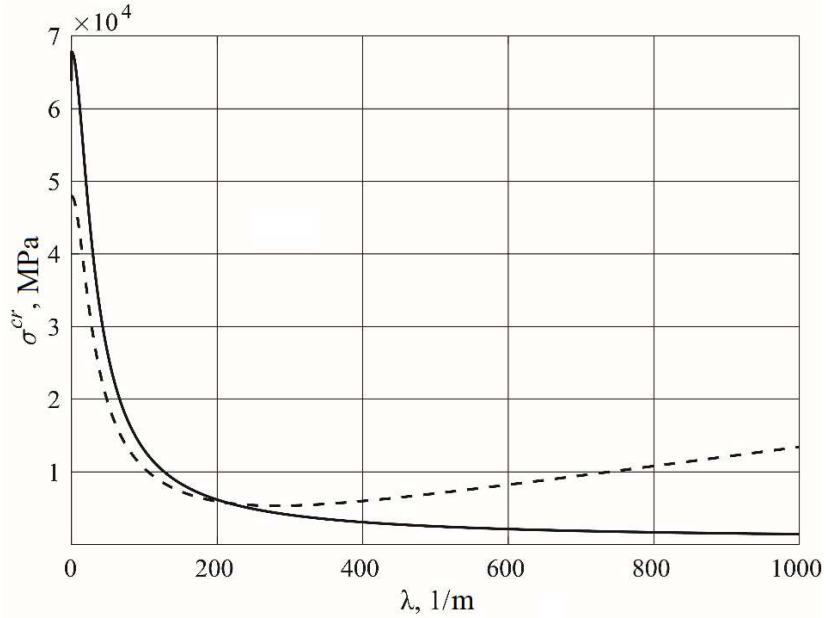


Fig.3. Dependence $\sigma^{cr} = \sigma^{cr}(\lambda)$ at $t = 8{,}48$ mm (σ_2^{cr} – solid line, σ_1^{cr} – dashed line)

A comparison of the experimental values of the load $P_э^{cr}$ and the stress σ^{cr}, at which failure of the specimens occurs due to the realization of a shear buckling mode under three-point bending, with the theoretical values σ_{*2}^{cr} obtained from formulas (37) and the values σ_{*3}^{cr} obtained from the solution presented in Section 3.2.1, for various values of a, is given in Table 2. Dependences $\sigma_2^{cr} = \sigma_2^{cr}(\lambda)$, $\sigma_3^{cr} = \sigma_3^{cr}(\lambda)$ are given in Fig.4.

Table 2

L, mm	h, mm	$P_э^{cr}$, kN	$\sigma_э^{cr}$, MPa	σ_{*2}^{cr}, MPa	σ_{*3}^{cr}, MPa		
					$a=1$ mm	$a=2$ mm	$a=4$ mm
170	9,07	2,3	801	796,7 as $\lambda_* \to \infty$	472,5 as $\lambda_*=23,6$	612,7 as $\lambda_*=16,3$	796,7 as $\lambda_* \to \infty$
50	8,48	5,4	634	796,7 as $\lambda_* \to \infty$	545,5 as $\lambda_*=41,6$	563,2 as $\lambda_*=27,7$	624,6 as $\lambda_*=18,6$

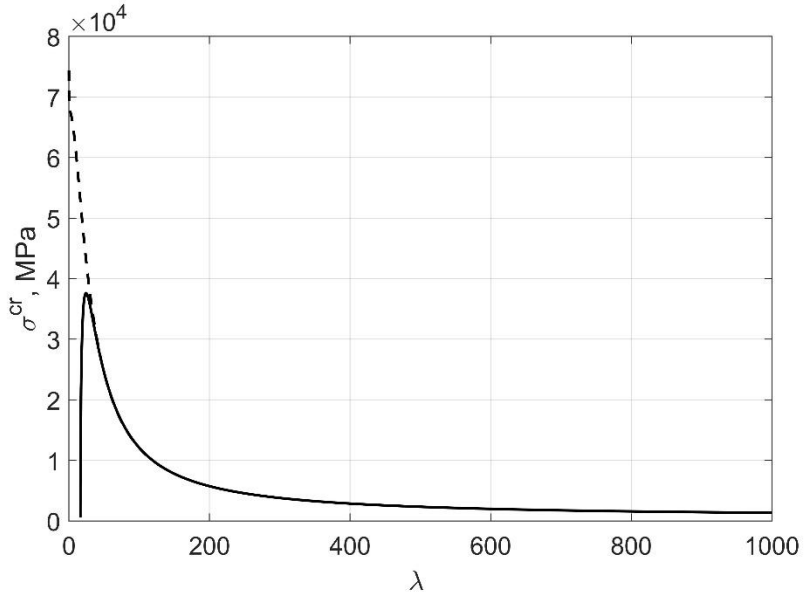


Fig.4. Dependences $\sigma_2^{cr}=\sigma_2^{cr}(\lambda)$, $\sigma_3^{cr}=\sigma_3^{cr}(\lambda)$ at h=9,07 mm, $P_э^{cr}$=2,3kN, $a=2$ mm (σ_3^{cr} – solid line, σ_2^{cr} – dashed line)

Table 3 presents the values of the compressive stress $\sigma_{33}^0=P_э/(aB)$ obtained for given values of $P_э^{cr}$ depending on the parameter a.

Table 3

h, mm	$P_э^{cr}$, kN	σ_{33}^0, MPa				
		$a=0,25$, mm	$a=0,5$, mm	$a=1$, mm	$a=2$, mm	$a=4$, mm
9,07	2,3	1022,22	511,11	255,56	127,78	63,89
8,48	5,4	2400,00	1200,00	600,00	300,00	150,00

Conclusion

An analysis of the obtained results shows that the theoretical results obtained using the second version of the theory are in good agreement with the experimental data for long specimens, while they are somewhat overestimated compared to the experimental results for short specimens. It is important to note that the value σ_{2*}^{cr} obtained from equations (33) is in nearly exact agreement with the value σ_{T*}^{cr} determined from formula (44) at $\tilde{G}_{13}=460$ MPa. Consequently, the

geometrically nonlinear equations based on the simplest refined shear model of the S.P. Timoshenko type accounting for transverse compression, while retaining the corresponding geometrically nonlinear terms (underlined, in particular, in equations (41)), possess a sufficiently high degree of accuracy and content, as can be seen from the results of [36 et al.].

Nevertheless, the results obtained and described above indicate the advisability of using, in the general case, only the equations of the second version of the theory [41] for formulating buckling problems of the class under consideration and for developing numerical methods for their solution taking into account geometric and physical nonlinearities, which make it possible to account for the nonuniform distribution of transverse shear stresses through the thickness, the length of the specimens, and the influence on the determined ultimate stress values of other stress components arising in the vicinities of the applied localized loads. Nevertheless, the results obtained and described above indicate that, in the general case, it is advisable to employ only the equations of the second version of the theory [41] for formulating bukcling problems of the considered class and for developing numerical methods for their solution that account for both geometric and physical nonlinearities. Such approaches make it possible to take into account the nonuniform distribution of transverse shear stresses through the thickness, the specimen length, as well as the influence of other stress components $\sigma_{33}^{0}, \sigma_{13}^{0}$ formed in the vicinity of the applied localized loads on the critical stress values σ^{cr}.

In light of the results obtained in Section 3.1, it should be noted that the use of equations based on approximations (8)–(10) (equations (13), based on the Reddy–Nemirovsky-type model) is not advisable for formulating and solving problems of non-classical (bending–shear and transverse-shear) buckling modes of composite structural elements subjected to localized loads that induce predominantly bending stress–strain states in the vicinity of their application points.

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