

Supplemental Material: Measurement-Cost Triage for Non-Markovian Entanglement Diagnostics in Hyperentangled Photonic Qubits

Matthias Jakob

May 2026

1 Local complementarity identity and defect

For a single qubit A with Bloch vector $\mathbf{r}_A$, the reduced state is $\rho_A = (\mathbb{I} + \mathbf{r}_A \cdot \boldsymbol{\sigma})/2$. In a fixed interferometric basis,

$$P_A = |\rho_{A,00} - \rho_{A,11}|, \quad V_A = 2|\rho_{A,01}|. \quad (1)$$

Thus $P_A^2 + V_A^2 = |\mathbf{r}_A|^2$ and

$$T_A = 2(1 - \text{Tr } \rho_A^2) = 1 - |\mathbf{r}_A|^2 = 4 \det \rho_A. \quad (2)$$

The exact local identity is $P_A^2 + V_A^2 + T_A = 1$. For an observed pair AB , the defect is defined as

$$D_A = T_A - C_{AB}^2, \quad (3)$$

which gives $P_A^2 + V_A^2 + C_{AB}^2 + D_A = 1$. In a purification ABR , $T_A = C_{A|BR}^2$ and therefore $D_A = C_{A|BR}^2 - C_{AB}^2$.

2 Structured pure-dephasing function

The frequency distribution is an unequal bimodal Gaussian,

$$f(\omega) = pg_\sigma(\omega - \Delta/2) + (1-p)g_\sigma(\omega + \Delta/2), \quad (4)$$

where $g_\sigma(x) = \exp[-x^2/(2\sigma^2)]/(\sigma\sqrt{2\pi})$. The characteristic function gives

$$\kappa(t) = \int d\omega f(\omega) e^{i\omega t} = e^{-\sigma^2 t^2/2} \left[\cos \frac{\Delta t}{2} + i(2p-1) \sin \frac{\Delta t}{2} \right]. \quad (5)$$

The time-local rate is $\gamma(t) = -d \ln |\kappa(t)|/dt$ for $|\kappa(t)| > 0$.

3 Four-qubit channel and basis order

The ordered basis is $|abcd\rangle$, with $a, b, c, d \in \{0, 1\}$. Pure dephasing acts on A . If i labels basis state $|a_i b_i c_i d_i\rangle$, then

$$\rho_{ij}(t) = \begin{cases} \rho_{ij}(0), & a_i = a_j, \\ \kappa(t)\rho_{ij}(0), & a_i = 0, a_j = 1, \\ \kappa^*(t)\rho_{ij}(0), & a_i = 1, a_j = 0. \end{cases} \quad (6)$$

This rule is used directly to construct the full 16×16 density matrix.

4 All-cut normalized negativity

For each bipartition $\alpha|\bar{\alpha}$, the partial transpose ρ^{T_α} is diagonalized and the negativity is computed from the absolute sum of negative eigenvalues, $\mathcal{N}_{\alpha|\bar{\alpha}} = \sum_{\lambda_i < 0} |\lambda_i|$. The figures use the normalized value $\mathcal{N}_{\alpha|\bar{\alpha}}^{\text{norm}} = \mathcal{N}_{\alpha|\bar{\alpha}}/\mathcal{N}_{\text{max},\alpha}$, where $\mathcal{N}_{\text{max},\alpha} = (d_{\text{min}} - 1)/2$ and d_{min} is the smaller Hilbert-space dimension of the two sides of the cut. The all-cut diagnostic is

$$N_{\text{min}}^{\text{all-cuts}}(\rho) = \min_{\alpha|\bar{\alpha}} \mathcal{N}_{\alpha|\bar{\alpha}}^{\text{norm}}(\rho), \quad (7)$$

with cuts $A|BCD$, $B|ACD$, $C|ABD$, $D|ABC$, $AB|CD$, $AC|BD$, and $AD|BC$. This is a simultaneous all-cut negative-partial-transpose (NPT) diagnostic, not a genuine multipartite-entanglement monotone and not a biseparability criterion.

5 Non-symmetric local-defect benchmark

To test whether the local-defect layer is more than a formal closure relation in the symmetric benchmark, we also use

$$|\Psi_{\eta,\chi}\rangle = \sqrt{\frac{1+\eta}{4}}(|0000\rangle + |0011\rangle) + \sqrt{\frac{1-\eta}{4}}(|1100\rangle + e^{i\chi}|1111\rangle). \quad (8)$$

The benchmark values are $\eta = 0.35$ and $\chi = \pi/2$. The parameter η biases the population of the $a = 0$ and $a = 1$ sectors, hence the reduced qubit state of A has nonzero predictability. The same pure-dephasing rule and the same all-cut negative-partial-transpose (NPT) diagnostics are applied. Closure is checked through

$$P_A^2 + V_A^2 + T_A = 1, \quad P_A^2 + V_A^2 + C_{AB}^2 + D_A = 1. \quad (9)$$

Under white-noise admixture, the defect D_A is retained as a local budget diagnostic, but is not interpreted as hidden entanglement by itself because white noise contributes local mixedness.

6 Locally rotated predictability–visibility benchmark

The non-symmetric benchmark activates local predictability. To also activate local visibility, we apply a local rotation to qubit A ,

$$|\Psi_{\eta,\chi,\theta}\rangle = (R_y(\theta)_A \otimes I_{BCD})|\Psi_{\eta,\chi}\rangle, \quad (10)$$

where $R_y(\theta) = \exp(-i\theta\sigma_y/2)$. For the benchmark values $\eta = 0.35$, $\chi = \pi/2$, and $\theta = \pi/4$, the initial local marginal has

$$P_A = |\eta \cos \theta|, \quad V_A = |\eta \sin \theta|, \quad (11)$$

so that $P_A^2 = V_A^2 = \eta^2/2 = 0.06125$. This benchmark is used only to test the operational role of the local complementarity layer; it does not change the pure-dephasing channel or the all-cut negative-partial-transpose (NPT)-diagnostic definition. The closure relations

$$P_A^2 + V_A^2 + T_A = 1, \quad P_A^2 + V_A^2 + C_{AB}^2 + D_A = 1 \quad (12)$$

are checked numerically for the visibility values $v = 1.0, 0.5$, and 0.25 .

7 State-perturbation stability test

The main benchmark in Eq. (5) of the manuscript is analytically useful because it gives closed-form pair/local relations and a transparent all-cut NPT threshold. To check that the qualitative diagnostic hierarchy is not tied to this exact amplitude pattern, we also perturb the initial state by small random complex components. For each perturbation strength ϵ , the normalized initial vector is generated as

$$|\Psi_\chi^{(\epsilon)}\rangle = \frac{|\Psi_\chi\rangle + \epsilon|\delta\psi\rangle}{\| |\Psi_\chi\rangle + \epsilon|\delta\psi\rangle \|}, \quad (13)$$

where $|\delta\psi\rangle$ is a complex Gaussian random vector in the same 16-dimensional basis, drawn independently for each sample and then normalized only through the final state normalization. The same pure-dephasing rule, concurrence reconstruction, local-defect calculation, and all-cut normalized-negativity diagnostic are then applied. The scan uses 24 random samples for each nonzero perturbation strength, 160 time points over the same time window, and the visibility values $v = 1.0, 0.5$, and 0.25 . The unperturbed line $\epsilon = 0$ is included as a reference.

Table 1: State-perturbation stability summary. Entries are medians over random perturbation samples. The low-visibility layer-separation fraction counts time points for which the Bell-sector flow is positive while the all-cut NPT diagnostic is inactive at $v = 0.25$.

ϵ	samples	median max C_{AB}^2	median max $N_{\min}^{\text{all-cuts}}$ at $v = 1$	separation fraction at $v = 0.25$
0.00	1	0.500	0.236	0.288
0.01	24	0.501	0.242	0.288
0.03	24	0.500	0.251	0.288
0.05	24	0.501	0.266	0.288
0.08	24	0.481	0.288	0.269
0.10	24	0.497	0.287	0.288

The perturbation test is intentionally modest: it does not replace a general classification over four-qubit state space and does not turn the all-cut NPT diagnostic into a genuine multipartite-entanglement test. Its purpose is to address the narrower robustness question relevant for the protocol: whether small preparation-level amplitude and phase deviations immediately collapse the separation between Bell-sector flow, pair/local diagnostics, and all-cut NPT activation. The summary in Table 1 and Figs. 1–3 show that this is not the case for the tested perturbation range.

8 Main-figure logic and supplemental diagnostic plots

The main text uses a six-figure argument chain: (i) hierarchy and measurement allocation, (ii) analytic pair/local separation, (iii) Bell-flow versus all-cut activation, (iv) visibility-thresholded all-cut NPT recovery, (v) controlled-phase justification of the benchmark point, and (vi) reservoir-parameter robustness. The remaining diagnostic plots are retained here to document the full numerical checks without overloading the main narrative.

9 Reproducibility and quick run

The project package contains:

```
programs/full_density_matrix_numerics.py
programs/chi_scan.py
programs/plot_results.py
programs/local_defect_predictability_visibility_benchmark.py
```

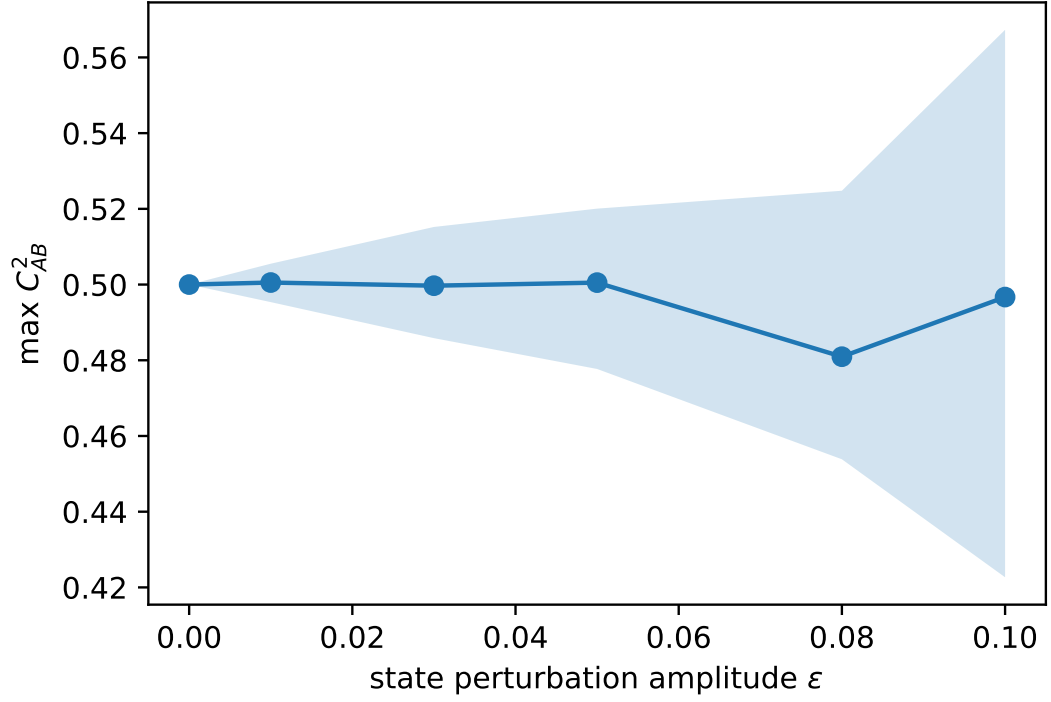


Figure 1: Pair-layer stability under random state perturbations. The maximum and mean values of C_{AB}^2 remain close to the unperturbed benchmark for small perturbation strengths, with increasing spread only at the largest tested perturbations.

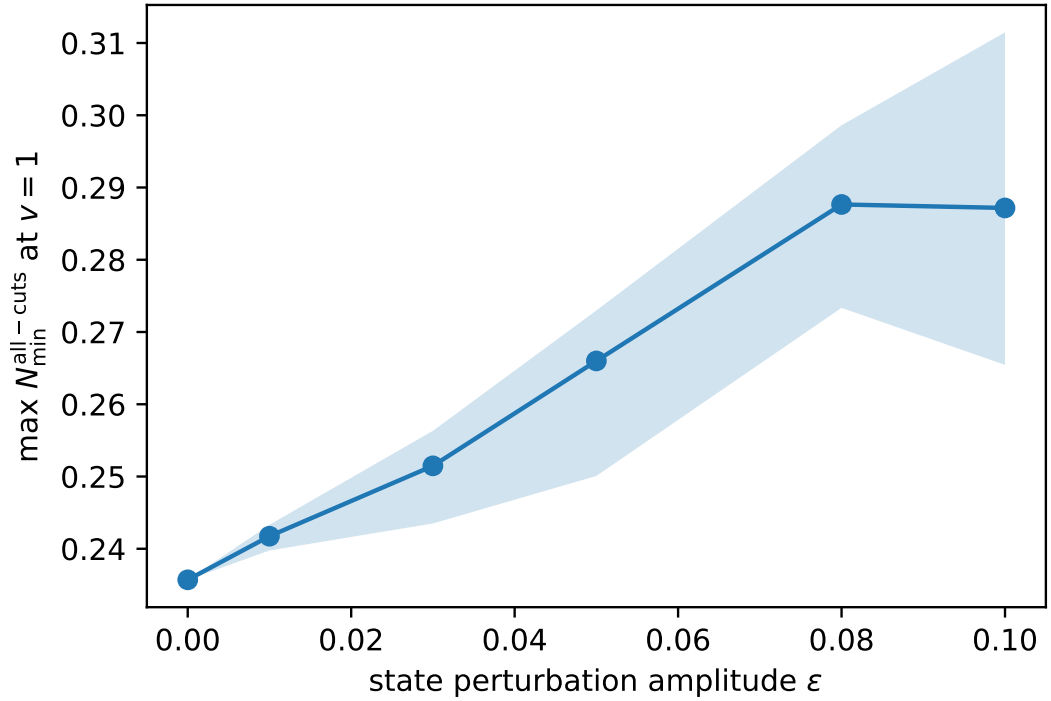


Figure 2: All-cut NPT stability under random state perturbations. The all-cut diagnostic remains active over the full time grid at $v = 1$ and shows the expected thresholded behavior at lower visibility.

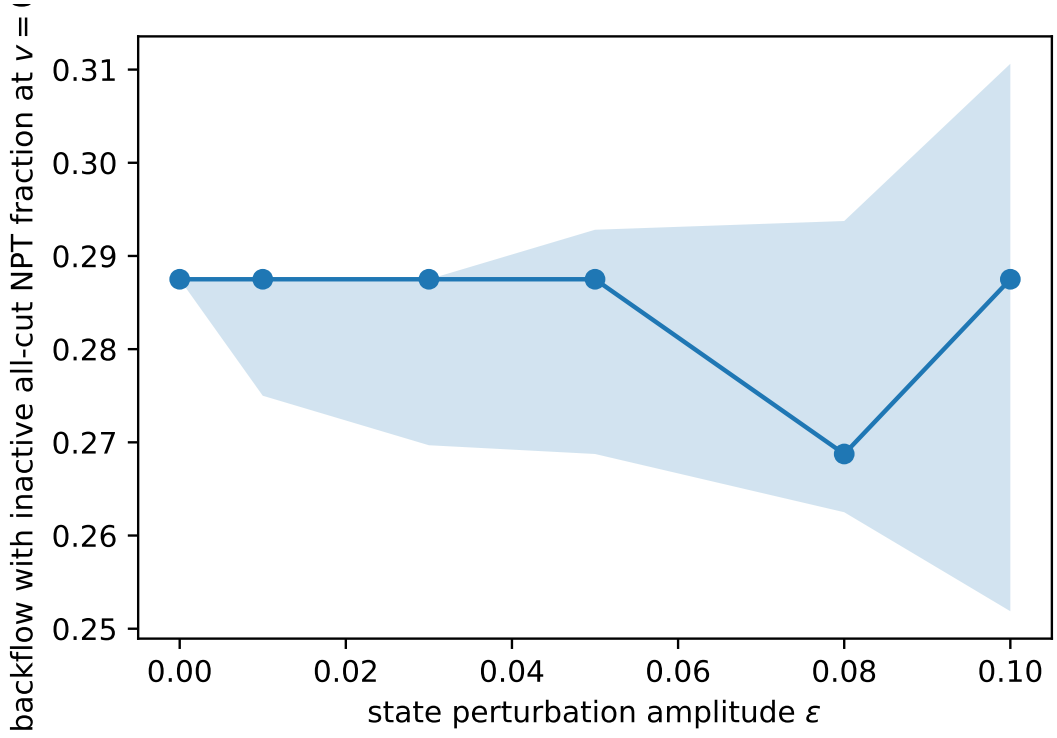


Figure 3: Layer-separation stability under random state perturbations. The separation fraction remains finite and comparable to the unperturbed value at low visibility, supporting the interpretation that Bell-sector backflow windows and all-cut NPT activation windows are distinct diagnostic objects.

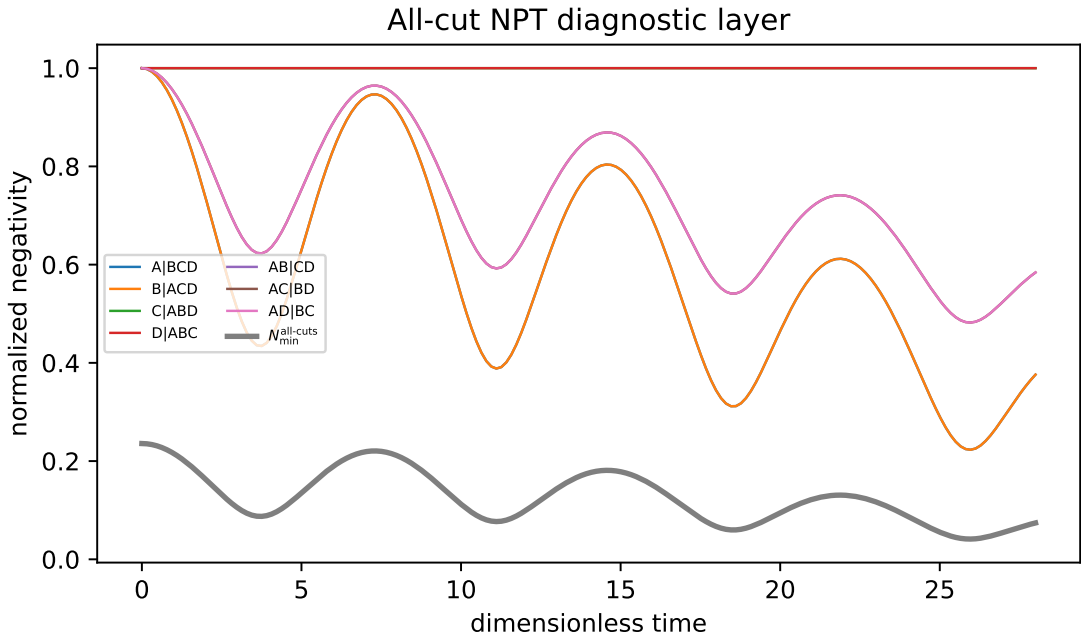


Figure 4: Individual normalized bipartition negativities for all seven selected cuts. The main text uses the minimum over these curves, $N_{\min}^{\text{all-cuts}}$, as the all-cut NPT diagnostic.

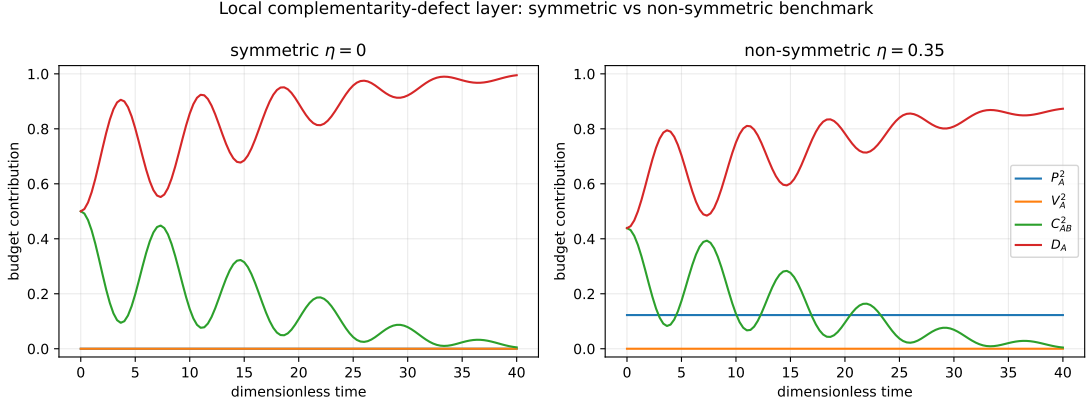


Figure 5: Symmetric versus non-symmetric local-defect benchmark. The non-symmetric family with $\eta = 0.35$ activates local predictability P_A^2 and tests whether the defect layer remains informative beyond the symmetric $P_A^2 = V_A^2 = 0$ limit.

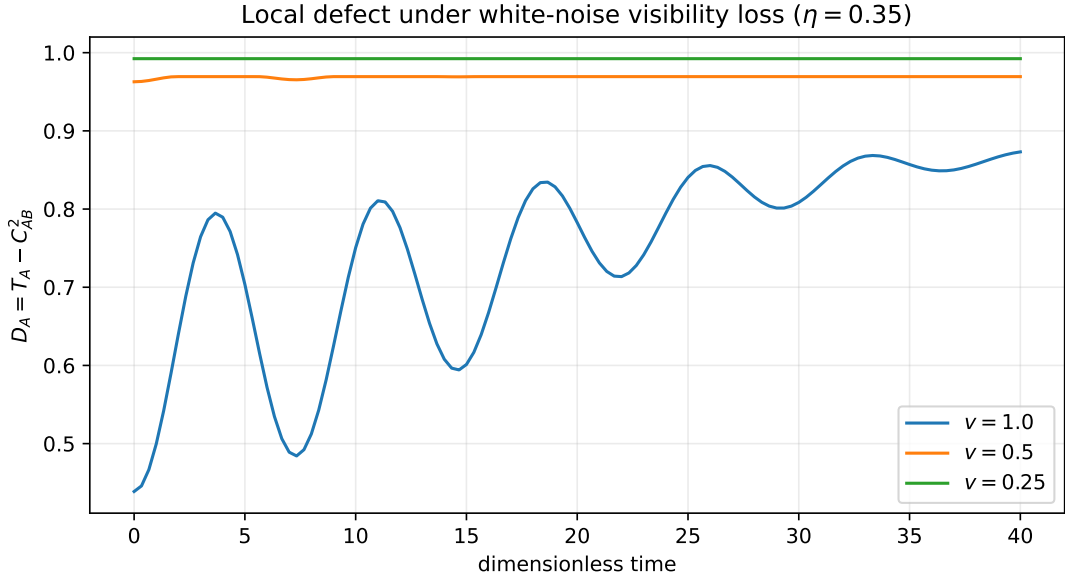


Figure 6: Non-symmetric defect layer under visibility loss. Under white-noise admixture D_A is retained as a local budget-defect diagnostic; it is not by itself a hidden-entanglement measure because local mixedness also contributes.

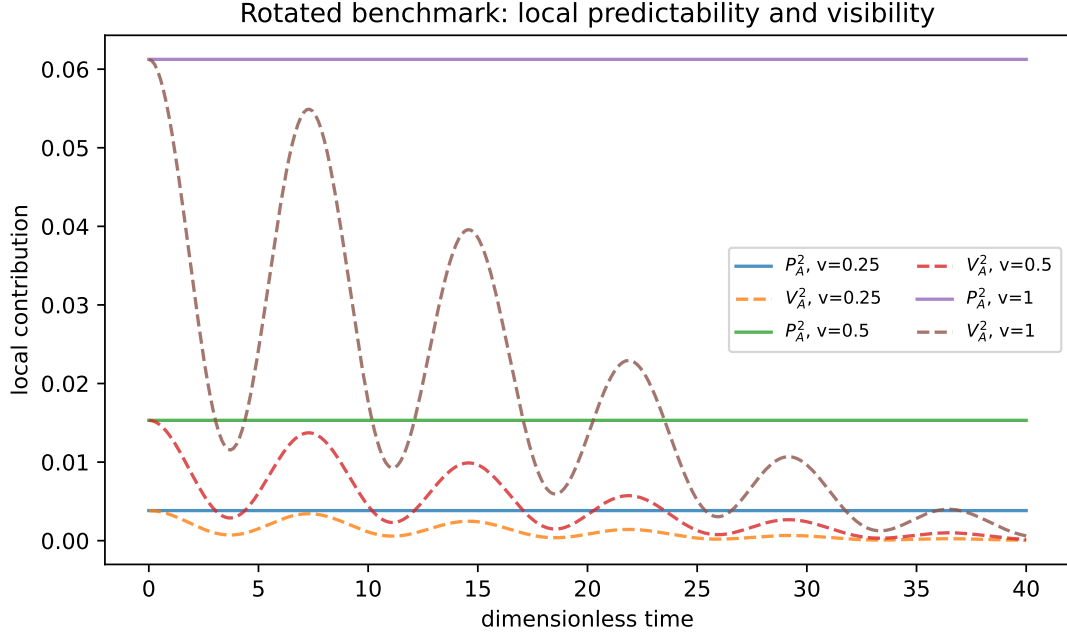


Figure 7: Locally rotated predictability–visibility benchmark. The local rotation $R_y(\pi/4)$ activates both P_A^2 and V_A^2 while the same full 16×16 dephasing dynamics is used.

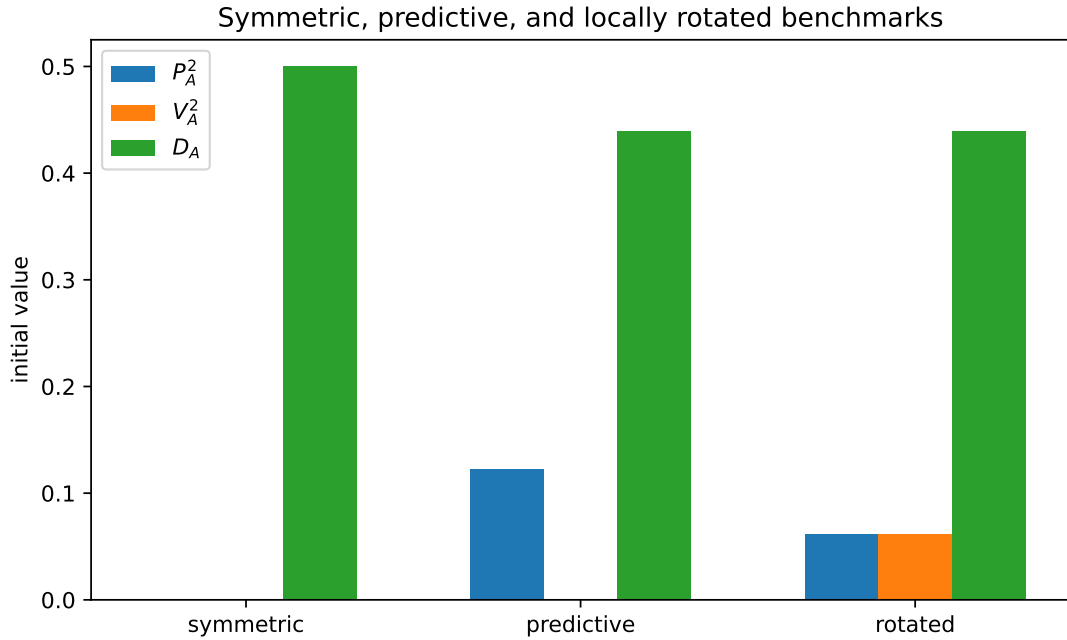


Figure 8: Comparison of local-defect benchmarks. The symmetric benchmark has $P_A^2 = V_A^2 = 0$, the predictive benchmark activates P_A^2 , and the locally rotated benchmark activates both P_A^2 and V_A^2 .

```
programs/state_perturbation_stability.py
programs/analytic_validation_checks.py
data/*.csv
figures/*.pdf
README.md
requirements.txt
environment.yml
manifest_sha256.txt
```

A quick reproducibility check is:

```
python programs/full_density_matrix_numerics.py --quick
python programs/chi_scan.py --quick
python programs/local_defect_predictability_visibility_benchmark.py --quick
python programs/state_perturbation_stability.py --quick
python programs/analytic_validation_checks.py --quick
python programs/plot_results.py
```

The full run regenerates the time series, all-cut negativities, visibility scans, controlled-phase scan, robustness map, the local predictability–visibility benchmark, and the state-perturbation stability test.