# Online Resource 1

**Title:** Symbolic derivation procedure for the explicit equations of motion of the human–exoskeleton system.

**Description:** This supplementary material provides the complete symbolic procedure used to derive the explicit equations of motion of the human–exoskeleton system from the Euler–Lagrange formulation. The document includes the detailed step-by-step construction of the inertia matrix $M(q)$, Coriolis and centrifugal matrix $C(q, \dot{q})$, gravity vector $G(q)$, generalized torque vector $\tau_i$, and the intermediate auxiliary variables employed in the disaggregated formulation. The resource is intended to improve reproducibility, facilitate implementation, and support readers interested in the full mathematical development of the proposed simulation framework.

**Article:** Dynamic modeling of human–exoskeleton interaction: A simulation framework for gait rehabilitation.

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# Symbolic derivation procedure for the explicit equations of motion of the human–exoskeleton system

The mathematical process for the explicit calculation of the equations of motion that describe the dynamics of the human-exoskeleton system is based on the Euler-Lagrange approach. The aim is to derive a state-space matrix representation of the system dynamics by exploiting the structure of the Euler-Lagrange equation, Equation (1), and vector operations.

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}+\frac{\partial D}{\partial \dot{q}}=\tau \tag{1}$$

Where $q \in \mathbb{R}^n$ denotes the joint angles of the lower limbs, $\dot{q}$ the joint angular velocities, $L \in \mathbb{R}$ the system Lagrangian, $D$ the system dissipation function and $\tau \in \mathbb{R}^n$ the torques non-conservative applied to the system with respect to the generalized coordinates. The human–exoskeleton system comprises six joints, each characterized by angular position, velocity, and acceleration. For the right leg: $[q_{m_1}, \dot{q}_{m_1}, \ddot{q}_{m_1}]$ – ankle, $[q_{m_2}, \dot{q}_{m_2}, \ddot{q}_{m_2}]$ – knee, and $[q_{m_3}, \dot{q}_{m_3}, \ddot{q}_{m_3}]$ – hip, and for the left leg: $[q_{m_6}, \dot{q}_{m_6}, \ddot{q}_{m_6}]$ – ankle, $[q_{m_5}, \dot{q}_{m_5}, \ddot{q}_{m_5}]$ – knee, and $[q_{m_4}, \dot{q}_{m_4}, \ddot{q}_{m_4}]$ – hip.

Therefore, six equations of motion are derived for each phase of the gait cycle, which means that 24 explicit equations describe the complete dynamic modeling of the human-exoskeleton system. The modeling by energies requires the use of the forward kinematics of the system in order to calculate the kinetic and potential energy in the COMs of the segments.

Thus, the kinetic energy, $E_{K\_com}$, in the COMs due to translational and rotational movements of each segment is Equation (2).

$$E_{K\_com} = E_{T\_com} + E_{R\_com} \quad [J] \tag{2}$$

Now, being $m_s$ the mass of the respective segment and $v_{com}$ the linear velocity of the COM, then Equation (3) and Equation (4) define the translational energy, $E_{T\_com}$, and the rotational energy, $E_{R\_com}$, in the COMs, respectively.

$$\begin{aligned} E_{T\_com} &= \frac{1}{2} m_s v^2, \quad \text{such that } v_{com} = \left(v_{com_x}{}^2 + v_{com_y}{}^2\right)^{\frac{1}{2}} \\ E_{T\_com} &= \frac{1}{2} m_s \left(v_{com_x}{}^2 + v_{com_y}{}^2\right) \quad [J] \end{aligned} \tag{3}$$

$$E_{R\_com} = \frac{1}{2}\dot{q}_m{}^2 \quad [J] \tag{4}$$

The Lagrangian, $L$, is the difference between the kinetic and potential energy, so that Equation (5) describes the potential energy, $E_{P\_com}$, in the COMs as a function of the gravitational acceleration, $g$.Then, $L$ is calculated for each COM of the segments from the Equation (6).

$$E_{P\_com} = m_s g y \ [J] \tag{5}$$

$$L = E_{K\_com} - E_{P\_com} \tag{6}$$

Thereby, the equation of motion for each joint is Equation (7).

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_m}\right) - \frac{\partial L}{\partial q_m} = \tau_{i_{m_i}} \tag{7}$$

Where $q_m = [q_{m_1}, q_{m_2}, q_{m_3}, q_{m_4}, q_{m_5}, q_{m_6}]^T$ denotes the measured joint angular positions; $\dot{q}_m = [\dot{q}_{m_1}, \dot{q}_{m_2}, \dot{q}_{m_3}, \dot{q}_{m_4}, \dot{q}_{m_5}, \dot{q}_{m_6}]^T$ the measured joint angular velocities; and $\tau_{i_m} = [\tau_{i_{m_1}}, \tau_{i_{m_2}}, \tau_{i_{m_3}}, \tau_{i_{m_4}}, \tau_{i_{m_5}}, \tau_{i_{m_6}}]^T$ the measured interaction torques at the joints $[Nm]$.

As mentioned above, a set of six nonlinear equations is derived for each model, where each equation represents a mathematical expression with a high computational cost. Therefore, the structure of the Euler-Lagrange equation is analyzed in order to derive the equations of motion explicitly through a matrix representation, Equation (8).

$$\begin{aligned} M(q_m)\ddot{q}_m + C(q_m, \dot{q}_m)\dot{q}_m + G(q_m) &= \tau_{m_i} \\ M(q_m)\ddot{q}_m + C(q_m, \dot{q}_m)\dot{q}_m + G(q_m) &= \tau_{m_{gearbox}} - \tau_f + \tau_h + \tau_{f_{ext}} \end{aligned} \tag{8}$$

Where, $n = 6$ DoF; $M(q) \in \mathbb{R}^{nxn}$ represents the symmetric positive-definite inertia matrix; $G(q, \dot{q}) \in \mathbb{R}^{nxn}$ the Coriolis and centrifugal force matrix; $G(q) \in \mathbb{R}^{nx1}$ the gravity matrix; $\ddot{q}_m = [\ddot{q}_{m_1}, \ddot{q}_{m_2}, \ddot{q}_{m_3}, \ddot{q}_{m_4}, \ddot{q}_{m_5}, \ddot{q}_{m_6}]^T$ the measured angular acceleration in the joints, $[deg/s^2]$; $\tau_{m_i} = \tau_{m_{gearbox}} - \tau_f + \tau_h + \tau_{f_{ext}}$ the interaction torques in the joints, $[Nm]$; $\tau_{m_{gearbox}} \in \mathbb{R}^{nx1}$ the torques at the motor gearbox output shaft; $\tau_f \in \mathbb{R}^{nx1}$ the torques due to motor gearbox frictions; $\tau_h \in \mathbb{R}^{nx1}$ the human torques; and $\tau_{f_{ext}} \in \mathbb{R}^{nx1}$ the torques due to external forces.

For the dynamic model in Equation (8), several properties are presented as follows, [41, 42]:

1. Matrix $\boldsymbol{M(q_m)}$ is symmetric and positive definite.
2. Matrix $\boldsymbol{\dot{M}(q_m) - 2C(q_m, \dot{q}_m)}$ is a skew-symmetric matrix so $\forall\ \boldsymbol{\varepsilon} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{\varepsilon^T}\left(\boldsymbol{\dot{M}(q_m) - 2C(q_m, \dot{q}_m)}\right)\boldsymbol{\varepsilon} = \boldsymbol{0}.$
3. There exist finites scalars $\boldsymbol{\delta_i > 0}$, $\boldsymbol{i = 1, 2, \ldots 4}$ such that $\|\boldsymbol{M(q_m)}\| \leq \boldsymbol{\delta_1}$, $\|\boldsymbol{C(q_m, \dot{q}_m)}\| \leq \boldsymbol{\delta_2}$, $\|\boldsymbol{G(q)}\| \leq \boldsymbol{\delta_3}$, $\left\|\boldsymbol{\tau_f + \tau_h + \tau_{f_{ext}}}\right\| \leq \boldsymbol{\delta_4}$, which means all items in dynamics model are bounded.

To explicitly derive the equations of motion in a matrix structure see the flowchart presented in Fig. 1. The following steps are explained:

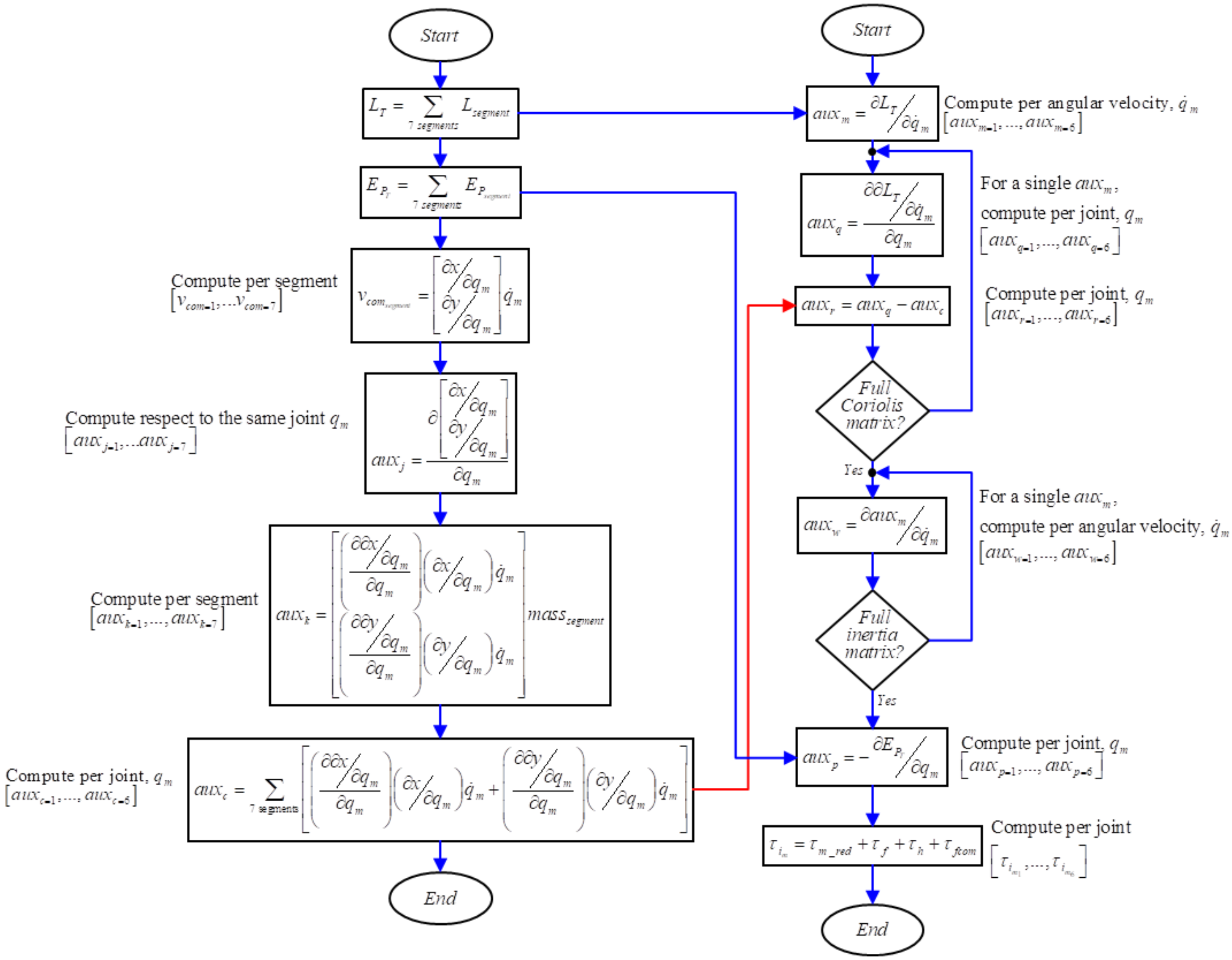


Fig. 1 Flowchart to derive the equations of motion in a matrix structure from the Euler-Lagrange equation

**Step No.1:** Calculate the total Lagrangian, $L_T$, by summing the Lagrangians per COM in the segments, as expressed in Equation (9). In this dynamic modeling, the whole body is represented by seven rigid segments $\{S_i\}_{i=1}^{7} = \{\text{right foot, right tibia, right femur, HAT (Head, Arms, and Trunk), left femur, left tibia, left foot}\}$. Here, $S_i$ denotes the $i$-th body segment considered in the model.

$$L_T = L_{LeftFoot_{com}} + L_{LeftTibia_{com}} + L_{LeftFemur_{com}} + L_{COM_{total}} + \ldots \\ + L_{RightFemur_{com}} + L_{RightTibia_{com}} + L_{RightFoot_{com}} \tag{9}$$

Where $[RightFoot_{com}, LeftFoot_{com}]$ denotes the positions of the COM in the right and the left foot, respectively; $[RightTibia_{com}, LeftTibia_{com}]$ the COM positions of the right and left tibia; $[RightFemur_{com}, LeftFemur_{com}]$ the COM positions of the right and left femur; and $[COM_{total}]$ the COM position of the whole body.

**Step No.2:** Calculate the total potential energy, $E_{P_T}$, through the sum of the potential energies per COM in the segments, Equation (10).

$$\begin{aligned} E_{P_T} = & E_{P_{LeftFoot_{xom}}} + E_{P_{LeftTibia_{xom}}} + E_{P_{LeftFemur_{xom}}} + E_{P_{COM_{total}}} + \ldots \\ & + E_{P_{RightFemur_{xom}}} + E_{P_{RightTibia_{xom}}} + E_{P_{RightFoot_{xom}}} \end{aligned} \tag{10}$$

**Step No.3:** Get the linear velocity, $v_{com_i}$, of each COM in the segments in a vector form, Equation (11). In this formulation, the Jacobian matrix of partial derivatives, $J_i$, with respect to the generalized angular velocity vector, $\dot{q}$, is not computed explicitly; instead, the mathematical expression is left in its explicit form. For compactness, the segments are indexed as $i = j = k \in [1,7]$.

$$\begin{aligned} & J_i = \begin{bmatrix} \frac{\partial x}{\partial q_{m_1}} & \frac{\partial x}{\partial q_{m_2}} & \frac{\partial x}{\partial q_{m_3}} & \frac{\partial x}{\partial q_{m_4}} & \frac{\partial x}{\partial q_{m_5}} & \frac{\partial x}{\partial q_{m_6}} \\ \frac{\partial y}{\partial q_{m_1}} & \frac{\partial y}{\partial q_{m_2}} & \frac{\partial y}{\partial q_{m_3}} & \frac{\partial y}{\partial q_{m_4}} & \frac{\partial y}{\partial q_{m_5}} & \frac{\partial y}{\partial q_{m_6}} \end{bmatrix} \\ & v_{com_i} = J_i \; [\dot{q}_{m_1}, \dot{q}_{m_2}, \dot{q}_{m_3}, \dot{q}_{m_4}, \dot{q}_{m_5}, \dot{q}_{m_6}]^T \\ & J_i \in \mathrm{R}^{2xn}, \; \dot{\mathrm{q}} \in \mathrm{R}^{nx1}, \; v_{com_i} \in \mathrm{R}^{2x1} \end{aligned} \tag{11}$$

**Step No.4:** Create an auxiliary variable, $aux_i$, to store the complete calculation of the linear velocity per COM in the segments, as given in Equation (12).

$$\begin{aligned} & aux_i = v_{com_i} \\ & aux_i \in \mathrm{R}^{2x1} \end{aligned} \tag{12}$$

**Step No.5:** Calculate the derivative of the matrices, $J_i$, with respect to the same joint, $q_m$, Equation (13).

$$\begin{aligned} & aux_j = \frac{\partial J_i}{\partial q_m} \\ & aux_j \in \mathrm{R}^{2xn} \end{aligned} \tag{13}$$

**Step No.6:** The following relationship contributes to the construction of the Coriolis matrix per segment, Equation (14).

$$\begin{aligned} & aux_k = mass_i \begin{bmatrix} \left[ row_1(aux_j) \right] aux_{i_{(1,1)}} \\ \left[ row_2(aux_j) \right] aux_{i_{(2,1)}} \end{bmatrix} \\ & aux_k \in R^{2xn} \end{aligned} \tag{14}$$

Where $row_1()$ denotes the elements of the first row in the known matrix, and $(1,1)$ represents the element at the intersection of its first row and first column.

**Step No.7:** The auxiliary variable, $aux_c$, computes a section of the Coriolis matrix per joint, Equation (15).

$$\begin{aligned} aux_c &= \sum_k \left( aux_{k_{(1,c)}} + aux_{k_{(2,c)}} \right) \\ aux_c &\in R^{1x1} \end{aligned} \tag{15}$$

Where $c = m = q = r = w = p \in [1, n]$ indexes the six joints considered (right ankle, right knee, right hip, left hip, left knee, and left ankle), and $(1, c)$ represents the element located at the intersection of the first row and the $c$-th column.

**Step No.8:** Calculate the derivative of $L_T$ with respect to each angular velocity in the joint, Equation (16).

$$\begin{aligned} aux_m &= \frac{\partial L_T}{\partial \dot{q}_{m_c}} \\ aux_m &\in R^{1x1} \end{aligned} \tag{16}$$

**Step No.9:** For a single given variable, $aux_m$, compute the derivative with respect to each angular position, $q_m$. Then, the same calculation is made for the other matrices $aux_m$ since the index $m$ corresponds to each joint. This calculation contributes to the remaining section of the Coriolis matrix by joint, Equation (17).

$$aux_q = \frac{\partial aux_m}{\partial q_{m_c}} \tag{17}$$

**Step No.10:** The auxiliary variable, $aux_r$, computes the section of the Coriolis matrix per joint, Equation (18). The variable $aux_r$ contains $r$ mathematical expressions that correspond to each component of one row of the Coriolis matrix per joint, for instance, $C_{(1,n)} = [aux_{r=1}\,, aux_{r=2}\,, \dots, aux_{r=n}]$, $C(q_m, \dot{q}_m) \in R^{nxn}$. In this way, the Coriolis and centrifugal force matrix is completely built.

$$\begin{aligned} aux_r &= aux_q - aux_c \\ aux_r &\in R^{1x1} \end{aligned} \tag{18}$$

**Step No.11:** The variables $aux_m$ contain the derivative of $L_T$ with respect to each angular velocity in the joint, Equation (16), that is, $aux_{m=1} = \partial L_T / \partial \dot{q}_{m_1}$, $aux_{m=2} = \partial L_T / \partial \dot{q}_{m_2}$, $aux_{m=n} = \partial L_T / \partial \dot{q}_{m_n}$. Therefore, for a single given variable, $aux_m$, compute the derivative with respect to each angular velocity, $\dot{q}_m$, Equation (19). The variable $aux_w$ contains $w$ mathematical expressions that correspond to each component of one row of the Inertial matrix per joint, for instance, $M_{(1,n)} = [aux_{w=1}\,, aux_{w=2}\,, \dots, aux_{w=n}]$, $M(q_m) \in R^{nxn}$. In this way, the Inertial matrix is completely built.

$$aux_w = \frac{\partial aux_m}{\partial \dot{q}_{m_c}} \tag{19}$$

**Step No.12:** The variable $aux_p$ compute the derivative of $E_{P_T}$, Equation (10), with respect to each angular position in the joint, $q_m$, Equation (20). The variable $aux_p$ contains $p$ mathematical expressions that correspond to each component of the Gravity matrix per motion equation, that is, $G(q_m) = [aux_{p=1}, aux_{p=2}, \ldots, aux_{p=n}]^T$, $G(q_m) \in R^{nx1}$. In this way, the Gravity matrix is completely built.

$$aux_p = -\frac{\partial E_{p_T}}{\partial q_m} \tag{20}$$

**Step No.13:** Equation (21) defines the vector of torques or generalized forces with respect to each joint.

$$\begin{aligned} \tau_{i_{m_c}} &= [\tau_{m_{gearbox_c}} - \tau_{f_c} + \tau_{h_c} + \tau_{fcom_c}] \\ \tau_{i_{m_c}} &\in R^{nx1} \end{aligned} \tag{21}$$

Where $\tau_{i_{m_c}}$ represents the vector of interaction or net joint torques, with $c = 1\ to\ n$; $\tau_{m_{gearbox_c}}$ corresponds to the torques at the motor gearbox output shaft; $\tau_{f_c}$ refers to the torques generated by gearbox friction; $\tau_{h_c}$ indicates the torques applied by the human; and $\tau_{f_{com_c}}$ accounts for the torques induced by external forces.