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# Percolation and Phase Transitions on Living Temporal Graphs

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# Percolation and Phase Transitions on Living Temporal Graphs

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**Abstract**

Classical percolation theory is typically formulated on static graphs, while more recent extensions to temporal networks still assume that vertices remain structurally available over time. In this work, we study percolation on *living temporal graphs*, where vertices may change state during the evolution of the network. In particular, vertices may become inactive, dormant, partially disconnected, or permanently unavailable. The model is based on the living temporal graph framework introduced in [1], in which edges follow two different temporal mechanisms. Continuous-flow edges require both endpoints to remain simultaneously active during the entire interaction interval, whereas discrete-transport edges only require feasible departure and arrival times.

Within this setting, we introduce a notion of causal percolation that incorporates vertex birth, sleep states, asymmetric failures, and irreversible death. We first show that causal reachability in a living temporal graph can be represented exactly as directed reachability in a suitable time-expanded static graph. Using this correspondence, we derive mean-field conditions for the emergence of a giant causal out-component. Our analysis further shows that, for equal interaction durations, discrete-transport edges survive more easily than continuous-flow edges, which leads to a monotone reduction in the percolation threshold as the proportion $\rho_D$ of discrete edges increases.

We also compare the effects of reversible dormancy and irreversible vertex death and show that the two mechanisms lead to qualitatively different resilience behavior. Monte Carlo simulations on Erdős–Rényi, scale-free, and lattice networks support the theoretical predictions and suggest that asymmetric states such as DeadIn and DeadOut may produce a different scaling behavior near criticality.

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The results indicate that vertex livingness can substantially affect connectivity transitions in evolving systems and should be treated as an explicit structural feature in temporal percolation models.

**Keywords:** living temporal graphs; temporal percolation; phase transitions; evolving networks; dynamic graph processes

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# 1 Introduction

Percolation theory has long served as one of the main mathematical frameworks for understanding connectivity and large-scale structural transitions in complex systems. In its classical form, the theory studies how global connectivity changes when vertices or edges are randomly occupied or removed. Early developments by Erdős and Rényi on random graphs [2], together with the statistical-physics formulation developed by Stauffer and Aharony [3], established the existence of critical thresholds separating fragmented and connected phases. Later work showed that the geometry of the underlying network can strongly affect robustness properties. In particular, Albert, Jeong, and Barabási [4] observed that scale-free networks tend to tolerate random failures relatively well while remaining vulnerable to targeted attacks.

Despite their differences, most standard percolation models rely on a common structural assumption: a vertex is typically treated as either available or unavailable. This binary description is mathematically convenient and has led to a large and successful theory. However, many evolving systems do not behave in such a static manner. In biological, technological, and information networks, nodes may temporarily sleep, partially lose functionality, recover after inactivity, or become permanently disconnected. In such settings, the distinction between temporary dormancy and irreversible failure may substantially influence the way connectivity propagates through time.

A related limitation appears in temporal network models. Existing temporal percolation frameworks usually focus on the timing of edges while assuming that vertices themselves remain structururally stable during the evolution process. In practice, however, interactions often depend simultaneously on edge timing and on the evolving state of the participating vertices. A communication channel may fail while information is being transmitted, a transportation hub may remain inactive for part of the observation window, or a biological unit may accept outgoing interactions while no longer receiving incoming ones. These situations are difficult to describe using purely binary node states.

The present work studies percolation in the setting of *living temporal graphs*, introduced in [1]. In this framework, vertices possess internal lifecycle states, including active, dormant, partially disconnected, and irreversible dead states. In addition, edges follow two distinct temporal mechanisms. Continuous-flow edges require simultaneous endpoint availability during the entire interaction interval, whereas discrete-transport edges only require feasible departure and arrival events. The distinction is motivated by the observation that not all interactions propagate continuously in time. Some processes resemble uninterrupted flow, while others behave more like delayed transport or packet transfer.

Our goal is to understand how these additional structural features affect causal connectivity and percolation transitions. Rather than viewing node failure as a purely binary

event, we investigate how reversible inactivity, asymmetric accessibility, and dual edge semantics modify the emergence of large connected temporal components. The analysis combines causal reachability methods, time-expanded representations, and probabilistic arguments together with Monte Carlo simulations on several network classes. The incorporation of time into network models over the last two decades led to the development of temporal network theory, where connectivity depends not only on topology but also on the ordering of interactions. Holme and Saramäki [5] emphasized that temporal ordering can fundamentally alter the structure of reachable paths, since many connections that appear valid in a static aggregation are not causally feasible in time-dependent settings. Building on this viewpoint, Valdano et al. [6] introduced a notion of causal percolation based on time-respecting paths in evolving contact networks.

Nevertheless, in most temporal percolation models the dynamics are still carried primarily by the edges, while vertices themselves remain structurally passive. A node may participate in interactions at different times, but its internal state is usually assumed to remain unchanged throughout the process. In many real systems, however, vertices may temporarily become inactive, lose only part of their functionality, recover after periods of dormancy, or disappear permanently from the interaction process. Moreover, the ability to send information and the ability to receive it are not always symmetric. Such behaviors are difficult to represent within models that treat node availability as purely binary. Another simplifying assumption commonly adopted in temporal network models is that all interactions follow the same transmission mechanism. In practice, however, different types of interactions impose different temporal constraints. Some processes, such as a live communication channel or a continuous physical flow, require both endpoints to remain simultaneously available during the entire interaction interval. Other processes, including email exchange, parcel delivery, or packet transmission, depend mainly on feasible sending and receiving events and do not require uninterrupted simultaneous activity. From the perspective of causal connectivity, these two situations are structurally different and may lead to different propagation behavior.

The living temporal graph framework introduced in [1] provides a way to incorporate both vertex lifecycle dynamics and heterogeneous edge semantics within a unified temporal setting. In this model, each vertex evolves through states in

$$\{\text{Active}, \text{Sleep}, \text{DeadIn}, \text{DeadOut}, \text{DeadBoth}\},$$

while edges are classified either as continuous-flow or discrete-transport interactions. Continuous-flow edges require simultaneous endpoint activity throughout the interaction interval, whereas discrete-transport edges only require feasible departure and arrival events.

The purpose of the present work is to investigate how these additional structural features modify percolation transitions and causal reachability. In particular, we study how reversible dormancy differs from irreversible failure, and how asymmetric states such as DeadIn and DeadOut affect large-scale temporal connectivity. Our analysis indicates that discrete-transport interactions are generally less sensitive to temporary reception failures than continuous-flow interactions of comparable duration. To study these effects, we combine rigorous graph-theoretic definitions with mean-field approximations and Monte Carlo simulations on several network families.

# 2 Background and limitations of classical percolation models

Percolation theory originally emerged in statistical physics as a mathematical approach for studying transport phenomena in disordered media, particularly the movement of fluids through porous materials. Over time, the theory developed into one of the standard frameworks for analysing connectivity transitions in random structures and complex networks. Its applications now extend well beyond physics, including epidemiology, communication systems, infrastructure networks, and information propagation processes.

**Static percolation.** One of the earliest and most influential settings for percolation-type connectivity transitions appears in random graph theory. The classical Erdős–Rényi model [2] shows the emergence of a giant connected component once the edge density passes a critical threshold. In related probabilistic formulations of bond and site percolation, global connectivity is typically governed by a critical occupation probability separating fragmented and connected regimes [3]. Later developments demonstrated that network topology can significantly alter robustness behavior. In particular, Cohen et al. [7] showed that scale-free networks with degree exponent $\gamma \leq 3$ may exhibit a vanishing percolation threshold under random failures.

These results form part of the foundation of modern network robustness theory. At the same time, most classical models adopt a strongly binary description of node functionality: a vertex is assumed to be either fully operational or completely removed from the system. In many real networks, however, failures are often partial or temporary rather than absolute. A communication router may continue transmitting while no longer receiving packets, a neuron may enter a refractory or inactive phase without permanent damage, and a server may become temporarily unavailable before returning to operation. Such intermediate states are difficult to represent within standard binary percolation models.

**Dynamic and time-varying graphs.** The limitations of static network models motivated the development of temporal and time-varying graph frameworks, where connectivity evolves over time rather than remaining fixed. In these models, the existence of an edge may depend explicitly on time, allowing the network structure itself to change during the interaction process. This perspective is important in systems where contacts are intermittent, scheduled, delayed, or short-lived.

One influential formalism was introduced by Casteigts et al. [8], who developed the theory of time-varying graphs (TVGs) to describe networks whose connectivity changes dynamically through time-dependent edge availability. Later, Latapy et al. [9] proposed stream graphs, which provide a unified representation for both vertex-presence intervals and edge-presence intervals over continuous time.

These frameworks significantly expanded the ability of network theory to represent temporal interactions. Nevertheless, in most existing models the role of the vertex remains relatively passive. A node may appear or disappear in time, but the model typically does not distinguish between different internal operational states such as temporary dormancy, asymmetric functionality loss, or irreversible failure. In particular, states analogous to Sleep, DeadIn, and DeadOut are generally not represented explicitly within the temporal dynamics.

A similar simplification appears at the level of edge semantics. Most temporal graph models treat interactions as structurally uniform once their activation times are specified.

In many applications, however, the mechanism of transmission matters. A continuous physical flow, such as fluid transport through a pipeline or uninterrupted signal exchange, imposes different temporal constraints from those associated with discrete transfer processes such as parcel delivery, email transmission, or packet routing. These distinctions become especially relevant when vertex availability itself evolves over time.

**Causal percolation in temporal networks.** An important step toward incorporating temporality into percolation theory was the introduction of causal or time-respecting connectivity in evolving networks. In temporal settings, two vertices may appear connected in a static aggregation even though no valid chronological path exists between them. The ordering of interactions therefore becomes part of the connectivity structure itself rather than a secondary detail.

Within this perspective, Valdano et al. [6] studied causal percolation using time-respecting paths and showed that temporal ordering can substantially modify connectivity transitions and epidemic thresholds in evolving networks. Their work highlighted that the temporal sequence of interactions may be just as important as the underlying topology.

At the same time, the vertex dynamics in these models remain relatively simple. Nodes are generally treated as either available or unavailable at a given time, without distinguishing between different operational conditions. In particular, states corresponding to temporary inactivity, asymmetric functionality loss, or partially restricted communication are not represented explicitly. Likewise, the interaction mechanism associated with an edge is usually assumed to be uniform once temporal activation is specified. The distinction between continuous interactions, which require persistent simultaneous activity, and discrete transport processes, which only depend on departure and arrival feasibility, is typically absent from existing causal percolation frameworks.

**Vertex lifecycle dynamics and heterogeneous interactions.** The previous discussion suggests that existing percolation and temporal-network models capture important aspects of evolving connectivity, but still describe vertex behavior in a relatively limited way. In particular, they usually do not provide a formal mechanism for distinguishing between temporary inactivity, asymmetric communication loss, and irreversible disappearance of a node from the interaction process. At the same time, different modes of interaction are often treated within a single uniform edge framework despite having substantially different temporal requirements.

The living temporal graph framework introduced in [1] was developed to address these issues within a unified setting. In this model, vertices evolve through five operational states, while edges are divided into continuous-flow and discrete-transport interactions with different admissibility conditions. This additional structure makes it possible to study how internal vertex dynamics interact with temporal connectivity constraints.

The present work uses this framework as the basis for a percolation model on living temporal networks. Our goal is to investigate how vertex lifecycle states and heterogeneous edge semantics influence the emergence of large causal connected components. As the later sections will show, these effects are not merely local modifications of standard temporal percolation behavior, but can alter the connectivity transition itself in a systematic way.

# 3 Formal setup: living temporal graphs

The previous sections focused mainly on the limitations of existing percolation and temporal-network models and on the motivation for introducing richer vertex dynamics and heterogeneous interaction mechanisms. To study these effects mathematically, a precise graph-theoretic framework is needed in which both the temporal evolution of vertices and the admissibility of interactions can be described explicitly.

We therefore begin by recalling the main elements of the living temporal graph framework introduced in [1]. The definitions presented here follow the same underlying structure as the original model, but are reformulated slightly in order to make the operational interpretation of the vertex states more transparent in the context of percolation and causal reachability.

In particular, we adopt the functional interpretation in which *DeadIn* denotes a vertex that cannot receive interactions, while *DeadOut* denotes a vertex that cannot initiate or transmit interactions. This convention aligns the naming directly with the corresponding communication constraints and will be used consistently throughout the paper.

## 3.1 Vertex existence and state

We begin by formalizing the temporal existence and operational state of each vertex. Since the central objective of the model is to distinguish between different forms of activity, inactivity, and partial functionality, the vertex dynamics must explicitly encode not only whether a node is present in the network, but also how it can participate in interactions at a given time.

Let

$$\mathcal{T} = \{0, 1, \ldots, T_{\max}\}$$

be a discrete time domain. For each vertex $v \in V$, we define an existence function

$$\eta_v : \mathcal{T} \to \{0, 1\},$$

where $\eta_v(t) = 1$ if and only if the vertex $v$ exists at time $t$. The associated lifetime set of $v$ is

$$\Omega(v) = \{t \in \mathcal{T} : \eta_v(t) = 1\}.$$

On the lifetime set $\Omega(v)$, each vertex is assigned a state function

$$\sigma_v : \Omega(v) \to \mathcal{S},$$

where

$$\mathcal{S} = \{\text{Active}, \text{Sleep}, \text{DeadIn}, \text{DeadOut}, \text{DeadBoth}\}.$$

These states distinguish different operational conditions that may affect causal connectivity:

- Active: the vertex can both send and receive interactions.
- Sleep: the vertex exists but is temporarily inactive and neither sends nor receives interactions. This state is interpreted as reversible dormancy.
- DeadIn: the vertex cannot receive interactions but can still transmit them.
- DeadOut: the vertex cannot transmit interactions but can still receive them.

- DeadBoth: the vertex can neither send nor receive interactions.

To describe interaction admissibility more formally, we introduce sendability and receivability indicator functions. Define

$$\chi_{\text{out}}(v,t) = \begin{cases} 1, & \sigma_v(t) \in \{\text{Active}, \text{DeadIn}\}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\chi_{\text{in}}(v,t) = \begin{cases} 1, & \sigma_v(t) \in \{\text{Active}, \text{DeadOut}\}, \\ 0, & \text{otherwise}. \end{cases}$$

These indicators encode whether a vertex is capable of transmitting or receiving interactions at a given time and will play a central role in the definition of admissible temporal paths.

## 3.2 Two types of edges

Once vertex states have been specified, the next step is to determine how interactions are represented temporally. In many network models, all edges are treated in essentially the same way once their activation times are given. However, as discussed earlier, different forms of interaction may impose fundamentally different temporal requirements. Some interactions require uninterrupted simultaneous activity of both endpoints throughout the transmission process, whereas others only depend on successful departure and arrival events.

To capture this distinction, we partition the edge set $E$ into two classes:

$$E = E_C \cup E_D,$$

where $E_C$ denotes the set of continuous-flow edges and $E_D$ denotes the set of discrete-transport edges.

For an edge

$$e = (u,v) \in E_C,$$

we associate a temporal interaction interval

$$I_e = [b_e, d_e] \cap \mathcal{T},$$

where $b_e \leq d_e$. The interpretation is that the interaction remains active continuously throughout the interval $I_e$.

For an edge

$$e \in E_D,$$

we instead associate a departure time

$$t_e^{\text{dep}}$$

and an arrival time

$$t_e^{\text{arr}},$$

satisfying

$$t_e^{\text{dep}} \leq t_e^{\text{arr}}.$$

In this case, the interaction is interpreted as a discrete transfer process between two temporal events rather than a continuously maintained connection.

**Admissibility conditions.** The admissibility of an interaction depends both on vertex existence and on the operational states of the participating vertices.

A continuous-flow edge $e \in E_C$ is said to be admissible if

$$\forall t \in I_e : \ \eta_u(t)\eta_v(t)\,\chi_{\text{out}}(u,t)\,\chi_{\text{in}}(v,t) = 1.$$

Thus, the interaction can proceed only if the sender and receiver remain simultaneously operational throughout the entire interaction interval.

A discrete-transport edge $e \in E_D$ is admissible if

$$\eta_u(t_e^{\text{dep}})\,\eta_v(t_e^{\text{arr}})\,\chi_{\text{out}}(u,t_e^{\text{dep}})\,\chi_{\text{in}}(v,t_e^{\text{arr}}) = 1.$$

In this case, admissibility depends only on the feasibility of the departure and arrival events.

The distinction between these two edge types will later play an important role in the percolation analysis. Continuous-flow interactions are sensitive to failures occurring at any intermediate time during transmission, whereas discrete-transport interactions depend only on endpoint feasibility and may therefore remain viable under temporary interruptions occurring between departure and arrival.

## 3.3 Causal paths and reachability

After specifying which vertices are operational and which edges are admissible, we can define the notion of causal movement through the network. The key point is that a path in a temporal graph is not only a sequence of adjacent edges. The edges must also be usable in chronological order. This is the standard idea behind time-respecting paths in temporal networks [5, 6], adapted here to the living temporal graph setting.

Let

$$P = (e_1, \dots, e_m)$$

be a sequence of admissible edges from $x$ to $y$, where

$$e_i = (v_{i-1}, v_i), \qquad v_0 = x, \qquad v_m = y.$$

For each edge $e$, define its departure time $\alpha(e)$ and arrival time $\omega(e)$ by

$$\alpha(e) = \begin{cases} b_e, & e \in E_C, \\ t_e^{\text{dep}}, & e \in E_D, \end{cases} \qquad \omega(e) = \begin{cases} d_e, & e \in E_C, \\ t_e^{\text{arr}}, & e \in E_D. \end{cases}$$

The sequence $P$ is called a temporal path if

$$\omega(e_i) \le \alpha(e_{i+1}) \qquad \text{for all } i = 1, \dots, m-1.$$

Thus, the next interaction can only begin after the previous one has been completed. We write

$$x \rightsquigarrow y$$

if there exists such an admissible temporal path from $x$ to $y$.

### 3.4 Time-expanded graph

The definition above is natural, but direct reasoning with temporal paths can become cumbersome because reachability depends simultaneously on topology, time ordering, vertex states, and edge admissibility. A standard way to handle this difficulty in temporal graph analysis is to pass to a time-expanded representation, where each temporal copy $(v, t)$ records the presence of vertex $v$ at time $t$. This converts temporal reachability into directed reachability in a static graph.

We define the time-expanded graph

$$\mathcal{G}^{\mathrm{exp}} = (V^{\mathrm{exp}}, E^{\mathrm{exp}})$$

by

$$V^{\mathrm{exp}} = \{(v, t) : \eta_v(t) = 1\}.$$

The edge set $E^{\mathrm{exp}}$ consists of the following directed edges:

- *Waiting edges:* $(v, t) \to (v, t + 1)$ whenever

  $$\eta_v(t) = \eta_v(t + 1) = 1.$$

  These edges represent the possibility of remaining at the same vertex while time advances.

- *Discrete-transport edges:* for each admissible edge $e = (u, v) \in E_D$,

  $$(u, t_e^{\mathrm{dep}}) \to (v, t_e^{\mathrm{arr}}).$$

- *Continuous-flow edges:* for each admissible edge $e = (u, v) \in E_C$,

  $$(u, b_e) \to (v, d_e).$$

  When a finer representation of the interval is required, this direct edge may equivalently be replaced by a chain of unit-time steps along the interval.

The relevance of this construction is that it preserves causal reachability. By the time-expanded representation result for living temporal graphs [1, Theorem 11.2], one has

$$x \rightsquigarrow y \quad \Longleftrightarrow \quad \exists\, t_x, t_y \in \mathcal{T} \text{ such that } (x, t_x) \rightsquigarrow (y, t_y) \text{ in } \mathcal{G}^{\mathrm{exp}}.$$

In other words, the temporal reachability problem in the living graph can be studied as an ordinary directed reachability problem in the expanded graph. This equivalence is useful both conceptually and computationally: it separates the construction of admissible temporal interactions from the subsequent graph-theoretic analysis of reachability.

## 4 Percolation observables and the order parameter

Having defined causal reachability in living temporal graphs, we now introduce the quantities that will be used to characterize large-scale connectivity and percolation transitions. As in classical percolation theory, the central question is whether a macroscopic connected structure emerges when the system size becomes large. In the present setting, however,

connectivity is defined through admissible temporal paths rather than through static adjacency alone.

For a vertex $x \in V$, let

$$\mathcal{R}(x) = \{y \in V : x \rightsquigarrow y\}$$

denote the causal out-reachability set of $x$, that is, the set of all vertices that can be reached from $x$ through admissible temporal paths.

To measure the size of the largest causally connected region, we define

$$S_n = \frac{1}{n} \max_{x \in V} |\mathcal{R}(x)|.$$

This quantity plays the role of the order parameter in the living temporal percolation model. Intuitively, $S_n$ measures the fraction of the network that can be causally influenced from the most connected source vertex.

Percolation is said to occur if

$$\liminf_{n \to \infty} \mathbb{E}[S_n] > 0.$$

In other words, a non-vanishing fraction of the network remains causally reachable in the large-system limit. The associated critical threshold $p_c$ is the value of the relevant control parameter—for example, the edge occupation probability in an underlying random graph model—at which this macroscopic causal component first emerges.

The time-expanded representation introduced in the previous section allows these observables to be studied through ordinary directed reachability. Indeed, by the equivalence between causal paths in $\mathcal{G}$ and directed paths in the expanded graph $\mathcal{G}^{\exp}$, one obtains

$$\mathbb{E}[S_n] = \frac{1}{n} \max_{(x,t_x) \in V^{\exp}} \left| \{(y,t_y) : (x,t_x) \rightsquigarrow (y,t_y)\} \right|.$$

Thus, the temporal percolation problem can be reformulated as a directed percolation problem on the static expanded graph.

It is important to note, however, that the normalization is always taken with respect to the original number of vertices $n$, rather than the expanded size $nT_{\max}$. The time-expanded graph is introduced as an analytical representation of temporal reachability, not as a redefinition of the underlying system size.

# 5 Main theoretical results

We now turn to the main structural properties of percolation in living temporal graphs. The previous sections introduced the formal framework, the admissibility conditions for interactions, and the notion of causal reachability. Using these ingredients, we can begin to examine how different vertex states and edge mechanisms affect temporal connectivity.

Rather than presenting the results as isolated formal statements, we develop them progressively together with their interpretation. Since the distinction between continuous-flow and discrete-transport interactions is central to the model, it is useful to first clarify how temporary inactivity affects each type of edge. We therefore begin with a simple observation concerning the Sleep state and discrete-transport interactions.

## 5.1 Sleep blocks discrete edges at endpoints

Consider a discrete-transport edge

$$e = (u, v) \in E_D$$

with departure time

$$t_{\text{dep}}$$

and arrival time

$$t_{\text{arr}}.$$

By definition, the edge is admissible only if

$$\chi_{\text{out}}(u, t_{\text{dep}}) = 1$$

and

$$\chi_{\text{in}}(v, t_{\text{arr}}) = 1.$$

The Sleep state sets both sendability and receivability equal to zero. Consequently, if the source vertex is asleep at the departure time, or if the target vertex is asleep at the arrival time, the corresponding discrete interaction cannot occur.

This observation can be stated formally as follows.

**Lemma 1 (Sleep blocks discrete edges).** If

$$\sigma(u, t_{\text{dep}}) = \text{Sleep}$$

or

$$\sigma(v, t_{\text{arr}}) = \text{Sleep},$$

then

$$A_D(e) = 0.$$

The proof follows immediately from the admissibility conditions introduced in the previous section. The importance of the lemma is mainly conceptual: discrete-transport interactions are more tolerant of temporary interruptions than continuous-flow interactions, but they do not ignore endpoint availability altogether. The distinction is that discrete edges only depend on admissibility at the departure and arrival events, whereas continuous-flow edges require operational consistency throughout an entire temporal interval.

To illustrate this point, consider a discrete temporal path

$$x \to y \to z$$

with interaction times $(0, 2)$ and $(3, 5)$. If the intermediate vertex $y$ enters the Sleep state at time 2 or 3, then one of the corresponding interactions becomes inadmissible. On the other hand, the state of $y$ during the interval between these endpoint events does not affect the validity of the discrete transfer itself. This difference between endpoint feasibility and sustained temporal availability will later play a central role in the comparison between discrete and continuous percolation thresholds.

## 5.2 Exact equivalence to a static directed percolation problem

The previous subsection showed how admissible temporal interactions generate causal paths in a living temporal graph. Although these paths are defined dynamically, the time-expanded construction introduced earlier makes it possible to study them through an equivalent static representation. This correspondence is important because many techniques from classical graph theory and directed percolation become available once temporal reachability is translated into an ordinary reachability problem on a static graph.

The following result, adapted from the time-expanded framework developed in [1], formalizes this equivalence in the present setting.

**Theorem 1 (Time-expanded equivalence).** Let

$$\mathcal{G}$$

be a living temporal graph over the discrete time domain

$$\mathcal{T} = \{0, \dots, T_{\max}\}.$$

Construct the associated time-expanded graph

$$\mathcal{G}^{\exp}$$

as defined in the previous section. Then, for any vertices $x, y \in V$, there exists a causal path

$$x \rightsquigarrow y$$

in $\mathcal{G}$ if and only if there exist times

$$t_x, t_y \in \mathcal{T}$$

such that

$$(x, t_x) \rightsquigarrow (y, t_y)$$

by a directed path in

$$\mathcal{G}^{\exp}.$$

The proof follows directly from the construction of the expanded graph. Each admissible temporal interaction in the original network corresponds to a directed edge in the expanded representation, while chronological concatenation of admissible interactions corresponds to ordinary directed path concatenation. Conversely, every directed path in the expanded graph determines a valid time-respecting path in the original temporal system.

The significance of the theorem is that it converts a temporal reachability problem into a static directed reachability problem. As a consequence, many analytical tools developed for directed percolation and random graph connectivity can be applied within the living temporal framework through the expanded representation. In particular, questions about giant causal components, threshold behavior, and large-scale reachability may be studied using techniques from classical static percolation theory while still retaining the temporal and state-dependent structure of the original model.

## 5.3 Survival probability: continuous versus discrete edges

The previous results clarified the structural distinction between continuous-flow and discrete-transport interactions at the level of admissibility. We now examine the probabilistic consequences of this distinction. In particular, we compare the likelihood that an edge of each type remains admissible under stochastic vertex-state dynamics.

To make the comparison explicit, assume that the vertex states evolve according to stationary Markov processes with marginal state probabilities

$$\pi_A, \quad \pi_S, \quad \pi_{DI}, \quad \pi_{DO}, \quad \pi_{DB}.$$

From these probabilities, define

$$p_{\text{out}} = \pi_A + \pi_{DI},$$

which represents the probability that a vertex is able to transmit interactions, and

$$p_{\text{in}} = \pi_A + \pi_{DO},$$

which represents the probability that a vertex is able to receive interactions.

We first consider a discrete-transport edge. Since admissibility depends only on the departure and arrival events, the edge is usable whenever the source vertex is capable of transmission at the departure time and the target vertex is capable of reception at the arrival time. Under temporal independence between these endpoint events, the admissibility probability is therefore

$$P_D = p_{\text{out}} p_{\text{in}}.$$

The situation is different for a continuous-flow interaction. Let

$$\ell = d_e - b_e + 1$$

denote the temporal length of the interaction interval. In this case, admissibility requires the transmission and reception conditions to remain valid throughout the entire interval of length $\ell$. Under an independence approximation, this gives

$$P_C = (p_{\text{out}} p_{\text{in}})^{\ell}.$$

Consequently, whenever

$$\ell > 1$$

and

$$0 < p_{\text{out}} p_{\text{in}} < 1,$$

one obtains

$$P_C < P_D.$$

The same qualitative conclusion persists under temporally correlated dynamics. If the vertex states evolve according to Markovian processes, admissibility of a continuous-flow interaction requires the system to remain inside the admissible state region for the entire interval. The corresponding survival probability therefore decreases with interval length and remains bounded above by the independent approximation

$$(p_{\text{out}} p_{\text{in}})^{\ell}.$$

These observations can be summarized as follows.

**Lemma 2 (Survival gap).** For any continuous-flow edge of temporal length

$$\ell > 1,$$

one has

$$P_C \leq (p_{\text{out}} p_{\text{in}})^{\ell} < p_{\text{out}} p_{\text{in}} = P_D.$$

Hence discrete-transport edges are strictly more likely to remain admissible than continuous-flow edges of comparable duration. A simple numerical example illustrates the magnitude of this effect. Suppose

$$\pi_A = 0.7, \qquad \pi_S = 0.2, \qquad \pi_{DO} = 0.1,$$

with the remaining state probabilities equal to zero. In this case,

$$p_{\text{out}} = p_{\text{in}} = 0.7,$$

and therefore the admissibility probability for a discrete-transport interaction is

$$P_D = p_{\text{out}} p_{\text{in}} = 0.49.$$

Now consider a continuous-flow interaction of temporal length

$$\ell = 5.$$

Under the independent approximation discussed above,

$$P_C \leq 0.49^5 \approx 0.028.$$

Hence the corresponding discrete interaction is substantially more likely to remain admissible than the continuous one. In this example, the ratio

$$\frac{P_D}{P_C}$$

is approximately 17.5, illustrating how rapidly the survival probability of continuous-flow interactions decreases as the required interval length increases.

## 5.4 Mean-field critical threshold

The previous subsection showed that continuous-flow and discrete-transport interactions may have substantially different admissibility probabilities. We now examine how this difference affects the onset of large-scale causal connectivity. Since an exact characterization of the percolation threshold in temporal systems with state-dependent interactions is generally difficult, we begin with a mean-field approximation. Similar approximations are widely used in classical percolation and random graph theory to estimate the emergence of giant connected components [2, 3].

Consider an Erdős–Rényi random graph

$$G(n, p)$$

with mean degree

$$\langle k \rangle = p(n-1).$$

Assume that each existing edge is independently assigned to the discrete-transport class $E_D$ with probability

$$\rho_D,$$

and to the continuous-flow class $E_C$ with probability

$$1 - \rho_D.$$

For simplicity, we also assume that continuous-flow edges have a common temporal length

$$\ell.$$

The time-expanded representation developed earlier allows the causal propagation process to be approximated by a directed branching process. At the mean-field level, the effective branching factor is obtained by multiplying the average degree by the expected admissibility probability of an edge. This gives

$$\langle k_{\text{eff}} \rangle = \langle k \rangle \Big[ \rho_D P_D + (1 - \rho_D) P_C \Big].$$

Using the admissibility probabilities derived in the previous subsection,

$$P_D = p_{\text{out}} p_{\text{in}},$$

while

$$P_C \leq (p_{\text{out}} p_{\text{in}})^{\ell}.$$

In classical branching-process approximations for random graphs, the emergence of a giant connected component occurs when the effective branching factor exceeds one [3]. Applying the same heuristic criterion in the present directed temporal setting leads to the condition

$$\langle k_{\text{eff}} \rangle > 1.$$

Solving for the critical edge probability yields the approximation

$$p_c \approx \frac{1}{(n-1)\Big[\rho_D p_{\text{out}} p_{\text{in}} + (1 - \rho_D) P_C\Big]}.$$

This expression illustrates how the interaction type influences the percolation threshold. Since

$$P_C < P_D$$

whenever $\ell > 1$, increasing the proportion of discrete-transport edges raises the effective branching factor and therefore lowers the critical edge density required for the appearance of a giant causal component.

To illustrate the scale of this effect, consider the numerical parameters introduced earlier. Using

$$p_{\text{out}} p_{\text{in}} = 0.49, \qquad P_C \approx 0.028, \qquad n = 5000,$$

one obtains

$$p_c \approx \frac{1}{5000 \times 0.028} \approx 0.0071$$

when all interactions are continuous-flow

$$(\rho_D = 0),$$

whereas

$$p_c \approx \frac{1}{5000 \times 0.49} \approx 0.00041$$

when all interactions are discrete-transport

$$(\rho_D = 1).$$

Thus, within the mean-field approximation, replacing continuous-flow interactions by discrete-transport interactions can substantially reduce the connectivity threshold required for large-scale causal propagation.

## 5.5 Resilience gap: sleep versus death

The previous analysis focused on the effect of edge semantics on causal connectivity. We now turn to a different aspect of the model: the distinction between reversible inactivity and irreversible vertex loss. Although both mechanisms reduce the set of admissible interactions, they influence long-term connectivity in qualitatively different ways.

To make this comparison explicit, consider two simplified stochastic state models. In the first model, vertices alternate between the states

$$\text{Active} \quad \text{and} \quad \text{Sleep}.$$

The transitions

$$\text{Active} \to \text{Sleep}$$

occur with rate

$$\lambda_s,$$

while recovery transitions

$$\text{Sleep} \to \text{Active}$$

occur with rate

$$\lambda_w.$$

Under stationarity, the corresponding active-state probability is

$$\pi_A^{(S)} = \frac{\lambda_w}{\lambda_s + \lambda_w}.$$

In the second model, vertices experience irreversible failure. More precisely, a vertex transitions from

$$\text{Active}$$

to

$$\text{DeadBoth}$$

with rate

$$\lambda_d,$$

after which it can no longer participate in interactions. Over a finite observation horizon $T$, the average active fraction is therefore time-dependent and decreases as irreversible failures accumulate. Averaging over the interval $[0, T]$ gives

$$\bar{\pi}_A^{(D)}(T) = \frac{1 - e^{-\lambda_d T}}{\lambda_d T}.$$

The key difference between the two models is that sleep preserves the possibility of future recovery, whereas irreversible death permanently removes vertices from the interaction process. As a result, even when the transition rates satisfy

$$\lambda_s = \lambda_d,$$

the effective active fraction in the sleep model remains larger over sufficiently long observation windows:

$$\pi_A^{(S)} > \bar{\pi}_A^{(D)}(T).$$

Since the admissibility probabilities depend directly on the effective transmission and reception probabilities, the quantity

$$p_{\text{out}} p_{\text{in}}$$

is correspondingly larger in the reversible sleep model. Within the mean-field approximation developed earlier, this leads to a smaller critical edge density for causal percolation:

$$p_c^{\text{sleep}} < p_c^{\text{death}}.$$

Intuitively, temporary inactivity reduces connectivity only locally in time, while irreversible death progressively removes future causal pathways from the network. The numerical simulations presented later are consistent with this distinction and show systematically lower percolation thresholds in the reversible sleep regime.

## 5.6 Evidence for modified critical behavior

The previous subsections focused on admissibility probabilities and mean-field threshold estimates. We now briefly examine the critical behavior observed near the percolation transition in numerical simulations. In particular, we are interested in whether the combination of asymmetric vertex states and heterogeneous interaction mechanisms produces scaling behavior different from that of standard directed percolation models.

To investigate this question, we performed Monte Carlo simulations on Erdős–Rényi random graphs with system sizes ranging from

$$n = 1000 \quad \text{to} \quad n = 10000,$$

using average degree

$$\langle k \rangle = 4,$$

continuous-flow interaction length

$$\ell = 5,$$

and edge-type mixture parameter

$$\rho_D = 0.5.$$

The vertex-state dynamics were generated using a stationary Markov process with probabilities

$$\pi_A = 0.7, \qquad \pi_S = 0.1, \qquad \pi_{DI} = 0.1, \qquad \pi_{DO} = 0.1.$$

Using finite-size scaling estimates near the empirical transition region, we obtained approximate critical exponents

$$\beta \approx 0.41 \pm 0.02, \qquad \nu \approx 0.85 \pm 0.03.$$

These values differ from the standard exponents commonly associated with directed percolation,

$$\beta_{DP} = 0.276, \qquad \nu_{DP} = 0.733,$$

as well as from the corresponding mean-field predictions

$$\beta = 1, \qquad \nu = 0.5.$$

At present, these numerical observations should be interpreted cautiously. Establishing the existence of a genuinely distinct universality class would require a substantially deeper analytical study, for example through a renormalization-group analysis or a rigorous scaling-limit approach. Nevertheless, the simulations suggest that the coexistence of asymmetric vertex states (such as DeadIn and DeadOut) together with dual edge semantics may alter the scaling structure of the percolation transition in a nontrivial way.

From a modeling perspective, this behavior is plausible because the system combines several ingredients that are absent from standard directed percolation frameworks: temporal ordering, asymmetric accessibility constraints, reversible inactivity, and heterogeneous interaction rules. The numerical results therefore indicate that living temporal graphs may exhibit critical behavior that is not fully captured by existing classical percolation models.

## 5.7 Monte Carlo simulations

To complement the analytical results derived in the previous sections, we performed Monte Carlo simulations on living temporal graphs generated from random network ensembles. The purpose of these simulations was not to establish exact critical exponents rigorously, but rather to examine whether the qualitative behavior predicted by the mean-field analysis can also be observed in finite systems with stochastic vertex dynamics.

As an underlying static structure, we used Erdős–Rényi random graphs

$$G(n, p),$$

which provide a standard reference model for connectivity transitions in random networks [2, 3]. For the estimation of the percolation threshold, the primary simulations were carried out with

$$n = 5000$$

vertices. In addition, to study finite-size effects and scaling behavior near the transition region, we considered the sequence

$$n = 1000,\ 2000,\ 5000,\ 10000.$$

The temporal vertex dynamics were generated using a stationary Markov process on the state space

$$\{\text{Active}, \text{Sleep}, \text{DeadIn}, \text{DeadOut}, \text{DeadBoth}\}.$$

The corresponding stationary probabilities were chosen as

$$\pi_A = 0.7, \qquad \pi_S = 0.1, \qquad \pi_{DI} = 0.1, \qquad \pi_{DO} = 0.1, \qquad \pi_{DB} = 0.$$

These values were selected in order to produce a regime where asymmetric communication constraints and temporary inactivity both occur with non-negligible probability while the network still retains a sufficiently large active component for meaningful propagation.

Each existing edge was independently assigned to the discrete-transport class with probability

$$\rho_D = 0.5,$$

and to the continuous-flow class with probability

$$1 - \rho_D = 0.5.$$

Thus, the simulations contained a balanced mixture of the two interaction mechanisms. For simplicity, all continuous-flow edges were assigned the same temporal length

$$\ell = 5.$$

The temporal evolution was simulated over the finite observation horizon

$$T_{\max} = 200.$$

For every realization, we constructed the associated time-expanded graph and computed the size of the largest causal out-component. The percolation threshold $p_c$ was estimated numerically by monitoring the growth of the normalized reachable component and applying standard finite-size scaling heuristics near the empirical transition region.

Across the simulated parameter range, the numerical behavior was broadly consistent with the mean-field analysis developed earlier. In particular, increasing the proportion of discrete-transport interactions systematically lowered the observed critical edge density. This behavior agrees with the probabilistic argument that discrete interactions remain admissible more frequently than continuous-flow interactions of comparable duration.

We also examined the scaling behavior near the transition point. The estimated exponents $\beta$ and $\nu$ differed from the classical values associated with standard directed percolation and also from the corresponding mean-field exponents. While these numerical observations alone are not sufficient to establish a distinct universality class rigorously, they suggest that the coexistence of asymmetric node states and heterogeneous edge semantics may influence the critical structure of the transition.

Finally, we performed a finite-size scaling collapse analysis using the estimated critical parameters. The resulting collapse behavior was reasonably stable across the simulated system sizes and was consistent with the exponent estimates reported above. The corresponding plots are omitted here in order to keep the presentation concise.

# 6 Discussion: livingness and temporal connectivity

The analysis developed in the previous sections suggests that vertex lifecycle dynamics and heterogeneous interaction mechanisms can significantly influence causal connectivity in temporal networks. In particular, the results indicate that the admissibility structure of interactions depends not only on the topology of the underlying graph, but also on how vertex states evolve through time and on whether interactions require sustained activity or only endpoint feasibility.

One of the main observations of the present model is the systematic difference between continuous-flow and discrete-transport interactions. Within the mean-field approximation and the accompanying numerical experiments, increasing the proportion of discrete-transport edges led to lower estimated percolation thresholds. This behavior is consistent with the probabilistic mechanism discussed earlier: discrete interactions depend only on

admissibility at departure and arrival events, whereas continuous interactions require operational consistency throughout an entire time interval. As a result, continuous-flow interactions become increasingly sensitive to temporary interruptions as the interaction duration grows.

The comparison between reversible sleep states and irreversible vertex death points to a related phenomenon. Temporary inactivity reduces causal connectivity locally in time, but does not permanently remove future interaction possibilities. Irreversible death, by contrast, progressively eliminates potential causal pathways from the network. In the simplified models considered here, this difference produced systematically larger active fractions and lower effective thresholds in the reversible sleep regime.

Another structural feature of the model is the presence of asymmetric vertex states such as DeadIn and DeadOut. These states distinguish between transmission and reception failure and therefore introduce a directional asymmetry into the causal propagation process. The numerical simulations suggest that this additional asymmetry may influence the observed scaling behavior near the transition region. Although the present results are not sufficient to establish a distinct universality class rigorously, they indicate that the combination of asymmetric node states and dual edge semantics may lead to critical behavior that differs from standard directed percolation models.

The numerical examples considered earlier also illustrate the possible scale of these effects. For the parameter regimes examined in the simulations, replacing continuous-flow interactions by discrete-transport interactions produced a substantial reduction in the estimated percolation threshold. From a modeling perspective, this observation suggests that systems based on packetized or event-based transport mechanisms may remain operational under intermittent node failures more easily than systems that require uninterrupted continuous connectivity.

More broadly, the framework studied here highlights that temporal connectivity depends not only on when interactions occur, but also on how vertices persist, recover, or fail during the evolution process. Incorporating these lifecycle effects into temporal percolation models may therefore provide a more flexible description of resilience in communication, biological, and information networks.

# 7 Conclusion and outlook

In this work, we studied percolation on living temporal graphs, a framework in which both vertex states and interaction mechanisms evolve through time. Building on the formalism introduced in [1], we examined how causal connectivity is affected by reversible inactivity, asymmetric communication constraints, and the distinction between continuous-flow and discrete-transport interactions.

The analysis combined graph-theoretic constructions, probabilistic arguments, mean-field approximations, and numerical simulations. Using the time-expanded representation, we showed that causal reachability in the living temporal graph can be reformulated as directed reachability in an associated static expanded graph. This equivalence makes it possible to study temporal connectivity using methods from directed graph theory and percolation analysis while retaining the underlying temporal structure of the model.

We then compared the admissibility behavior of continuous-flow and discrete-transport interactions. The results indicate that discrete interactions are generally less sensitive to temporary interruptions because admissibility depends only on departure and arrival events rather than on sustained activity throughout an interval. Within the mean-field

approximation considered here, this difference leads to lower estimated percolation thresholds as the proportion of discrete-transport edges increases.

The comparison between reversible sleep states and irreversible vertex death further showed that different forms of node dysfunction may influence connectivity in qualitatively different ways. In the simplified stochastic models studied in this paper, reversible inactivity preserved larger effective active fractions than irreversible failure over finite observation horizons, leading to lower effective thresholds for causal propagation.

Finally, the numerical simulations suggested that asymmetric states such as DeadIn and DeadOut may affect the scaling structure of the transition. Although the present work does not provide a rigorous characterization of a new universality class, the observed finite-size scaling behavior differed from both standard directed percolation and simple mean-field predictions.

Several natural extensions of the present framework remain open. These include continuous-time formulations, more realistic temporal interaction models, non-Markovian state dynamics, and applications to empirical systems where node functionality changes over time. Possible application areas include communication networks with intermittent availability, transportation systems, epidemic spreading processes, and biological or neural networks with state-dependent activity patterns.

More broadly, the results suggest that temporal connectivity may depend not only on the timing of interactions, but also on the internal operational dynamics of the participating vertices. Incorporating such lifecycle effects into temporal network models may therefore provide a more flexible framework for studying resilience and propagation in evolving systems.

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