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Asymptotic Analysis for Pure Dominated Strategy in Random Games

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Research Article

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Xihao Song

Abstract

We study the asymptotic existence and prevalence of dominated strategies in large random games, where players face many possible actions and payoffs are drawn from an underlying distribution. Our main contribution is the introduction and analysis of *q-Portion dominated strategies*, a generalization of classical single-strategy dominance, which considers the elimination of a nontrivial fraction of a player's strategies. We identify sharp thresholds governing the existence of such *q-Portion* dominated strategies. Results on classical single-strategy dominance are covered, providing a unified view of strategic elimination in large games and characterizing how these thresholds depend on the underlying payoff distribution via ρ, the reciprocal of the probability that one strategy outperforms another. Additionally, we propose a simple, distribution-free algorithm that detects dominated strategies more efficiently than naive brute-force search by exploiting payoff extrema. Our results clarify the relationship between single-strategy elimination and large-scale strategic pruning, providing both theoretical insights and practical computational tools.

Keywords: Dominated strategy; Random game; *q-Portion* dominated strategies; Asymptotic probability thresholds; Algorithmic dominance elimination
JEL Classification: C70

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1. Introduction

The concept of a pure dominated strategy is fundamental in game theory. Fixing a player, a strategy is strictly dominated if there exists another strategy that, in every possible scenario, yields a strictly higher payoff ((Gale 1953; Luce and Raiffa 1957; Moulin 1979)). In many real-world decision environments, decision makers face an overwhelming number of possible actions. Identifying and eliminating dominated strategies is, therefore, not only a theoretical exercise but also a practical tool for simplifying large and complex strategic spaces.

In practice, however, identifying strictly dominated strategies or computing undominated sets can be computationally demanding (Gilboa, Kalai, and Zemel (1993); Yu and Zeleny (1975)). Throughout, M denotes the number of strategies available to the row player, N those of the column player, and $q \in (0,1)$ the target fraction of strategies to be eliminated. The naive brute-force approach requires $O(M^2N)$ operations, which becomes prohibitive for large strategy spaces. Moreover, the existence of a single dominated strategy carries limited practical relevance when players face vast strategy spaces. This paper introduces a practically perspective: the existence of a nontrivial fraction of dominated strategies—what we call a *q-portion*—in large random games.

Random games is a game, where payoffs are drawn independently and identically from a given distribution (Goldman 1957). Random games provide a tractable yet rich model of strategic uncertainty, relevant to settings ranging from rapidly changing competitive environments to social and biological systems where precise payoff prediction is impossible (Fudenberg and Harris 1992; May 2001; Gross et al. 2009). Within this framework, we address three interrelated questions:

(a) What is the asymptotic threshold for the existence of a *q-Portion* of strictly dominated strategies as the players' strategy spaces grow?

(b) How do these thresholds relate to the classical question of whether a game contains any strictly dominated strategy?

(c) How to practically efficiently identify the dominated strategy in a fixed game?

We establish sharp asymptotic conditions for both positive and negative existence results. The negative result is obtained via binomial coefficient approximations, while the positive result relies on Chernoff bounds. Specifically, for the row player with M strategies, the probability that a *q-Portion* of strategies is dominated converges to zero as $M, N \to \infty$ whenever

$$N \geq \frac{M}{(\ln M)^{\alpha}}, \quad \text{for some } \alpha > 0,$$

and converges to one whenever

$$M \gg \left(\frac{N}{1-\delta-q}\right)^{N}, \quad \text{for some } \delta > 0.$$

Notably, the positive result is a stronger statement, as we require the *q-Portion* to be dominated by a *single* strategy. The threshold might be sharpened further if this requirement is relaxed.

Alongside this main contribution, we revisit the single-dominated-strategy problem studied by Alon, Rudov, and Yariv (2021) and others. They conjecture that for growth rates $m = o(\ln n)$or $m = \log_2(n) + \omega(1)$, the probability of a strictly dominated strategy exhibits a sharp transition. We confirms their conjecture: there exists a threshold function $f(N) \in \Theta(\ln N)$ such that, as $M, N \to \infty$ with $M > f(N)$, the probability that a game contains no dominated strategy approaches one, and hence the probability of dominance solvability converges to zero. For example, in an $M \times N$ random game with uniformly distributed payoffs, if the growth rate $M(N)$ lies within

$$\left[(2+\delta)\log_2(N),\ 2^{\frac{N}{2+\delta}}\right],$$

for some $\delta > 0$, our results imply that the probability of a strictly dominated strategy, denoted $P(\text{SD})$, approaches zero. Consequently, the probability of dominance solvability also tends to zero, as illustrated in Figure 1 (yellow region).

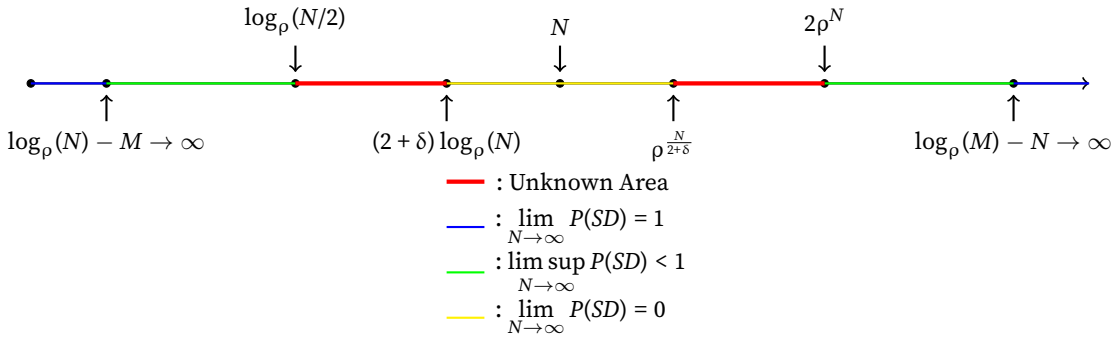


FIGURE 1. Summary of Results on dominated strategy

Note: This figure is as M varies for fixed N

This paper also analyzes the complementary area beyond the thresholds identified above. As illustrated in Figure 1, two additional asymptotic domains emerge as M varies for fixed N. First, when M grows either too quickly or too slowly relative to N (blue regions), the probability of strictly dominated strategies converges to one. Second, in the surrounding intermediate zones (green regions), the probability remains bounded away from one but does not vanish, leaving only two narrow unresolved boundaries (red segments).

A further contribution of this study is to quantify these results in terms of the underlying payoff distribution, captured by ρ in Figure 1. Here, ρ measures how likely one strategy is to outperform another: formally, it is the reciprocal of the probability that a randomly drawn payoff from one strategy exceeds that of another. For non-atomic distributions, $\rho = 2$, while distributions with atoms satisfy $\rho > 2$.

Finally, we propose a simple, distribution-free algorithm for detecting dominated strategies that improves on naive brute-force search by a constant factor. The problem of efficiently identifying

dominated strategies has attracted interest in economics and computer science since Yu and Zeleny (1975), who introduced a more efficient algorithm under special constraints. In the general case, the conventional computational complexity is $O(M^2N)$ for detecting strictly dominated strategies of the row player, since each payoff vector must be compared element-wise across strategies. Our method leverages payoff-vector extrema and range-containment properties to avoid unnecessary comparisons in roughly half of all cases in large games, making it a practical complement to elimination procedures in computational game theory. Beyond the reduction in comparison count, new Algorithm exhibits a substantially more cache-friendly memory access pattern than the baseline method, a consideration of practical importance when the payoff matrix exceeds the capacity of the processor's last-level cache.

The basic insight is straightforward: if the maximum of one payoff vector is smaller than the minimum of another, then the first strategy is strictly dominated. Although such cases become rare as the number of strategies grows, we exploit an additional observation: when the maximum of one vector exceeds the minimum of another, but the range of one payoff vector is strictly contained in the range of the other, neither strategy dominates the other. The probability of this containment event approaches 1/2 as the strategy spaces grow large. In all remaining cases, element-wise comparison is necessary.

In practice, this leads to significant efficiency gains. Since about half of all dominance checks can be resolved without full comparisons, the algorithm avoids unnecessary computation in the majority of large-game instances. Moreover, by precomputing the minimum and maximum of each strategy's payoff vector, subsequent comparisons require no rescanning. By Knuth (1997), the min and max of a set of n elements can be obtained with $3n/2 - 2$ comparisons when n is even and $(3n/2 - 3/2)$ when n is odd. Accordingly, in the efficient cases, at most $M \times (3N/2 - 2) + M(M - 1)$ comparisons are required, rather than $\frac{M(M-1)}{2} \times N$ in the brute-force baseline.

While the asymptotic worst-case complexity of our Algorithm 2 remains $O(M^2N)$, the constant-factor savings and frequent early termination yield substantial practical improvements. First, the minimum and maximum values of each strategy's payoff vector are computed once as a preprocessing step, enabling rapid comparisons later without scanning every element repeatedly. Second, these min/max bounds allow many pairwise dominance checks to be skipped—approximately half of the comparisons can be resolved without full element-wise evaluation—substantially improving computational efficiency in practice.

2. Literature Review

To best of our knowledge, Goldman (1957) is the first study to consider random games in the context of zero-sum games. As random games became a common topic of interest, many other concepts in this framework were investigated, including: value of a random zero-sum game (Thrall and Falk (1965),Thomas (1965),Cover (1966)), Nash Equilibrium (Goldberg, Goldman, and Newman (1968), Dresher (1970), Powers (1990), Rinott and Scarsini (2000), Bárány, Vempala, and Vetta (2007),

McLennan (2005)), best-response dynamics (Galla and Farmer (2013), Sanders, Farmer, and Galla (2018), Pangallo, Heinrich, and Doyne Farmer (2019), Heinrich et al. (2021), Amiet et al. (2021)), or equilibrium properties in evolutionary games (Gokhale and Traulsen (2010), Han, Traulsen, and Gokhale (2012), Galla and Farmer (2013), Gokhale and Traulsen (2014), Duong and Han (2016b), Duong and Han (2016a)). Also, modeling payoffs as independent random variables enables tractable, average-case analysis across diverse settings. For example, using random games is useful to model and understand social and biological systems in which very limited information is available, or where the environment changes so rapidly and frequently that one cannot predict the payoffs of their inhabitant (Fudenberg and Harris (1992), Han, Traulsen, and Gokhale (2012), May (2001), Gross et al. (2009)). In addition, some experimentalists randomly select games to test theories(Erev et al. (2007)).

Despite this broad literature, strictly dominated strategies themselves have rarely been the primary focus. The closest one is by Pei and Takahashi (2019). They examine point-rationalizable and rationalizable strategies in random games, showing that all strategies are rationalizable with probability one as n approaches infinity in both symmetric and asymmetric games when the two players' strategy spaces are of the same order. Specifically, they show that all strategies survive iterated elimination of strategies strictly dominated by pure strategies with probability close to one when the strategy space is large in a symmetric random game. Our work complements theirs by identifying explicit imbalance thresholds between players' action sets that ensure the existence of dominated strategies, and by extending the scope from "any" dominated strategy to q-portions. Notably, we analyze the upper bound of the probability that a strategy is rationalizable, as a rationalizable strategy cannot be strictly dominated by any pure strategy. Our main contribution is determining the threshold of how "imbalanced" a game must be to ensure the existence of strictly dominated strategies or "q-portion" dominated strategies.

The other relevant paper is Alon, Rudov, and Yariv (2021), which demonstrated that the probability of a two-person random game being dominance-solvable decreases as the number of actions increases, aligning closely with our results. They prove that the probability of dominance solvability tends to zero when strategy spaces are balanced, and conjecture that for growth rates $m = o(\ln n)$or $m = \log_2(n) + \omega(1)$, the probability of a strictly dominated strategy exhibits a sharp transition. We confirm their conjecture for certain growth regimes, providing a precise cutoff for the existence of at least one dominated strategy, and generalize the question to the existence of q-portions—offering a more practically relevant diagnostic for large games.

Finally, from a computational perspective, the identification of dominated strategies has long been recognized as a bottleneck in game-theoretic analysis (Yu and Zeleny 1975). While specialized algorithms can exploit structural constraints to improve efficiency, our proposed distribution-free elimination shortcut offers a general-purpose improvement without assuming special structure—making it applicable across the random-game framework and beyond.

3. Preliminary

Throughout this paper, we restrict attention to pure strategies. The extension to mixed-strategy dominance is discussed in the conclusion as a direction for future work.

A *two-player finite game* is defined as

$$G = \{S, T, u_1, u_2\},$$

where S and T are the strategy sets for players 1 and 2 with cardinalities $|S| = M$ and $|T| = N$, and $u_i : S \times T \to \mathbb{R}$ denotes the payoff function for player $i = 1, 2$. The game can equivalently be represented by a pair of payoff matrices

$$R_{M\times N} = \{R^1_{M\times N}, R^2_{M\times N}\},$$

where any entry $(u_1(s_i, t_j), u_2(s_i, t_j)) \in R_{M\times N}$ corresponds to the payoffs of players 1 and 2 under the strategy profile $\mathbf{s} = (s_i, t_j)$:

$$R_{M\times N} = \left\{ \begin{bmatrix} u_1(s_1,t_1) & \cdots & u_1(s_1,t_N) \\ \vdots & \ddots & \vdots \\ u_1(s_M,t_1) & \cdots & u_1(s_M,t_N) \end{bmatrix}, \begin{bmatrix} u_2(s_1,t_1) & \cdots & u_2(s_1,t_N) \\ \vdots & \ddots & \vdots \\ u_2(s_M,t_1) & \cdots & u_2(s_M,t_N) \end{bmatrix} \right\}. \tag{1}$$

In this work, we consider a *random game* G, where the entries of $R^1_{M\times N}$ and $R^2_{M\times N}$ are realizations of i.i.d. random variables $(X^i_{\mathbf{s}})_{i=1,2,\mathbf{s}\in S\times T}$ drawn from a common distribution on $[0, 1]$. We define

$$\rho = \left[P(X^i_{\mathbf{s}} > X^i_{\mathbf{s}'})\right]^{-1},$$

for any pair of random variables $X^i_{\mathbf{s}}$ and $X^i_{\mathbf{s}'}$. For instance, $\rho = 2$ for a non-atomic distribution, and $\rho > 2$ for distributions with atoms.

A strategy $s_i \in S$ (or $t_j \in T$) is said to be *strictly dominated* for player 1 (or 2) if there exists another strategy $s'_i \in S$ (or $t'_j \in T$) such that

$$u_1(s'_i, t) > u_1(s_i, t) \quad \forall t \in T \quad (\text{or } u_2(s, t'_j) > u_2(s, t_j) \quad \forall s \in S).$$

We denote strict dominance by $s'_i \succ_1 s_i$for player 1; similarly, $t'_j \succ_2 t_j$ for player 2.

Let $SD(G)$ denote the set of strictly dominated strategies in game G, with cardinality $\mathbf{card}(SD(G))$. For brevity, we use SD^1 and SD^2 to denote the number of strictly dominated strategies for players 1 and 2, respectively. We define

$$P(\neg SD) = P(\mathbf{card}(SD(G)) = 0)$$

as the probability that no strictly dominated strategy exists. Due to independence of $(X^i_{\mathbf{s}})_{i=1,2,\mathbf{s}\in S\times T}$,

we have

$$P(\neg SD) = P(SD^1 = 0)\, P(SD^2 = 0).$$

We note a conceptual distinction: while individual payoffs may be unpredictable in rapidly changing environments, the analyst may nonetheless know the distribution from which payoffs are drawn—a weaker informational requirement. Our model assumes knowledge of the distribution but not of realized payoffs, consistent with the random-game framework of Goldman (1957) and subsequent literature.

We will focus on asymptotic behavior of strictly dominated strategies because the specific distribution of strictly dominated strategies is far too cumbersome to calculate regarding M and N (see Hammett and Pittel (2008); Gunby and Pálvölgyi (2019); Alon, Rudov, and Yariv (2021)). The reason it's difficult is that the domination relationships are complex. Strategy A might dominate Strategy B. Strategy C might also dominate Strategy B. Determining the final set of undominated strategies requires considering all these relationships simultaneously. Notably, even for a relatively simple case (M=3), just finding the probability that the other player (Column) has no strictly dominated strategies is equivalent to a known difficult mathematical problem ("permutation avoidance") that doesn't have an explicit, simple formula (Alon, Rudov, and Yariv (2021) Online Appendix).

4. Asymptotic Existence of *q-Portion* Dominated Strategies

In very large games, it is often more practical and meaningful to eliminate a fraction of strategies rather than a single strategy. Accordingly, we focus on the existence or nonexistence of *q-Portion* dominated strategies. In this section, we consider only the ordinal comparison between payoffs, so the results hold under any continuous distribution. For simplicity, the subsequent arguments and proofs are presented assuming a uniform distribution.

DEFINITION 1. *For any subset $S' \subseteq S$, we say that S' is* strictly dominated *if for every $s' \in S'$ there exists some $s \in S$ such that $s \succ s'$.*

For any $x \in \mathbb{R}^+$, let $\lceil x \rceil = \min\{z \in \mathbb{Z}^+ : x \leq z\}$. Whenever $q \in (0,1)$, define $P^1_{q,M,N}$ as the probability that there exists a subset $S_{qM} \subset S$ of size $\lceil qM \rceil$ that is strictly dominated for player 1 in the game $R_{M,N}$.

For a fixed choice of S_{qM}, let

$$E^k(S_{qM}) = \left\{ \exists\ {}_kS \subseteq S \setminus S_{qM} \ :\ |{}_kS| = k \text{ and } {}_kS \text{ strictly dominates } S_{qM} \right\}.$$

By definition, the probability that there exists some subset $S_{qM} \subset S$ which is strictly dominated by another subset of strategies can be expressed as

$$P\left[\exists S_{qM} \subset S,\ \exists S' \subset S \text{ s.t. } S_{qM} \text{ is strictly dominated by } S'\right] = \binom{M}{qM} \bigcup_{k=1}^{qM} P\left[E^k(S_{qM})\right].$$

In particular, let $P^1_{q,M,N,\downarrow}$ denote the probability that a *specific block* of $\lceil qM \rceil$ strategies (for example, the bottom rows) is strictly dominated for player 1 in the game $R_{M,N}$.

PROPOSITION 1. *In an $M \times N$ random game, for any $q \in (0,1)$ and any M:*

(a) $P^1_{q,M,N,\downarrow} \to 0$ *as* $M, N \to \infty$.

(b) $P^1_{q,M,N} \to 0$ *as* $M, N \to \infty$ *whenever* $N \geq M/(\ln M)^\alpha$ *for some fixed* $\alpha > 0$.

Proposition 1 highlights two distinct but related points. First, if we fix the location of the candidate block (e.g., always selecting the bottom $\lceil qM \rceil$ strategies), the probability that these are all strictly dominated vanishes, regardless of how M and N grow. The intuition is straightforward: if a set of rows is strictly dominated, then in every column the maximal payoff cannot come from that set. This immediately bounds $P^1_{q,M,N,\downarrow}$, independent of the growth rate of M and N.

Second, if the location of the $\lceil qM \rceil$ block is unrestricted, the existence of *q-Portion* dominated strategies depends critically on the relative growth rates of M and N. Specifically, when $N \geq M/(\ln M)^\alpha$ for some fixed $\alpha > 0$,, the probability of such dominated portions vanishes. Importantly, this condition covers, among others, all linear and faster growth regimes of N relative to M.

Lastly, the argument relies only on ordinal comparisons, the result holds for any continuous payoff distribution, showing that it captures a structural feature of large random games rather than an artifact of a particular model. Compared to the literature on striclty domianted strategy(Pei and Takahashi (2019), Alon, Rudov, and Yariv (2021), etc), our focus is deliberately narrower but relevant in large game—existence of (fractions of) dominated strategies—yet the order-independence of iterated deletion under strict dominance in finite games ensures that "one-shot" removal of currently dominated actions yields the same reduced game as sequential deletion.

The proof proceeds by characterizing the event $E^k(S_{qM})$ via partitions of S_{qM}. For each partition, we examine whether a corresponding subset ${}_kS$ strictly dominates it, and then derive upper bounds on $P[E^k(S_{qM})]$. Applying logarithmic transformations and approximating via Shannon entropy yields that $P^1_{q,M,N} \to 0$ under the stated growth condition.

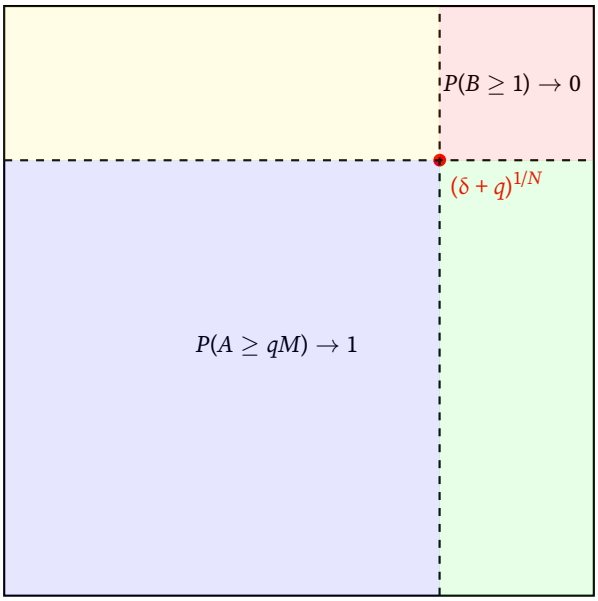


FIGURE 2. Chernoff Bound of Event A and B

Finally, note that when M grows much faster, the situation changes dramatically: there will almost surely exist a block of size qM that is strictly dominated by a single strategy. Figure 2 illustrates this intuition. Interpreting the entries of the payoff matrix as i.i.d. random draws, the problem reduces to showing that with high probability one strategy falls in the red region while at least qM others fall in the purple region, namely

$$x_{ij} \leq (\delta + q)^{1/N}, \qquad \forall j = 1, \ldots, N.$$

That is,

$$P\Big(\exists S' \subseteq S,\ |S'| \geq qM,\ \forall x \in S',\ x_{ij} \leq (\delta + q)^{1/N},\ \forall j = 1, \ldots, N\Big).$$

We now quantify the probability of finding a q-fraction of dominated strategies using concentration inequalities. (Let $X = \sum_{i=1}^{n} X_i$, let all X_i are independent and $X_i = 1$ with probability p_i. Then Chernoff bound gives us 1) $P(X \geq (1+\delta)E[X]) \leq e^{-\frac{\delta^2}{2+\delta}E[X]}$ for all $\delta > 0$; 2) $P(X \leq (1-\delta)E[X]) \leq e^{-\frac{\delta^2 E[X]}{2}}$ for all $0 < \delta < 1$.) The key idea is to separate two events: (i) the existence of at least qM "weak" rows, whose entries all fall below the threshold $(\delta + q)^{1/N}$, and (ii) the existence of at least one "strong" row, whose entries all exceed this threshold. Together, these two events imply the existence of a block of qM strategies strictly dominated by a single strategy.

For each row i, let

$$A_i = \{x_{ij} \leq (\delta + q)^{1/N},\ \forall j = 1, \ldots, N\}, \quad B_i = \{x_{ij} > (\delta + q)^{1/N},\ \forall j = 1, \ldots, N\}.$$

Define the counts

$$A = \sum_{i=1}^{M} \mathbf{1}_{A_i}, \quad B = \sum_{i=1}^{M} \mathbf{1}_{B_i},$$

where $\mathbf{1}_{A_i}, \mathbf{1}_{B_i}$ are indicator variables. By independence across entries,

$$\mathbb{E}[A] = M(\delta + q), \quad \mathbb{E}[B] = M\big(1 - (\delta + q)^{1/N}\big)^N.$$

PROPOSITION 2. *In an $M \times N$ random game with i.i.d. uniform payoffs, for any $q \in (0,1)$ and any M:*

(a) $P(A \geq qM) \to 1$ *as* $M, N \to \infty$.

(b) $P(B \geq 1) \to 1$ *as* $M \to \infty$, $N \to \infty$, *provided* $M \gg (N/(1-\delta-q))^N$ *for some fixed* $\delta > 0$.

Proposition 2 shows that with high probability, as $M, N \to \infty$ and M grows sufficiently fast relative to N, there exists a block of qM rows (strategies) that are uniformly weak, all strictly dominated by a single uniformly strong row. Thus, a nontrivial q-fraction of strategies can be eliminated in one step of strict dominance. This conceptual contrast fits naturally into the q-portion framework introduced earlier. By focusing on the fraction of strategies rather than individual ones, our analysis captures both extremes of the asymptotic behavior and provides a unified perspective on strategic pruning in large random games. It emphasizes that even partial elimination of dominated strategies can meaningfully simplify strategic analysis, bridging the gap between theoretical results and practical applicability.

The two propositions highlight a clear difference in the behavior of q-portion dominated strategies. Together, these results show two clear asymptotic regimes: one where q-portion domination is almost impossible, and one where it occurs with high probability (Figure 3). In the black area, no *q-Portion* dominated strategies exist for player 1, regardless of the value of q. It is important to note that this holds even when M grows faster than a linear growth rate with respect to N. In the green area, *q-Portion* dominated strategies always exist for player 1, and they are dominated by a single strategy. Similarly, the red area remains unknown. By symmetric, we can easily obtain the partial characterization of the growth rate required for the existence of *q-Portion* dominated strategies for player 2. The intermediate regime, where M and N grow at a similar rate, is more subtle and requires further analysis.

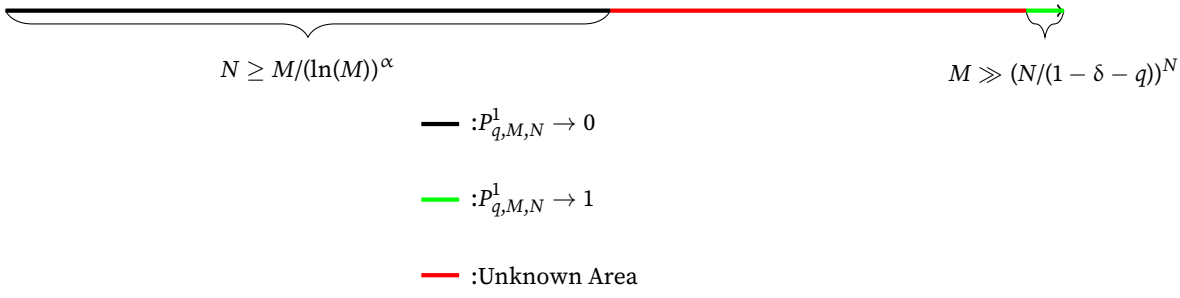


FIGURE 3. Summary of Results on *q-Portion* dominated strategies

Note: This figure is as M varies for fixed N

By the order-independence of iterated strict dominance (Dufwenberg and Stegeman (2002)), one may apply $q-$portion elimination iteratively. In each round, the current game has $M' = (1-q)M$ strategies for the row player. Our threshold conditions apply to each reduced game. Provided the game remains sufficiently unbalanced after each round—i.e., $M' \gg (N/(1-\delta-q))^N$-*q-Portion* elimination succeeds again with high probability. This yields a sequence of reductions that terminates in $O(log_{\frac{1}{1-q}} M)$ rounds, exponentially collapsing the strategy space.

Our result establishes that with high probability, the *q-portion* is dominated by a single strategy. This is a stronger structural statement than mere existence of domination by some subset, and provides a sufficient condition that is easier to verify computationally. Relaxing the single-dominator requirement to domination by any subset would sharpen the positive threshold and we conjecture the threshold lies strictly below our current condition. Characterizing this boundary remains an open problem.

5. On the Existence of a Strictly Dominated Strategy with any probability distribution

Having analyzed the specific case where a fixed fraction of strategies are dominated (the *q-Portion* framework), we now turn to the more general question: under what conditions does a game contain any dominated strategy at all? Formally, recall that $P(SD)$ denotes the probability that a dominated strategy exists, and $P(\neg SD) = 1 - P(SD)$ denotes the probability that no dominated strategy exists. This probability is central for both theory and applications: in game theory, it speaks to the robustness of iterative elimination as a solution concept; in economics and management science, it reflects whether decision makers can expect unambiguously inferior choices to be eliminated in large, complex environments.

Two observations provide immediate bounds:

(a) If there exists a pair (s_i, s_j) with $s_i \succ s_j$, then $P(\neg SD) = 0$. Hence,

$$P(\neg SD) \leq 1 - P(\text{for some } s_i,\ \exists s_j,\ s_i \succ s_j).$$

(b) Conversely, if no pair of strategies is related by strict domination, then no dominated strategy exists. Hence,

$$\max\left\{1 - \sum_{s_i \not\succ s_j} P(s_i \succ s_j), 0\right\} \cdot \max\left\{1 - \sum_{t_i \not\succ t_j} P(t_i \succ t_j), 0\right\} \leq P(\neg SD).$$

This motivates three natural questions: (i) under what growth conditions on M and N do these bounds converge to one? (ii) when does the upper bound fail to converge to one? (iii) when do both bounds converge to zero?

PROPOSITION 3. *In a $M \times N$ random game,*

(a) $P(SD) \to 0$ *if* $M, N \to \infty$ *with* $M = N$.

(b) $P(SD) \to 0$ *if* $M, N \to \infty$ *with* $(2+\delta) log_\rho(N) \leq M \leq \rho^{\frac{N}{2+\delta}}$ *for some* $\delta > 0$.

Proposition 3 shows that as the game grows, the probability of observing any dominated strategy vanishes under broad conditions on the relative growth of M and N. Intuitively, in sufficiently large and balanced games, the "curse of dimensionality" makes it unlikely that one strategy uniformly outperforms another across all opponents' choices. The first statement is a special case of the second.

For instance, in the square case ($M = N$) with uniform payoffs, the probability that one strategy strictly dominates another is $1/2^N$. Hence,

$$P(\neg SD) \;\geq\; \left(\max\left\{1 - N^2 \cdot \frac{1}{2^N}, 0\right\}\right)^2,$$

which converges to one as $N \to \infty$. The second statement generalizes this reasoning to broader growth rates, using bounds based on ρ, the distributional likelihood ratio defined earlier.

A natural corollary is that *dominance solvability* itself becomes vanishingly rare in large games.

COROLLARY 1. *In a* $M \times N$ *random game,* $P($*Dominance Solvability*$) \to 0$ *if* $M, N \to \infty$ *with* $(2+\delta) log_\rho(N) \leq M \leq \rho^{\frac{N}{2+\delta}}$, *for some* $\delta > 0$.

This result emphasizes that the classical dominance-solvability paradigm, while elegant in small games, becomes increasingly uninformative in large random environments. Iterative elimination cannot be expected to simplify the game substantially when M and N grow at comparable or moderate rates.

To further refine these results, we establish the near-tightness of the above thresholds.

PROPOSITION 4. *In a* $M \times N$ *random game,*

(a) $\limsup P(SD) < 1$ *if* $M, N \to \infty$ *with* $N \leq \log_\rho(M/2)$.

(b) $P(SD) \to 1$ *if* $M, N \to \infty$ *with* $\log_\rho(M) - N \to \infty$.

The first part shows that if the number of opponent strategies grows slowly relative to M, then the probability of dominated strategies remains bounded away from one. The second part shows the opposite: if M grows sufficiently faster than N, then dominated strategies are almost certain to appear. Together, Propositions 3 and 4 sharply delineate the parameter regions where dominated strategies vanish, persist, or become inevitable. Figure 1 summarizes this landscape. Notice, there is a small uncovered area in the line as shown in Figure 1.

To prove the first statement in Proposition 4, we rely on the inclusion and exclusion equation of $P(\neg SD) \leq 1 - P$ (for some s_i, s_i is dominated)= $\lim_{M,N\to\infty}(1 - \binom{N-1}{1}/\rho^M + \binom{N-1}{2}/(\rho^M)^2 - \cdots - (-1)^{N-2}\binom{N-1}{N-1}/(\rho^M)^{N-1})$, where $\rho = [P(X^i_{\mathbf{s}} > X^i_{\mathbf{s}'})]^{-1}$. We focus on the function $f(k) = \binom{N-1}{k}/(\rho^M)^k$ and find the condition such that $f(k+1)/f(k) \leq \alpha < 1$ for each k and $\liminf_{M,N\to\infty} f(1) > 0$. As a result, the probability will be positive and less than 1. To prove the second statement in Proposition 4, we

will divide the game into Z equal-sized subgames such that all subgames will satisfy the conditions given in part (a).

By the first statement, each such subgame has at most probability $\eta < 1$ of lacking any strictly dominated strategy, Since these subgames are generated independently, the probability that all Z subgames simultaneously lack a strictly dominated strategy is at most η^Z. Hence, the probability that the original large game contains no strictly dominated strategy approaches zero as well.

Now, we simulate random normal-form games of all sizes from 2×2 up to 20×20. For each size, we generated 100,000 payoff matrices at random, and used a 'brute-force' algorithm to test whether either player has a strictly dominated strategy (the algorithm 1 in Section 6). Finally, for each game size, we recorded the proportion of games in which a strictly dominated strategy was present.

Figure 4 displays the result. In the right graph, the x-axis represents the number of strategies for player 2 (n), the y-axis represents the number of strategies for player 1 (m), and the z-axis represents the probability of strictly dominated strategies. It can be observed that the $P(\neg SD)$ approaches 1 with increasing values of n and m within the yellow region. However, $P(\neg SD)$ does not approach 1 in other areas of the graph. In the left one, we point out the specific growth rate $N = 2.01l\,og_2(M)$ and $N = l\,og_2(M/2)$, similar to the row player. The probability will not approach 1 in the large game if the growth rate is slower than $N = l\,og_2(M/2)$ and will approach 1 if the growth rate is faster than $N = 2.01l\,og_2(M)$, which is consistent with our analysis. Note that Pei and Takahashi (2019) also stimulate the symmetric (Figure 4 and Proposition 5), which is exactly the '45 degree line' of this graph.

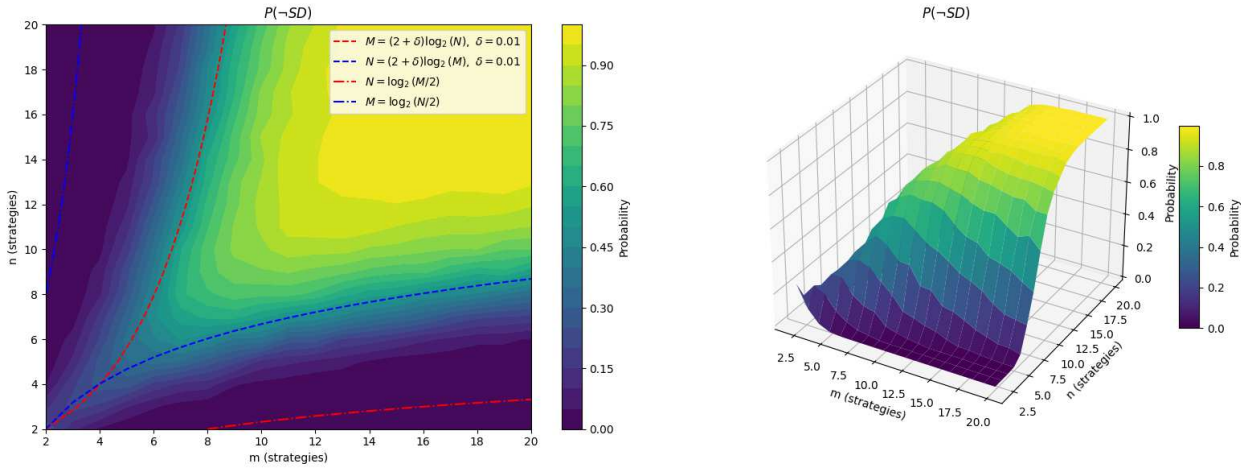


FIGURE 4. Simulation Result for Propositions 1 and 2

6. Computational Methods and Practical Efficiency

While Sections 4–5 establish when dominated strategies exist asymptotically, Section 6 addresses how to detect them efficiently in finite games, hence we focus on the setting in Section 5. Algorithm 1 detects the existence of any strictly dominated strategy (the setting of Section 5), while the threshold

results of Section 4 govern when q-portion elimination succeeds. Algorithm 2 is designed for the Section 5 setting.

To determine the existence of strictly dominated strategies in a game, a comparison of each strategy's payoffs against all others for every player is required. This process factors in every strategy of the opponents as in Algorithm 1, and it's the $O(M^2 \times N)$ problem. We aim to find a simple algorithm that is faster, at least by a constant factor, than the most direct brute-force algorithms. Such an algorithm could potentially be combined with existing ones to achieve even greater efficiency.

Algorithm 1 Baseline Method

Input: Payoff matrix of a game $R_{M\times N}$

Output: A boolean value indicating if the game has a strictly dominated strategy or not

1: ▷ For player 1 (or 2), for any strategy $s_i \in S$ (or $t_i \in T$), for each other strategy $s_j \in S$ (or $t_j \in T$), compare the payoffs of strategy s_i, s_j (or t_i, t_j) for all $t \in T$ (or $s \in S$).

2: ▷ If strategy $s_i \in S$ (or $t_i \in T$) is not dominated by all the other strategies, then denote $s_i \in S$ (or $t_i \in T$) as an undominated strategy.

3: ▷ If all strategies are undominated strategies, then the game does not have a strictly dominated strategy.

4: **Return** A boolean value indicating if the game has a strictly dominated strategy or not

Assume we have the matrix $R^1_{M\times N}$, and we mark the maximum as red and the minimum as green. Comparing the maximum and minimum of any two rows, i and j, we can claim Proposition 5:

(2)

$$R^1_{M\times N} = \begin{bmatrix} R^1_{M\times N}(s_1) \\ R^1_{M\times N}(s_2) \\ \vdots \\ R^1_{M\times N}(s_M) \end{bmatrix} = \begin{bmatrix} \dots & u_1(s_1,t_p) & \dots & u_1(s_1,t_q) & \dots & u_1(s_1,t_k) & \dots & u_1(s_1,t_l) & \dots \\ \dots & u_1(s_2,t_p) & \dots & u_1(s_2,t_q) & \dots & u_1(s_2,t_k) & \dots & u_1(s_2,t_l) & \dots \\ \vdots & & & & \vdots & & & & \vdots \\ \dots & u_1(s_M,t_p) & \dots & u_1(s_M,t_q) & \dots & u_1(s_M,t_k) & \dots & u_1(s_M,t_l) & \dots \end{bmatrix}$$

PROPOSITION 5. *In above problem, for any two different $s_i, s_j \in S$, we have*

(a) $P(s_i \succ_1 s_j \,|\, \min(s_i) > \max(s_j)) = P(s_i \succ_1 s_j \,|\, \min(s_j) > \max(s_i)) = 0$. *The union of those two events will happen with probability $2/\binom{2N}{N}$ and will go to 0 as N go to infinity.*

(b) $P(s_i \sim_1 s_j \,|\, \max(s_i) > \max(s_j) > \min(s_j) > \min(s_i)) = P(s_i \sim_1 s_j \,|\, \max(s_j) > \max(s_i) > \min(s_i) > \min(s_j)) = 1$. *The union of two events will happen with probability $(2N-2)/(4N-2)$ and will go to $1/2$ as N go to infinty.*

(c) $P(\max(s_i) > \max(s_j) > \min(s_i) > \min(s_j)) = P(\max(s_j) > \max(s_i) > \min(s_j) > \min(s_i)) = N/(4N-2) - 1/\binom{2N}{N}$. *The union of those two events will go to $1/2$ as N go to infinity.*

Proposition 5 states that if $\min(s_i) > \max(s_j)$, then s_i must strictly dominate s_j for any strategies s_i and s_j. If $\max(s_i) > \max(s_j)$ and $\min(s_j) > \min(s_i)$, then there is no domination between s_i and s_j.

However, in the last situation, we cannot conclude strict domination. Since this result relies only on the ordinal comparison between the elements of s_i and s_j, regardless of the underlying probability distribution, we can show that the last case occurs with a probability of 1/2. Hence, there is a half chance that we can tell the dominance relation between two vectors without comparing element by element. The key here is the combinatorics involved in the ordering of elements in the vectors, and how the relationships between their maximum and minimum elements affect these probabilities. The general intuition is that the probability of these specific orderings of elements in the vectors decreases as the vector size N grows. As N approaches infinity, the probabilities tend to the limits stated in this proposition.

Algorithm 2 MaxMin and MinMax

Input: Payoff matrix of a game
Output: A boolean value indicating if the game has a strictly dominated strategy or not

1: ▷ For player 1 (or player 2), for any strategy $s_i \in S$ (or $t_i \in T$), for each other strategy $s_j \in S$ (or $t_j \in T$),
2:

- If $\min(s_i) > \max(s_j)$ or $\min(s_j) > \max(s_i)$, then return there exist strictly dominated strategy and the algorithm stops.
- If $\max(s_i) > \max(s_j) > \min(s_j) > \min(s_i))$ or $\max(s_j) > \max(s_i) > \min(s_i) > \min(s_j)$, then there is no dominance relation between s_i and s_j.
- Otherwise, compare the payoffs of strategy s_i, s_j (or t_i, t_j) for all $t \in T$ (or $s \in S$). If strategy $s_i \in S$ (or $t_i \in T$) is not dominated by all the other strategies, then denote $s_i \in S$ (or $t_i \in T$) as an undominated strategy.

3: ▷ If all strategies are undominated strategies, then the game does not have a strictly dominated strategy.
4: **Return** A boolean value indicating if the game has a strictly dominated strategy or not

Algorithm 2 is deterministic that it requires no distributional assumptions and produces exact answers. Although the last case will dominate the time complexity, Algorithm 2 is more efficient in practice because of its potential for early termination via the MinMax/MaxMin checks, leading to fewer comparisons on average. While a constant-factor improvement does not alter worst-case complexity class, its practical significance is substantial in the relevant parameter regime. For any comparison of two strategies, there is approximately a 50% chance of the event that we do not need to compare them element by element. For player 1, Algorithm 2 checks the minimum and maximum of each row in the payoff matrix with $3N/2 - 2$ comparisons when N is even and with $3(N-1)/2$ comparisons when N is odd (Knuth (1997)). Therefore, this algorithm requires less computation because 1) we compare $M \times (3N/2 - 2) + M(M-1)$ times, 2) and we compare $M(M-1)/2 \times N$ times in the last case (in 50%).

This is primarily due to two factors. First, the computation of the minimum and maximum payoff values for each strategy vector requires $\Theta(N)$ operations per vector and is performed once as a preprocessing step. These precomputed values enable constant-time comparisons during subsequent pairwise dominance checks, effectively amortizing the initial cost over multiple comparisons. This contrasts with the baseline method, which requires up to $\Theta(N)$ comparisons for each pair without such amortization.

Second, leveraging the min/max bounds allows for early pruning of roughly 50% of the pairwise comparisons, as dominance can often be established or ruled out without full element-wise evaluation. This early termination substantially reduces the average number of payoff comparisons, leading to notable improvements in computational efficiency in practice.

Beyond the reduction in comparison count, Algorithm 2 exhibits a substantially more cache-friendly memory access pattern than the baseline method, a consideration of practical importance when the payoff matrix exceeds the capacity of the processor's last-level cache. The baseline algorithm, when evaluating a pair (s_i, s_j), performs interleaved reads from rows i and j of the payoff matrix. When $|i - j|$ is large, these rows reside in non-adjacent memory locations separated by $|i - j| \times N \times 8$ bytes (for double-precision entries), resulting in cache misses that can dominate actual runtime at scale. Algorithm 2, by contrast, separates computation into two structurally distinct phases. In the preprocessing phase, each row is scanned sequentially once to compute its minimum and maximum; sequential memory access patterns are optimal for hardware prefetching and incur minimal cache miss overhead. In the comparison phase, approximately half of all pairwise checks—those falling under case (b) of Proposition 5—require only four scalar lookups from the precomputed min[·] and max[·] arrays. For M strategies, these arrays occupy $2M \cdot 8$ bytes. When $M = 10^4$, this is 160 KB, well within the L2 cache capacity of contemporary processors. Consequently, for this half of pairwise comparisons, the algorithm accesses no matrix entries whatsoever. The practical implication is that the effective memory bandwidth requirement of Algorithm 2 is approximately half that of the baseline in the comparison phase, which can yield runtime improvements exceeding the theoretical $2\times$ comparison reduction in large-game instances where memory latency rather than arithmetic throughput is the binding constraint.

7. Conclusion

This paper provides a systematic analysis of strictly dominated strategies in large random games. While an explicit distribution of dominated strategies cannot be derived in full generality, we established asymptotic bounds that clarify when dominated strategies are absent with high probability. In particular, Proposition 3 shows that the probability of having no strictly dominated strategy approaches one as both players' strategy sets grow large. This further implies sharp results on the likelihood of dominance solvability. Complementing this, Proposition 4 identifies regions in which strictly dominated strategies are guaranteed to appear with positive probability, highlighting the role of the underlying payoff distribution in shaping these thresholds.

Beyond the existence of pure dominated strategies, we introduced the novel concept of *q-Portion dominated strategies*, where a nontrivial fraction of strategies is dominated. This generalization better approximates realistic decision environments, where players face large action spaces and can more easily identify the portion of dominated strategies. We show in Proposition 1 that *q-Portion* dominated strategies fail to exist even when the number of strategies grows faster than linearly, but in Proposition 2 we established that such strategies are dominated by a single strategy

when the game becomes sufficiently unbalanced. Together, these results delineate the parameter regions where large-scale strategic pruning is possible, extending classical results on single-strategy dominance.

Our analysis also contributes to the algorithmic literature on identifying dominated strategies. Since Yu and Zeleny (1975), efficient elimination procedures have been of interest in both economics and computer science. While earlier results required structural constraints, we provide a simple algorithm that avoids such assumptions and reduces the time complexity compared to the baseline brute-force approach. This has implications for both theoretical game analysis and applied fields such as operations research, management science, and economics, where the ability to prune strategies efficiently can improve decision-making in complex environments.

Our findings open several promising directions. From a theoretical standpoint, refining the thresholds for the existence of *q-Portion* dominated strategies remains an important challenge, particularly when strategies are dominated by more than one alternative. From a practical perspective, our results provide a foundation for simplifying strategic environments in management and economics, where decision makers often face overwhelming complexity. Relaxing the assumption of independent and identically distributed payoffs, or extending the analysis to random extensive-form games, would further broaden applicability. Finally, improving algorithmic efficiency for detecting large sets of dominated strategies remains an essential task, with potential impact on computational economics, operations research, and real-world strategic planning.

In summary, our results advance both the theoretical understanding and computational tractability of dominated strategies in random games. By linking asymptotic probability bounds, distributional properties, and algorithmic improvements, this work provides new tools for analyzing strategy elimination in economics, management, and related fields where decision-makers face large and uncertain strategic environments.

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Appendix A. Mathematical Appendix

A.1. Proof for Proposition 1

PROOF. (a)

LEMMA A1. *For any $S' \subset S$, if S' is strictly dominated, then for any $j \in N$,* $\arg\max_{i \in M} a_{ij} \notin S'$.

PROOF. We show if there exist some j that $\max_{i \in M} a_{ij} \in S'$, then S' is not strictly dominated. Denote $\hat{s}' \in S'$ be the strategy contains at least one maximum, then it can not be strictly dominated by any strategy in S, thus S' is not strictly dominated.

□

COROLLARY A1. *$P^1_{q,M,N,\downarrow}$ is bounded by $(1-q)^N$ and it will go to 0 when $N.M$ go to infinity.*

PROOF. Given M, N, the probability that the maximum in the random matrix R^1_{MxN} in column j is in S/S_{qM} is $1-q$. Then by i.i.d., we have $P^1_{q,M,N,\downarrow} \leq (1-q)^N$. Hence, $P^1_{q,M,N,\downarrow}$ will go to 0 when $N.M$ go to infinity. □

(b)

LEMMA A2. *For all $1 \leq k \leq qM$, $P\left[E^k(S_{qM})\right] \leq \sum_{\boldsymbol{a}} \frac{(M-qM)!(qM)!}{(M-qM-k)!k!} \frac{1}{\prod_{i=1} \boldsymbol{a}_i!(\boldsymbol{a}_i+1)^N}$, where $\boldsymbol{a} \in \mathbb{Z}^k_+$ s.t. $\sum_{i=1}^{k} \boldsymbol{a}_i = qM$.*

PROOF. For each event $E^k(S_{qM})$, we have the following result

$$E^k(S_{qM}) = \bigcup_{\mathbf{a}} \left\{ \begin{array}{l} \{\text{pick } \mathbf{a}_1 \text{ strategies } (\equiv T_1) \text{ from } S_{qM}, \quad \text{s.t. first strategy in } {}_kS \text{ strictly dominates } T_1\} \\ \bigcap \{\text{pick } \mathbf{a}_2 \text{ strategies } (\equiv T_2) \text{ from } S_{qM} \setminus T_1, \quad \text{s.t. second strategy in } {}_kS \text{ strictly dominates } T_2\} \\ \vdots \\ \bigcap \{\text{k-th strategy in } {}_kS \text{ strictly dominates remaining } T_k\} \end{array} \right\}$$

We obtain

$$\begin{aligned} P\left[E^k(S_{qM})\right] &\leq \binom{M-qM}{k} \sum_{\mathbf{a}} \binom{qM}{\mathbf{a}_1} \frac{1}{(\mathbf{a}_1+1)^N} \cdot \binom{qM-\mathbf{a}_1}{\mathbf{a}_2} \frac{1}{(\mathbf{a}_2+1)^N} \cdot \binom{qM-\mathbf{a}_1-\mathbf{a}_2}{\mathbf{a}_3} \cdots \frac{1}{(\mathbf{a}_k+1)^N} \\ &= \binom{M-qM}{k} \sum_{\mathbf{a}} \binom{qM}{\mathbf{a}_1, \ldots, \mathbf{a}_k} \cdot \left(\prod_i (\mathbf{a}_i+1)\right)^{-N} \end{aligned}$$

After simplification, we have

$$P\left[E^k(S_{qM})\right] \leq \sum_{\mathbf{a}} \frac{(M-qM)!(qM)!}{(M-qM-k)!k!} \frac{1}{\prod_{i=1}^{k} \mathbf{a}_i!(\mathbf{a}_i+1)^N}.$$

□

For example, if $k = 1$ then $P\left[E^1(S_{qM})\right] = \frac{M-qM}{(qM+1)^N}$ which is the probability that pick one strategy, $s_i^* \in_1 S$ from $S \setminus S_{qM}$ such that $s_i^* \succ s_i$ for all $s_i \in S_{qM}$. If $k = qM$, then $P\left[E^{qM}(S_{qM})\right] = \frac{(M-qM)!}{(M-qM-qM)!2^{qMN}} = \binom{M-qM}{qM}\frac{(qM)!}{2^{qMN}}$ which is the probability that pick qM strategies, S^*, from $S \setminus S_{qM}$ and pick any possible (one to one) matches with ${}_{qM}S$ and S_{qM}, we have $s' \succ s$ where $s' \in {}_{qM}S$ and $s \in S_{qM}$.

Lemma A3. *For any $q \leq \frac{1}{3}$, $\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \to 0$ for any when N, M goes to infinity with $N \geq \frac{M}{(\ln(M))^\alpha}$ for some $0 < \alpha < 1$.*

Proof. We can rewrite upper bound of $P\left[E^k(S_{qM})\right]$ with Binomial coefficients and Multinomial coefficients and obtain:

$$\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \leq \sum_k \binom{M-qM}{k} \sum_{\mathbf{a}} \binom{qM}{\mathbf{a}_1, \ldots, \mathbf{a}_k} \cdot (\prod_i (\mathbf{a}_i+1))^{-N}$$

Also, given k and q, there are $\binom{qM-1}{k-1}$ distinct $\mathbf{a}$ which are positive integer solutions to $\sum_{i=1}^k \mathbf{a}_i = qM$. Now, we have

$$\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \leq qM \binom{M-qM}{k} \binom{qM-1}{k-1} \binom{qM}{\mathbf{a}_1, \ldots, \mathbf{a}_k} (\prod_i (\mathbf{a}_i+1))^{-N}$$

For binomial coefficients, after logarithmic transformations and approximations,

$$\ln(\binom{n}{m}) = n\ln(n) - m\ln(m) - (n-m)l\,n(n-m) \approx n[-(\frac{m}{n})\ln(\frac{m}{n}) - (1-\frac{m}{n})\ln(1-\frac{m}{n})] = n\mathcal{H}(\frac{m}{n})$$

where $\mathcal{H}(\frac{m}{n}) = -\frac{m}{n}\ln(\frac{m}{n}) - (1-\frac{m}{n})\ln(1-\frac{m}{n})$ is the Shannon entropy.

For multinomial coefficient, we make logarithm and according to Lemma 2.2 in Csiszár, Shields et al. (2004), we have $\ln(\binom{qM}{\mathbf{a}_1,\ldots,\mathbf{a}_k}) \approx qM\mathcal{H}(\frac{\mathbf{a}_1}{qM}, \ldots, \frac{\mathbf{a}_k}{qM})$ where $\mathcal{H}$ is also the Shannon entropy $\mathcal{H}$ for a distribution of values $\left(\frac{\mathbf{a}_1}{qM}, \ldots, \frac{\mathbf{a}_k}{qM}\right)$ as follows:

$$\mathcal{H}\left(\frac{\mathbf{a}_1}{qM}, \ldots, \frac{\mathbf{a}_k}{qM}\right) = -\sum_{i=1}^{k} \frac{\mathbf{a}_i}{qM} \ln\left(\frac{\mathbf{a}_i}{qM}\right).$$

By Shannon (1948), $H(p)$ will be maximized when all observations have the same probability of occurrence $p_1 = p_2 = \cdots = p_n$. Hence, we have

$$\ln(\binom{qM}{\mathbf{a}_1, \ldots, \mathbf{a}_k}) \approx qM\mathcal{H}\left(\frac{\mathbf{a}_1}{qM}, \ldots, \frac{\mathbf{a}_k}{qM}\right) \leq qM\ln(k)$$

and

$$\ln(\binom{n}{m}) \approx n\mathcal{H}(\frac{m}{n}) \leq n\ln(2)$$

Lastly, for term $\frac{1}{\prod_i(\mathbf{a}_i+1))^N}$, we have

$$\frac{1}{\prod_i(\mathbf{a}_i+1))^N} \leq \frac{1}{(qM+1)^N 2^{k(N-1)}}$$

Hence, we obtain

$$\begin{aligned}
&\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right]) \\
&\leq \ln(k) + \ln(\binom{M-qM}{k}) + \ln(\binom{qM-1}{k-1}) + \ln(\binom{qM}{\mathbf{a}_1, \ldots, \mathbf{a}_k}) + \ln((\prod_i(\mathbf{a}_i+1))^{-N}) \\
&\leq \ln(k) + (M-qM)\ln(2) + (qM-1)\ln(2) + qM\ln(k) - N\ln(qM+1) - k(N-1)\ln(2) \\
&\approx M\ln(2) - N\ln(qM) + \ln(\frac{k^{qM}}{2^{kN}})
\end{aligned}$$

For function $\ln(\frac{k^{qM}}{2^{kN}})$, by F.O.C, it will be maximized when $k = \frac{qM}{N\ln(2)}$. Hence

$$\begin{aligned}
\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right]) &\leq M\ln(2) - N\ln(qM) + qM\ln(\frac{qM}{N\ln(2)}) - qM \\
&= M(\ln(2) - q) + qM\ln(\frac{qM}{N\ln(2)}) - N\ln(qM)
\end{aligned}$$

If $N \geq \frac{M}{(\ln(M))^\alpha}$ for some $0 < \alpha < 1$, then
we have $N\ln((qM)) \geq M(\ln(M))^{1-\alpha}$ and $qM\ln(\frac{qM}{N\ln(2)}) \leq qM\alpha\ln(\ln(M)) - qM\ln(\frac{q}{\ln(2)})$. Hence

$$\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right]) \leq M(\ln(2) - q - q\ln(\frac{q}{\ln(2)}) + q\alpha\ln(\ln(M))) - M(\ln(M))^{1-\alpha}$$

Since $\ln(M)$ goes faster than $\ln(\ln(M))$ and $\ln(2) - q - q\ln(\frac{q}{\ln(2)})$ is a constant number between 0 and 1 given $0 < q < \frac{1}{3}$. We conclude $\ln(\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right])$ goes to negative infinity when N, M

goes to infinity with $N \geq \frac{M}{(\ln(M))^{\alpha}}$ for some $0 < \alpha < 1$. It implies $\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \to 0$.

□

From Lemma A3, we have $\sum_k \binom{M}{qM} P\left[E^k(S_{qM})\right] \to 0$ when N, M goes to infinity with $N \geq \frac{M}{(\ln(M))^{\alpha}}$ for some $\alpha > 0$ and for any $q \leq \frac{1}{3}$, for any $k = \{1, .., qM\}$. If the probability that a small ratio of strategies is strictly dominated for player 1 will go to 0 in a large game, then it is also less likely to have the larger ratio of strategies being strictly dominated, therefore this is also true for all $q \in (\frac{1}{3}, 1)$.

□

A.2. Proof for Proposition 2

PROOF. (a) We apply the Chernoff bound to estimate the probability that at most $(q + \delta + \epsilon)M$ rows satisfy the condition. The Chernoff bound gives:

$$P(A \leq qM) = P(A \leq (1 - \frac{\delta}{q+\delta})(q+\delta)M) \leq e^{-\frac{(\frac{\delta}{q+\delta})^2 E[A]}{2}} = e^{-\frac{\delta^2 M}{2(q+\delta)}}$$

where $\frac{\delta}{q+\delta} > 0$ for some $\delta > 0$.

Hence $\lim_{M,N\to\infty} P(A \geq qM) \geq \lim_{M,N\to\infty} P(A > qM) = 1 - \lim_{M,N\to\infty} P(A \leq qM) = 1$.

(b) By definition,

$$P(B_i) = \prod_{j=1}^{N} P(x_{ij} \geq (\delta + q)^{1/N}) = \left(1 - (\delta + q)^{1/N}\right)^N.$$

Then $P(B = 0) = (1 - P(B_i))^M = (1 - (1 - (\delta + q)^{\frac{1}{N}})^N)^M$.

Since for $0 < r < 1$, $\frac{1-r}{N} < 1 - r^{1/N}$, then $1 - (\delta + q)^{\frac{1}{N}} > \frac{1-\delta-1}{N}$. Hence

$$P(B = 0) \leq (1 - (\frac{1 - \delta - q}{N})^N)^M$$

Also $(1 - r)^N \leq e^{-Nr}$ for $0 < r < 1$ and all integer N. we have

$$P(B = 0) \leq e^{-M(\frac{1-\delta-q}{N})^N}$$

So, if M grows much faster than $(\frac{N}{1-\delta-q})^N$, then $-M(\frac{1-\delta-q}{N})^N$ goes to negative infinity as N goes to infinity. Hence $\lim_{M,N\to\infty} P(B = 0) = 0$. Immediately, we have $\lim_{M,N\to\infty} P(B \geq 1) = 1$.

□

A.3. Proof for Proposition 3

PROOF. By definition of strictly dominated strategy, we have:

$$\begin{aligned}\lim_{N,M\to\infty} P(\neg SD) &= \lim_{N,M\to\infty} P(SD^1 = 0) \lim_{N,M\to\infty} P(SD^2 = 0) \\ &\geq \lim_{N,M\to\infty} \max\{1 - \sum_{s_i \not\succ s_j} P(s_i \succ s_j), 0\} \lim_{N,M\to\infty} \max\{1 - \sum_{t_i \not\succ t_j} P(t_i \succ t_j), 0\} \\ &= \lim_{M,N\to\infty} \max\{1 - M(M-1)(\frac{1}{\rho})^N, 0\} \lim_{M,N\to\infty} \max\{1 - N(N-1)(\frac{1}{\rho})^M, 0\}\end{aligned} \tag{A1}$$

(a) If $M = N$,

$$\lim_{N,M\to\infty} P(\neg SD) \geq \lim_{N\to\infty} ((1 - N^2(\frac{1}{\rho})^N)^2 = 1 \tag{A2}$$

(b) Simply, let start with $M = (2+\delta) log_\rho(N)$, for some $\delta > 0$,

$$\lim_{N,M\to\infty} P(\neg SD) \geq \lim_{N\to\infty} (1 - \frac{N(N-1)}{\rho^N})(1 - \frac{N(N-1)}{N^{2+\delta}}) = 1 \tag{A3}$$

Then, if $(2+\delta) log_\rho(N) \leq M \leq N$, for some $\delta > 0$,

$$\lim_{N,M\to\infty} P(\neg SD) \geq \lim_{N\to\infty} ((1 - N^2(\frac{1}{\rho})^N) \lim_{M\to\infty} (1 - \rho^{\frac{2M}{2+\delta} - M}) = 1 \tag{A4}$$

The case $M \leq N \leq \rho^{\frac{M}{2+\delta}}$ for some $\delta > 0$ follows similarly because of symmetry.

□

A.4. Proof for Proposition 4

LEMMA A4. *In a $M \times N$ random game,* $\limsup_{M,N\to\infty} P(\neg SD) \leq 1 - (1-\alpha)2\alpha$ *if* $N = \log_\rho(M) - \log_\rho(2\alpha)$ *for some* $\alpha \in (0, 1]$.

PROOF. By definition of strictly dominated strategy, we have:

$$
\begin{aligned}
&\lim_{M,N\to\infty} P(SD^1 = 0)) \\
&\leq \lim_{M,N\to\infty} (1 - P(\exists j, s_1 \succ s_j))) \\
&= \lim_{M,N\to\infty} (1 - \sum_j P(s_1 \succ s_j) + \sum_{1<j<k} P([s_1 \succ s_j] \cap [s_1 \succ s_k]) - \cdots - (-1)^{M-2} P(\sum_{j\neq 1} s_1 \succ s_j)) \qquad \text{(A5)} \\
&= \lim_{M,N\to\infty} (1 - \frac{\binom{M-1}{1}}{\rho^N} + \frac{\binom{M-1}{2}}{(\rho^N)^2} - \cdots - (-1)^{M-2} \frac{\binom{M-1}{M-1}}{(\rho^N)^{M-1}})
\end{aligned}
$$

Similarly.

$$
\lim_{M,N\to\infty} P(SD^2 = 0)) \leq \lim_{M,N\to\infty} \left(1 - \frac{\binom{N-1}{1}}{\rho^M} + \frac{\binom{N-1}{2}}{(\rho^M)^2} - \cdots - (-1)^{N-2} \frac{\binom{N-1}{N-1}}{(\rho^M)^{N-1}}\right) \qquad \text{(A6)}
$$

Now focus on the function $f(k) = \frac{\binom{M-1}{k}}{(\rho^N)^k}$, we will show at first $\frac{f(k+1)}{f(k)} \leq \alpha < 1, \forall k \in \mathbb{N}^+$ if $N = \log_\rho(M) - \log_\rho(2\alpha)$ for some $\alpha \in (0, 1]$. We observe $\frac{f(k+1)}{f(k)} = \frac{M-k-1}{(k+1)\rho^N}, \forall k$ and $\frac{M-k-1}{k+1}$ is decreasing on k, so we have $\frac{f(2)}{f(1)} = \frac{M-2}{2\rho^N} < \frac{M}{2\rho^N} \leq \alpha \Rightarrow \frac{f(k+1)}{f(k)} \leq \frac{f(2)}{f(1)} = \alpha < 1$.

Then if $M = 2\alpha\rho^N \Leftrightarrow N = \log_\rho(M) - \log_\rho(2\alpha)$, we have

$$
\begin{aligned}
&\limsup_{N\to\infty} P(SD^1 = 0)) \\
&\leq \limsup_{N\to\infty} (1 - f(1) + f(2) - \cdots - (-1)^{M-2} f(M-1)) \\
&\leq \limsup_{N\to\infty} (1 - f(1) + f(2)) \\
&= 1 - \limsup_{N\to\infty} ((1-\alpha) f(1)) \qquad \text{(A7)} \\
&= 1 - \limsup_{N\to\infty} ((1-\alpha) \frac{M-1}{\rho^N})) \\
&\leq 1 - (1-\alpha) 2\alpha < 1
\end{aligned}
$$

To sum up, we have $\limsup_{M,N\to\infty} P(\neg SD) \leq \limsup_{M,N\to\infty} P(\neg SD^1) < 1$ when $N = \log_\rho(M) - \log_\rho(2\alpha)$. □

Now we prove the Proposition 4.

PROOF. (a) Fix some $0 < \alpha < 1$. Given M, N, with $N \leq log_\rho(\frac{M}{2})$, then if we set $N' = log_\rho(\frac{M}{2\alpha})$, then $N' \geq N$. The probability of no strictly dominated strategy for Player 1 is greater in the $M \times N'$ game than in the $M \times N$ game, and in the game $M \times N'$, we have $\limsup_{M,N'\to\infty} P(\neg SD)$ as $N' = log_\rho(\frac{M}{2\alpha})$ is bounded from above by $1-(1-\alpha)2\alpha$; hence, $\lim_{M,N\to\infty} P(\neg SD)$ as $N \leq log_\rho(\frac{M}{2})$ is bounded from above by $1-(1-\alpha)2\alpha$.

(b) Fix Z arbitrarily large, we may assume $\log_\rho(M) \geq N + Z$. Then we divid the rows into $\frac{Z}{2}$ groups, each forming a matrix of $M' = \frac{2M}{Z}$ rows and N columns. In the below, we will denote G_z as a subgame for all positive integer $z \leq Z$. If there is no ambiguity, we denote $G[s_i]$ as the subgame that strategy s_i belongs to.

$$G = \begin{bmatrix} G1 \\ \hline \vdots \\ \hline G_z \\ \hline \vdots \\ \hline G_Z \end{bmatrix}$$

Now, let's denote two events:

SD_z: The subgame G_z has a strategy that is strictly dominated by a strategy in G_z.
$SD_z^{'}$: The subgame G_z has a strategy that is strictly dominated by a strategy in G.

Therefore, $SD_z \subset SD_z^{'}$ for all z and SD_1, SD_2, ..., SD_Z are independent.

Claim: $P(\neg SD) \leq \prod_s (\neg SD_z)$**.**

$$P(\neg SD) = 1 - P(SD) = 1 - P(\cup_z SD_z^{'}) \leq 1 - P(\cup_z SD_z) = P(\cap_z \neg SD_z) = \prod_z P(\neg SD_z)$$

From Proposition 4 (1), there exist $\eta \in (0,1)$ such that in each subgame G_z, when $N \leq \log_\rho(M) - Z \leq \log_\rho(\frac{M}{Z}) = \log_\rho(\frac{M'}{2})$, then $\lim_{N,M\to\infty} P(\neg SD_z) \leq \eta$ for all z. Then we have

$$\lim_{N,M\to\infty} P(\neg SD) \leq \lim_{N,M\to\infty} \prod_z P(\neg SD_z) = \lim_{N,M\to\infty} P(\neg SD_1)^Z \leq \lim_{N,M\to\infty} \eta^Z = 0$$

□

A.5. Proof for Proposition 5

Notice that Proposition 5 depends only on the ordinal comparison between elements and the minimum/maximum values of the vectors. We can arrange the elements of two vectors into a single sequence ordered from largest to smallest. The total number of distinct arrangements of two vectors of size N each is given by $\binom{2N}{N}$.

PROOF. (a) Given two payoff vectors s_i and s_j, if $\min(s_i) > \max(s_j)$, then s_j must be strictly dominated by s_i.

Moreover, it requires that all elements of s_i appear to the left of all elements of s_j in this ordered sequence. If all elements are randomly selected, there is only one arrangement that satisfies

this condition.

$$P(\min(s_i) > \max(s_j)) = \frac{1}{\binom{2N}{N}}$$

and it will approach 0 as N go to infinity. Same analysis for the event $\min(s_j) > \max(s_i)$.

(b) Given two payoff vectors s_i and s_j, suppose $\max(s_i) > \max(s_j) > \max(s_j) > \min(s_i)$. We denote $t^* = \arg\max(s_i)$ as the index of the column of the maximum of vector s_i. Then we have $\max(s_i) > \max(s_j) > u(s_j, t^*)$. Moreover, we denote $t_* = \arg\min(s_i)$ as the index of the column of the minimum of vector s_i. Then we have $u(s_j, t_*) > \min(s_j) > \min(s_i)$. Hence there is no dominance between s_i and s_j.

Similar to statement a), there are totally $\binom{2N-2}{N}$ distinct arrangements out of $\binom{2N}{N}$, we have

$$P(\max(s_i) > \max(s_j) > \min(s_j) > \min(s_i)) = \frac{\binom{2N-2}{N}}{\binom{2N}{N}} = \frac{N-1}{4N-2}$$

and it will approach $\frac{1}{4}$ as N go to infinity. Same analysis for the event $\max(s_j) > \max(s_i) > \min(s_i) > \min(s_j)$.

(c) Given two payoff vectors s_i and s_j, suppose $\max(s_i) > \max(s_j) > \min(s_i) > \min(s_j)$. There are totally $\binom{2N-2}{N-1} - 1$ distinct arrangements out of $\binom{2N}{N}$, we have

$$P(\max(s_i) > \max(s_j) > \min(s_i) > \min(s_j)) = \frac{\binom{2N-2}{N-1}}{\binom{2N}{N}} - \frac{1}{\binom{2N}{N}} = \frac{N}{4N-2} - \frac{1}{\binom{2N}{N}}$$

and it will approach $\frac{1}{4}$ as N go to infinity. Same analysis for the event $\max(s_j) > \max(s_i) > \min(s_j) > \min(s_i)$.

□

Appendix B. Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used Chatgpt in order to polish English and generate the plots. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.