

Bilevel Split Inverse Problems: Theory and Applications to Multi-Class Traffic Equilibrium

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BILEVEL SPLIT INVERSE PROBLEMS: THEORY AND APPLICATIONS TO MULTI-CLASS TRAFFIC EQUILIBRIUM

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ABSTRACT. This paper introduces a novel equilibrium model called the *Multi-Space Bilevel Split Variational Inequality Problem* (MSBSVIP). The model integrates hierarchical decision-making with split equilibrium constraints across multiple output spaces, providing a powerful framework for analysing complex systems involving both user equilibrium behaviour and multiple policy objectives. Motivated primarily by transportation network management with heterogeneous user classes, the MSBSVIP captures the interaction between travellers' route-choice responses (lower level) and a planner's multi-criteria policy goals such as system efficiency, equity, revenue generation, and environmental sustainability (upper level). A projection-type extragradient algorithm is proposed to solve the MSBSVIP. Under standard assumptions—bounded linear split operators with adjoints, Lipschitz continuity and pseudomonotonicity of the lower-level operators, and Lipschitz continuity together with strong monotonicity of the upper-level operators—the strong convergence of the generated sequence to a solution of the problem is established. Comprehensive numerical experiments on both static and dynamic multi-class traffic networks demonstrate the effectiveness, robustness, and computational efficiency of the proposed algorithm. The results illustrate how varying policy weights influence equilibrium flows and the implied pricing or control structures, confirming the practical applicability of the MSBSVIP framework.

1. INTRODUCTION

1.1. Background.

Transportation systems are inherently hierarchical and multi-agent in nature. At the lower level, travellers of different classes (commuters, business travellers, freight operators, electric vehicle users, etc.) make route choices to minimise their individual perceived travel costs, leading to some form of user equilibrium. At the upper level, network authorities and planners seek to influence these equilibria through pricing, capacity control, or regulatory measures in order to achieve broader societal objectives such as minimising total system travel time, improving equity across income groups, generating revenue, or reducing environmental externalities. Modelling this interplay accurately requires a mathematical framework that simultaneously handles multi-class user equilibria, split structures induced by class-specific perceptions, and multiple policy criteria.

Traditional traffic assignment models based on Wardrop's user equilibrium principle [26] have been successfully cast as variational inequalities [10, 23]. However, these classical models typically assume a single homogeneous user class and a single system-wide objective. Modern applications demand greater expressiveness: heterogeneous cost perceptions across user classes, policy interventions that act differently on each class, and constraints that must hold simultaneously in a primary flow space and several transformed (image) spaces. The Multi-Space Bilevel Split Variational Inequality Problem (MSBSVIP) introduced in this paper addresses these challenges by unifying bilevel variational inequalities with multi-set split structures.

1.2. Literature Review.

The foundations of variational inequality theory were laid by Hartman and Stampacchia [13] and Lions and Stampacchia [16]. Subsequent developments provided comprehensive treatments of finite-dimensional variational inequalities and their applications to traffic equilibrium problems [12, 14]. In the context of transportation, Dafermos [10] and Smith [23] were among the first to establish the equivalence between Wardrop equilibria and variational inequalities. Over the decades, numerous computational methods have been proposed, with the extragradient method of Korpelevich [15] and Tseng's forward-backward-forward algorithm [25] serving as cornerstones for solving monotone variational inequalities.

Key words and phrases. Bilevel optimization, Bilevel split inverse problems, Variational inequalities, Traffic Equilibrium Problems, Extragradient Method

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The concept of split inverse problems originated with the split feasibility problem introduced by Censor and Elfving [8] and popularised by Byrne [3]. This framework models situations where a solution must satisfy constraints in both a domain space and a range space linked by a bounded linear operator. The split variational inequality problem (SVIP), which combines variational inequalities with split structures, was formalised by Censor, Gibali and Reich [9]. Extensions to multiple output sets were systematically studied by Reich et al. [19]. This advancement has since given rise to a substantial and rapidly expanding body of research on split inverse problems with multiple output sets [2, 7, 18, 20, 21, 27].

Bilevel optimisation and bilevel variational inequalities have also attracted significant attention due to their ability to model hierarchical decision processes [11, 17]. In transportation, bilevel models have been applied to toll design, network design, and Stackelberg-type routing problems [6, 29]. However, most existing bilevel VI models assume a single upper-level criterion and do not incorporate split structures across multiple output spaces corresponding to heterogeneous user classes.

The MSBSVIP proposed in this paper bridges these research streams by combining:

- multi-class split user equilibrium constraints (lower level),
- a global upper-level variational inequality over the equilibrium set,
- multiple class-specific upper-level variational inequalities in the respective image spaces.

To the best of our knowledge, this unified multi-space bilevel split structure has not been previously investigated in the literature, particularly in the context of both static and dynamic traffic equilibrium problems with multiple policy objectives.

1.3. Contribution of this Paper.

The main contributions of this work are as follows:

- (1) We introduce the Multi-Space Bilevel Split Variational Inequality Problem (MSBSVIP), a new mathematical model that generalises existing split variational inequalities and bilevel variational inequalities.
- (2) We propose a projection-type extragradient algorithm and prove its strong convergence under some mild conditions.
- (3) We provide a detailed traffic-theoretic formulation of the MSBSVIP for both static and dynamic multi-class networks.
- (4) We conduct extensive numerical experiments demonstrating the practical applicability and numerical efficiency of the proposed method.

The remainder of the paper is organised as follows. Section 2 provides the definitions and lemmas required for the convergence analysis of the proposed algorithm. Section 3 presents the precise mathematical formulation of the MSBSVIP. Section 4 introduces the projection-type algorithm and establishes its strong convergence. Section 5 applies the main results to related problems. Section 6 reports numerical results on traffic networks. Finally, concluding remarks are given in Section 7.

2. TOOL BOX

In what follows, it is assumed that H is a Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$.

Definition 2.1. [1] An operator $A : H \rightarrow H$ is said to be

- (i) α -strongly monotone, if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H;$$

- (ii) monotone, if

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x, y \in H;$$

- (iii) pseudomonotone, if

$$\langle Ay, x - y \rangle \geq 0 \implies \langle Ax, x - y \rangle \geq 0, \quad \forall x, y \in H,$$

- (iv) L -Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \quad \forall x, y \in H.$$

Remark 2.2. It is known that the following implications hold: (i) $\implies$ (ii) $\implies$ (iii) but the converses are not generally true.

Lemma 2.3. [22] Suppose $\{a_m\}$ is a sequence of nonnegative real numbers, $\{\alpha_m\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\{b_m\}$ is a sequence of real numbers. Assume that

$$a_{m+1} \leq (1 - \alpha_m)a_m + \alpha_m b_m \quad \text{for all } m \geq 1.$$

If $\limsup_{k \rightarrow \infty} b_{m_k} \leq 0$ for every subsequence $\{a_{m_k}\}$ of $\{a_m\}$ satisfying $\liminf_{k \rightarrow \infty} (a_{m_{k+1}} - a_{m_k}) \geq 0$, then $\lim_{m \rightarrow \infty} a_m = 0$.

Lemma 2.4. [24] Suppose $\{\lambda_m\}$ and $\{\theta_m\}$ are two nonnegative real sequences such that

$$\lambda_{m+1} \leq \lambda_m + \theta_m, \quad \forall m \geq 1.$$

If $\sum_{n=1}^{\infty} \theta_n < +\infty$, then $\lim_{m \rightarrow \infty} \lambda_m$ exists.

Lemma 2.5. [4] Let H be a real Hilbert space. Then the following results hold for all $x, y \in H$ and $\delta \in \mathbb{R}$:

- (i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$;
- (ii) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$;
- (iii) $\|\delta x + (1 - \delta)y\|^2 = \delta\|x\|^2 + (1 - \delta)\|y\|^2 - \delta(1 - \delta)\|x - y\|^2$.

Lemma 2.6. [5] Consider the variational inequality problem (VIP)

$$\text{Find } x^* \in C \text{ such that } \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C$$

where C is a nonempty, closed and convex subset of a real Hilbert space H and $A : C \rightarrow H$ is pseudomonotone and continuous. Then q is a solution of the VIP if and only if

$$\langle Ax, x - q \rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.7. [28] Suppose C is a closed and convex subset of a real Hilbert space H , and $F : H \rightarrow H$ is K -Lipschitz and δ -strongly monotone over C . Then, the VIP

$$\text{Find } x^* \in C \text{ such that } \langle Fx^*, y - x^* \rangle \geq 0, \quad \forall y \in C$$

has a unique solution $x^* \in C$.

3. PROBLEM FORMULATION

In this section, the concept of the bilevel split inverse problem (BSIP) is first considered to motivate the model, followed by the precise formulation of the *Multi-Space Bilevel Split Variational Inequality Problem* (MSBSVIP).

3.1. Bilevel Split Inverse Problem (BSIP).

Let H_1 and H_2 be real Hilbert spaces. Let IP_1 and IP_2 be two inverse problems formulated in H_1 and H_2 , with solution sets denoted by $\text{Sol}(\text{IP}_1)$ and $\text{Sol}(\text{IP}_2)$, respectively. Let $T : H_1 \rightarrow H_2$ be a bounded linear operator with adjoint T^* .

The *split inverse problem* (SIP) consists of finding a point $x^* \in H_1$ satisfying the following two conditions simultaneously:

$$x^* \in \text{Sol}(\text{IP}_1), \tag{3.1}$$

and

$$Tx^* \in \text{Sol}(\text{IP}_2). \tag{3.2}$$

The solution set of this split problem is denoted by Ω :

$$\Omega = \{x \in H_1 : x \in \text{Sol}(\text{IP}_1) \text{ and } Tx \in \text{Sol}(\text{IP}_2)\}. \tag{3.3}$$

Building upon this structure, the *bilevel split inverse problem* (BSIP) is formulated as follows: find a point $u^* \in \Omega$ such that

$$u^* \in \text{Sol}(\text{IP}_3), \tag{3.4}$$

and

$$Tu^* \in \text{Sol}(\text{IP}_4), \tag{3.5}$$

where IP_3 and IP_4 are inverse problems formulated in H_1 and H_2 , respectively.

Equations (3.1)–(3.5) define the BSIP. In the language of optimisation, the pair (3.1)–(3.2) constitutes the *lower-level problem*, while (3.4)–(3.5) represents the *upper-level problem*. This hierarchical split structure provides a natural mathematical framework for modelling situations in which decisions made in one space induce feasible solutions in another linked space.

3.2. Multi-Space Bilevel Split Variational Inequality Problem (MSBSVIP).

The above idea is now specialised to variational inequalities and extended to the multi-space setting with multiple output sets. Let H and H_i ($i = 1, 2, \dots, N$) be real Hilbert spaces. Let $C \subset H$ and $C_i \subset H_i$ ($i = 1, \dots, N$) be nonempty, closed and convex subsets. Let $T_i : H \rightarrow H_i$ ($i = 1, \dots, N$) be bounded linear operators with adjoints $T_i^* : H_i \rightarrow H$.

Let $A : H \rightarrow H$ and $A_i : H_i \rightarrow H_i$ ($i = 1, \dots, N$) be the lower-level operators. The solution set of a variational inequality $\text{VI}(K, B)$ is defined as

$$\text{VI}(K, B) = \{x \in K : \langle B(x), y - x \rangle \geq 0 \quad \forall y \in K\}.$$

3.2.1. *Lower-Level Constraints (Multi-Class User Equilibrium)*. The lower level of the MSBSVIP consists of finding $y^* \in C$ such that

$$\langle A(y^*), y - y^* \rangle \geq 0 \quad \forall y \in C, \quad (3.6)$$

and, for each user class $i = 1, 2, \dots, N$,

$$\langle A_i(T_i y^*), z_i - T_i y^* \rangle \geq 0 \quad \forall z_i \in C_i. \quad (3.7)$$

The set of all flows satisfying the entire lower-level system is denoted by Ψ :

$$\Psi := \{y \in C : y \in \text{VI}(C, A) \text{ and } T_i y \in \text{VI}(C_i, A_i) \text{ for all } i = 1, \dots, N\}. \quad (3.8)$$

The set Ψ represents the collection of all feasible multi-class user equilibria.

3.2.2. *Upper-Level Criteria (Policy Objectives)*. Let $F : H \rightarrow H$ and $F_i : H_i \rightarrow H_i$ ($i = 1, \dots, N$) be the upper-level operators. The *Multi-Space Bilevel Split Variational Inequality Problem* (MSBSVIP) consists of finding a point $x^* \in \Psi$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \Psi, \quad (3.9)$$

and, for each $i = 1, 2, \dots, N$,

$$\langle F_i(T_i x^*), z_i - T_i x^* \rangle \geq 0 \quad \forall z_i \in \text{VI}(C_i, A_i). \quad (3.10)$$

Equations (3.6)–(3.10) constitute the MSBSVIP. In the transportation context, Equation (3.9) represents the planner's global system efficiency objective, while Equations (3.10) encode class-specific policy goals (equity, revenue, environmental protection, etc.) evaluated in the perceived image spaces of each user class.

Remark 3.1. The MSBSVIP generalises several existing models:

- When all $F_i \equiv F \equiv 0$, it reduces to the split variational inequality problem with multiple output sets.
- When $N = 1$ and $F_i \equiv 0$, it reduces to a standard bilevel split variational inequality.
- When all T_i are the identity and $F_i \equiv F$, it collapses to a classical bilevel variational inequality.
- The multi-space structure with distinct F_i allows modelling heterogeneous policy objectives across user classes in a natural way.

4. PROPOSED ALGORITHM AND CONVERGENCE ANALYSIS

Now, the proposed iterative scheme for solving the MSBSVIP (3.6)–(3.10) is presented. For convenience, the notation $[N] = \{0, 1, \dots, N\}$ is used. The convergence of the proposed algorithm is analysed under the following assumptions.

For each $i \in [N]$, let C_i be a nonempty, closed, and convex subset of a real Hilbert space H_i , and let $T_i : H_0 \rightarrow H_i$ be a bounded linear operator with adjoint T_i^* . Let $F_i : H_i \rightarrow H_i$ be a K_i -Lipschitz continuous and δ^i -strongly monotone operator, and suppose that $A_i : H_i \rightarrow H_i$ is an L_i -Lipschitz continuous and pseudomonotone operator satisfying the following condition:

$$\text{whenever } \{T_i x_m\} \subset C_i \text{ and } T_i x_m \rightarrow T_i y, \text{ then } \|A_i T_i y\| \leq \liminf_{m \rightarrow \infty} \|A_i T_i x_m\|, \quad i \in [N]. \quad (4.1)$$

Here, $H_0 = H$, $C_0 = C$, $F_0 = F$, $A_0 = A$, and $T_0 = I^{H_0} = I^H$. Furthermore, it is assumed that the solution set

$$\Psi := \left\{ u^* \in C : u^* \in \bigcap_{i \in [N]} T_i^{-1} VI(C_i, A_i) \right\}$$

is nonempty, and that the control parameters satisfy the following conditions:

Assumption A:

- (A1) $0 < k^i < k^{i'} < 1$, $\lim_{m \rightarrow \infty} k_m^i = 0$, $\lambda_0^i > 0$, $0 < \mu^i < \frac{2\delta^i}{K_i^2}$, $\mu = \mu^0 \geq \mu^i$, $i \in [N]$;
- (A2) $\gamma^i = 1 - \sqrt{1 - \mu^i(2\delta^i - \mu^i K_i^2)} \in (0, 1)$, $\gamma = \gamma^0 > \mu^0$, $\eta^i \in (0, \frac{1}{L_i})$, $L_i = \|T_i T_i^*\|$, $i \in [N]$;
- (A3) $0 < \alpha_m < \min \{1, \min_{i \in [N]} \{\frac{2\gamma^i - \mu^i}{\gamma^{i2}}\}\}$, $\lim_{m \rightarrow \infty} \alpha_m = 0$, $\sum_{m=1}^{\infty} \alpha_m = +\infty$;
- (A4) $\{\phi_m^i\} \subset \mathbb{R}_+$, $\sum_{m=1}^{\infty} \phi_m^i < +\infty$, $0 < a_i \leq \beta_m^i \leq b_i < 1$, $\sum_{i \in [N]} \beta_m^i = 1$.

Now, the algorithm is presented as follows:

Algorithm 4.1.

Initialization: Choose an initial point $x_0 \in H$ and set $m = 0$.

Iterative Steps: Compute x_{m+1} as follows:

Step 1. Compute

$$v_m^i = (I^{H_i} - \alpha_m \mu^i F_i) T_i x_m.$$

Step 2. Compute

$$y_m^i = P_{C_i}(v_m^i - \lambda_m^i A_i v_m^i).$$

Step 3. Compute

$$z_m^i = y_m^i - \lambda_m^i (A_i y_m^i - A_i v_m^i),$$

$$\lambda_{m+1}^i = \begin{cases} \min \left\{ \frac{(k_i + k_m^i) \|v_m^i - y_m^i\|}{\|A_i v_m^i - A_i y_m^i\|}, \lambda_m^i + \phi_m^i \right\}, & \text{if } A_i v_m^i - A_i y_m^i \neq 0, \\ \lambda_m^i + \phi_m^i, & \text{otherwise.} \end{cases}$$

Step 4. Compute

$$u_m = \sum_{i \in [N]} \beta_m^i (x_m + \eta^i T_i^* (z_m^i - T_i x_m)).$$

Step 5. Compute

$$x_{m+1} = (I^H - \alpha_m \mu F) u_m.$$

Set $m := m + 1$ and return to **Step 1**.

4.1. Convergence Analysis.

In this subsection, the convergence of the proposed iterative method is analysed. The presentation begins with several auxiliary lemmas that are essential for proving the strong convergence of the method.

Lemma 4.2. For each $i \in [N]$, suppose $\{\lambda_m^i\}$ is the sequence generated by Algorithm 4.1. Then $\{\lambda_m^i\}$ is well defined for each $i \in [N]$ and $\lim_{m \rightarrow \infty} \lambda_m^i = \lambda^i \in [\min\{\frac{k_i}{L_i}, \lambda_0^i\}, \lambda_0^i + \Phi^i]$, where $\Phi^i = \sum_{m=1}^{\infty} \phi_m^i$.

Proof. Since A_i is L_i -Lipschitz continuous for each $i \in [N]$, it follows that, in the case where $A_i v_m^i - A_i y_m^i \neq 0$ for all $m \geq 0$, the following holds:

$$\frac{(k_i + k_m^i) \|v_m^i - y_m^i\|}{\|A_i v_m^i - A_i y_m^i\|} \geq \frac{(k_i + k_m^i) \|v_m^i - y_m^i\|}{L_i \|v_m^i - y_m^i\|} \geq \frac{k_i}{L_i}.$$

Hence, by the definition of λ_{m+1}^i , the sequence $\{\lambda_m^i\}$ has lower bound $\min\{\frac{k_i}{L_i}, \lambda_0^i\}$ and upper bound $\lambda_0^i + \Phi^i$ for each $i \in [N]$. By Lemma 2.4, the $\lim_{m \rightarrow \infty} \lambda_m^i$ exists and denoted by $\lambda^i = \lim_{m \rightarrow \infty} \lambda_m^i$. It is obvious that $\lambda^i \in [\min\{\frac{k_i}{L_i}, \lambda_0^i\}, \lambda_0^i + \Phi^i]$.

□

Lemma 4.3. *Suppose Assumption A of Algorithm 4.1 is satisfied. Then, there exists a positive integer M such that $\frac{\lambda_m^i(k^i + k_m^i)}{\lambda_{m+1}^i} \in (0, 1)$ for all $m \geq M$.*

Proof. Observe that since $0 < k^i < k^{i'} < 1$, $\lim_{m \rightarrow \infty} k_m^i = 0$ and $\lim_{m \rightarrow \infty} \lambda_m^i = \lambda^i$ for each $i \in [N]$, it follows that

$$\lim_{m \rightarrow \infty} \left(1 - \frac{\lambda_m^i(k^i + k_m^i)}{\lambda_{m+1}^i}\right) = 1 - k^i > 1 - k^{i'} > 0.$$

Therefore, for each $i \in [N]$ there exists a positive integer M_i such that

$$1 - \frac{\lambda_m^i(k^i + k_m^i)}{\lambda_{m+1}^i} > 0, \quad \forall m \geq M_i.$$

Now, by letting $M = \max\{M_i : i \in [N]\}$ the desired result is obtained.

□

Lemma 4.4. *Suppose $\{z_m^i\}$ is a sequence generated by Algorithm 4.1 for each $i \in [N]$. Then, the following inequality holds for all $q \in \Psi$:*

$$\|z_m^i - T_i q\|^2 \leq \|v_m^i - T_i q\|^2 - \left(1 - \frac{\lambda_m^i}{\lambda_{m+1}^i} \frac{(k^i + k_m^i)^2}{2}\right) \|v_m^i - y_m^i\|^2.$$

Proof. First, by the definition of λ_{m+1}^i , it follows that

$$\|A_i v_m^i - A_i y_m^i\| \leq \frac{(k^i + k_m^i)}{\lambda_{m+1}^i} \|v_m^i - y_m^i\|, \quad \forall m \in \mathbb{N}, i \in [N]. \quad (4.2)$$

It is easy to see that (4.2) is valid both for $A_i v_m^i - A_i y_m^i \neq 0$ and $A_i v_m^i - A_i y_m^i = 0$ for each $i \in [N]$. Let $q \in \Psi$. Then, it implies that $T_i q \in VI(C_i, A_i)$ for each $i \in [N]$. By the definition of z_m^i and the application of Lemma 2.5, this yields

$$\begin{aligned} \|z_m^i - T_i q\|^2 &= \|y_m^i - \lambda_m^i(A_i y_m^i - A_i v_m^i) - T_i q\|^2 \\ &= \|y_m^i - T_i q\|^2 + \lambda_m^i{}^2 \|A_i y_m^i - A_i v_m^i\|^2 - 2\lambda_m^i \langle y_m^i - T_i q, A_i y_m^i - A_i v_m^i \rangle \\ &= \|v_m^i - T_i q\|^2 + \|y_m^i - v_m^i\|^2 + 2\langle y_m^i - v_m^i, v_m^i - T_i q \rangle + \lambda_m^i{}^2 \|A_i y_m^i - A_i v_m^i\|^2 \\ &\quad - 2\lambda_m^i \langle y_m^i - T_i q, A_i y_m^i - A_i v_m^i \rangle \\ &= \|v_m^i - T_i q\|^2 + \|y_m^i - v_m^i\|^2 - 2\langle y_m^i - v_m^i, y_m^i - v_m^i \rangle + 2\langle y_m^i - v_m^i, y_m^i - T_i q \rangle \\ &\quad + \lambda_m^i{}^2 \|A_i y_m^i - A_i v_m^i\|^2 - 2\lambda_m^i \langle y_m^i - T_i q, A_i y_m^i - A_i v_m^i \rangle \\ &= \|v_m^i - T_i q\|^2 - \|y_m^i - v_m^i\|^2 + 2\langle y_m^i - v_m^i, y_m^i - T_i q \rangle + \lambda_m^i{}^2 \|A_i y_m^i - A_i v_m^i\|^2 \\ &\quad - 2\lambda_m^i \langle y_m^i - T_i q, A_i y_m^i - A_i v_m^i \rangle. \end{aligned} \quad (4.3)$$

Since $y_m^i = P_{C_i}(v_m^i - \lambda_m^i A_i v_m^i)$ and $T_i q \in VI(C_i, A_i)$, $i \in [N]$, it follows from the property of the projection map that

$$\langle y_m^i - v_m^i + \lambda_m^i A_i v_m^i, y_m^i - T_i q \rangle \leq 0,$$

and this is equivalent to

$$\langle y_m^i - v_m^i, y_m^i - T_i q \rangle \leq -\lambda_m^i \langle A_i v_m^i, y_m^i - T_i q \rangle. \quad (4.4)$$

Moreover, since for each $i \in [N]$ $y_m^i \in C_i$, it follows that

$$\langle A_i T_i q, y_m^i - T_i q \rangle \geq 0.$$

By the pseudomonotonicity of A_i , $\langle A_i y_m^i, y_m^i - T_i q \rangle \geq 0$. Since $\lambda_m^i > 0$ for each $i \in [N]$, it follows that

$$\lambda_m^i \langle A_i y_m^i, y_m^i - T_i q \rangle \geq 0. \quad (4.5)$$

Next, upon applying (4.2), (4.4) and (4.5) in (4.3), one obtains

$$\begin{aligned} \|z_m^i - T_i q\|^2 &\leq \|v_m^i - T_i q\|^2 - \|y_m^i - v_m^i\|^2 - 2\lambda_m^i \langle A_i v_m^i, y_m^i - T_i q \rangle + (k^i + k_m^i)^2 \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} \|v_m^i - y_m^i\|^2 \\ &\quad - 2\lambda_m^i \langle y_m^i - T_i q, A_i y_m^i - A_i v_m^i \rangle \\ &= \|v_m^i - T_i q\|^2 - \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k^i + k_m^i)^2\right) \|v_m^i - y_m^i\|^2 - 2\lambda_m^i \langle y_m^i - T_i q, A_i y_m^i \rangle \\ &\leq \|v_m^i - T_i q\|^2 - \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k^i + k_m^i)^2\right) \|v_m^i - y_m^i\|^2, \end{aligned} \quad (4.6)$$

which is the required result. $\square$

Lemma 4.5. *The sequence $\{x_m\}$ generated by the proposed Algorithm 4.1 is bounded.*

Proof. It is known that by Lemma 4.3, there exists a positive integer M such that $1 - \frac{\lambda_m^i}{\lambda_{m+1}^i} (k_i + k_m^i) > 0$, $\forall m \geq M$, $i \in [N]$. Therefore, it follows from Lemma 4.4 that for all $m \geq M$ and $i \in [N]$

$$\|z_m^i - T_i q\|^2 \leq \|v_m^i - T_i q\|^2. \quad (4.7)$$

On the other hand, by the definition of v_m^i and the application of Lemma 2.5, it follows that

$$\begin{aligned} \|v_m^i - T_i q\|^2 &= \|(I^{H_i} - \alpha_m \mu^i F_i) T_i x_m - T_i q\|^2 \\ &= \|(I^{H_i} - \alpha_m \mu^i F_i) T_i x_m - (I^{H_i} - \alpha_m \mu^i F_i) T_i q - \alpha_m \mu^i F_i T_i q\|^2 \\ &\leq \|(I^{H_i} - \alpha_m \mu^i F_i) T_i x_m - (I^{H_i} - \alpha_m \mu^i F_i) T_i q\|^2 - 2\alpha_m \mu^i \langle F_i T_i q, v_m^i - T_i q \rangle \\ &\leq (1 - \alpha_m \gamma^i)^2 \|T_i x_m - T_i q\|^2 + 2\alpha_m \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle \\ &\leq (1 - \alpha_m \gamma^i)^2 \|T_i x_m - T_i q\|^2 + \alpha_m \mu^i \|F_i T_i q\|^2 + \alpha_m \mu^i \|T_i q - v_m^i\|^2. \end{aligned} \quad (4.8)$$

Observe that, by applying the conditions on the control parameters, the last inequality yields

$$\begin{aligned} \|v_m^i - T_i q\|^2 &\leq \frac{(1 - \alpha_m \gamma^i)^2}{1 - \alpha_m \mu^i} \|T_i x_m - T_i q\|^2 + \frac{\alpha_m \mu^i}{1 - \alpha_m \mu^i} \|F_i T_i q\|^2 \\ &\leq \|T_i x_m - T_i q\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \|F_i T_i q\|^2. \end{aligned} \quad (4.9)$$

It is known that the function $\|\cdot\|^2$ is convex. By applying this property together with the definition of u_m yields

$$\begin{aligned} \|u_m - q\|^2 &= \left\| \sum_{i \in [N]} \beta_m^i (x_m + \eta^i T_i^* (z_m^i - T_i x_m)) - q \right\|^2 \\ &\leq \sum_{i \in [N]} \beta_m^i \|x_m + \eta^i T_i^* (z_m^i - T_i x_m) - q\|^2. \end{aligned} \quad (4.10)$$

From (4.10) and by applying Lemma 2.5 together with (4.7) and (4.9), it follows that

$$\begin{aligned}
 \|x_m + \eta^i T_i^*(z_m^i - T_i x_m) - q\|^2 &= \|x_m - q\|^2 + \eta^{i^2} \|T_i^*(z_m^i - T_i x_m)\|^2 + 2\eta^i \langle x_m - q, T_i^*(z_m^i - T_i x_m) \rangle \\
 &= \|x_m - q\|^2 + \eta^{i^2} \langle z_m^i - T_i x_m, T_i T_i^*(z_m^i - T_i x_m) \rangle + 2\eta^i \langle x_m - q, T_i^*(z_m^i - T_i x_m) \rangle \\
 &\leq \|x_m - q\|^2 + \eta^{i^2} L_i \|z_m^i - T_i x_m\|^2 + 2\eta^i \langle T_i x_m - T_i q, z_m^i - T_i x_m \rangle \\
 &= \|x_m - q\|^2 + \eta^{i^2} L_i \|z_m^i - T_i x_m\|^2 + \eta^i [\|z_m^i - T_i q\|^2 - \|T_i x_m - T_i q\|^2 - \|z_m^i - T_i x_m\|^2] \\
 &\leq \|x_m - q\|^2 + \eta^{i^2} L_i \|z_m^i - T_i x_m\|^2 - \eta^i \|z_m^i - T_i x_m\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \eta^i \|F_i T_i q\|^2 \\
 &= \|x_m - q\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \eta^i \|F_i T_i q\|^2 - \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|. \tag{4.11}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \|x_m - q\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \eta^i \|F_i T_i q\|^2 - \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|. \tag{4.12}
 \end{aligned}$$

Substituting (4.12) in (4.10) yields

$$\|u_m - q\|^2 \leq \|x_m - q\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \sum_{i \in [N]} \beta_m^i \eta^i \|F_i T_i q\|^2 - \sum_{i \in [N]} \beta_m^i \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|. \tag{4.13}$$

Now, using the definition of x_{m+1} together with (4.13) gives

$$\begin{aligned}
 \|x_{m+1} - q\|^2 &= \|(I^H - \alpha_m \mu F)u_m - q\|^2 \\
 &= \|(I^H - \alpha_m \mu F)u_m - (I^H - \alpha_m \mu F)q - \alpha_m \mu Fq\|^2 \\
 &\leq \|(I^H - \alpha_m \mu F)u_m - (I^H - \alpha_m \mu F)q\|^2 - 2\alpha_m \mu \langle Fq, x_{m+1} - q \rangle \\
 &\leq (1 - \alpha_m \gamma)^2 \|u_m - q\|^2 + 2\alpha_m \mu \langle Fq, q - x_{m+1} \rangle \\
 &\leq (1 - \alpha_m \gamma)^2 \|u_m - q\|^2 + \alpha_m \mu \|Fq\|^2 + \alpha_m \mu \|x_{m+1} - q\|^2. \tag{4.14}
 \end{aligned}$$

Observe that, upon applying (4.13) together with the conditions on the control parameters, the last inequality implies that, for all $m \geq M$

$$\begin{aligned}
 \|x_{m+1} - q\|^2 &\leq \frac{(1 - \alpha_m \gamma)^2}{1 - \alpha_m \mu} \|u_m - q\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \|Fq\|^2 \\
 &\leq \frac{(1 - \alpha_m \gamma)^2}{1 - \alpha_m \mu} \|x_m - q\|^2 + \frac{(1 - \alpha_m \gamma)^2}{1 - \alpha_m \mu} \frac{\alpha_m \mu}{1 - \alpha_m \mu} \sum_{i \in [N]} \beta_m^i \eta^i \|F_i T_i q\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \|Fq\|^2 \\
 &\leq \frac{(1 - \alpha_m \gamma)^2}{1 - \alpha_m \mu} \|x_m - q\|^2 + \frac{\alpha_m \mu}{1 - \alpha_m \mu} \left[\sum_{i \in [N]} \beta_m^i \eta^i \|F_i T_i q\|^2 + \|Fq\|^2 \right] \\
 &\leq \left[1 - \frac{\alpha_m (\gamma - \mu)}{1 - \alpha_m \mu} \right] \|x_m - q\|^2 + \frac{\alpha_m (\gamma - \mu)}{1 - \alpha_m \mu} \frac{\mu}{\gamma - \mu} \left[\sum_{i \in [N]} \eta^i \|F_i T_i q\|^2 + \|Fq\|^2 \right] \\
 &\leq \max \left\{ \|x_m - q\|^2, \quad \frac{\mu}{\gamma - \mu} \left[\sum_{i \in [N]} \eta^i \|F_i T_i q\|^2 + \|Fq\|^2 \right] \right\} \\
 &\vdots \\
 &\leq \max \left\{ \|x_M - q\|^2, \quad \frac{\mu}{\gamma - \mu} \left[\sum_{i \in [N]} \eta^i \|F_i T_i q\|^2 + \|Fq\|^2 \right] \right\}.
 \end{aligned}$$

From this, it follows that $\{\|x_m - q\|\}$ is bounded. Consequently, the sequence $\{x_m\}$ is bounded as claimed. $\square$

Lemma 4.6. *Let $\{x_m\}$ be a sequence generated by Algorithm 4.1 and suppose $q \in \Psi$. Then, the following inequality holds:*

$$\begin{aligned} \|x_{m+1} - q\|^2 &\leq (1 - \alpha_m \gamma) \|x_m - q\|^2 + \alpha_m \gamma \left[\frac{2\mu}{\gamma} \langle Fq, q - x_{m+1} \rangle + \frac{2(1 - \alpha_m \gamma)^2}{\gamma} \sum_{i \in [N]} \beta_m^i \eta^i \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle \right] \\ &\quad - (1 - \alpha_m \gamma)^2 \left[\sum_{i \in [N]} \beta_m^i \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|^2 + \sum_{i \in [N]} \beta_m^i \eta^i \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k_i + k_m^i)^2 \right) \|v_m^i - y_m^i\|^2 \right]. \end{aligned}$$

Proof. Let $q \in \Psi$. Observe that the application of (4.8) in Lemma 4.4 yields

$$\|z_m^i - T_i q\|^2 \leq \|T_i x_m - T_i q\|^2 + 2\alpha_m \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle - \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k_i + k_m^i)^2 \right) \|v_m^i - y_m^i\|^2. \quad (4.15)$$

Next, substituting (4.15) in (4.11) gives

$$\begin{aligned} \|x_m + \eta^i T_i^* (z_m^i - T_i x_m) - q\|^2 &\leq \|x_m - q\|^2 + \eta^i{}^2 L_i \|z_m^i - T_i x_m\|^2 - \eta^i \|z_m^i - T_i x_m\|^2 \\ &\quad + 2\alpha_m \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle - \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k_i + k_m^i)^2 \right) \|v_m^i - y_m^i\|^2 \\ &= \|x_m - q\|^2 + 2\alpha_m \eta^i \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle - \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|^2 \\ &\quad - \eta^i \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k_i + k_m^i)^2 \right) \|v_m^i - y_m^i\|^2. \end{aligned} \quad (4.16)$$

Applying (4.16) in (4.10) yields

$$\begin{aligned} \|u_m - q\|^2 &\leq \|x_m - q\|^2 + 2\alpha_m \sum_{i \in [N]} \beta_m^i \eta^i \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle - \sum_{i \in [N]} \beta_m^i \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|^2 \\ &\quad - \sum_{i \in [N]} \beta_m^i \eta^i \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k_i + k_m^i)^2 \right) \|v_m^i - y_m^i\|^2. \end{aligned} \quad (4.17)$$

Now, from (4.14) and by using (4.17), it follows that

$$\begin{aligned} \|x_{m+1} - q\|^2 &\leq (1 - \alpha_m \gamma)^2 \|x_m - q\|^2 + 2\alpha_m (1 - \alpha_m \gamma)^2 \sum_{i \in [N]} \beta_m^i \eta^i \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle \\ &\quad - (1 - \alpha_m \gamma)^2 \sum_{i \in [N]} \beta_m^i \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|^2 \\ &\quad - (1 - \alpha_m \gamma)^2 \sum_{i \in [N]} \beta_m^i \eta^i \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k_i + k_m^i)^2 \right) \|v_m^i - y_m^i\|^2 \\ &\quad + 2\alpha_m \mu \langle Fq, q - x_{m+1} \rangle \\ &\leq (1 - \alpha_m \gamma) \|x_m - q\|^2 + \alpha_m \gamma \left[\frac{2\mu}{\gamma} \langle Fq, q - x_{m+1} \rangle + \frac{2(1 - \alpha_m \gamma)^2}{\gamma} \sum_{i \in [N]} \beta_m^i \eta^i \mu^i \langle F_i T_i q, T_i q - v_m^i \rangle \right] \\ &\quad - (1 - \alpha_m \gamma)^2 \left[\sum_{i \in [N]} \beta_m^i \eta^i (1 - \eta^i L_i) \|z_m^i - T_i x_m\|^2 \right. \\ &\quad \left. + \sum_{i \in [N]} \beta_m^i \eta^i \left(1 - \frac{\lambda_m^i{}^2}{\lambda_{m+1}^i} (k_i + k_m^i)^2 \right) \|v_m^i - y_m^i\|^2 \right], \end{aligned}$$

which is the required inequality. $\square$

Lemma 4.7. *Suppose $\{x_m\}$ and $\{u_m\}$ are two sequences generated by Algorithm 4.1 with subsequences $\{x_{m_k}\}$ and $\{u_{m_k}\}$, respectively, such that $\lim_{k \rightarrow \infty} \|x_{m_k} - u_{m_k}\| = 0$. In addition, suppose $x_{m_k} \rightharpoonup p \in H$, then $p \in \Psi$.*

Proof. From (4.13), it follows that

$$\begin{aligned} \sum_{i \in [N]} \beta_{m_k}^i \eta^i (1 - \eta^i L_i) \|z_{m_k}^i - T_i x_{m_k}\| &\leq \|x_{m_k} - q\|^2 - \|u_{m_k} - q\|^2 + \frac{\alpha_{m_k} \mu}{1 - \alpha_{m_k} \mu} \sum_{i \in [N]} \beta_{m_k}^i \eta^i \|F_i T_i q\|^2 \\ &\leq \|x_{m_k} - u_{m_k}\|^2 + 2\|x_{m_k} - u_{m_k}\| \|u_{m_k} - q\| \\ &\quad + \frac{\alpha_{m_k} \mu}{1 - \alpha_{m_k} \mu} \sum_{i \in [N]} \beta_{m_k}^i \eta^i \|F_i T_i q\|^2. \end{aligned} \quad (4.18)$$

By the hypothesis of the lemma and the fact that $\lim_{k \rightarrow \infty} \alpha_{m_k} = 0$, from (4.18) it follows that

$$\sum_{i \in [N]} \beta_{m_k}^i \eta^i (1 - \eta^i L_i) \|z_{m_k}^i - T_i x_{m_k}\| \rightarrow 0, \quad k \rightarrow \infty.$$

Consequently, it can be concluded that

$$\|z_{m_k}^i - T_i x_{m_k}\| \rightarrow 0, \quad k \rightarrow \infty, \quad \forall i \in [N]. \quad (4.19)$$

It follows from this that

$$\|T_i^*(z_{m_k}^i - T_i x_{m_k})\| \leq \|T_i^*\| \|z_{m_k}^i - T_i x_{m_k}\| = \|T_i\| \|z_{m_k}^i - T_i x_{m_k}\| \rightarrow 0, \quad k \rightarrow \infty, \quad \forall i \in [N]. \quad (4.20)$$

From (4.15), it follows that

$$\begin{aligned} \left(1 - \frac{\lambda_{m_k}^i}{\lambda_{m_k+1}^i} (k_i + k_{m_k}^i)^2\right) \|v_{m_k}^i - y_{m_k}^i\|^2 &\leq \|T_i x_{m_k} - T_i q\|^2 - \|z_{m_k}^i - T_i q\|^2 + 2\alpha_{m_k} \mu^i \langle F_i T_i q, T_i q - v_{m_k}^i \rangle \\ &\leq \|T_i x_{m_k} - z_{m_k}^i\| (\|T_i x_{m_k} - T_i q\| + \|z_{m_k}^i - T_i q\|) \\ &\quad + 2\alpha_{m_k} \mu^i \langle F_i T_i q, T_i q - v_{m_k}^i \rangle. \end{aligned} \quad (4.21)$$

Using the fact that $\lim_{k \rightarrow \infty} \alpha_{m_k} = 0$ together with (4.19), it follows from (4.21) that

$$\left(1 - \frac{\lambda_{m_k}^i}{\lambda_{m_k+1}^i} (k_i + k_{m_k}^i)^2\right) \|v_{m_k}^i - y_{m_k}^i\|^2 \rightarrow 0, \quad k \rightarrow \infty, \quad i \in [N],$$

which implies that

$$\|v_{m_k}^i - y_{m_k}^i\| \rightarrow 0, \quad k \rightarrow \infty, \quad i \in [N], \quad (4.22)$$

and by the continuity of A_i yields

$$\|A_i v_{m_k}^i - A_i y_{m_k}^i\| \rightarrow 0, \quad k \rightarrow \infty, \quad \forall i \in [N]. \quad (4.23)$$

Furthermore, by the definition of $z_{m_k}^i$ and by applying (4.22) and (4.23), it follows that

$$\begin{aligned} \|z_{m_k}^i - v_{m_k}^i\| &= \|y_{m_k}^i - \lambda_{m_k}^i (A_i y_{m_k}^i - A_i v_{m_k}^i) - v_{m_k}^i\| \\ &\leq \|y_{m_k}^i - v_{m_k}^i\| + \lambda_{m_k}^i \|A_i y_{m_k}^i - A_i v_{m_k}^i\| \rightarrow 0, \quad k \rightarrow \infty, \quad i \in [N]. \end{aligned} \quad (4.24)$$

From (4.19), (4.22) and (4.24), this yields

$$\|y_{m_k}^i - T_i x_{m_k}\| \leq \|y_{m_k}^i - v_{m_k}^i\| + \|v_{m_k}^i - z_{m_k}^i\| + \|z_{m_k}^i - T_i x_{m_k}\| \rightarrow 0, \quad k \rightarrow \infty, \quad i \in [N]. \quad (4.25)$$

By the definition of $y_{m_k}^i$ together with the property of the projection map, it follows that

$$\langle v_{m_k}^i - \lambda_{m_k}^i A_i v_{m_k}^i - y_{m_k}^i, T_i y - y_{m_k}^i \rangle \leq 0, \quad \forall T_i y \in C_i, \quad i \in [N],$$

and this implies that

$$\frac{1}{\lambda_{m_k}^i} \langle v_{m_k}^i - y_{m_k}^i, T_i y - y_{m_k}^i \rangle \leq \langle A_i v_{m_k}^i, T_i y - y_{m_k}^i \rangle, \quad \forall T_i y \in C_i, \quad i \in [N].$$

This yields

$$\frac{1}{\lambda_{m_k}^i} \langle v_{m_k}^i - y_{m_k}^i, T_i y - y_{m_k}^i \rangle + \langle A_i v_{m_k}^i, y_{m_k}^i - v_{m_k}^i \rangle \leq \langle A_i v_{m_k}^i, T_i y - v_{m_k}^i \rangle, \quad \forall T_i y \in C_i, \quad i \in [N]. \quad (4.26)$$

Moreover, by using the fact that $\lim_{k \rightarrow \infty} \lambda_{m_k}^i = \chi_i > 0$ together with (4.22), it follows from (4.26) that

$$\liminf_{k \rightarrow \infty} \langle A_i v_{m_k}^i, T_i y - v_{m_k}^i \rangle \geq 0, \quad \forall T_i y \in C_i, \quad i \in [N]. \quad (4.27)$$

Observe that

$$\langle A_i y_{m_k}^i, T_i y - y_{m_k}^i \rangle = \langle A_i y_{m_k}^i - A_i v_{m_k}^i, T_i y - v_{m_k}^i \rangle + \langle A_i v_{m_k}^i, T_i y - v_{m_k}^i \rangle + \langle A_i y_{m_k}^i, v_{m_k}^i - y_{m_k}^i \rangle. \quad (4.28)$$

Furthermore, by applying (4.22) and (4.23), it follows from (4.27) and (4.28) that

$$\liminf_{k \rightarrow \infty} \langle A_i y_{m_k}^i, T_i y - y_{m_k}^i \rangle \geq 0, \quad \forall T_i y \in C_i, \quad i \in [N]. \quad (4.29)$$

Now, suppose $\{\theta_k^i\}$ is a decreasing sequence of positive numbers which tends to 0 for each $i \in [N]$. For each k , denote by N_k the smallest positive integer such that

$$\langle A_i y_{n_j}^i, T_i y - y_{n_j}^i \rangle + \theta_k^i \geq 0, \quad \forall j \geq N_k, \quad T_i y \in C_i, \quad i \in [N], \quad (4.30)$$

where the existence of N_k is guaranteed by (4.29). Since $\{\theta_k^i\}$ is decreasing, it follows that $\{N_k\}$ is increasing. Moreover, since for each k the sequence $\{y_{N_k}^i\} \subset C_i$, it may be assumed that $A_i y_{N_k}^i \neq 0$ (otherwise, $y_{N_k}^i \in VI(C_i, A_i)$, $i \in [N]$), and set

$$z_{N_k}^i = \frac{A_i y_{N_k}^i}{\|A_i y_{N_k}^i\|^2}.$$

Observe that $\langle A_i y_{N_k}^i, z_{N_k}^i \rangle = 1$ for each k , $i \in [N]$. It can be deduced from (4.30) that

$$\langle A_i y_{N_k}^i, T_i y + \theta_k^i z_{N_k}^i - y_{N_k}^i \rangle \geq 0, \quad \forall T_i y \in C_i, \quad i \in [N].$$

Since A_i is pseudomonotone for each $i \in [N]$, it follows that

$$\langle A_i(T_i y + \theta_k^i z_{N_k}^i), T_i y + \theta_k^i z_{N_k}^i - y_{N_k}^i \rangle \geq 0, \quad \forall T_i y \in C_i, \quad i \in [N],$$

and this implies that

$$\langle A_i T_i y, T_i y - y_{N_k}^i \rangle \geq \langle A_i T_i y - A_i(T_i y + \theta_k^i z_{N_k}^i), T_i y + \theta_k^i z_{N_k}^i - y_{N_k}^i \rangle - \theta_k^i \langle A_i T_i y, z_{N_k}^i \rangle, \quad \forall T_i y \in C_i, \quad i \in [N]. \quad (4.31)$$

To complete the proof, it remains to show that $\lim_{k \rightarrow \infty} \theta_k^i z_{N_k}^i = 0$ for each $i \in [N]$. Observe that since $x_{m_k} \rightharpoonup z$ and T_i is a bounded linear operator for each $i \in [N]$, then $T_i x_{m_k} \rightharpoonup T_i p$, $\forall i \in [N]$. Thus, from (4.25) it follows that $y_{m_k}^i \rightharpoonup T_i p$, $\forall i \in [N]$. Observe that since $\{y_{m_k}^i\} \subset C_i$, then $T_i p \in C_i$, $i \in [N]$. If $A_i T_i p = 0$, $\forall i \in [N]$, then $T_i p \in VI(C_i, A_i)$, $\forall i \in [N]$ and this would imply that $p \in \Psi$. Suppose on the contrary $A_i T_i p \neq 0$, $\forall i \in [N]$. Since A_i satisfies condition (4.1), it follows that for all $i \in [N]$

$$0 < \|A_i T_i p\| \leq \liminf_{k \rightarrow \infty} \|A_i y_{n_k, i}\|.$$

By using the facts that $\{y_{N_k}^i\} \subset \{y_{n_k}^i\}$ and $\theta_k^i \rightarrow 0$ as $k \rightarrow \infty$, $i \in [N]$, this yields

$$0 \leq \limsup_{k \rightarrow \infty} \|\theta_k^i z_{N_k}^i\| = \limsup_{k \rightarrow \infty} \left(\frac{\theta_k^i}{\|A_i y_{n_k, i}\|} \right) \leq \frac{\limsup_{k \rightarrow \infty} \theta_k^i}{\liminf_{k \rightarrow \infty} \|A_i y_{n_k, i}\|} = 0,$$

which implies that $\limsup_{k \rightarrow \infty} \theta_k^i z_{N_k}^i = 0$. Moreover, by applying the facts that $\{y_{N_k}^i\}$ and $\{z_{N_k}^i\}$ are bounded, $\lim_{k \rightarrow \infty} \theta_k^i z_{N_k}^i = 0$ together with the continuity of A_i for each $i \in [N]$, it follows from (4.31) that

$$\liminf_{k \rightarrow \infty} \langle A_i T_i y, T_i y - y_{N_k}^i \rangle \geq 0, \quad \forall T_i p \in C_i, \quad i \in [N].$$

It follows from this that

$$\langle A_i T_i y, T_i y - T_i p \rangle = \lim_{k \rightarrow \infty} \langle A_i T_i y, T_i y - y_{N_k}^i \rangle = \liminf_{k \rightarrow \infty} \langle A_i T_i y, T_i y - y_{N_k}^i \rangle \geq 0, \quad \forall T_i y \in C_i, \quad i \in [N].$$

By Lemma 2.6, it follows that

$$T_i p \in VI(C_i, A_i), \quad i \in [N],$$

and this implies that

$$p \in T_i^{-1}(VI(C_i, A_i)), \quad i \in [N],$$

Thus, $p \in \bigcap_{i \in [N]} T_i^{-1}(VI(C_i, A_i))$, which implies that $p \in \Psi$, as asserted. $\square$

Theorem 4.8. *The sequence $\{x_m\}$ generated by the proposed Algorithm 4.1 converges strongly to a point $v \in \Psi$, where v is the unique solution of the MSBSVIP (3.6)-(3.10).*

Proof. Observe that for each $i \in [N]$, $VI(C_i, A_i)$ is a closed and convex subset of H_i . Then, by Lemma 2.7 there exists a unique solution $v \in \Psi$ to the MSBSVIP (3.6)-(3.10). From Lemma 4.6, it follows that

$$\|x_{m+1} - v\|^2 \leq (1 - \alpha_m \gamma) \|x_m - v\|^2 + \alpha_m \gamma \left[\frac{2\mu}{\gamma} \langle Fv, v - x_{m+1} \rangle + \frac{2(1 - \alpha_m \gamma)^2}{\gamma} \sum_{i \in [N]} \beta_m^i \eta^i \mu^i \langle F_i T_i v, T_i v - v_m^i \rangle \right].$$

By setting $\Upsilon_m = \|x_m - v\|^2$ and $\Theta_m = \frac{2\mu}{\gamma} \langle Fv, v - x_{m+1} \rangle + \frac{2(1 - \alpha_m \gamma)^2}{\gamma} \sum_{i \in [N]} \beta_m^i \eta^i \mu^i \langle F_i T_i v, T_i v - v_m^i \rangle$, then the last inequality becomes

$$\Upsilon_{m+1} \leq (1 - \alpha_m \gamma) \Upsilon_m + \alpha_m \gamma \Theta_m. \quad (4.32)$$

It is now claimed that the sequence $\{\Upsilon_m\}$ converges to zero. In order to establish this claim, by Lemma 2.3 it suffices to show that $\limsup_{k \rightarrow \infty} \Theta_{m_k} \leq 0$ for every subsequence $\{\Upsilon_{m_k}\}$ of $\{\Upsilon_m\}$ satisfying

$$\liminf_{k \rightarrow \infty} (\Upsilon_{m_k+1} - \Upsilon_{m_k}) \geq 0. \quad (4.33)$$

Suppose that $\{\Upsilon_{m_k}\}$ is a subsequence of $\{\Upsilon_m\}$ such that (4.33) is satisfied. Again, from Lemma 4.6 it follows that

$$\begin{aligned} \limsup_{k \rightarrow \infty} & \left[(1 - \alpha_{m_k} \gamma)^2 \left[\sum_{i \in [N]} \beta_{m_k}^i \eta^i (1 - \eta^i L_i) \|z_{m_k}^i - T_i x_{m_k}\|^2 \right. \right. \\ & \left. \left. + \sum_{i \in [N]} \beta_{m_k}^i \eta^i \left(1 - \frac{\lambda_{m_k}^i}{\lambda_{m+1}^i} \frac{2}{2} (k^i + k_{m_k}^i)^2 \right) \|v_{m_k}^i - y_{m_k}^i\|^2 \right] \right] \\ & \leq \limsup_{k \rightarrow \infty} \left[(1 - \alpha_{m_k} \gamma) \Upsilon_{m_k} - \Upsilon_{m_k+1} + \alpha_{m_k} \gamma \Theta_{m_k} \right] \\ & \leq \limsup_{k \rightarrow \infty} [\Upsilon_{m_k} - \Upsilon_{m_k+1}] \\ & = -\liminf_{k \rightarrow \infty} [\Upsilon_{m_k+1} - \Upsilon_{m_k}] \\ & \leq 0. \end{aligned}$$

This yields

$$\lim_{k \rightarrow \infty} \|z_{m_k}^i - T_i x_{m_k}\| = 0, \quad \lim_{k \rightarrow \infty} \|v_{m_k}^i - y_{m_k}^i\| = 0, \quad i \in [N]. \quad (4.34)$$

Thus, it follows that

$$\|T_i^*(z_{m_k}^i - T_i x_{m_k})\| \leq \|T_i^*\| \|z_{m_k}^i - T_i x_{m_k}\| = \|T_i\| \|z_{m_k}^i - T_i x_{m_k}\| \rightarrow 0, \quad k \rightarrow \infty, i \in [N]. \quad (4.35)$$

Applying (4.35) together with the definition of u_m yields

$$\begin{aligned} \|u_{m_k} - x_{m_k}\| &= \left\| \sum_{i \in [N]} \beta_{m_k}^i (x_{m_k} + \eta^i T_i^* (z_{m_k}^i - T_i x_{m_k})) - x_{m_k} \right\| \\ &\leq \|x_{m_k} - x_{m_k}\| + \sum_{i \in [N]} \beta_{m_k}^i \eta^i \|T_i^* (z_{m_k}^i - T_i x_{m_k})\| \rightarrow 0, \quad k \rightarrow \infty. \end{aligned} \quad (4.36)$$

Consequently, it follows from (4.24) and (4.34) that

$$\|v_{m_k}^i - T_i x_{m_k}\| \rightarrow 0, \quad k \rightarrow \infty, i \in [N]. \quad (4.37)$$

Moreover, by the definition of x_{m+1} and the application of (4.36) together with the fact that $\lim_{m \rightarrow \infty} \alpha_m = 0$, it follows that

$$\begin{aligned} \|x_{m_k+1} - x_{m_k}\| &= \|(I^H - \alpha_{m_k} \mu F)u_{m_k} - (I^H - \alpha_{m_k} \mu F)x_{m_k} - \alpha_{m_k} \mu F x_{m_k}\| \\ &\leq (1 - \alpha_{m_k} \gamma) \|u_{m_k} - x_{m_k}\| + \alpha_{m_k} \mu \|F x_{m_k}\| \rightarrow 0, \quad k \rightarrow \infty. \end{aligned} \quad (4.38)$$

Since $\{x_m\}$ is bounded, $w_\omega(x_m) \neq \emptyset$. Let $u^* \in w_\omega(x_m)$ be chosen arbitrarily. Then, there exists a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $x_{m_k} \rightharpoonup u^*$. In view of Lemma 4.7 and by applying (4.36), it follows that $u^* \in \Omega$. Since $u^* \in w_\omega(x_m)$ was chosen arbitrarily, it can be concluded that $w_\omega(x_m) \subset \Psi$.

Also, by the boundedness of $\{x_{m_k}\}$, there exists a subsequence $\{x_{m_{k_j}}\}$ of $\{x_{m_k}\}$ such that $x_{m_{k_j}} \rightharpoonup z$ and

$$\limsup_{k \rightarrow \infty} \langle Fv, v - x_{m_k} \rangle = \lim_{j \rightarrow \infty} \langle Fv, v - x_{m_{k_j}} \rangle = \langle Fv, v - z \rangle. \quad (4.39)$$

Moreover, since T_i is a bounded linear operator for each $i \in [N]$ and $x_{m_{k_j}} \rightharpoonup z$, it follows that $T_i x_{m_{k_j}} \rightharpoonup T_i z$. Consequently, (4.37) implies that $v_{m_{k_j}}^i \rightharpoonup T_i z$ and

$$\limsup_{k \rightarrow \infty} \langle F_i T_i v, T_i v - v_{m_k}^i \rangle = \lim_{j \rightarrow \infty} \langle F_i T_i v, T_i v - v_{m_{k_j}}^i \rangle = \langle F_i T_i v, T_i v - T_i z \rangle. \quad (4.40)$$

Since v is the unique solution of the MSBSVIP (3.6)-(3.10), it follows from (4.38) - (4.40) that

$$\limsup_{k \rightarrow \infty} \langle Fv, v - x_{m_k+1} \rangle \leq \limsup_{k \rightarrow \infty} \langle Fv, v - x_{m_k} \rangle + \limsup_{k \rightarrow \infty} \langle Fv, x_{m_k} - x_{m_k+1} \rangle = \langle Fv, v - z \rangle \leq 0 \quad (4.41)$$

and

$$\limsup_{k \rightarrow \infty} \langle F_i T_i v, T_i v - v_{m_k}^i \rangle = \langle F_i T_i v, T_i v - T_i z \rangle \leq 0, \quad i \in [N]. \quad (4.42)$$

Thus, by (4.41) and (4.42) it follows that $\limsup_{k \rightarrow \infty} \Theta_{m_k} \leq 0$. Consequently, by applying Lemma 2.3, it can be deduced from (4.32) that $\{\|x_m - v\|\}$ converges to zero as required. $\square$

5. APPLICATIONS

In this section, the main results developed in the previous sections are applied to study a related class of split inverse problems. In particular, a distributed extension of the bilevel split variational inequality problem is introduced and analysed.

5.1. Distributed Bilevel Split Variational Inequality Problem.

A new class of split inverse problems, referred to as the *distributed bilevel split variational inequality problem* (DBSVIP), is considered. Let $\mathbb{C}_i$ be nonempty, closed, and convex subsets of real Hilbert spaces H_i , for $i = 1, 2, \dots, N$, and let $\mathbb{T}_i : H_i \rightarrow H_{i+1}$, for $i = 1, 2, \dots, N-1$, be bounded linear operators such that $\mathbb{T}_i \neq 0$. Furthermore, let $\mathbb{A}_i : H_i \rightarrow H_i$ and $\mathbb{F}_i : H_i \rightarrow H_i$, for $i = 1, 2, \dots, N$, be single-valued operators.

The DBSVIP is formulated as follows:

$$\text{Find } \hat{x} \in VI(\mathbb{C}_1, \mathbb{A}_1) \text{ such that } \langle \mathbb{F}_1 \hat{x}, x - \hat{x} \rangle \geq 0 \quad \forall x \in VI(\mathbb{C}_1, \mathbb{A}_1), \quad (5.1)$$

and for each $i = 2, 3, \dots, N$,

$$\hat{\mathbb{T}}_{i-1} \hat{x} \in VI(\mathbb{C}_i, \mathbb{A}_i) \text{ satisfies } \langle \mathbb{F}_i \hat{\mathbb{T}}_{i-1} \hat{x}, x_i - \hat{\mathbb{T}}_{i-1} \hat{x} \rangle \geq 0 \quad \forall x_i \in VI(\mathbb{C}_i, \mathbb{A}_i). \quad (5.2)$$

Here, the operator $\hat{\mathbb{T}}_{i-1}$ is defined recursively by

$$\hat{\mathbb{T}}_{i-1} = \mathbb{T}_{i-1}\mathbb{T}_{i-2}\cdots\mathbb{T}_0, \quad \text{with } \mathbb{T}_0 = I^{H_1}.$$

The sets $VI(\mathbb{C}_1, \mathbb{A}_1)$ and $VI(\mathbb{C}_i, \mathbb{A}_i)$, for $i = 2, 3, \dots, N$, denote the solution sets of the following variational inequality problems:

$$\text{Find } \hat{y} \in \mathbb{C}_1 \text{ such that } \langle \mathbb{A}_1 \hat{y}, y - \hat{y} \rangle \geq 0 \quad \forall y \in \mathbb{C}_1, \quad (5.3)$$

and for each $i = 2, 3, \dots, N$,

$$\hat{\mathbb{T}}_{i-1} \hat{y} \in \mathbb{C}_i \text{ satisfies } \langle \mathbb{A}_i \hat{\mathbb{T}}_{i-1} \hat{y}, y_i - \hat{\mathbb{T}}_{i-1} \hat{y} \rangle \geq 0 \quad \forall y_i \in \mathbb{C}_i. \quad (5.4)$$

It is assumed that the solution set

$$\Omega = \left\{ \hat{y} \in \mathbb{C}_1 : \hat{y} \in VI(\mathbb{C}_1, \mathbb{A}_1) \cap \left(\bigcap_{i=1}^N \hat{\mathbb{T}}_{i-1}^{-1} VI(\mathbb{C}_i, \mathbb{A}_i) \right) \right\}$$

is nonempty.

It is important to observe that, by setting

$$C = \mathbb{C}_1, \quad C_i = \mathbb{C}_{i+1}, \quad F = \mathbb{F}_1, \quad F_i = \mathbb{F}_{i+1}, \quad A = \mathbb{A}_1, \quad A_i = \mathbb{A}_{i+1}, \quad 1 \leq i \leq N-1,$$

and defining the operators

$$T_1 = \mathbb{T}_1, \quad T_2 = \mathbb{T}_2\mathbb{T}_1, \quad \dots, \quad T_{N-1} = \mathbb{T}_{N-1}\mathbb{T}_{N-2}\cdots\mathbb{T}_1,$$

the MSBSVIP formulation (3.6)–(3.10) reduces to the DBSVIP defined in (5.1)–(5.4).

Consequently, the strong convergence results established for the MSBSVIP framework can be directly applied to the DBSVIP, yielding the following convergence theorem for determining its solution in Hilbert spaces.

Theorem 5.1. *Suppose that, for each $i = 1, 2, \dots, N$, the operator $\mathbb{F}_i : H_i \rightarrow H_i$ is K_i -Lipschitz continuous and δ_i -strongly monotone, and the operator $\mathbb{A}_i : H_i \rightarrow H_i$ is L_i -Lipschitz continuous and pseudomonotone, satisfying condition (4.1). Assume further that Assumption A of Theorem 4.8 holds and that $\Omega \neq \emptyset$. Then, the sequence $\{x_m\}$ generated by the following algorithm converges strongly to a point $u \in \Omega$, where u is the unique solution of the DBSVIP (5.1)–(5.4).*

Algorithm 5.2.

Initialization: Choose an initial point $x_0 \in H$. Let $\hat{\mathbb{T}}_{i-1} = \mathbb{T}_{i-1}\mathbb{T}_{i-2}\cdots\mathbb{T}_0$, $\hat{\mathbb{T}}_{i-1}^* = \hat{\mathbb{T}}_0^*\hat{\mathbb{T}}_1^*\cdots\hat{\mathbb{T}}_{i-1}^*$, $i = 1, 2, \dots, N$ and set $m = 0$.

Iterative Steps: Compute x_{m+1} as follows:

Step 1. Compute

$$v_m^i = (I^{H_i} - \alpha_m \mu^i \mathbb{F}_i) \hat{\mathbb{T}}_{i-1} x_m.$$

Step 2. Compute

$$y_m^i = P_{C_i}(v_m^i - \lambda_m^i \mathbb{A}_i v_m^i).$$

Step 3. Compute

$$z_m^i = y_m^i - \lambda_m^i (\mathbb{A}_i y_m^i - \mathbb{A}_i v_m^i),$$

$$\lambda_{m+1}^i = \begin{cases} \min \left\{ \frac{(k_i + k_m^i) \|v_m^i - y_m^i\|}{\|\mathbb{A}_i v_m^i - \mathbb{A}_i y_m^i\|}, \lambda_m^i + \phi_m^i \right\}, & \text{if } \mathbb{A}_i v_m^i - \mathbb{A}_i y_m^i \neq 0, \\ \lambda_m^i + \phi_m^i, & \text{otherwise.} \end{cases}$$

Step 4. Compute

$$u_m = \sum_{i=1}^N \beta_m^i (x_m + \eta^i \hat{\mathbb{T}}_{i-1}^* (z_m^i - \hat{\mathbb{T}}_{i-1} x_m)).$$

Step 5. Compute

$$x_{m+1} = (I^H - \alpha_m \mu^1 \mathbb{F}_1) u_m.$$

Set $m := m + 1$ and return to **Step 1**.

6. NUMERICAL EXPERIMENTS

This section evaluates the practical performance of the proposed Algorithm 4.1 on the multi-space bilevel split variational inequality problem (MSBSVIP) using transportation network applications. Two static (finite-dimensional) experiments and two dynamic (infinite-dimensional) experiments formulated in $L^2([0, T]; \mathbb{R}^{|\mathcal{P}|})$ are presented. All implementations are carried out in Python 3. Convergence is monitored through the successive-iterate error

$$e_m := \|x_{m+1} - x_m\|,$$

with plots showing both e_m versus iteration number and e_m versus CPU time.

For each experiment, we report (i) baseline results with a tuned parameter setting, and (ii) sensitivity analyses with respect to initial points and key algorithmic parameters. The stopping criterion is $e_m < 10^{-6}$.

In all baseline runs, the following parameter schedules are employed:

$$\alpha_m = \frac{\alpha_0}{m+1}, \quad \varphi_m^i = \frac{1}{(m+1)^2}, \quad k_m^i = \frac{1}{(m+1)^2}, \quad \beta_m^i = \frac{1}{N+1}.$$

The fixed parameters are $\mu_i = \mu_0 = 0.02$, $k_i = 0.5$, $\alpha_0 = 0.5$, and initial step sizes $\lambda_0^i = 0.1$ for all i .

6.1. Common Setting for Static Experiments 6.1–6.2.

Both static experiments adopt a path-based formulation. Let $\mathcal{W}$ be the set of origin–destination (O–D) pairs and $\mathcal{P}_w$ the set of paths connecting pair w . The path flow space is the Hilbert space $H = \mathbb{R}^{|\mathcal{P}|}$. The feasible sets are defined by non-negativity and O–D demand conservation:

$$C = \left\{ f \in \mathbb{R}^{|\mathcal{P}|} \mid f \geq 0, \sum_{p \in \mathcal{P}_w} f_p = d_w, \forall w \in \mathcal{W} \right\},$$

and similarly for each class i ,

$$C_i = \left\{ z \in \mathbb{R}^{|\mathcal{P}|} \mid z \geq 0, \sum_{p \in \mathcal{P}_w} z_p = d_w^{(i)}, \forall w \in \mathcal{W} \right\}.$$

Link flows are obtained via the link-path incidence matrix δ_{ap} . Travel times on each link follow the Bureau of Public Roads (BPR) function:

$$B_a(x) = t_a^0 \left(1 + 0.20 \left(\frac{x_a}{\text{cap}_a} \right)^4 \right),$$

with free-flow times $t_a^0 = [10, 12, 15, 10, 5]$ and capacities $\text{cap}_a = [20, 15, 25, 18, 40]$.

The lower-level operator $A : H \rightarrow H$ maps path flows f to path costs $c(f)$, where $c_p(f) = \sum_a \delta_{ap} B_a(x(f))$. Class-specific perception is incorporated through

$$T_i f = \frac{d_i}{D} f, \quad A_i(z) = \text{VOT}_i c\left(\frac{D}{d_i} z\right).$$

The upper-level operators are defined using the marginal social cost $\text{MSC}(f) = c(f) + \nabla c(f) \cdot f$ with regularization:

$$F(f) = \text{MSC}(f) + \rho_0 f, \quad F_i(z) = \chi_i \frac{d_i}{D} \text{MSC}\left(\frac{D}{d_i} z\right) + \rho_i z,$$

where χ_i represents policy weight and $\rho_0 = \rho_i = 10^{-3}$.

Experiment 6.1. Environmental Sustainability vs. Revenue Generation (Static)

We consider a corridor network with three user classes: - Low-income/eco-conscious: demand $d_1 = 8.0$, $\text{VOT}_1 = 1.0$, $\chi_1 = 0.6$, - Business travellers: demand $d_2 = 6.0$, $\text{VOT}_2 = 2.5$, $\chi_2 = 1.3$, - Freight: demand $d_3 = 9.0$, $\text{VOT}_3 = 3.0$, $\chi_3 = 1.8$, with total demand $D = 23$.

The lower level models a multi-class user equilibrium. The upper level balances environmental protection (lower weight on low-income class) against revenue generation (higher weights on business and freight).

The algorithm converged in **1064 iterations** to the solution

$$x^* \approx [15.0868, 7.9120, -4.54 \times 10^{-4}],$$

with total flow $\sum x_p^* \approx 22.9983 \approx D$. The equilibrium flow is concentrated on two dominant paths. The policy weights successfully produce a compromise between environmental protection and revenue generation.

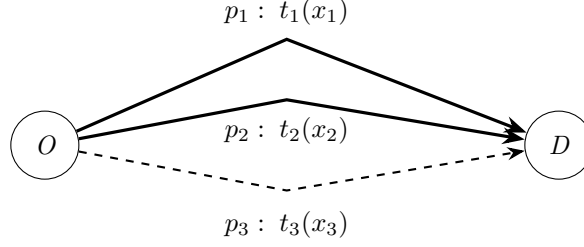


FIGURE 1. Corridor network with three alternative paths between O and D for Experiment 6.1. Equilibrium flow concentrates on two dominant paths.

Experiment 6.2. Multi-Modal Policy Design in a Grid Network (Static)

This experiment considers a grid network with four user classes: - Low-income: $d_1 = 7.0$, $\text{VOT}_1 = 1.0$, $\chi_1 = 0.5$, - Business: $d_2 = 8.0$, $\text{VOT}_2 = 2.5$, $\chi_2 = 1.2$, - Freight: $d_3 = 5.0$, $\text{VOT}_3 = 3.0$, $\chi_3 = 1.6$, - Public transport: $d_4 = 6.0$, $\text{VOT}_4 = 0.5$, $\chi_4 = 0.8$, with total demand $D = 26$.

The algorithm converged in **1180 iterations** to a solution with strong concentration on a single dominant path:

$$x^* \approx [-2.35 \times 10^{-4}, -2.84 \times 10^{-4}, 25.9991, -5.63 \times 10^{-4}],$$

with total flow $\approx 25.9980 \approx D$. This indicates that network topology and cost structure dominate policy-induced diversification in this setting, while still satisfying all equilibrium and policy conditions.

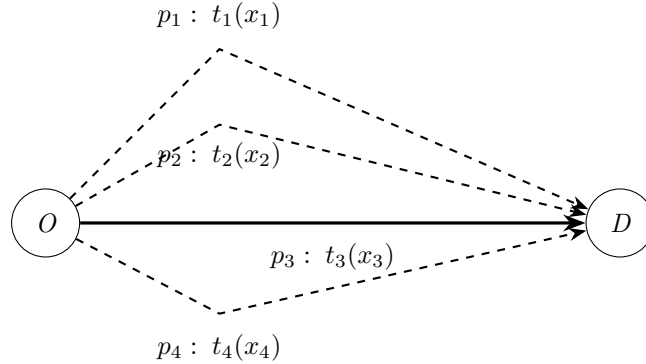


FIGURE 2. Grid network with four alternative paths between O and D for Experiment 6.2. Equilibrium flow concentrates on a single dominant path.

6.2. Common Setting for Dynamic Experiments 6.3–6.4.

In the dynamic case, all quantities are time-dependent functions belonging to the Hilbert space $H = L^2([0, T]; \mathbb{R}^{|\mathcal{P}|})$. The feasible sets C and C_i enforce O–D demand conservation almost everywhere in time. The BPR function and all operators (A, A_i, T_i, F, F_i) are applied pointwise in time.

The network topology for the dynamic experiments (Experiments 6.3 and 6.4) is identical to that used in the corresponding static cases. Hence, the previously presented network diagrams are reused, with the understanding that flows and costs are time-dependent.

Experiment 6.3. Dynamic Incident Management

We consider a corridor network over $[0, 3]$ hours ($M = 361$ discretization points) with time-varying demand $d_i(t) = d_i(1 + 0.5 \sin(\pi t/T))$. A temporary 40% capacity reduction occurs on one link during $[1, 2]$ hours. Class parameters are identical to Experiment 6.1.

The algorithm converged in **1678 iterations**. The solution $x^*(t)$ captures realistic temporal rerouting in response to the incident while balancing environmental, economic, and freight policy objectives. Flow conservation holds at every discretized time step.

Experiment 6.4. Dynamic Eco-Routing with Subsidies

This experiment uses a grid network over $[0, 12]$ hours ($M = 721$ points) with bimodal demand $d_i(t) = d_i(1 + 0.6 \sin^2(\pi t/T))$ and class parameters identical to Experiment 6.2.

The algorithm converged in **1524 iterations**. The solution exhibits strong concentration on a dominant path across the entire horizon, demonstrating temporal stability despite demand fluctuations and subsidy policies.

6.3. Sensitivity Analysis.

We examined the robustness of Algorithm 4.1 with respect to initial points and key algorithmic parameters. Results are summarized in Tables 1–5 and Figures 3–7.

The initial points were generated using five different strategies: a uniform distribution, three concentrated initialisations biased towards individual paths, and a Dirichlet distribution to enforce structured randomness. These choices allow us to assess the robustness of the algorithm with respect to both unstructured and path-dependent initialisations.

 TABLE 1. Sensitivity of Algorithm 4.1 to different initial points x_0

x_0	Experiment 6.1		Experiment 6.2		Experiment 6.3		Experiment 6.4	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Uniform	1064	0.3467	1180	0.6244	1678	67.20	2302	302.78
Conc. p1	1064	0.3341	1180	0.6216	1678	66.22	2302	387.82
Conc. p2	1064	0.3292	1180	0.6116	1678	64.53	2302	391.36
Conc. p3	1064	0.3438	1180	0.5806	1678	104.22	2302	380.01
Dirichlet	1064	0.3428	1180	0.5701	1678	118.81	2302	409.32

 TABLE 2. Sensitivity of Algorithm 4.1 to different values of α_0

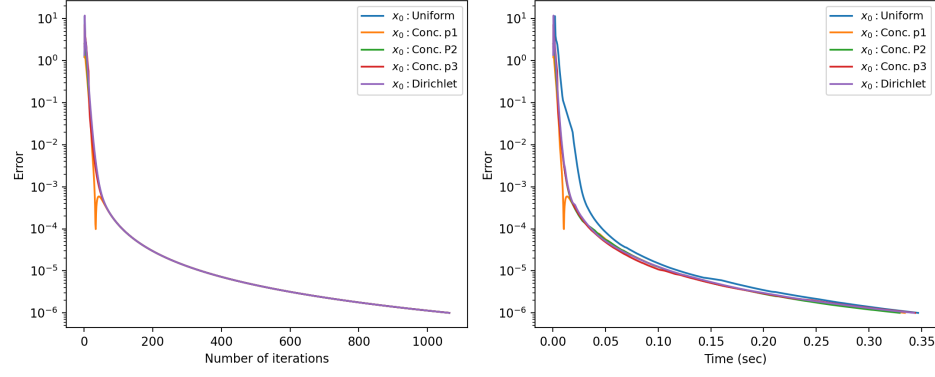
α_0	Experiment 6.1		Experiment 6.2		Experiment 6.3		Experiment 6.4	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
0.2	675	0.3236	750	0.3945	1065	87.47	1460	266.68
0.35	891	0.4645	989	0.5373	1406	97.57	1927	342.86
0.5	1064	0.5762	1180	0.6255	1678	114.00	2302	355.65
0.75	1301	0.8079	1444	0.7335	2053	147.81	2816	518.01
0.95	1464	0.8192	1624	0.9039	2309	165.11	3168	560.66

 TABLE 3. Sensitivity of Algorithm 4.1 to different values of λ_0^i

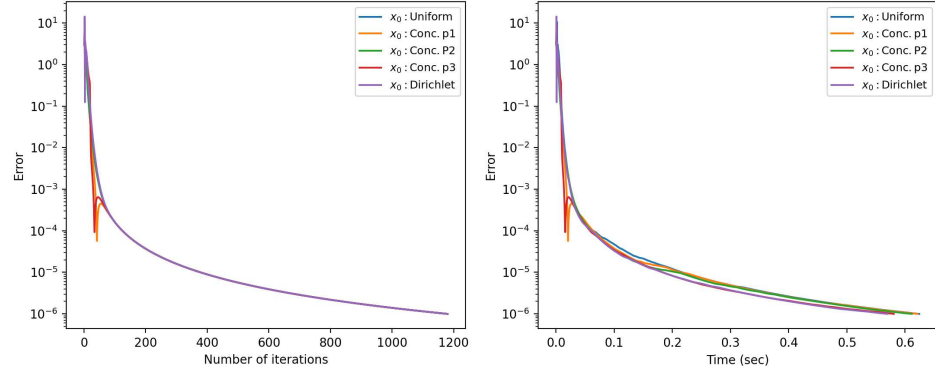
λ_0^i	Experiment 6.1		Experiment 6.2		Experiment 6.3		Experiment 6.4	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
0.05	1064	0.4780	1180	0.6151	1678	116.80	2302	382.96
0.1	1064	0.5632	1180	0.5773	1678	114.17	2302	376.60
0.3	1064	0.7084	1180	0.6789	1678	116.34	2302	387.30
0.5	1064	0.5585	1180	0.6196	1678	117.25	2302	415.22
0.8	1064	0.4977	1180	0.5834	1678	112.58	2302	436.33

 TABLE 4. Sensitivity of Algorithm 4.1 to different values of k_m^i

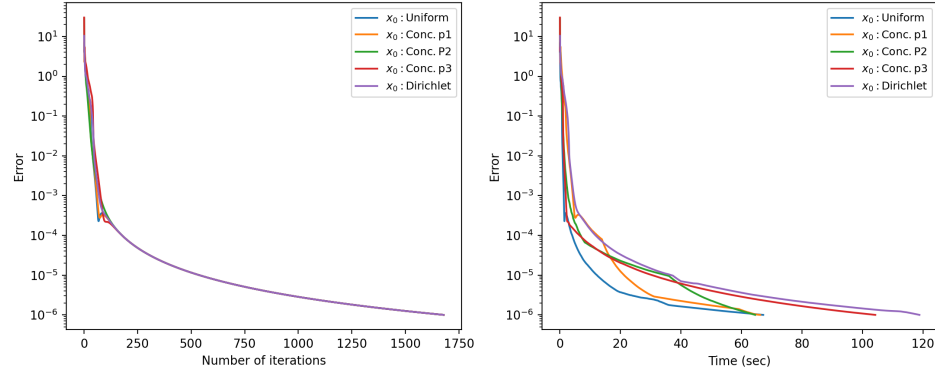
k_m^i	Experiment 6.1		Experiment 6.2		Experiment 6.3		Experiment 6.4	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
$1/(m+1)$	1063	0.4879	1179	0.5853	1676	122.96	2300	406.32
$1/(m+1)^2$	1064	0.6873	1180	0.6694	1678	123.25	2302	378.71
$1/\sqrt{m+1}$	1050	0.5000	1153	0.5872	1626	130.44	2255	366.63
$1/\log(m+2)$	1039	0.4753	1105	0.5886	1496	113.50	2133	347.93
$1/(m+1)^{0.75}$	1060	0.4956	1175	0.6055	1668	132.19	2293	402.79



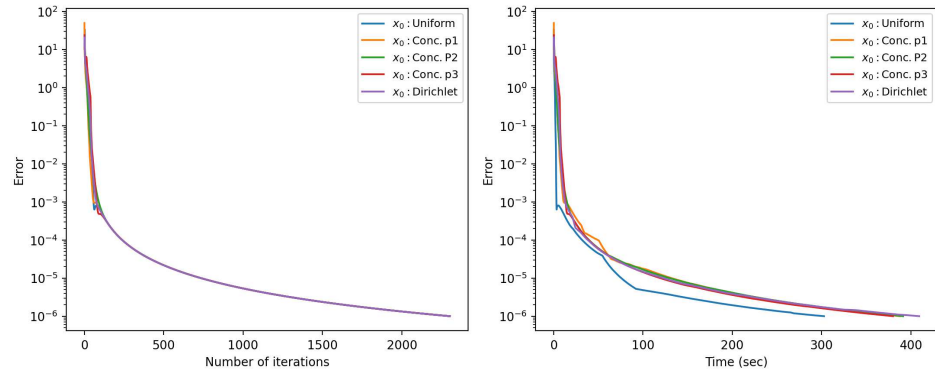
(A) Experiment 6.1



(B) Experiment 6.2

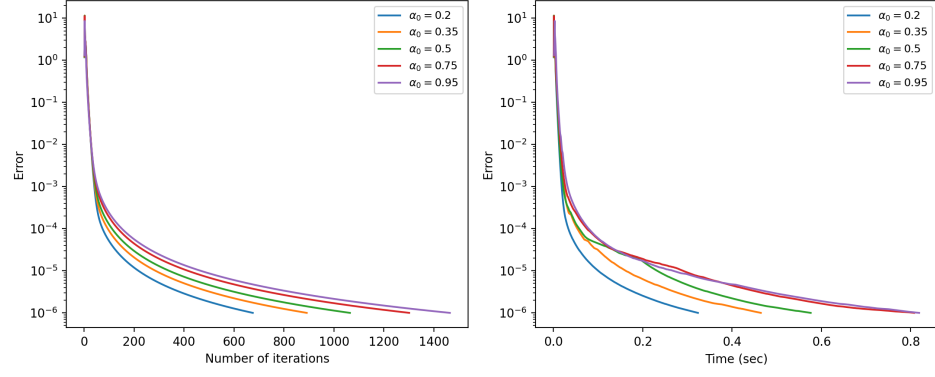


(C) Experiment 6.3

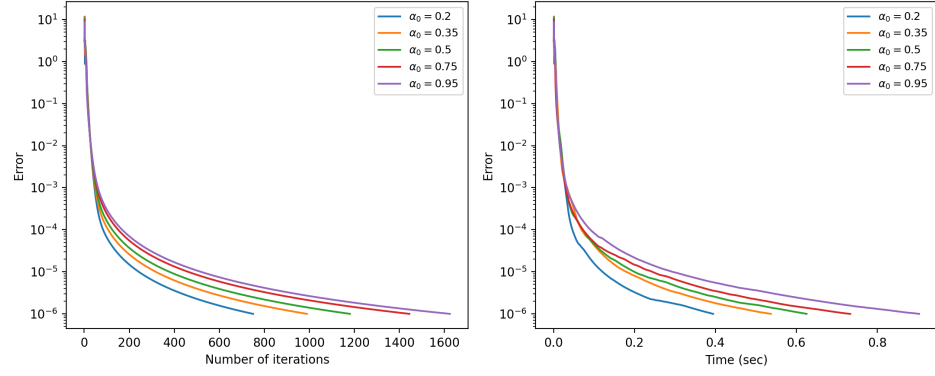


(D) Experiment 6.4

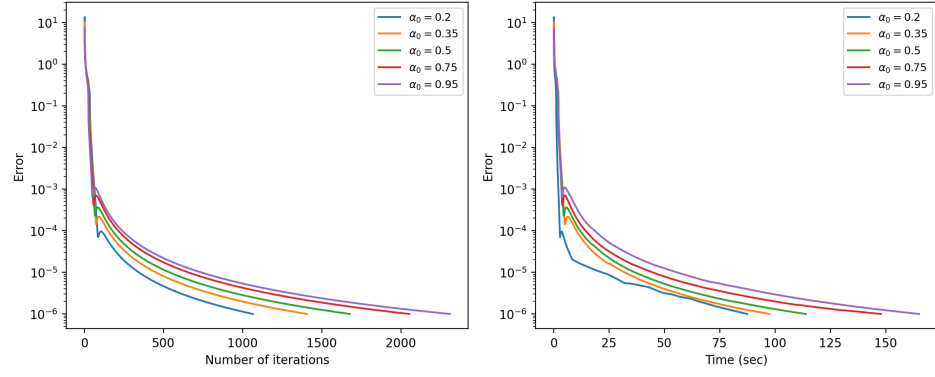
FIGURE 3. Convergence profiles (error vs. iterations and CPU time) for five choices of initial points.



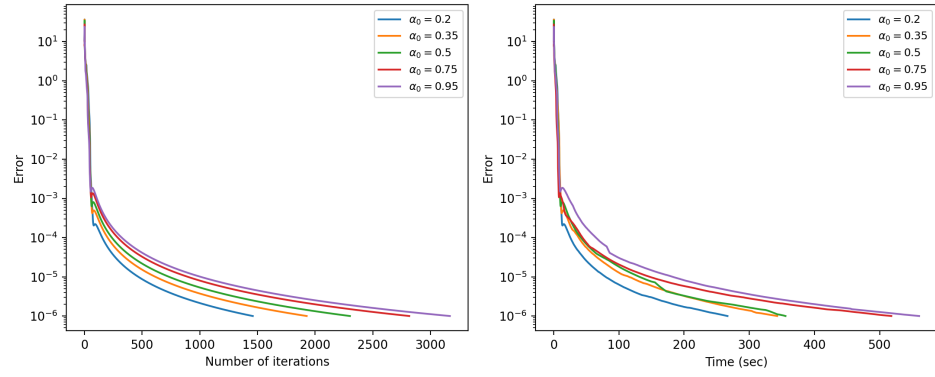
(A) Experiment 6.1



(B) Experiment 6.2

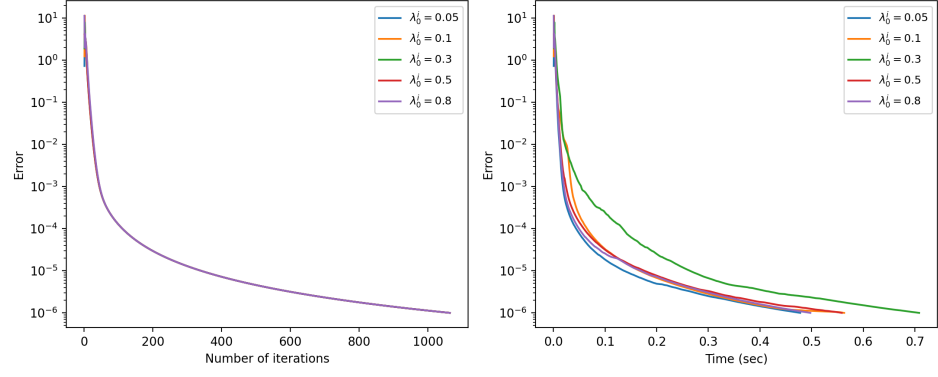


(C) Experiment 6.3

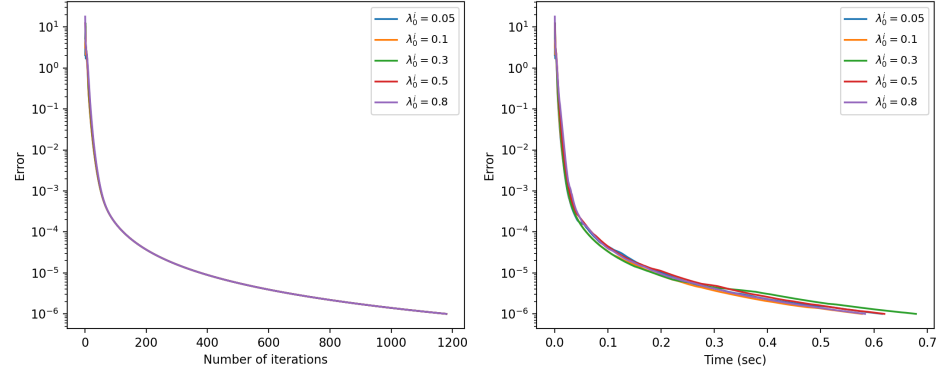


(D) Experiment 6.4

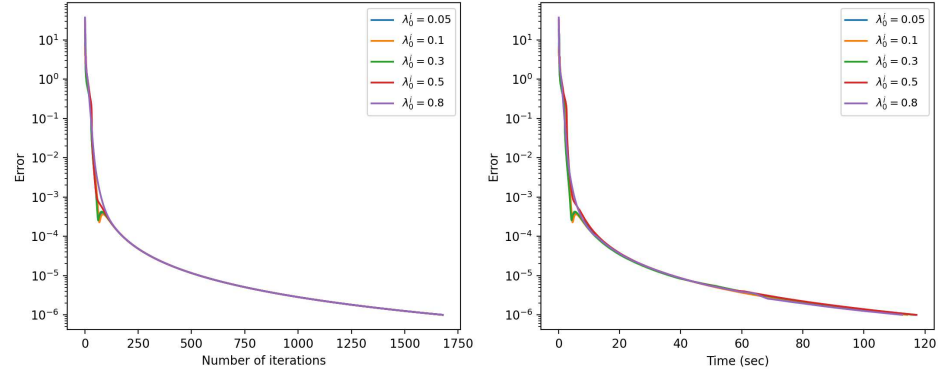
 FIGURE 4. Convergence profiles (error vs. iterations and CPU time) for five choices of α_0 .



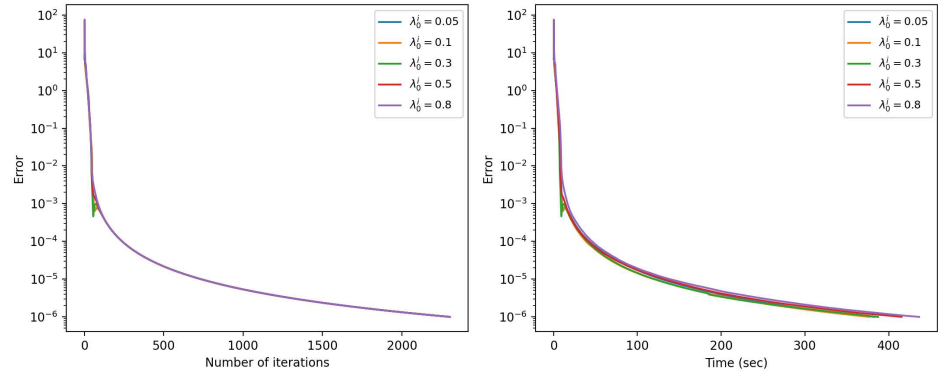
(A) Experiment 6.1



(B) Experiment 6.2

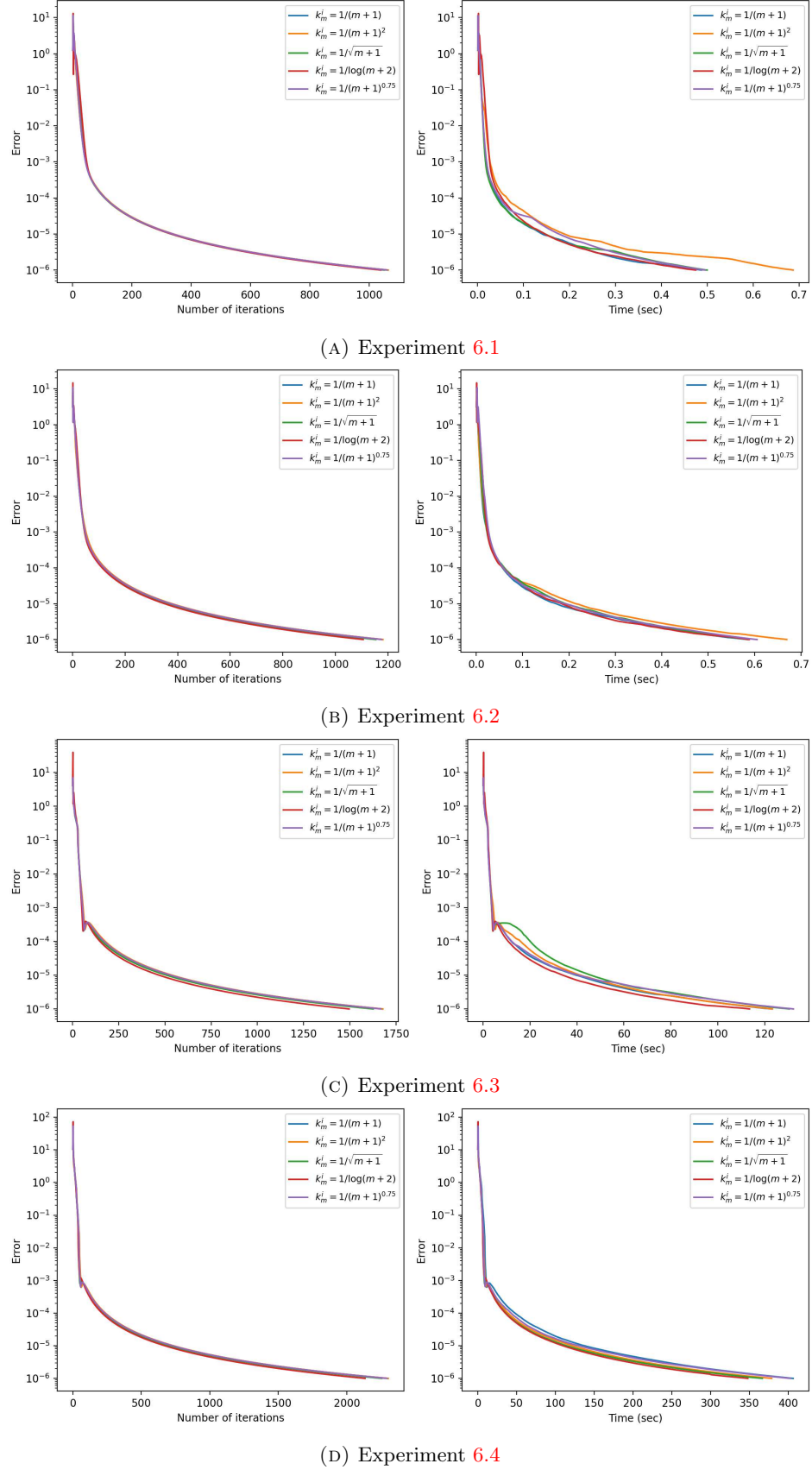


(C) Experiment 6.3



(D) Experiment 6.4

FIGURE 5. Convergence profiles (error vs. iterations and CPU time) for five choices of λ_0^i .


 FIGURE 6. Convergence profiles (error vs. iterations and CPU time) for five choices of k_m^i .

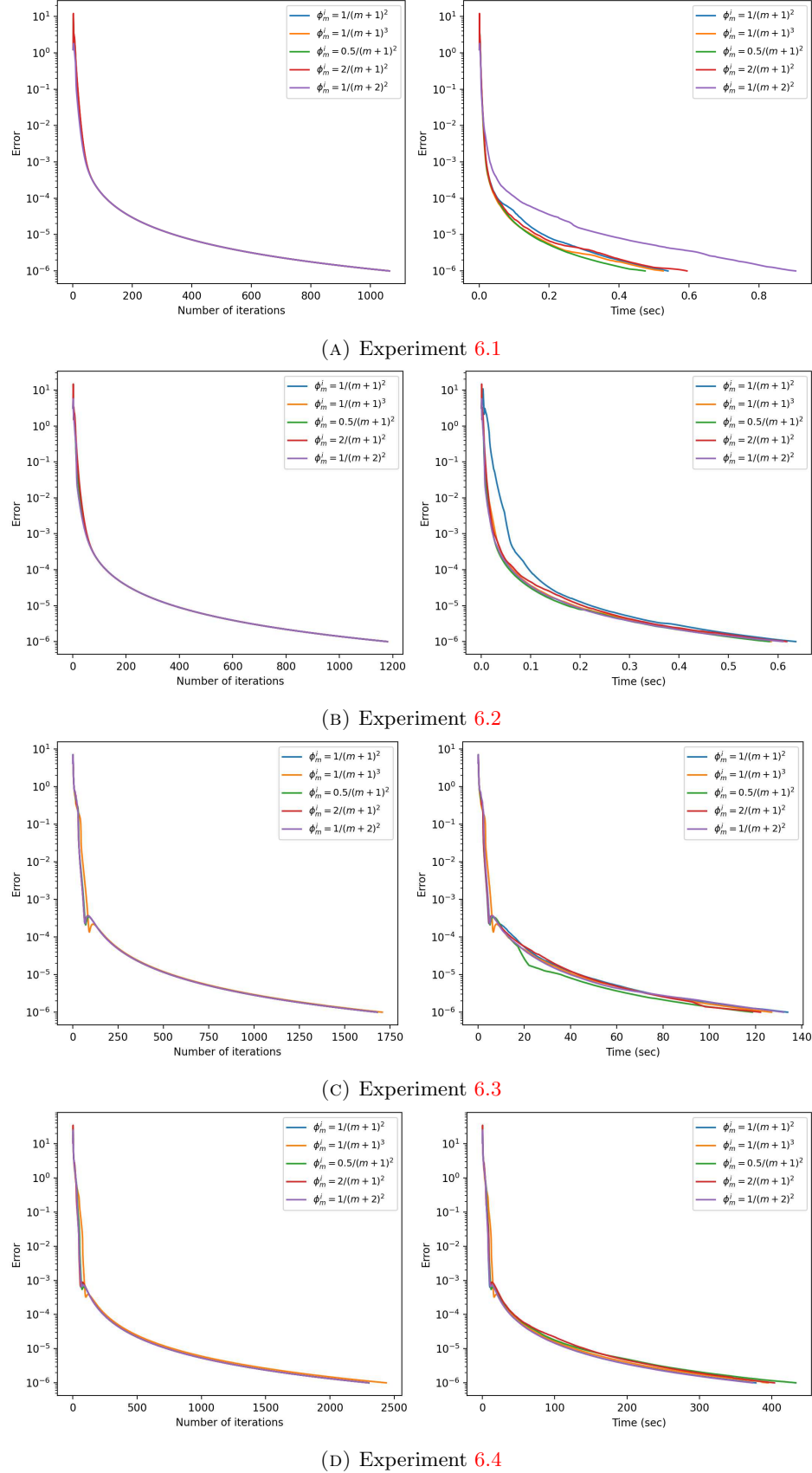

 FIGURE 7. Convergence profiles (error vs. iterations and CPU time) for five choices of ϕ_m^i .

TABLE 5. Sensitivity of Algorithm 4.1 to different values of ϕ_m^i

ϕ_m^i	Experiment 6.1		Experiment 6.2		Experiment 6.3		Experiment 6.4	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
$1/(m+1)^2$	1064	0.5401	1180	0.6360	1678	133.90	2302	377.86
$1/(m+1)^3$	1064	0.5261	1180	0.5861	1707	126.96	2438	394.87
$0.5/(m+1)^2$	1064	0.4746	1180	0.5825	1678	118.68	2302	433.10
$2/(m+1)^2$	1064	0.5944	1180	0.6177	1678	122.14	2302	403.55
$1/(m+2)^2$	1064	0.9053	1180	0.6099	1678	132.08	2302	373.37

6.4. Overall Observations.

Across all experiments, the proposed algorithm consistently converges to solutions that satisfy flow conservation (up to numerical precision) and the associated equilibrium conditions. The results demonstrate strong numerical stability across both static and dynamic settings, with iteration counts remaining largely unaffected by variations in initialisations and algorithmic parameters.

In contrast, computational time exhibits greater sensitivity to parameter choices and problem characteristics, particularly in larger-scale instances. This indicates that while the convergence behaviour of the algorithm is robust in terms of iteration complexity, its runtime performance can vary depending on parameter selection and problem structure.

The influence of policy weights and algorithmic parameters is further reflected in the resulting equilibrium flows, confirming the model’s ability to capture trade-offs in hierarchical multi-class traffic networks. Overall, these findings reinforce the practical applicability of the MSBSVIP framework and demonstrate the reliability and efficiency of the proposed projection-type algorithm under varying experimental conditions.

7. CONCLUSION

In this paper, the Multi-Space Bilevel Split Variational Inequality Problem (MSBSVIP) is introduced as a new and flexible modelling framework that unifies bilevel variational inequalities with multi-set split structures. The model effectively captures hierarchical decision processes in which a central planner seeks to achieve multiple, often competing policy objectives while accounting for the equilibrium responses of heterogeneous user classes in a shared transportation network. A projection-type extragradient algorithm is developed, and its strong convergence is established under standard monotonicity and Lipschitz conditions. The theoretical analysis shows that the proposed method generates a sequence that converges strongly to a solution of the MSBSVIP. Extensive numerical experiments on both static and dynamic multi-class traffic networks validate the theoretical findings and demonstrate the practical utility of the framework. The results show that the algorithm exhibits robust and efficient convergence behaviour across various network configurations, demand patterns, and policy weight combinations. Furthermore, the model reflects the trade-offs between system efficiency, equity, revenue generation, and environmental considerations through the class-specific upper-level operators.

The MSBSVIP provides a powerful and unified mathematical tool for addressing complex hierarchical equilibrium problems in transportation and beyond. Future research directions include large-scale network implementations and applications of the model to other domains such as energy markets, supply chain management, and multi-agent systems. Overall, this work advances the theory and computation of hierarchical split variational inequalities and offers a practical methodology for designing balanced and sustainable transportation policies in multi-class environments.

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