Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Dark state number in the Tavis-Cummings model

Theerthagiri L

tgiri9157@gmail.com

Indian Institute of Technology Madras

Rajesh Narayanan

Indian Institute of Technology Madras

Ganesh R

Brock University

Research Article

Keywords:

Posted Date: June 30th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-9754244/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Dark state number in the Tavis-Cummings model

L. Theerthagiri,[1] Rajesh Narayanan,[1] and R. Ganesh[2,*]

[1] *Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India*
[2] *Department of Physics, Brock University, St. Catharines, Ontario L2S 3A1, Canada*

(Dated: May 25, 2026)

Quantum mechanics allows for light-matter setups that hold excitations without releasing them as light. Arising from destructive interference processes, they are best seen in a Tavis-Cummings-like setup where two-level atoms (or qubits) are placed within a lossy cavity. If the system is initialized with some qubits excited and some in the ground state, there is a non-zero likelihood of no photons being emitted. This can be framed as a Stern-Gerlach measurement, with a detector to measure if one or more photons leave the cavity. If no photons are detected, the qubits collapse onto a dark state. This can be viewed as heralding of a dark state based on zero photon detection. Building upon this idea, we propose a protocol to measure the number of independent dark states. Remarkably, we show that this quantity is robust to arbitrary levels of disorder in the qubit-photon coupling constants. This provides a rare example of an observable that remains unchanged with any amount of disorder. We then discuss a phase transition where the number of dark states plays the role of an order parameter. This provides a rare example of a phase transition that is completely insensitive to disorder.

I. INTRODUCTION

Dark states are long-lived quantum states that hold excitations without releasing them as light[1–3]. They are closely related to the phenomenon of subradiance. They have the potential to serve as quantum memories that can store information. Motivated by this technological promise, they have been explored in several theoretical and experimental studies. The underlying notions may be traced back to a seminal thought experiment of Dicke from 1954 with two emitters[4]. This was experimentally realized in 2014 using superconducting qubits in a microwave cavity[5]. A key feature of this experiment was the lossy-cavity limit, where the time scale for a qubit-photon coupling is much longer than that for a photon to leave the cavity. This allows an emitted photon to 'immediately' leave the cavity and be detected outside.

Inspired by the lossy-cavity setup, two of the current authors have proposed Stern-Gerlach-like protocols to synthesize dark states[6, 7]. A measurement is performed by a photon detector placed outside the lossy mirror. A null result (zero photons detected) is a possible outcome. As with any quantum measurement, the null result will collapse the qubits within onto a dark state. The resulting state is entangled, expressible as a resonating valence bond wavefunction. In this article, we build on these ideas to (i) develop a protocol to measure N_{dark}, the number of dark states, (ii) establish that N_{dark} is robust to arbitrary levels of disorder, and (iii) describe a phase transition where N_{dark} serves as the order parameter.

A crucial ingredient in our arguments is the role of a zero-outcome in photon measurement[8–10]. It is well known that detection of a single photon can herald a specific quantum state in a second channel[11, 12]. Studies have explored the possibility of a zero-measurement doing the same[6, 9, 10]. Here, we view a cavity-QED setup as two channels – one of qubits and one of photons. Non-detection in the photon channel (a zero-measurement) can place the qubits in a specific state. We exploit this notion to generate and even count dark states.

The Tavis-Cummings model[13, 14] has been realized in ultracold atomic gases, circuit-QED setups[2, 15–17], cavity-QED[18–20], nitrogen-vacancy centres[21, 22], cold atom gases[23–26] etc. Any experimental setup will involve some level of disorder. In the context of the Tavis-Cummings model, disorder can enter in diagonal or off-diagonal terms. The former corresponds to randomness in qubit energies (frequencies) while the latter pertains to the coupling of each qubit to the photon mode. Diagonal disorder has been discussed in the context of persistent subradiance in Ref. [27]. We restrict our attention to off-diagonal disorder below.

The impact of disorder on phases and phase transitions of quantum systems has been extensively studied [28, 29]. Quenched or "frozen-in" disorder can have drastic effects. In low dimensions ($d \leq 2$) [30, 31], quenched disorder rounds first order transitions. It may destroy a putative critical point via smearing [32–34]. It may produce entirely new critical behaviour, e.g., an Infinite Randomness Fixed Point (IRFP) with attendant Griffiths phases[35–37]. These ideas have been experimentally explored in superconductors[34, 38–41] and magnets [42, 43]. In the context of light-matter interactions, there have been relatively few studies on the effects of disorder on phase transitions[44–46]. Here, we use the disorder-insensitive character of N_{dark} to devise an intriguing phase transition. Qualitatively, the transition occurs between a bright state (where N_{dark} vanishes) and a dark state. Remarkably, the phase transition here is unchanged with any amount of disorder. This gives an interesting example of a phase transition that is completely insensitive to disorder.

* r.ganesh@brocku.ca

II. DARK STATES IN THE UNIFORM LIMIT

In the interest of completeness, we first briefly summarize the problem in the absence of disorder. The results here have been demonstrated in Refs. [6] and [7]. In Sec. IV below, we show that these results can be extended to a class of disordered systems. We consider the Tavis-Cummings model with N qubits coupled to a single photon mode. Within the rotating wave approximation, the Hamiltonian is given by

$$H_{TC} = \omega(a^\dagger a + S^z_{(tot.)}) + g\left\{S^+_{(tot.)}a + S^-_{(tot.)}a^\dagger\right\}, \quad (1)$$

where $S^\alpha_{(tot.)} = \sum_{j=1}^N S^\alpha_j$ represents a collective spin operator ($\alpha = z, +, -$). The qubits are assumed to be resonant with the photon mode. The state of the qubits can be expressed using total angular momentum variables. For example, a state where all qubits are excited can be characterized as ($S_{(tot.)} = N/2$; $m_{(tot.)} = N/2$). In this language, the photon-emission process encoded by the term $\{S^-_{(tot.)}a^\dagger\}$ preserves the quantum number $S_{(tot.)}$, but lowers $m_{(tot.)}$ by unity. We define a dark state as one that cannot emit a photon (even if it may absorb photons). From the allowed values of angular momentum, any state with ($S_{(tot.)} = \lambda$; $m_{(tot.)} = -\lambda$; $n_{photon} = 0$) can be labelled as dark. Such a state cannot emit a photon as $m_{(tot.)}$ is at the minimum possible value and cannot be lowered further.

A crucial result emerges from the quantum addition-of-angular-momenta rules. The number of dark states with s qubits in the excited state and $(N - s)$ qubits in the ground state is given by a simple formula. These states are characterized $m_{(tot.)} = -S_{(tot.)} = s - N/2$. The number of such states is

$$N_{dark,N,s} = \begin{cases} 0, & \text{if } s > N/2, \\ \binom{N}{s} - \binom{N}{s-1}, & \text{if } s \leq N/2. \end{cases} \quad (2)$$

A derivation is given in Appendix. A.

To realize such states in experiments, a Stern-Gerlach-like protocol can be employed[6]. A direct-product initial state is placed within a lossy cavity. The state is prepared with s qubits excited and $(N - s)$ qubits in the ground state, corresponding to $m_{(tot.)} = s - N/2$. A photon detector is placed outside at the lossy mirror. If one or more photons is detected, the run is discarded and the protocol repeated. Non-detection of photons (over a time scale set by the qubit-photon coupling) collapses the qubits onto a dark state. The precise wavefunction is obtained by projecting the initial state onto the sector with $S_{(tot.)} = (-)(s - N/2)$.

Ref. 7 describes a bright-to-dark phase transition that emerges upon taking the total number of qubits N to infinity. The tuning parameter is s, equivalent to the polarization in the initial state. The order parameter is the number of photons emitted — see Fig. 7 in Ref. 7. Below, in Sec. VII, we describe a modified version of this phase transition that remains robust to arbitrary levels of disorder.

III. DISORDER IN LIGHT-MATTER COUPLING

We now introduce disorder in the Tavis-Cummings model, with the Hamiltonian,

$$H_{dis.TC} = \omega(a^\dagger a + S^z_{(tot.)}) + \sum_{j=1}^N \left\{g_j^* S^+_j a + g_j S^-_j a^\dagger\right\}, \quad (3)$$

where g_j's are independent, possibly non-uniform, coupling constants at each qubit. Note that we allow for disorder in the qubit-photon coupling constants, but not in the qubit frequencies. Experimental realizations of the Tavis-Cummings model have been achieved using superconducting qubits, gases of sodium atoms, etc. Disorder of the type encoded in Eq. 3 above is inevitable in such realizations. The position of each qubit within the cavity will affect the strength of the electric field as well as the local phase of the photon mode. In turn, these will affect the magnitude and phase of the coupling g_j.

IV. ROBUSTNESS OF DARK STATES WITH DISORDER

We define dark states as long-lived states that carry excitations but do not emit photons. The requirements for such a state are (i) it must be stationary, i.e., it must be an eigenstate of the Hamiltonian, (ii) it must not contain photons as any photons will eventually be emitted, and (iii) it must be annihilated by

$$\hat{O}_{dark} = \sum_j g_j S_{j,-}, \quad (4)$$

the operator that appears alongside $a^\dagger$ in the Hamiltonian of Eq. 3. A state that satisfies these requirements may possibly absorb a photon if one is available from the environment, e.g., see discussion on subradiant states in Ref. 2. We persist with our definition of a dark state as one that does *not* emit, as photon absorption is highly unlikely without external pumping.

The action of $\hat{O}_{dark}$ can be separated into various sectors,

$$\hat{O}_{dark} = \sum_{s=0}^N \hat{O}_s, \quad (5)$$

where s represents the number of qubits that are initially excited. As $\hat{O}_{dark}$ is a linear combination of spin-lowering operators, the final state after the action of $\hat{O}_{dark}$ has $s-1$ qubits excited. We note that each s corresponds to a different sector of the Hilbert space.

We may write $\hat{O}_s$ as a matrix of dimensions $\binom{N}{s-1} \times \binom{N}{s}$. This can be viewed as a linear transformation from the s-sector to the $(s-1)$-sector. Based on arguments presented in Appendix. B, we find the rank of $\hat{O}_s$ to be

$$\text{Rank}\{\hat{O}_s\} = \begin{cases} \binom{N}{s}, & \text{if } s > N/2, \\ \binom{N}{s-1}, & \text{if } s \leq N/2. \end{cases} \tag{6}$$

Using the rank-nullity theorem[47], we have

$$\text{Nullity}\{\hat{O}_s\} = \begin{cases} 0, & \text{if } s > N/2, \\ \binom{N}{s} - \binom{N}{s-1}, & \text{if } s \leq N/2. \end{cases} \tag{7}$$

Here, nullity represents the number of eigenstates of $\hat{O}_s$ with eigenvalue zero. Such states (with no photons in the cavity mode) satisfy all requirements for a dark state: (i) they are eigenstates of the Hamiltonian in Eq. 3, as they have a well-defined $S^z_{(tot.)}$ quantum number, (ii) they are annihilated by the $\hat{O}_s$ operator which accompanies $a^\dagger$, and (iii) they are annihilated by the operator a as there are no photons. This guarantees a long-lived state that does not emit photons.

The expression in Eq. 7 is the same as that in Eq. 2. It yields the number of independent dark states regardless of whether the couplings are uniform or not. Remarkably, this result holds for arbitrary variations in g_j's, as long as all g_j's are non-zero. This result has been obtained in Ref. 48 for the case where g_j's are positive numbers. Here, we provide a simpler demonstration that holds for arbitrary complex values. Physically, complex values could arise in a cavity-QED setup due to the spatial variation of the electric field. In the section below, we demonstrate the validity of this assertion by explicitly constructing dark states in one particular sector.

V. DARK STATES WITH A SINGLE EXCITATION

We now present analytic forms for dark states in the single-excitation sector, i.e., with $s = 1$. In this sector, $\hat{O}_{dark}$ of Eq. 4 takes the form of a $1 \times N$ matrix,

$$\hat{O}^{(s=1)}_{dark} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{N-1} & g_N \end{pmatrix}. \tag{8}$$

Each entry is obtained as $\langle \downarrow \ldots \downarrow | \sum_j g_j S_{j,-} | e_j \rangle$, where $|e_j\rangle = |\downarrow_1 \ldots \uparrow_j \ldots \downarrow_N\rangle$ represents a single-excitation-state where the j^{th} qubit alone is excited.

From Eq. 7, we have $N-1$ dark states in the $(s=1)$ sector. The simple structure of $\hat{O}_{dark}$ allows us to write analytic expressions for the wavefunctions of these dark-states. We write

$$|d_j\rangle = \frac{|g_j|}{\sqrt{|g_j|^2 + |g_N|^2}} \left\{ -\left(\frac{g_N}{g_j}\right) |e_j\rangle + |e_N\rangle \right\} \otimes |n_{ph} = 0\rangle, \tag{9}$$

where $j = 1, 2, \ldots N-1$. Here, $|n_{ph} = 0\rangle$ denotes that the photon sector is empty, with no photons in the cavity mode. We readily see that each $|d_j\rangle$ is annihilated by $\hat{O}^{(s=1)}_{dark}$. Moreover, it is an eigenstate of the $S^z_{(tot.)}$ operator. As a result, each $|d_j\rangle$ is an eigenstate of the Hamiltonian in Eq. 3. It follows that $|d_j\rangle$'s represent long-lived states that contain a single excitation, but do not release it as light. While $|d_j\rangle$ states are not orthogonal to one another, they can be easily seen to be linearly independent. We thus have an $(N-1)$-dimensional dark subspace. Expressions for dark states in the single-excitation sector can also be found in Ref. 49.

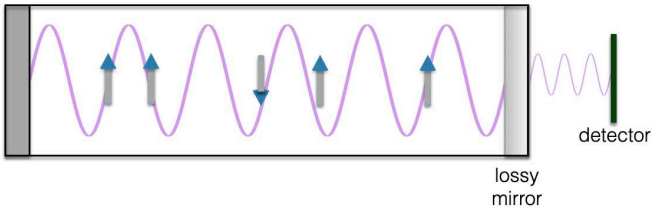


FIG. 1. Schematic setup required for protocol to measure number of dark states. Qubits are placed within a lossy cavity, with a photon detector outside. We may have variations in the amplitude of the photon mode from one qubit position to the next. This can produce randomness in the qubit-photon coupling constants, which does not affect the outcome.

VI. NULLITY AS AN OBSERVABLE

We now describe a protocol to experimentally measure the number of dark states for a given s. It can be used to experimentally verify that dark-state-number is robust to disorder of arbitrary strength, as argued in Sec. IV. More generally, this protocol demonstrates that dark-state-number is an observable quantity. This can be useful in quantum memory applications, where dark-state-number can, in principle, be read out.

We assume a light-matter setup that realizes the Tavis-Cummings model with or without disorder. Qubits are placed within a cavity with a lossy mirror. In the lossy-cavity limit[5], the time-scale for a photon to leave the cavity is much shorter than that associated with qubit-photon coupling. If a photon is emitted by the qubits, it 'immediately' leaves the cavity via the lossy mirror. We propose the following protocol:

- An initial direct product state is synthesized where s qubits are excited and $N-s$ are in the ground state. This initial state is placed within the cavity.

- A photon detector (assumed to be 100% efficient) is placed outside the lossy mirror and monitored. If the detector clicks, it signals the emission of one or more photons. This is recorded as an emission event.
- If the detector does not click over a sufficiently long time (set by the weakest qubit-photon coupling), we conclude that the initial state has collapsed onto a dark state. This is recorded as a null-emission event.
- The above steps are repeated several times to obtain a reliable estimate of the probability of null emission.
- The entire process is repeated over all rearrangements of the initial state, i.e., over all possible arrangements of N qubits where s are excited. Upon *adding* the probability of null emission from each rearrangement of N qubits, we obtain the number of dark states.

In mathematical terms, the outcome of this protocol can be described as follows. We first define $\hat{\mathcal{P}}_{dark,N,s}$, an operator that projects onto the dark subspace — the space spanned by the null eigenstates of $\hat{O}_s$. We have

$$\hat{\mathcal{P}}_{dark,N,s} = \sum_{j=1}^{N_{dark,N,s}} |d_j\rangle\langle d_j|, \tag{10}$$

where $|d_j\rangle$ runs over all dark states in the s-sector. For a given initial state $|\psi_{init.}\rangle$, the probability of null emission is given by $\langle\psi_{init.}|\hat{\mathcal{P}}_{dark,N,s}|\psi_{init.}\rangle$.

We next define a permutation operator, $\hat{\pi}_{k,N,s}$. Here, k runs over the $\begin{pmatrix} N \\ s \end{pmatrix}$ ways to select s qubits out of N. The s-sector is spanned by $\begin{pmatrix} N \\ s \end{pmatrix}$ basis states that can be expressed as

$$|\psi_k\rangle = \hat{\pi}_{k,N,s}|\psi_{ref.}\rangle, \tag{11}$$

where $|\psi_{ref.}\rangle$ is a reference state, say with the first s qubits excited while the rest are in the ground state. The $|\psi_k\rangle$'s represent all rearrangements of the excited qubits.

The final outcome of the protocol can be expressed as

$$\begin{aligned} D(s) &= \sum_k \Big\{\langle\psi_{ref.}|\hat{\pi}_{k,N,s}\Big\}\hat{\mathcal{P}}_{dark,N,s}\Big\{\hat{\pi}_{k,N,s}|\psi_{ref.}\rangle\Big\} \\ &= \sum_k \langle\psi_k|\hat{\mathcal{P}}_{dark,N,s}|\psi_k\rangle = \mathrm{Tr}_s\big\{\hat{\mathcal{P}}_{dark,N,s}\big\} \\ &= N_{dark,N,s}. \end{aligned} \tag{12}$$

We have used the fact that $|\psi_k\rangle$'s are basis states that span the s-sector. In the last step, we have replaced the sum by a trace of $\mathcal{P}_{dark,N,s}$ over the s-sector. As $\mathcal{P}_{dark,N,s}$ is a projection operator, this operation directly yields the number of independent dark states.

We have demonstrated that the number of dark states (the dimensionality of the dark subspace) can be directly measured using a suitably designed experiment. It is noteworthy that this quantity is robust to arbitrary levels of disorder in the qubit-photon coupling constants, as shown in Sec. IV above.

VII. DARK-BRIGHT PHASE TRANSITION

A phase transition requires three notions: a tuning parameter, an order parameter and a definition of the thermodynamic limit. As the tuning parameter, we propose

$$\alpha_{s,N} = s/N, \tag{13}$$

which represents the fraction of excited qubits in the initial state. It can also be viewed as the polarization of the initial state. Note that $\alpha_{s,N}$ can take values from zero to unity.

As the order parameter, we propose

$$o_{s,N} \;=\; N_{dark,N,s} / \begin{pmatrix} N \\ s \end{pmatrix}. \tag{14}$$

which represents the fraction of dark states in the s-sector. The numerator, $N_{dark,N,s}$, represents the number of dark states – a measurable quantity as shown in Sec. VI above. The denominator represents the total number of states within the s-sector. Using the expression for $N_{dark,N,s}$ in Eq. 2, we may write

$$o_{s,N} = \begin{cases} 0, & \text{if } \alpha_{s,N} > 1/2, \\ \frac{N-2s+1}{N-s+1}, & \text{if } \alpha_{s,N} \leq 1/2. \end{cases} \tag{15}$$

As the thermodynamic limit, we propose taking the limit $N \to \infty$, while preserving the ratio $\alpha = s/N$. We obtain

$$o_{N\to\infty} = \begin{cases} 0, & \text{if } \alpha > 1/2, \\ \frac{1-2\alpha}{1-\alpha}, & \text{if } \alpha \leq 1/2. \end{cases} \tag{16}$$

where α represents the tuning parameter in the thermodynamic limit (with subscripts s, N suppressed).

This expression matches the profile of an order parameter at a continuous phase transition. An 'ordered' phase appears for $\alpha < 1/2$, where $o_{N\to\infty}$ takes a finite value. This indicates a 'dark' phase where there is a non-zero likelihood that no photons will be emitted. This probability may be substantial, approaching unity for $\alpha \to 0$. A 'disordered' phase appears for $\alpha > 1/2$ where $o_{N\to\infty}$ vanishes. This indicates a 'bright' phase where photon emission is inevitable as there are no dark states. The critical point corresponds to $\alpha = 1/2$, where the order parameter goes to zero. Note that the order parameter itself varies continuously, but its derivative is singular at the transition. This is a characteristic feature of second-order phase transitions.

In Fig. 2, we plot the order parameter as a function of the tuning parameter. The plot shows data for a few

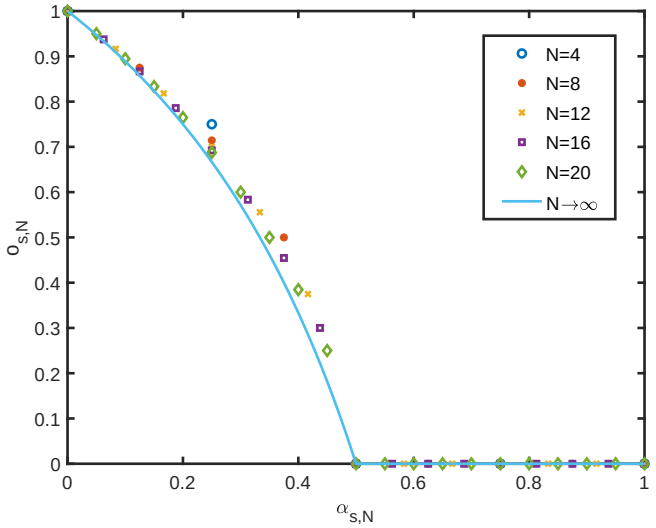


FIG. 2. The dark-to-bright phase transition. On the X axis, we have the tuning parameter $\alpha_{s,N}$, which corresponds to varying the polarization of initial states. The Y axis plots $o_{s,N}$, the fraction of dark states. The markers show values obtained for $N = 4, 8, 12, 16, 20$. The solid line is the result expected in the $N \to \infty$ limit.

values of N, ranging from 4 to 20. It also shows the result expected in the thermodynamic $N \to \infty$ limit. A clear second-order phase transition emerges, with small finite-size corrections. Crucially, this phase transition is completely insensitive to disorder in the qubit-photon coupling constants (g_j's). The order parameter and its behaviour do not vary as g_j's are varied.

VIII. DISCUSSION

Our central result is that dark states are robust to arbitrary disorder in the Tavis-Cummings model. The number of dark sates remains unchanged as each qubit-photon coupling is tuned. We prove this assertion and support it with an experimental protocol to measure the number of dark states in a Tavis-Cummings setup. This demonstrates that dark-state-number is an observable quantity. It can be experimentally determined to verify robustness against disorder.

Implementing the protocol in experiments can pose several challenges. The first is the need for a perfect photon detector. While the required technology exists at visible frequencies[50], efforts are underway to achieve perfect detection in the microwave regime[51–54]. The second is the requirement to create direct-product states with various arrangements of excited- and ground-state-qubits. This can be achieved in circuit-QED setups with a few qubits. Even if all rearrangements cannot be accessed, a limited number of trials could set a lower bound for the number of dark states. The third involves effects beyond the standard Tavis-Cummings model such as inter-qubit coupling[55], dephasing[56], counter-rotating terms[57], etc.

We have demonstrated that our results are completely insensitive to disorder in the qubit-photon coupling constants. In realistic setups, disorder may also appear in the qubit energies. The effect of such disorder on photon-number-dynamics has been studied[58]. The effect on dark states has also been studied, but only within the single-excitation sector[49]. Future studies could examine their role in all sectors.

The key motivation for studying dark states is that they may serve as quantum memories. At the same time, dark character makes it difficult to produce or to manipulate such states. Our result provides a way to experimentally determine the number of dark states. This can pave the way for utilizing a Tavis-Cummings setup as a register with N_{dark} possible states. Future studies can explore ways to create each state dark individually, as well to transform a given dark state to another.

ACKNOWLEDGMENTS

LT thanks Pachaiyappan N for helpful discussions. RG thanks IIT Madras for warm hospitality and acknowledges support from National Sciences and Engineering Research Council of Canada (NSERC) through Discovery Grant 2022-05240. RN and LT acknowledge funding from the Center for Quantum Information Theory in Matter and Space-time, IIT Madras, and from the Department of Science and Technology, Government of India, under Grant No. DST/ICPS/QuST/Theme-3/2019/Q69, as well as support from the Mphasis F1 Foundation via the Centre for Quantum Information, Communication, and Computing (CQuICC).

DECLARATIONS

Consent to Publish declaration: Not applicable.
Consent to Participate declaration: Not applicable.

Appendix A: Number of dark states in the uniform limit

With our system of N qubits, consider the space of states where s qubits are excited while $(N - s)$ are in the ground state. All states in this space have $m_{(tot.)} = s/2 - (N - s)/2 = s - N/2$. For any of these states to be 'dark', we must have $S_{(tot.)} = N/2 - s$ so that $m_{(tot.)} = -S_{(tot.)}$. This is not possible for $s > N/2$, as $S_{(tot.)}$ cannot be negative. Below, we focus on $s \leq N/2$ and determine the number of dark states.

The space with s excited qubits is spanned by $\begin{pmatrix} N \\ s \end{pmatrix}$ elements. The quantum number $S_{(tot.)}$ can take any of

the following values: $N/2-s, N/2-s+1, \ldots, N/2$. We seek to exclude those with $(S_{(tot.)} > N/2-s)$ so that we are left with the dark states in this sector. We may do this as follows. Consider the sector with $s-1$ qubits excited while $(N-s+1)$ are in the ground state. These states have $m_{(tot.)} = s-1-N/2$, a negative value. They have $S_{(tot.)} = N/2-s+1, N/2-s+2, \ldots, N/2$. Upon acting the $S^+_{(tot.)}$ operator on any of these states, we obtain states with $S_{(tot.)} > N/2-s$ but with $m_{(tot.)} = s-N/2$. These are precisely the states that must be excluded from the sector with s qubits excited. The number of states to be excluded is given by $\begin{pmatrix} N \\ s-1 \end{pmatrix}$. We conclude that the number of dark states in the sector with s qubits excited is given by Eq. 2 in the main text.

Appendix B: Rank of $\hat{O}_s$

As defined in the main text, $\hat{O}_s$ is a matrix of dimensions $\begin{pmatrix} N \\ s-1 \end{pmatrix} \times \begin{pmatrix} N \\ s \end{pmatrix}$. We first consider the case $s \leq N/2$, where we have fewer rows than columns. The rank of $\hat{O}_s$ has an upper bound of $\begin{pmatrix} N \\ s-1 \end{pmatrix}$. We label the basis vectors of the source space as $|s_j\rangle$, where $j = 1, \ldots, \begin{pmatrix} N \\ s \end{pmatrix}$. These represent direct-product-states with s qubits excited and $(N-s)$ qubits in the ground state. We represent the target states as $|t_k\rangle$, where $k = 1, \ldots, \begin{pmatrix} N \\ s-1 \end{pmatrix}$. These states span the subspace with $(s-1)$ qubits excited. Denoting the entries in $\hat{O}_s$ as $a_{k,j}$, they are defined as

$$a_{k,j} = \langle t_k | \left(\sum_i g_i S_i^- \right) |s_j\rangle. \tag{B1}$$

If the rows in $\hat{O}_s$ are linearly dependent, there must exist $\{b_k\}$ such that

$$\sum_k b_k a_{k,j} = 0, \tag{B2}$$

for every j. Using the definition of $a_{k,j}$, we have

$$\sum_k b_k \langle t_k | \left(\sum_i g_i S_i^- \right) |s_j\rangle = 0, \tag{B3}$$

Taking the complex conjugate, we have

$$\langle s_j | \left(\sum_i g_i^* S_i^+ \right) \left\{ \sum_k b_k^* |t_k\rangle \right\} = 0, \tag{B4}$$

This equation must hold for any j. As $|s_j\rangle$'s are independent basis vectors for the s-subspace, this is only possible if

$$\left(\sum_i g_i^* S_i^+ \right) \left\{ \sum_k b_k^* |t_k\rangle \right\} = 0. \tag{B5}$$

We argue that this equation can only hold true if every b_k vanishes. In other words, the rows of $\hat{O}_s$ must be linearly independent. To see this, we observe that the raising operators S_i^+ act on $|t_k\rangle$'s to produce states with s qubits excited ($|s_j\rangle$'s). As every raising operator acts on the $|t_k\rangle$'s, their action will produce all $|s_j\rangle$'s, which number $\begin{pmatrix} N \\ s \end{pmatrix}$. This is greater than the number of free parameters, b_k's, which is given by $\begin{pmatrix} N \\ s-1 \end{pmatrix}$. As a result, Eq. B5 cannot have a non-trivial solution. All b_k's must be zero, showing that the rows of $\hat{O}_s$ are linearly independent. It follows that $\hat{O}_s$ has maximal rank, given by $\begin{pmatrix} N \\ s-1 \end{pmatrix}$ for $s \leq N/2$.

We next consider $s > N/2$, with fewer columns than rows. In this case, the rank of $\hat{O}_s$ has an upper bound given by $\begin{pmatrix} N \\ s \end{pmatrix}$. Suppose the columns are linearly dependent, we must have $\{c_j\}$ such that

$$\sum_j a_{k,j} c_j = 0 \tag{B6}$$

must hold for all k. That is,

$$\langle t_k | \left(\sum_i g_i S_i^- \right) \left\{ \sum_j c_j |s_j\rangle \right\} = 0. \tag{B7}$$

This can only be possible if

$$\left(\sum_i g_i S_i^- \right) \left\{ \sum_j c_j |s_j\rangle \right\} = 0. \tag{B8}$$

Here, the action of S_i^- operators on the $|s_j\rangle$'s will produce all states of the target space. This corresponds to $\begin{pmatrix} N \\ s-1 \end{pmatrix}$ independent states. This exceeds the number of free parameters, c_j's, in Eq. B8, which is $\begin{pmatrix} N \\ s \end{pmatrix}$. It follows that all c_j's must vanish, indicating that the columns are linearly independent. The matrix $\hat{O}_s$ has rank $\begin{pmatrix} N \\ s \end{pmatrix}$ when $s > N/2$.

[1] R. Holzinger, R. Gutiérrez-Jáuregui, T. Hönigl-Decrinis, G. Kirchmair, A. Asenjo-Garcia, and H. Ritsch, Control of localized single- and many-body dark states in waveg-

uide QED, Phys. Rev. Lett. **129**, 253601 (2022).
[2] Z. Wang, H. Li, W. Feng, X. Song, C. Song, W. Liu, Q. Guo, X. Zhang, H. Dong, D. Zheng, H. Wang, and D.-W. Wang, Controllable switching between superradiant and subradiant states in a 10-qubit superconducting circuit, Phys. Rev. Lett. **124**, 013601 (2020).
[3] C. J. Villas-Boas, C. E. Máximo, P. J. Paulino, R. P. Bachelard, and G. Rempe, Bright and dark states of light: The quantum origin of classical interference, Phys. Rev. Lett. **134**, 133603 (2025).
[4] R. H. Dicke, Coherence in spontaneous radiation processes, Phys. Rev. **93**, 99 (1954).
[5] J. A. Mlynek, A. A. Abdumalikov, C. Eichler, and A. Wallraff, Observation of Dicke superradiance for two artificial atoms in a cavity with high decay rate, Nature Communications **5**, 5186 (2014).
[6] R. Ganesh, L. Theerthagiri, and G. Baskaran, Generating resonating-valence-bond states through Dicke subradiance, Phys. Rev. A **96**, 033829 (2017).
[7] R. Ganesh, L. Theerthagiri, and G. Baskaran, Resonating valence bonds and spinon pairing in the Dicke model, SciPost Phys. **4**, 044 (2018).
[8] C. M. Nunn, J. D. Franson, and T. B. Pittman, Heralding on the detection of zero photons, Phys. Rev. A **104**, 033717 (2021).
[9] E. A. Cryer-Jenkins, K. D. Major, J. Clarke, G. Enzian, M. Szczykulska, J. Zhang, A. Gupta, A. C. Leung, H. Rathee, A. O. Svela, A. K. C. Tan, A. Beige, K. Mølmer, and M. R. Vanner, Enhanced laser cooling of a mechanical resonator via zero-photon detection, Phys. Rev. Lett. **134**, 073601 (2025).
[10] J. Clarke, E. A. Cryer-Jenkins, A. Gupta, K. D. Major, J. Zhang, G. Enzian, M. Szczykulska, A. C. Leung, H. Rathee, A. O. Svela, A. K. C. Tan, A. Beige, K. Mølmer, and M. R. Vanner, Theoretical framework for enhancing or enabling cooling of a mechanical resonator via the anti-stokes or stokes interaction and zero-photon detection, Phys. Rev. A **111**, 023516 (2025).
[11] A. L. Migdall, D. Branning, and S. Castelletto, Tailoring single-photon and multiphoton probabilities of a single-photon on-demand source, Phys. Rev. A **66**, 053805 (2002).
[12] T. B. Pittman, B. C. Jacobs, and J. D. Franson, Single photons on pseudodemand from stored parametric down-conversion, Phys. Rev. A **66**, 042303 (2002).
[13] M. Tavis and F. W. Cummings, Exact solution for an n-molecule—radiation-field Hamiltonian, Phys. Rev. **170**, 379 (1968).
[14] A. Wierzchucka, F. Piazza, and P. W. Claeys, Integrability, multifractality, and two-photon dynamics in disordered Tavis-Cummings models, Phys. Rev. A **109**, 033716 (2024).
[15] J. M. Fink, R. Bianchetti, M. Baur, M. Göppl, L. Steffen, S. Filipp, P. J. Leek, A. Blais, and A. Wallraff, Dressed collective qubit states and the Tavis-Cummings model in circuit QED, Phys. Rev. Lett. **103**, 083601 (2009).
[16] M. Feng, Y. P. Zhong, T. Liu, L. L. Yan, W. L. Yang, J. Twamley, and H. Wang, Exploring the quantum critical behaviour in a driven Tavis–Cummings circuit, Nature Communications **6**, 7111 (2015).
[17] S. Filipp, A. F. van Loo, M. Baur, L. Steffen, and A. Wallraff, Preparation of subradiant states using local qubit control in circuit QED, Phys. Rev. A **84**, 061805 (2011).
[18] K.-Y. Kim, J. H. Lee, W. B. Jeon, D. H. Park, S. I. Park, J. D. Song, C. Lee, and J.-H. Kim, Cavity-mediated collective emission from steady-state subradiance, Nature Communications **16**, 6346 (2025).
[19] K. N. Tolazzi, B. Wang, C. Ianzano, J. Neumeier, C. J. Villas-Boas, and G. Rempe, Continuous quantum light from a dark atom, Communications Physics **4**, 57 (2021).
[20] B. Sundar, D. Barberena, A. M. Rey, and A. P. n. Orioli, Squeezing multilevel atoms in dark states via cavity superradiance, Phys. Rev. Lett. **132**, 033601 (2024).
[21] Y. Kubo, F. R. Ong, P. Bertet, D. Vion, V. Jacques, D. Zheng, A. Dréau, J.-F. Roch, A. Auffeves, F. Jelezko, J. Wrachtrup, M. F. Barthe, P. Bergonzo, and D. Esteve, Strong coupling of a spin ensemble to a superconducting resonator, Phys. Rev. Lett. **105**, 140502 (2010).
[22] V. Roopini and R. Radhakrishnan, Implementation of Tavis Cummings model in solid-state defect qubits: Diamond nitrogen-vacancy center, Materials Today: Proceedings **27**, 446 (2020), first International Conference on Recent Advances in Materials and Manufacturing 2019.
[23] A. Johnson, M. Blaha, A. E. Ulanov, A. Rauschenbeutel, P. Schneeweiss, and J. Volz, Observation of collective superstrong coupling of cold atoms to a 30-m long optical resonator, Phys. Rev. Lett. **123**, 243602 (2019).
[24] S. J. Roof, K. J. Kemp, M. D. Havey, and I. M. Sokolov, Observation of single-photon superradiance and the cooperative Lamb shift in an extended sample of cold atoms, Phys. Rev. Lett. **117**, 073003 (2016).
[25] M. O. Araújo, I. Krešić, R. Kaiser, and W. Guerin, Superradiance in a large and dilute cloud of cold atoms in the linear-optics regime, Phys. Rev. Lett. **117**, 073002 (2016).
[26] W. Guerin, M. O. Araújo, and R. Kaiser, Subradiance in a large cloud of cold atoms, Phys. Rev. Lett. **116**, 083601 (2016).
[27] N. Leppenen and A. Poddubny, Persistent subradiant correlations in a random driven dicke model, arXiv preprint arXiv:2507.19467 (2025).
[28] T. Vojta, Phases and phase transitions in disordered quantum systems, AIP Conference Proceedings **1550**, 188 (2013).
[29] T. Vojta, Disorder in quantum many-body systems, Ann. Rev. Condens. Mat. Phys. **10**, 233 (2019).
[30] M. Aizenman and J. Wehr, Rounding of first-order phase transitions in systems with quenched disorder, Phys. Rev. Lett. **62**, 2503 (1989).
[31] K. Hui and A. N. Berker, Random-field mechanism in random-bond multicritical systems, Phys. Rev. Lett. **62**, 2507 (1989).
[32] J. A. Hoyos and T. Vojta, Dissipation effects in random transverse-field Ising chains, Phys. Rev. B **85**, 174403 (2012).
[33] J. A. Hoyos and T. Vojta, Theory of smeared quantum phase transitions, Phys. Rev. Lett. **100**, 240601 (2008).
[34] S. Kaur, H. Kumar Kundu, S. Kumar, A. Dogra, R. Narayanan, T. Vojta, and A. Bid, Novel emergent phases in a two-dimensional superconductor, New Journal of Physics **26**, 083001 (2024).
[35] D. S. Fisher, Random transverse field ising spin chains, Phys. Rev. Lett. **69**, 534 (1992).
[36] J. A. Hoyos, C. Kotabage, and T. Vojta, Effects of dissipation on a quantum critical point with disorder, Phys. Rev. Lett. **99**, 230601 (2007).
[37] T. Vojta, C. Kotabage, and J. A. Hoyos, Infinite-

randomness quantum critical points induced by dissipation, Phys. Rev. B **79**, 024401 (2009).
[38] Y. Xing, H.-M. Zhang, H.-L. Fu, H. Liu, Y. Sun, J.-P. Peng, F. Wang, X. Lin, X.-C. Ma, Q.-K. Xue, *et al.*, Quantum Griffiths singularity of superconductor-metal transition in Ga thin films, Science **350**, 542 (2015).
[39] S. Shen, Y. Xing, P. Wang, H. Liu, H. Fu, Y. Zhang, L. He, X. C. Xie, X. Lin, J. Nie, and J. Wang, Observation of quantum Griffiths singularity and ferromagnetism at the superconducting $LaAlO_3/SrTiO_3$ interface, Phys. Rev. B **94**, 144517 (2016).
[40] N. A. Lewellyn, I. M. Percher, J. Nelson, J. Garcia-Barriocanal, I. Volotsenko, A. Frydman, T. Vojta, and A. M. Goldman, Infinite-randomness fixed point of the quantum superconductor-metal transitions in amorphous thin films, Phys. Rev. B **99**, 054515 (2019).
[41] C. Zhang, Y. Fan, Q. Chen, T. Wang, X. Liu, Q. Li, Y. Yin, and X. Li, Quantum Griffiths singularities in TiO superconducting thin films with insulating normal states, NPG Asia Materials **11**, 1 (2019).
[42] S. Ubaid-Kassis, T. Vojta, and A. Schroeder, Quantum Griffiths phase in the weak itinerant ferromagnetic alloy $Ni_{1-x}V_x$, Phys. Rev. Lett. **104**, 066402 (2010).
[43] R. Wang, A. Gebretsadik, S. Ubaid-Kassis, A. Schroeder, T. Vojta, P. J. Baker, F. L. Pratt, S. J. Blundell, T. Lancaster, I. Franke, J. S. Möller, and K. Page, Quantum Griffiths phase inside the ferromagnetic phase of $Ni_{1-x}V_x$, Phys. Rev. Lett. **118**, 267202 (2017).
[44] H. Goto and K. Ichimura, Quantum phase transition in the generalized dicke model: Inhomogeneous coupling and universality, Phys. Rev. A **77**, 053811 (2008).
[45] P. Kirton and J. Keeling, Suppressing and restoring the dicke superradiance transition by dephasing and decay, Phys. Rev. Lett. **118**, 123602 (2017).
[46] P. Das, S. Wüster, and A. Sharma, Dicke model with disordered spin-boson couplings, Phys. Rev. A **109**, 013715 (2024).
[47] S. Friedberg, A. Insel, and L. Spence, *Linear Algebra* (Pearson Education, 2014).
[48] Y. I. Ozhigov, Space of dark states in tavis-cummings model, arXiv preprint arXiv:1606.08483 (2019).
[49] M. A. Zeb, Analytical solution of the disordered Tavis-Cummings model and its Fano resonances, Phys. Rev. A **106**, 063720 (2022).
[50] L. Wang, Y. Ye, D. Kong, T. Bai, X. Yao, S. Yuan, P. Zou, W. Zhai, and M. Xia, Advances and perspectives in single photon detectors: Principles, materials, cooling systems, and applications, Advanced Optical Materials **13**, 2500138 (2025).
[51] K. Inomata, Z. Lin, K. Koshino, W. D. Oliver, J.-S. Tsai, T. Yamamoto, and Y. Nakamura, Single microwave-photon detector using an artificial Λ-type three-level system, Nature Communications **7**, 12303 (2016).
[52] W. Khan, P. P. Potts, S. Lehmann, C. Thelander, K. A. Dick, P. Samuelsson, and V. F. Maisi, Efficient and continuous microwave photoconversion in hybrid cavity-semiconductor nanowire double quantum dot diodes, Nature Communications **12**, 5130 (2021).
[53] A. L. Pankratov, L. S. Revin, A. V. Gordeeva, A. A. Yablokov, L. S. Kuzmin, and E. Il'ichev, Towards a microwave single-photon counter for searching axions, npj Quantum Information **8**, 61 (2022).
[54] E. Chatterjee, W. Pan, and D. Soh, Ultra-high-precision detection of single microwave photons based on a hybrid system between a majorana zero mode and a quantum dot, Phys. Rev. Res. **5**, 013034 (2023).
[55] C. Gonzalez-Ballestero, J. Feist, E. Gonzalo Badía, E. Moreno, and F. J. Garcia-Vidal, Uncoupled dark states can inherit polaritonic properties, Phys. Rev. Lett. **117**, 156402 (2016).
[56] E. Davidsson and M. Kowalewski, The role of dephasing for dark state coupling in a molecular Tavis–Cummings model, The Journal of Chemical Physics **159**, 044306 (2023).
[57] S. Agarwal, S. M. H. Rafsanjani, and J. H. Eberly, Tavis-Cummings model beyond the rotating wave approximation: Quasidegenerate qubits, Phys. Rev. A **85**, 043815 (2012).
[58] K. Sun, C. Dou, M. F. Gelin, and Y. Zhao, Dynamics of disordered Tavis–Cummings and Holstein–Tavis–Cummings models, The Journal of Chemical Physics **156**, 024102 (2022).
[59] A. E. Lita, A. J. Miller, and S. W. Nam, Counting near-infrared single-photons with 95% efficiency, Opt. Express **16**, 3032 (2008).
[60] A. Imamoglu, O. Cotlet, and R. Schmidt, Exciton-polarons in two-dimensional semiconductors and the Tavis-Cummings model, Comptes Rendus. Physique **22**, 89 (2021).