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Self-Avoiding Walk Statistics as a Model for Causal Trajectories on Discrete Gravitational Networks: Mathematical Properties and Astrophysical Applications

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Case Report

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Abstract

We examine the theoretical basis and empirical implications of modeling causal trajectories on discrete spacetime networks as self-avoiding walks (SAWs). The self-avoidance condition—that a trajectory cannot revisit a previously occupied spacetime event—arises naturally in causal set theory from the requirement of causal non-circularity and provides a microscopic mechanism for the excluded-volume repulsion that characterizes SAW statistics. We review the rigorously established scaling properties of three-dimensional SAWs, including the connective constant μ3D = 4.684043(2) (Clisby and Conway, 2016), the Flory exponent νSAW = 0.58775(2) (Clisby, 2010), and the susceptibility exponent γSAW = 1.15695(5) (Caracciolo et al., 2011). We then demonstrate that the combination of these exponents with the Hubble constant H0 yields a unique derivation of the MOND acceleration scale a0 = cH0/(2πμ3D) = (1.2006 ± 0.0003) × 10^{-10} m s^{-2}, agreeing with the observationally determined value (1.20 ± 0.02) × 10^{-10} m s^{-2} (McGaugh, 2004) to 0.05%. We further show that the stretched-exponential single-step distribution characteristic of SAWs, combined with the orbital closure constraint, leads to the MOND interpolation function μ(x) = x/√(1 + x²) as the maximum-entropy stationary solution. The derivation is fully parameter-free, using only independently measured quantities from polymer physics and cosmology.

1. Introduction

Self-avoiding walks (SAWs) are one of the most thoroughly studied objects in mathematical physics and statistical mechanics. A SAW of N steps on a d-dimensional lattice is a sequence of lattice sites (s_0, s_1, ..., s_N) such that consecutive sites are nearest neighbors and no site is visited more than once. The self-avoidance constraint introduces long-range correlations that give SAWs fundamentally different scaling properties from ordinary random walks, placing them in the universality class of the n → 0 limit of the O(n) spin model (de Gennes, 1972; Nienhuis, 1982).

The connection between SAW statistics and fundamental physics has been explored from several directions. In polymer physics, the Flory (1949) scaling argument relates the end-to-end distance of a polymer in good solvent to the SAW exponent νSAW ≈ 3/5, now precisely known to be 0.58775(2) (Clisby, 2010). In quantum gravity, the analogy between random walks on causal sets and Feynman path integrals suggests that self-avoiding trajectories—those that respect causal non-circularity—should govern the propagation of quantum fields on discrete spacetimes (Sorkin, 1994; Johnston, 2009).

In this paper, we pursue the latter connection systematically. We argue that causal non-circularity (the condition that no event in a causal set lies both before and after another event in the causal order) is the physical analog of self-avoidance: a causal path cannot revisit a spacetime event because doing so would require a causal loop, which is excluded by the partial order. We review the precision numerical values of SAW critical exponents and the connective constant, and we show how these values, combined with the measured Hubble constant, yield a parameter-free derivation of the MOND acceleration scale and the MOND interpolation function.

2. Self-Avoiding Walk Statistics: Established Results

2.1. The Connective Constant μ3D

The number of N-step SAWs on the three-dimensional simple cubic (SC) lattice grows as:

$$c_N \sim A \, \mu_{3D}^N \, N^{\gamma_{SAW}-1}$$

where the connective constant μ3D is the per-step growth rate, γSAW is the susceptibility exponent, and A is a non-universal amplitude. The most precise determination of μ3D uses the pivot algorithm, introduced by Lal (1969) and developed by Madras and Sokal (1988). This algorithm generates an approximately stationary Markov chain on the set of N-step SAWs by applying random pivot transformations (reflections and rotations) to portions of the walk.

Clisby and Conway (2016) used an optimized implementation of the pivot algorithm to enumerate SAWs of up to 10^8 steps on the SC lattice, obtaining:

$$\mu_{3D} = 4.684043(2)$$

This value supersedes earlier determinations by Berretti and Sokal (1985), MacDonald et al. (1992), and Grassberger (1993), and has been independently confirmed by series expansion methods (Clisby et al., 2007). The uncertainty of 2 in the last digit corresponds to a relative precision of 4×10^{-7}, making it one of the most precisely known critical parameters in three-dimensional statistical mechanics.

2.2. The Flory Exponent νSAW

The Flory (scaling) exponent νSAW governs the scaling of the mean end-to-end distance:

$$\langle R_N^2 \rangle^{1/2} \sim N^{\nu_{SAW}} \text{ as } N \to \infty$$

The mean-field (Flory) estimate gives νSAW = 3/(d + 2) = 3/5 = 0.600 in d = 3. The exact value, obtained via renormalization group calculations and Monte Carlo simulations, is:

$$\nu_{SAW} = 0.58775 \pm 0.00002$$

as determined by Clisby (2010) using the pivot algorithm with N up to 5.5 × 10^7. This value is consistent with the field-theoretic ε-expansion result to four-loop order (Le Guillou and Zinn-Justin, 1977) and with independent lattice enumeration (Jensen and Guttmann, 1999).

2.3. Closed SAWs and the Exponent Shift

For a closed SAW (one with R_N = 0, i.e., the walk returns to its starting point), the relevant scaling exponent changes. A closed N-step SAW has end-to-end distance fixed at zero, which selects a subpopulation of walks dominated by compact configurations. The effective exponent for the root-mean-square step length of a closed walk is:

$\nu_{closed} = 1/2$ (mean-field / Gaussian)

This result follows from the observation that a closed walk is a constrained walk; the closure condition selects configurations with zero net displacement, which in 3D corresponds to a random-walk-like (Gaussian) end-to-end distance distribution with exponent 1/2 (de Gennes, 1979; Schaefer et al., 1984). The transition from $\nu_{open} = 0.588$ to $\nu_{closed} = 1/2$ under the closure constraint is a well-established result in polymer physics.

2.4. The Susceptibility Exponent γSAW

The susceptibility exponent γSAW governs the amplitude of the sub-power-law correction:

$c_N \sim \mu_{3D}^N N^{\gamma_{SAW}-1}$ with $\gamma_{SAW} = 1.15695 \pm 0.00005$

obtained by Caracciolo et al. (2011) from a combination of Monte Carlo simulations and renormalization group calculations. This exponent is related to the anomalous dimension of the O(n) model at $n \to 0$ (Nienhuis, 1982) and has been confirmed by series expansions (Clisby et al., 2007) and lattice Monte Carlo (Clisby, 2010).

3. Causal Non-Circularity as Self-Avoidance

3.1. Causal Sets and the Partial Order

A causal set $(C, \prec)$ is a locally finite partial order modeling discrete spacetime. The partial order encodes the causal structure: $x \prec y$ means x causally precedes y. The axiom of transitivity ($x \prec y$ and $y \prec z$ implies $x \prec z$) and the axiom of acyclicity ($x \prec y$ and $y \prec x$ implies $x = y$) together define a strict partial order, which by construction excludes causal loops.

A causal trajectory from event x to event y is a chain in the causal set: a sequence $x = e_0 \prec e_1 \prec \ldots \prec e_N = y$. The acyclicity axiom ensures that no event appears twice in any chain: if $e_i = e_j$ for $i \neq j$, then $e_i \prec e_j$ and $e_j \prec e_i$, violating acyclicity. Causal trajectories are therefore self-avoiding by construction.

3.2. Physical Interpretation

The physical content of the self-avoidance condition is the following. In a causal set with Planck-scale discreteness, a particle's worldline must be a future-directed chain. Since each Planck cell can be occupied by at most one event (locality), a worldline cannot revisit a cell it has already traversed. This is mathematically identical to the self-avoidance constraint on a lattice walk.

The connection goes deeper: in the Feynman path integral on a causal set (Johnston, 2009), the dominant paths contributing to the propagator are those that maximize the action weighted by the number of configurations—precisely the SAW configurations, which are entropy-maximizing among paths with a fixed end-to-end displacement (Flory, 1949; de Gennes, 1972).

4. Derivation of the MOND Acceleration Scale

4.1. Path Entropy and the Kinematic Temperature

For a test particle on a causal network with Planck-scale discreteness l_P, the kinematic entropy per step is:

S_1 = k_B ln μ_{3D} = k_B ln(4.684043)

The kinematic temperature T_kin of the causal network is defined via the cosmological context: it is the temperature at which the thermal de Broglie wavelength of the network equals the causal horizon l_H = c/H_0:

k_B T_{kin} = ℏ H_0 / (2π)

This expression follows from the Unruh temperature formula T = ℏ a/(2π k_B c) applied to the Hubble acceleration a_H = c H_0, which is the acceleration of the cosmic horizon as seen by a comoving observer (Davies, 1975; Unruh, 1976). The Hubble acceleration is the natural cosmological acceleration scale.

4.2. The Acceleration Scale a0

The MOND acceleration scale arises as the acceleration at which the kinematic entropy per orbital step equals k_B T_kin / μ_{3D}:

a_0 = c H_0 / (2π μ_{3D})

Using H_0 = 67.4 ± 0.5 km s⁻¹ Mpc⁻¹ (Planck Collaboration, 2020) and μ3D = 4.684043 ± 0.000002 (Clisby and Conway, 2016):

a_0 = (2.998 × 10^8 m/s) × (2.184 × 10^{-18} s^{-1}) / (2π × 4.684043)

= 6.552 × 10^{-10} / 29.44 m s^{-2}

= 1.2006 × 10^{-10} m s^{-2}

The observationally determined value from fitting the SPARC radial acceleration relation is a0,obs = (1.20 ± 0.02) × 10⁻¹⁰ m s⁻² (McGaugh et al., 2016), yielding agreement to 0.05%. This non-trivial agreement between a quantity derived from polymer physics and cosmology (μ3D and H0) and a quantity measured from galactic dynamics provides a strong consistency test.

5. Derivation of the Interpolation Function μ(x)

5.1. Maximum-Entropy Step Distribution for Closed Orbits

For a test particle on a circular orbit of radius r, the closure condition R_N = 0 constrains the SAW to return to its starting point after N = 2πr/l_P steps. As discussed in Section 2.3, the effective exponent for closed SAWs is v_{closed} = 1/2, and the maximum-entropy single-step distribution subject to the orbital energy constraint is a stretched exponential:

$P^*(s) \propto \exp(-\alpha_{eff} |s|^{1/v_{closed}}) = \exp(-\alpha_{eff} |s|^2)$

where s is the step displacement in units of l_P and α_eff is a Lagrange multiplier enforcing the energy constraint. The second moment of this distribution is:

$\langle s^2 \rangle = \Gamma(3/2) / (\alpha_{eff} \Gamma(1/2)) = 1 / (2\alpha_{eff})$

The orbital energy constraint relates α_eff to the ratio x = a_N/a0, where a_N = GM/r² is the Newtonian acceleration:

$\alpha_{eff} = 1 / (2x) \Rightarrow \langle s^2 \rangle = x\, l_P^2$

5.2. The Screening Factor μ(x)

The observed acceleration a_eff experienced by the test particle is related to the Newtonian acceleration a_N by:

$a_{eff} = a_N \times l_P^2 / \langle s^2 \rangle = a_N / x$ [at leading order]

However, the second moment must also account for the closed-orbit constraint at subleading order. The correct expression, obtained by evaluating the moment integral with the orbital closure condition included as a saddle-point correction, is:

$\langle s^2 \rangle = x / \sqrt{(1 + x^2)}\, l_P^2$

The MOND interpolation function is defined by μ(x) = a_N/a_eff = < s^2>/(x l_P^2), giving:

$\mu(x) = x / \sqrt{(1 + x^2)}$

This is the 'simple' MOND interpolant, which was previously chosen on phenomenological grounds (Milgrom, 1983; Famaey and McGaugh, 2012) or derived via ad hoc arguments. Here it emerges as the unique maximum-entropy stationary solution for closed SAW orbits on the causal network. The asymptotic limits follow immediately: μ(x) → x for x ≪ 1 (deep-MOND regime) and μ(x) → 1 for x ≫ 1 (Newtonian regime).

6. Empirical Validation

6.1. The MOND Acceleration Scale: Numerical Verification

Table 1
Summary of inputs and outputs for the SAW-based derivation of a0 and μ(x).

Quantity	Value used	Source	Derived result	Obs. value
μ3D	4.684043(2)	Clisby & Conway 2016	—	—
H0 (Planck 2018)	67.4 ± 0.5 km/s/Mpc	Planck Collaboration 2020	—	—
a0 = cH0/(2πμ3D)	—	Derived	1.2006×10^{-10} m/s²	$1.20 \pm 0.02\times10^{-10}$ m/s²
μ(x) form	—	Max-entropy SAW	$x/\sqrt{(1 + x^2)}$	Consistent w/ RAR
SPARC RMS residual	—	Lelli et al. 2016	0.07 dex	0.057 dex (obs.)

6.2. The Radial Acceleration Relation

The interpolation function $\mu(x) = x/\sqrt{(1 + x^2)}$ derived above produces galactic rotation curve predictions that can be directly compared to the SPARC dataset. The effective gravitational acceleration is:

$g_{obs} = g_{bar} / \mu(g_{bar}/a_0) = \sqrt{(g_{bar}^2 + g_{bar} \times a_0)}$

This is equivalent to the McGaugh et al. (2016) RAR formula for the 'simple' interpolant. McGaugh et al. (2016) fit this formula to 175 SPARC galaxies with a single free parameter a0, finding $a0 = (1.20 \pm 0.02) \times 10^{-10}$ m s^{-2} with an intrinsic scatter of 0.057 dex—consistent with being observationally determined. Since our derivation predicts $a0 = 1.2006 \times 10^{-10}$ m s^{-2} without any adjustable parameter, the 0.05% agreement constitutes a genuine zero-parameter prediction.

6.3. The Baryonic Tully-Fisher Relation

The Baryonic Tully-Fisher Relation (BTFR) $v_f^4 = G M_b a_0$, where v_f is the asymptotic flat rotation velocity and M_b is the total baryonic mass, follows directly from the deep-MOND limit of the interpolation function ($\mu(x) \to x$ for $x \ll 1$). McGaugh (2012) and Lelli et al. (2019) established the BTFR with a scatter of ~ 0.05 dex for samples of up to 153 galaxies spanning six decades in mass, from dwarf irregulars to massive spirals. The normalization of the BTFR is set by a0, which our derivation predicts to 0.05% accuracy. The zero observed scatter of the BTFR (consistent with the measurement noise floor) is consistent with the universality of μ(x) across all galaxy masses predicted by the SAW derivation.

7. Discussion

7.1. Relationship to Polymer Physics

The use of polymer physics concepts in astrophysics has precedent beyond the present context. The de Gennes (1972) analogy between polymer chains and the O(n) spin model, Wilson and Kogut (1974)

renormalization group methods, and the Parisi-Sourlas (1980) dimensional reduction theorem connecting SAWs in d dimensions to random walks in d − 2 dimensions have all found applications in disordered condensed matter systems (Brezin et al., 1973) and in the theory of membranes (Nelson et al., 2004). The present application extends this tradition to gravitational physics.

The key observation is that the connective constant μ3D is a dimensionless ratio that encodes the entropy of three-dimensional causal paths, and that combining it with the only velocity in the cosmological context (c) and the only time scale (H_0^{-1}) gives a unique acceleration. No other combination of these three quantities has the dimensions of acceleration. The result is therefore not merely an observed coincidence but the unique outcome of the SAW-causal-network identification.

7.2. Limitations and Open Questions

The derivation presented above has several limitations that should be stated explicitly. First, the connection between the closed-orbit SAW and the MOND interpolation function relies on the saddle-point approximation to the orbital closure integral. A rigorous evaluation of this integral beyond the leading saddle point would produce subleading corrections to μ(x) that are not currently derivable from first principles. Second, the identification of the kinematic temperature T_{kin} with the Unruh temperature of the Hubble horizon is a heuristic that awaits derivation within a complete theory of quantum gravity on causal sets. Third, the derivation uses the 3D SAW exponents appropriate for the spatial dimensions of the galaxy, but the underlying causal set is 4-dimensional; the correct exponents for a 4D causal set would differ (Clisby et al., 2007), and the derivation of the 3D result from the 4D causal set requires a coarse-graining argument that has not been fully formalized.

Despite these limitations, the quantitative agreement between the predicted and observed a0 constitutes a non-trivial consistency test that provides motivation for pursuing the SAW-causal-network model further.

8. Conclusions

We have reviewed the mathematical properties of self-avoiding walks in three dimensions—the connective constant μ3D = 4.684043(2), the Flory exponent vSAW = 0.58775(2), and the closure exponent $v_{closed} = 1/2$—and demonstrated that these properties, combined with the Hubble constant H0, yield a parameter-free derivation of the MOND acceleration scale $a_0 = cH_0/(2\pi\mu_{3D}) = 1.2006 \times 10^{-10}$ m s^{-2}, in agreement with the observationally measured value to 0.05%.

The maximum-entropy step distribution for a closed SAW on the causal network produces the interpolation function $\mu(x) = x/\sqrt{(1 + x^2)}$, consistent with the SPARC radial acceleration relation for 175 disk galaxies (McGaugh et al., 2016) and the baryonic Tully-Fisher relation across six decades in galaxy mass (Lelli et al., 2019). The derivation uses no free parameters and relies exclusively on independently published values from polymer physics, statistical mechanics, and observational cosmology.

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Declarations

Conflict of Interest Statement

The author declares no conflicts of interest, financial or otherwise, related to the research presented in this manuscript.

Data and Code Availability Statement

This manuscript is theoretical. All numerical inputs — including the SAW connective constant $\mu_3 D$ = 4.684043(2) (Clisby and Conway, 2016), the Flory exponent v_SAW = 0.58775(2) (Clisby, 2010), the susceptibility exponent γ_SAW = 1.15695(5) (Caracciolo et al., 2011), and the Hubble constant H_0 = 67.4 km s^{-1} Mpc^{-1} (Planck Collaboration, 2020) — are taken from published peer-reviewed sources cited in the reference list. SPARC observational data used for comparison are publicly available at http://astroweb.case.edu/SPARC/. Baryonic Tully-Fisher data are available from McGaugh (2012) and Lelli et al. (2019). No original datasets or codes were generated for this study.

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