Supplementary Material for
"Hamiltonian Dynamics of a Geometric Scalar Field on Superspace"

Avadhut V. Purohit[1,*]

[1]*Department of Physics, Visvesvaraya National Institute of Technology, Nagpur 440010, India*

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Contents

SI Stress-energy tensor

We derive the effective spacetime stress-energy tensor associated with the geometric scalar field Φ, which is defined as follows:

$$T_{ab} = -\frac{2}{\sqrt{q}}\frac{\delta S}{\delta q^{ab}}, \tag{S1}$$

where S is the action for the free geometric scalar field Φ discussed in the main text and q_{ab} is the induced metric on the spatial hypersurface given by

$$q_{ab} = \mathcal{G}_{\mu\nu}\frac{\partial \zeta^\mu}{\partial x^a}\frac{\partial \zeta^\nu}{\partial x^b}. \tag{S2}$$

The variation of the action with respect to q^{ab} is computed as follows:

$$\frac{\delta S}{\delta q^{ab}} = \int d^{d+1}\zeta \left[\frac{\delta\sqrt{-\mathcal{G}}}{\delta q^{ab}}\mathcal{L} + \sqrt{-\mathcal{G}}\frac{\delta \mathcal{L}}{\delta q^{ab}}\right] = \int d^{d+1}\zeta \left[-\frac{1}{2}\sqrt{-\mathcal{G}}q_{ab}\mathcal{L} + \sqrt{-\mathcal{G}}\frac{\delta \mathcal{L}}{\delta q^{ab}}\right]. \tag{S3}$$

Since we are mainly interested in the flat spacetime limit, we set $\mu^2 = 0$ and the Lagrangian density reduces to the following form:

$$\mathcal{L} = -\frac{1}{2}\mathcal{G}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi. \tag{S4}$$

Thus, the variation of the Lagrangian density with respect to q^{ab} is given by

$$\frac{\delta \mathcal{L}}{\delta q^{ab}} = -\frac{1}{2}\frac{\delta \mathcal{G}^{\mu\nu}}{\delta q^{ab}}\partial_\mu\Phi\partial_\nu\Phi. \tag{S5}$$

Since $\mathcal{G}^{00} = -1$, $\mathcal{G}^{AB} = \frac{32}{3}\zeta^{-2}\bar{\mathcal{G}}^{AB}$ and $\zeta = \sqrt{\frac{32}{3}}\,(\det q_{ab})^{\frac{1}{4}}$, the above expression can be computed as follows:

$$\frac{\delta \mathcal{L}}{\delta q^{ab}} = -\frac{16}{3}\zeta^{-2}\left(\frac{\delta \bar{\mathcal{G}}^{AB}}{\delta q^{ab}} + \frac{1}{2}q_{ab}\bar{\mathcal{G}}^{AB}\right)\partial_A\Phi\partial_B\Phi. \tag{S6}$$

* avdhoot.purohit@gmail.com

Here, the second term is obtained using the following identity:

$$\frac{\delta \zeta^{-2}}{\delta q^{ab}} = \frac{1}{2} q_{ab} \zeta^{-2}. \tag{S7}$$

Then, collecting all the terms, we obtain the following expression for the stress-energy tensor:

$$T_{ab} = -\frac{2}{\zeta^2} \int d\zeta \int d\zeta^A \sqrt{-\mathcal{G}} \left[\frac{1}{4} q_{ab} \mathcal{G}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{16}{3} \zeta^{-2} \left(\frac{\delta \bar{\mathcal{G}}^{AB}}{\delta q^{ab}} + \frac{1}{2} q_{ab} \bar{\mathcal{G}}^{AB} \right) \partial_A \Phi \partial_B \Phi \right]. \tag{S8}$$

Since we are interested in the ζ-dependence of the stress-energy tensor, we solve the KG-like equation for the free geometric scalar field Φ in the flat spacetime background, which is given by

$$\left[\frac{\partial^2}{\partial \zeta^2} + \frac{d}{\zeta} \frac{\partial}{\partial \zeta} - \frac{\lambda}{\zeta^2} \right] \Phi = 0, \quad \text{where} \quad \frac{32}{3} \frac{1}{\sqrt{\bar{\mathcal{G}}}} \partial_A \left(\sqrt{\bar{\mathcal{G}}} \bar{\mathcal{G}}^{AB} \partial_B \right) \Phi = \lambda \Phi. \tag{S9}$$

The solution to the above equation has the following form:

$$\Phi \sim \zeta^{-\frac{(d-1)}{2} \pm \sqrt{\left(\frac{d-1}{2}\right)^2 - 4\lambda}} f(\zeta^A). \tag{S10}$$

If anisotropies are such that λ is large and positive, the second-half part of the solution is oscillatory, and the dominating part of the solution is $\Phi \sim \zeta^{-\frac{(d-1)}{2}}$. For $\lambda < 0$ and large in magnitude, the decaying branch becomes strongly suppressed at large ζ, while the growing branch is excluded on physical grounds (the positive part is unphysical and the negative part survives). In general, λ determines the ζ-dependence of the geometric modes.

Substituting the asymptotic solution into Eq. (S8), we can determine the leading ζ-dependence of the stress-energy tensor. The leading ζ-dependence of the stress-energy tensor is given by

$$T_{ab} = -\frac{2}{\zeta^2} \int d\zeta \int d\zeta^A \sqrt{-\mathcal{G}} \left[\frac{1}{4} q_{ab} \left((\partial_\zeta \Phi)^2 - \frac{32}{3\zeta^2} \bar{\mathcal{G}}^{AB} \partial_A \Phi \partial_B \Phi \right) - \frac{16}{3\zeta^2} \left(\frac{\delta \bar{\mathcal{G}}^{AB}}{\delta q^{ab}} + \frac{1}{2} q_{ab} \bar{\mathcal{G}}^{AB} \right) \partial_A \Phi \partial_B \Phi \right]. \tag{S11}$$

Using the leading order solution $\Phi \sim \zeta^{-\frac{(d-1)}{2}} f(\zeta^A)$, we obtain the following leading ζ-dependence of the stress-energy tensor:

$$\begin{aligned} T_{ab} &\approx -\frac{2}{\zeta^2} \int d\zeta \int d\zeta^A \sqrt{\bar{\mathcal{G}}} \left(\frac{3}{32}\right)^{\frac{d}{2}} \zeta^d \left[\frac{1}{4} q_{ab} \zeta^{-(d+1)} \left(f^2 - \frac{32}{3} \bar{\mathcal{G}}^{AB} \partial_A f \partial_B f \right) - \frac{16}{3} \zeta^{-(d+1)} \left(\frac{\delta \bar{\mathcal{G}}^{AB}}{\delta q^{ab}} + \frac{1}{2} q_{ab} \bar{\mathcal{G}}^{AB} \right) \partial_A f \partial_B f \right] \\ &\approx -\frac{2}{\zeta^2} \ln \zeta \int d\zeta^A \sqrt{\bar{\mathcal{G}}} \left(\frac{3}{32}\right)^{\frac{d}{2}} \left[\frac{1}{4} q_{ab} \left(f^2 - \frac{32}{3} \bar{\mathcal{G}}^{AB} \partial_A f \partial_B f \right) - \frac{16}{3} \left(\frac{\delta \bar{\mathcal{G}}^{AB}}{\delta q^{ab}} + \frac{1}{2} q_{ab} \bar{\mathcal{G}}^{AB} \right) \partial_A f \partial_B f \right] \sim q_{ab} \zeta^{-2}. \end{aligned} \tag{S12}$$

Since the logarithmic dependence varies slowly at large ζ, the dominant asymptotic scaling is $T_{ab} \sim \zeta^{-2}$.

SII Riccati Equation

We discuss the uniqueness of the structure of the Riccati Eq. (**??**) given in the main text. Its general form is given by

$$\omega(\zeta, \vec{\zeta})^2 - \frac{1}{\alpha(\zeta)} \nabla_A \left(n^A \omega(\zeta, \vec{\zeta}) \right) = \mu(\zeta, \vec{\zeta})^2, \ \alpha(\zeta) = \sqrt{\frac{3\zeta^2}{32}} \ \text{ and } \ \nabla_A V^A = \frac{1}{\sqrt{\bar{\mathcal{G}}}} \partial_A \left(\sqrt{\bar{\mathcal{G}}}\, V^A \right). \tag{S13}$$

SII.A Uniqueness of Solution

Applying identity $\nabla_A (n^A \omega) = n^A \nabla_A \omega + (\nabla_A n^A)\, \omega$, we rewrite the above equation as follows:

$$n^A \nabla_A \omega + \theta\, \omega = \alpha(\zeta) \left(\omega^2 - \mu^2 \right) \ \text{ and } \ \theta := \nabla_A n^A. \tag{S14}$$

Let $x^A(s)$ be integral curves of n^A, defined by

$$\frac{dx^A}{ds} = n^A. \tag{S15}$$

Then the equation reduces along each curve to the ordinary differential equation given by

$$\frac{d\omega}{ds} = F(s, \omega), \quad \text{where} \quad F(s, \omega(\zeta(s)^\mu, s)) = \alpha(\zeta(s)) \left(\omega(\zeta(s)^\mu, s)^2 - \mu(\zeta(s)^\mu, s)^2 \right) - \theta(s)\, \omega(\zeta(s)^\mu, s). \tag{S16}$$

a. Local Uniqueness: Since the function $F(s,\omega)$ is continuous in s and smooth in ω, its derivative with respect to ω:

$$\frac{\partial F}{\partial \omega} = 2\alpha(\zeta(s))\,\omega - \theta(s), \tag{S17}$$

is continuous for all finite ω. Therefore, $F(s,\omega)$ is locally Lipschitz in ω. By the Picard–Lindelöf theorem [1, 2], for any initial condition $\omega(s_0) = \omega_0$, there exists a unique local solution along each integral curve of n^A.

b. Global Uniqueness: The solution extends uniquely as long as $\omega(s)$ remains finite, and this finiteness is assumed within the physically admissible sector considered in the present framework.

c. Conclusion: For smooth background fields n^A, ζ, and μ, the above Riccati equation admits a unique local solution along each integral curve of n^A, given initial data.

SII.B Structure of the Solution

We introduce the following transformation:

$$\omega(s) = -\frac{1}{\alpha(\zeta(s))}\frac{u'(s)}{u(s)}, \tag{S18}$$

to reduce the above nonlinear equation to a second-order linear equation as follows:

$$u''(s) + \theta(s)\,u'(s) + \alpha(\zeta(s))^2\,\mu(\zeta(s)^\mu, s)^2\,u(s) = 0. \tag{S19}$$

a. General Solution: The evolution equation along integral curves of n^A is given by

$$\frac{d\omega(s)}{ds} = \alpha(s)\,\omega(s)^2 - \theta(s)\,\omega(s) - \alpha(s)\,\mu(s)^2, \tag{S20}$$

where we denote:

$$\alpha(s) := \sqrt{\frac{3\zeta(s)^2}{32}} \quad \text{and} \quad \theta(s) := \nabla_A n^A. \tag{S21}$$

b. Exact Reduction: To account for the s-dependence of $\alpha(s)$, we use the following transformation:

$$\omega(s) = -\frac{1}{\alpha(s)}\frac{u'(s)}{u(s)}. \tag{S22}$$

Substituting and simplifying, we obtain the linear second-order equation as follows:

$$u''(s) + \left(\theta(s) - \frac{\alpha'(s)}{\alpha(s)}\right)u'(s) + \alpha(s)^2\mu(s)^2\,u(s) = 0. \tag{S23}$$

c. General Solution: Let $u_1(s)$ and $u_2(s)$ be two linearly independent solutions of the above equation. Then the general solution is given by

$$u(s) = C_1 u_1(s) + C_2 u_2(s), \tag{S24}$$

and the corresponding solution for $\omega(s)$ is given by

$$\omega(s) = -\frac{1}{\alpha(s)}\frac{d}{ds}\ln\left[C_1 u_1(s) + C_2 u_2(s)\right]. \tag{S25}$$

d. Solution in Integral Form: Alternatively, the Riccati equation may be written in integral form as follows:

$$\omega(s) = \omega_0 + \int_{s_0}^{s} ds'\,\left[\alpha(s')\big(\omega(s')^2 - \mu(s')^2\big) - \theta(s')\,\omega(s')\right], \tag{S26}$$

which provides an implicit representation suitable for iterative or numerical construction.

e. Special Case: If α, θ, μ are constant along a given integral curve, the equation reduces to

$$\frac{d\omega}{ds} = \alpha\,\omega^2 - \theta\,\omega - \alpha\,\mu^2, \tag{S27}$$

which admits the explicit solution

$$\omega(s) = \frac{\omega_+ - C\,\omega_-\,e^{\alpha(\omega_+ - \omega_-)(s-s_0)}}{1 - C\,e^{\alpha(\omega_+ - \omega_-)(s-s_0)}}, \tag{S28}$$

where

$$\omega_\pm = \frac{\theta \pm \sqrt{\theta^2 + 4\alpha^2\mu^2}}{2\alpha} \quad \text{and} \quad C = \frac{\omega_0 - \omega_+}{\omega_0 - \omega_-}. \tag{S29}$$

f. Remarks: The above expressions provide the complete solution along each integral curve of n^A. The nonlinear evolution of ω is, thus, fully determined by the geometry through $\theta(s)$, the scale factor $\zeta(s)$, and the effective mass function $\mu(s)$.

[1] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, 1955).
[2] V. I. Arnold, *Ordinary Differential Equations* (Springer, 1992).