SUPPLEMENTARY INFORMATION
Exponential Measurement Error Mitigation in Quantum Sampling

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I. PROOF OF THEOREM 1: ROBUST ERROR SUPPRESSION UNDER NOISY ENCODING

In this section, we provide a detailed proof of Theorem 1 in the main text. We first analyze the effective bit-flip errors induced by noisy encoding gates, then combine them with measurement errors, and finally derive the logical error probability after majority-vote decoding.

Throughout this section, we consider the chain-structured repetition encoding circuit

$$\mathrm{CNOT}_{1\to2}, \mathrm{CNOT}_{2\to3}, \ldots, \mathrm{CNOT}_{k-1\to k},$$

where the first qubit is the data qubit and the remaining qubits are ancilla qubits initialized in $|0\rangle$.

We assume that each CNOT gate is followed by a two-qubit depolarizing channel with error rate e_g,

$$\mathcal{E}(\rho) = (1-e_g)\rho + \frac{e_g}{15}\sum_{P\in\mathcal{P}_2\setminus\{I\}} P\rho P, \tag{1}$$

where

$$\mathcal{P}_2 = \{I, X, Y, Z\}^{\otimes 2}.$$

We further assume that each physical qubit measurement flips with probability e_m.

Lemma 1: Effective bit-flip probability induced by noisy encoding

Lemma 1 (Bit-flip probability induced by noisy encoding)**.** *Consider the chain-structured repetition encoding circuit described above. Then, for any physical qubit participating in the encoding circuit, the probability of acquiring a bit-flip component before measurement satisfies*

$$p_{\mathrm{flip}} \le \frac{16}{15}e_g + O(e_g^2). \tag{2}$$

Proof. We first analyze how a noisy CNOT gate contributes to bit-flip errors.

Under the two-qubit depolarizing channel, each nontrivial two-qubit Pauli operator occurs with probability $\frac{e_g}{15}$. The full set of non-identity two-qubit Pauli operators is

$$\begin{gathered} XI,\ YI,\ ZI, \\ IX,\ IY,\ IZ, \\ XX,\ XY,\ XZ, \\ YX,\ YY,\ YZ, \\ ZX,\ ZY,\ ZZ. \end{gathered}$$

Under computational-basis measurement, only Pauli operators containing an X or Y component on the measured qubit contribute to a bit-flip error. This follows because:

- X flips the computational basis states,

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle;$$

- Y also flips the computational basis states up to a phase,

$$Y|0\rangle = i|1\rangle, \qquad Y|1\rangle = -i|0\rangle;$$

- Z contributes only a phase and therefore does not affect computational-basis measurement outcomes.

Consider a particular physical qubit participating in one noisy CNOT gate. There are exactly eight two-qubit Pauli operators that contain an X or Y component acting on this qubit:

$$XI,\ XX,\ XY,\ XZ,\ YI,\ YX,\ YY,\ YZ.$$

Since each occurs with probability $e_g/15$, the probability that a single noisy CNOT induces a bit-flip on that qubit equals

$$\frac{8}{15}e_g. \tag{3}$$

Next, we analyze how many noisy CNOT gates can directly affect a given physical qubit in the chain encoding circuit.

Boundary qubits. The first qubit q_1 participates only in

$$\mathrm{CNOT}_{1\to 2},$$

while the last qubit q_k participates only in

$$\mathrm{CNOT}_{k-1\to k}.$$

Therefore, for boundary qubits,

$$p_{\mathrm{flip}} = \frac{8}{15}e_g + O(e_g^2).$$

Intermediate qubits. Each intermediate qubit

$$q_i, \qquad 2 \le i \le k-1,$$

participates in exactly two CNOT gates:

$$\mathrm{CNOT}_{i-1\to i}, \qquad \mathrm{CNOT}_{i\to i+1}.$$

Define the events

$$A_i = \{\text{the first adjacent noisy CNOT induces a bit-flip on } q_i\},$$

$$B_i = \{\text{the second adjacent noisy CNOT induces a bit-flip on } q_i\}.$$

Using Eq. (3),

$$P(A_i) = P(B_i) = \frac{8}{15}e_g.$$

The probability that at least one of these two noisy CNOTs induces a bit-flip is

$$P(A_i \cup B_i).$$

Using the union formula,

$$P(A_i \cup B_i) = P(A_i) + P(B_i) - P(A_i \cap B_i).$$

Substituting the probabilities gives

$$P(A_i \cup B_i) = \frac{8}{15}e_g + \frac{8}{15}e_g - P(A_i \cap B_i).$$

Since the two noisy CNOT events are independent,

$$P(A_i \cap B_i) = \left(\frac{8}{15} e_g\right)^2 .$$

Therefore,

$$P(A_i \cup B_i) = \frac{16}{15} e_g + O(e_g^2). \tag{4}$$

Finally, note that contributions from non-adjacent CNOT gates require at least two or more simultaneous fault events in order to propagate to qubit q_i. Such processes therefore contribute only at order at least $O(e_g^2)$.

Combining all contributions, we obtain

$$p_{\text{flip}} \le \frac{16}{15} e_g + O(e_g^2).$$

This completes the proof.

□

Lemma 2: Combination of gate-induced and measurement errors

Lemma 2 (Effective measurement error probability)**.** *Suppose a physical qubit experiences*

1. *a pre-measurement bit-flip error with probability* p_{flip}*,*
2. *a measurement error with probability* e_m*.*

Then the final observed measurement outcome is incorrect with probability

$$e_m^{\text{eff}} = p_{\text{flip}} + e_m - 2p_{\text{flip}} e_m. \tag{5}$$

Consequently,

$$e_m^{\text{eff}} \le e_m + \frac{16}{15} e_g + O(e_g^2, e_m e_g). \tag{6}$$

Proof. Define the events

$$F = \{\text{a pre-measurement bit-flip occurs}\},$$

$$M = \{\text{the measurement outcome flips}\}.$$

By assumption,

$$P(F) = p_{\text{flip}}, \qquad P(M) = e_m.$$

The final observed measurement outcome is incorrect if and only if exactly one of the two events occurs:

1. the qubit flips before measurement, but the measurement itself is correct;
2. the qubit state is correct, but the measurement outcome flips.

Therefore,

$$e_m^{\text{eff}} = P(F \cap \overline{M}) + P(\overline{F} \cap M).$$

Using independence,

$$\begin{aligned} e_m^{\text{eff}} &= P(F)P(\overline{M}) + P(\overline{F})P(M) \\ &= p_{\text{flip}}(1 - e_m) + (1 - p_{\text{flip}})e_m. \end{aligned}$$

Expanding the expression gives

$$\begin{aligned} e_m^{\text{eff}} &= p_{\text{flip}} - p_{\text{flip}}e_m + e_m - p_{\text{flip}}e_m \\ &= p_{\text{flip}} + e_m - 2p_{\text{flip}}e_m. \end{aligned}$$

Substituting the result of Theorem 1,

$$p_{\text{flip}} \leq \frac{16}{15}e_g + O(e_g^2),$$

we obtain

$$e_m^{\text{eff}} \leq e_m + \frac{16}{15}e_g + O(e_g^2) + O(e_m e_g).$$

Therefore,

$$e_m^{\text{eff}} \leq e_m + \frac{16}{15}e_g + O(e_g^2, e_m e_g).$$

This completes the proof.

$\square$

Proof of Theorem 1

Proof of Theorem 1. Consider one encoded logical qubit consisting of k physical qubits. Let

$$X = \text{number of erroneous physical measurement outcomes.}$$

Since each physical qubit independently experiences an effective error with probability e_m^{eff}, the random variable X follows a binomial distribution:

$$\Pr(X = i) = \binom{k}{i} \left(e_m^{\text{eff}}\right)^i \left(1 - e_m^{\text{eff}}\right)^{k-i}.$$

Majority-vote decoding fails whenever at least half of the physical qubits are incorrect, namely when

$$X \geq \lceil k/2 \rceil.$$

Therefore, the logical error probability satisfies

$$\begin{aligned} e_{\text{err}} &= \Pr(X \geq \lceil k/2 \rceil) \\ &= \sum_{i=\lceil k/2 \rceil}^{k} \Pr(X = i) \\ &= \sum_{i=\lceil k/2 \rceil}^{k} \binom{k}{i} \left(e_m^{\text{eff}}\right)^i \left(1 - e_m^{\text{eff}}\right)^{k-i}, \end{aligned}$$

where $e_m^{\text{eff}} \leq e_m + \frac{16}{15}e_g$. This proves the first statement of the theorem.

Next, we derive the low-error asymptotic behavior.

As

$$e_m^{\text{eff}} \to 0,$$

the dominant contribution comes from the smallest power of e_m^{eff}, which corresponds to $i = \lceil k/2 \rceil$.

Separate this leading-order contribution:

$$e_{\mathrm{err}} = \binom{k}{\lceil k/2 \rceil} \left(e_m^{\mathrm{eff}}\right)^{\lceil k/2 \rceil} \left(1 - e_m^{\mathrm{eff}}\right)^{k - \lceil k/2 \rceil} + \sum_{i=\lceil k/2 \rceil + 1}^{k} \binom{k}{i} \left(e_m^{\mathrm{eff}}\right)^{i} \left(1 - e_m^{\mathrm{eff}}\right)^{k-i}.$$

Now expand

$$\left(1 - e_m^{\mathrm{eff}}\right)^{k - \lceil k/2 \rceil}.$$

Using the binomial expansion,

$$(1-x)^r = 1 - rx + O(x^2),$$

with

$$x = e_m^{\mathrm{eff}} \ll 1, \qquad r = k - \lceil k/2 \rceil,$$

we obtain

$$\left(1 - e_m^{\mathrm{eff}}\right)^{k - \lceil k/2 \rceil} = 1 + O(e_m^{\mathrm{eff}}).$$

Therefore,

$$\binom{k}{\lceil k/2 \rceil} \left(e_m^{\mathrm{eff}}\right)^{\lceil k/2 \rceil} \left(1 - e_m^{\mathrm{eff}}\right)^{k - \lceil k/2 \rceil} = \binom{k}{\lceil k/2 \rceil} \left(e_m^{\mathrm{eff}}\right)^{\lceil k/2 \rceil} + O\Big(\left(e_m^{\mathrm{eff}}\right)^{\lceil k/2 \rceil + 1}\Big).$$

For all remaining terms,

$$i \geq \lceil k/2 \rceil + 1,$$

so each term is already of order

$$O\Big(\left(e_m^{\mathrm{eff}}\right)^{\lceil k/2 \rceil + 1}\Big).$$

Combining all contributions yields

$$e_{\mathrm{err}} = \binom{k}{\lceil k/2 \rceil} \left(e_m^{\mathrm{eff}}\right)^{\lceil k/2 \rceil} + O\Big(\left(e_m^{\mathrm{eff}}\right)^{\lceil k/2 \rceil + 1}\Big).$$

This completes the proof.

□

II. DETAILS OF NUMERICAL SIMULATIONS ON MEASUREMENT-BASED QUANTUM COMPUTATION (MBQC)

A. Bell-state generation from a one-dimensional cluster state

We consider a one-dimensional cluster state consisting of n qubits initialized in the state $|+\rangle^{\otimes n}$, followed by controlled-Z (CZ) gates applied between neighboring qubits,

$$|C_n\rangle = \prod_{i=1}^{n-1} CZ_{i,i+1} \, |+\rangle^{\otimes n}. \tag{7}$$

The resulting state forms a linear cluster state,

$$1—2—\cdots—n. \tag{8}$$

To generate a Bell pair between the two end qubits, qubits $2, \ldots, n-1$ are sequentially measured in the X basis. Let the corresponding measurement outcomes be

$$m_i \in \{0, 1\}, \qquad i = 2, \ldots, n-1. \tag{9}$$

Using the stabilizer properties of cluster states, the post-measurement state of qubits 1 and n can be written as

$$|\Phi_{\{m_i\}}\rangle = X_n^{s_X} Z_n^{s_Z} |\Phi^+\rangle, \tag{10}$$

where

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{11}$$

is the Bell state, and the correction parities are determined by the measurement outcomes,

$$s_X = \bigoplus_{i\in\mathcal{E}} m_i, \qquad s_Z = \bigoplus_{i\in\mathcal{O}} m_i. \tag{12}$$

Here $\oplus$ denotes addition modulo 2, while $\mathcal{E}$ and $\mathcal{O}$ denote the sets of even and odd indexed measurement qubits, respectively.

Thus, the final logical state depends explicitly on the parity of the intermediate measurement outcomes through feedforward Pauli corrections. In the ideal case, applying the correction operator

$$Z_n^{s_Z} X_n^{s_X} \tag{13}$$

recovers the target Bell state exactly.

This protocol is intrinsically sensitive to measurement errors. A single incorrect readout outcome changes the inferred correction parity and therefore propagates directly into a logical Pauli error on the final Bell pair. Consequently, measurement errors accumulate throughout the MBQC process, making this setting a particularly suitable testbed for evaluating measurement error mitigation schemes.

B. Numerical simulation settings

In the numerical simulations, our goal is to isolate and characterize the effect of measurement errors and the performance of the proposed mitigation scheme. To this end, we assume that the preparation of the one-dimensional cluster state is ideal and noiseless. Consequently, the only noise sources considered in the simulation arise from measurement processes.

Specifically, we consider two types of measurement-related errors:

1. direct single-qubit readout errors with probability e_m in the unmitigated case;
2. effective readout errors generated by the repetition-based mitigation scheme, including both measurement errors and gate errors introduced during the encoding circuit.

For the repetition-based encoding, each logical measurement qubit is encoded into a distance-k repetition code immediately prior to measurement. The encoding circuit consists of CNOT gates arranged in the chain structure discussed in the main text. After encoding, all physical qubits are measured independently in the computational basis, and the logical outcome is determined via majority-vote decoding.

To incorporate realistic imperfections in the encoding process, we assume that each CNOT gate is followed by an independent two-qubit depolarizing channel with error rate e_g, while each physical qubit measurement is subject to an independent bit-flip readout error with probability e_m.

Under these assumptions, the final Bell-state fidelity is evaluated after applying feedforward corrections inferred from the noisy measurement outcomes. The Bell-state infidelity shown in Fig.3 (main text) is computed as

$$1 - F, \tag{14}$$

where

$$F = \langle\Phi^+|\,\rho_{\mathrm{out}}\,|\Phi^+\rangle \tag{15}$$

denotes the fidelity between the simulated output state ρ_{out} and the ideal Bell state.

This simplified setting allows a direct comparison between unmitigated readout and repetition-based mitigation, thereby clearly revealing the scaling behavior and robustness of the proposed scheme against measurement noise.

III. DETAILS OF NUMERICAL SIMULATIONS ON RANDOM CIRCUIT SAMPLING (RCS)

In this section, we describe the fidelity estimation method used in the numerical simulations of random circuit sampling (RCS). Following Ref. [1–5], the circuit fidelity can be approximately estimated as the product of the fidelities of individual operations,

$$F = \prod_{g\in G_1}(1-e_g)\prod_{g\in G_2}(1-e_g)\prod_{q\in Q}(1-{e_m}^q), \tag{16}$$

where e_g denotes the Pauli error rate of individual quantum gates, G_1 and G_2 are the sets of single-qubit and two-qubit gates, respectively, Q is the set of qubits, and ${e_m}^q$ denotes the measurement error associated with each qubit.

In the present work, our focus is specifically on the effect of measurement error mitigation. We therefore isolate the contribution of measurement errors while keeping the remaining circuit fidelity unchanged. Let e_m denote the physical measurement error rate and e_m^{eff} the effective logical measurement error rate after repetition-based mitigation. For an n-qubit sampling task, the measurement contribution to the fidelity changes from

$$(1-e_m)^n \tag{17}$$

to

$$(1-e_m^{\mathrm{eff}})^n. \tag{18}$$

Accordingly, the fidelity improvement factor induced by measurement error mitigation is estimated as

$$\eta = \frac{(1-e_m^{\mathrm{eff}})^n}{(1-e_m)^n}. \tag{19}$$

As the encoding distance k increases, the effective logical measurement error e_m^{eff} is exponentially suppressed and approaches zero. In this limit, the maximal achievable fidelity improvement becomes

$$\eta_{\max} = \frac{1}{(1-e_m)^n}. \tag{20}$$

This expression defines the theoretical upper bound of measurement-error mitigation in RCS.

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