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Random Variable Transform deployed in Computing Probability Density Function for Probabilistic Solutions of a Second Difference Equation

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Abstract

The main aim of this study is to compute the marginal probability density function (p.d.f) based on random variable transform (R.V.T) by finding the solutions of the non-homogeneous second difference equation via z-transform. The performance of the marginal p.d.f. is to illustrate the distribution of the entry values by the image values, setting the variables of the non-homogeneous second difference equation.

Keywords: z-transform, marginal p.d.f, random variable transform (R.V.T).

1 Introduction

Transformations, particularly the Laplace transform and z-transform, are widely used in engineering and mathematics [1–5]. The z-transform is especially useful in discrete mathematics, such as discrete system theory. At the same time, the Laplace transform is commonly applied in continuous systems like signal processing and information flow [6]. The delimitation of the continuous and discrete systems is flexible based on the use of the transform in specific problems [7]. Random variable transform is a one-to-one mapping between two n-dimensional random vectors. Meanwhile, R.V.T is the

mechanism of random variable transform established by domain elements corresponding to the image's members. Based on the input values and mapping functions, the dense distribution of the image is computed concerning the input values [8]. For the correlation visualization, the probability density function is calculated and illustrated relying on the input values and test functions to understand the distribution of the image to the domain input values [9].
The critical parameters and correlation coefficients are established using transform results to make the variance-covariance matrix used to compute the marginal probability density function (p.d.f) [10]. Direct calculation from the inputs and output, the probability is the criterion to understand the reliability of prediction [11]. By comparing the input and output values' correlation, the probability performs the safe region and failure region that prediction attained [12]. To make reliable predictions, confidence in a prediction and the interpretation of the data is necessary to connect the populous value performance in SVM, or K-nearest neighbor regression [13]. The role of gradient-boosting prediction is based on the gradient-based study investigating the behavior transferred of the output features concerning the input features. The structure of prediction belongs to multiple variables and one-to-one mapping is established to seek the distribute function [14].
This paper will focus on the z-transform and its application in solving linear difference equations. Additionally, we introduce the Random Variable Transform (R.V.T), a method for mapping n-dimensional random vectors to compute dense distributions and marginal probability density functions (p.d.f.). To make reliable predictions, it is crucial to understand the relationship between input and output values. This involves computing correlation coefficients and the variance-covariance matrix, which help in estimating the marginal p.d.f. The probability distribution function is a key tool in this analysis, defined as [15]:

$$f_x\left(x_1,...,x_N\right)=\frac{1}{\left(2\pi\right)^{\frac{N}{2}}\left|\Sigma\right|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(x-\mu\right)^T\Sigma^{-1}\left(x-\mu\right)\right]. \tag{1.1}$$

where N : input variables, Σ : variance-covariance matrix, T : transpose of the matrix, Σ^{-1} : inverse of the matrix, $|\Sigma|$: determinant of the variance-covariance matrix, x : vector of the x_k values, μ : vector of the means of the k distributions. $X\left(t\right)$ denotes the random function of time of the processing prediction. The reliability of prediction is set within the range of lower limit X_L and upper limit X_U. The failure criterion is defined in the complement of the safe region [16]. The probability of reliable prediction has been applied in estimating the safe region of prediction. The reliable prediction is based on the computation of correlation values between the input values and the prediction values. The probability of a safe region is procured as follows:

$$R\left(t\right)=\Pr\left(X_L<X\left(t\right)<X_U\right).$$

The probability of failure P of the system is

$$P\left(t\right)=\Pr\left(X\left(t\right)>X_U\right)+\Pr\left(X\left(t\right)<X_L\right).$$

These practical applications highlight the versatility and significance of the R.V.T method in solving second-order difference equations with uncertainties. By providing a probabilistic perspective, the R.V.T approach enhances the accuracy, reliability, and robustness of models across diverse domains, from engineering systems to financial modeling, epidemiology, and signal processing. The integration of uncertainty quantification through the R.V.T method opens up new avenues for analysis and decision-making in these fields, enabling more informed and effective strategies.

2 The z-transform

The z-transform supports a mathematical tool to transform functions of discrete values with the quantity k into continuous functions with variable z. This section shows how the z-transform method may attain solutions to a linear non-homogeneous difference equation with constant coefficients. This method provides the property of transforming a linear non-homogeneous difference equation into an algebraic equation, which will be controlled by applying the familiar rules of algebra.

Definition 1. *Let $\{y_k\}$ be a sequence of numbers such as $y_k = 0$ for $k < 0$. The $z-$transform of this sequence is the series [17], p.146:*

$$Z(y_k) = F(z) = \sum_{k=0}^{\infty} \frac{y_k}{z^k},$$

where Z is the transform variable, and $Z(y_k)$ is a function of $z, y_k \leq c^k$.

Table 1 Selected $z-$transform pairs

y_k	$F(z) = Z(y_k)$
1	$\frac{1}{z-1}$
k	$\frac{z}{(z-1)^2}$
a^k	$\frac{z}{z-a}$
ka^k	$\frac{az}{(z-a)^2}$
y_{k+2}	$z^2F(z) - z^2y_0 - zy_1$
y_{k+1}	$zF(z) - zy_0$

2.1 Application z-transform for solving second difference equation

Considering a second difference equation

$$ay_{k+2} + by_{k+1} + cy_k = d_k, y_0 = \alpha, y_1 = \beta. \tag{2.1}$$

Assume a sequence $\{y_k\}$ is a solution of Eq. (2.1). Taking z-transform on both sides of Eq. (2.1), we have

$$az^2F(z) - az^2y_0 - azy_1 + bzF(z) - bzy_0 + cF(z) = \frac{z}{s-d}. \tag{2.2}$$

Simplify on both sides, Eq. (2.2) become

$$\left(az^2 + bz + c\right)F(z) = \alpha\left(az^2 + bz\right) + a\beta z + \frac{z}{z-d}. \tag{2.3}$$

Using the formula establishment: $a^2 + bz + c = a(z - z_1)(z - z_2)$; $z_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$; $z_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\Delta = b^2 - 4ac$. ; From Eq. (2.3), we apply to the transform:

Case 1: If $\Delta = 0$, and $z_1 = \frac{b}{2a}$, we have

$$\begin{aligned} F(z) &= \frac{az^2 + bz}{a(z - z_1)^2}\alpha + \frac{az}{a(z - z_1)^2}\beta + \frac{z}{a(z-d)(z - z_1)^2} \\ &= \frac{az^2 + bz}{a(z - z_1)^2}\alpha + \frac{z}{(z - z_1)^2}\beta + \frac{z}{a(z-d)(z - z_1)^2}. \end{aligned}$$

By partial fraction, and taking the inverse z-transform using Table 1, the general solution of Eq. (2.1) is

$$\begin{aligned} y_k = k&\left(3\alpha + \frac{2a\beta}{b} - \frac{4a}{b(2ad - b)}\right)\left(\frac{b}{2a}\right)^k \\ &+ \left(\alpha - \frac{4a}{(b - 2d)^2}\right)\left(\frac{b}{2a}\right)^k + \frac{4a}{(b - 2d)^2}d^k, d \neq \frac{b}{2}. \end{aligned} \tag{2.4}$$

Case 2: If $\Delta > 0$, $z_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $z_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$, we have

$$\begin{aligned} F(z) &= \frac{az^2 + bz}{az^2 + bz + c}\alpha + \frac{az}{az^2 + bz + c}\beta + \frac{z}{(z-d)(az^2 + bz + c)} \\ &= \frac{(az + b)z}{a(z - z_1)(z - z_2)}\alpha + \frac{z}{(z - z_1)(z - z_2)}\beta \\ &+ \frac{z}{a(z-d)(z - z_1)(z - z_2)}. \end{aligned} \tag{2.5}$$

By partial fractions, and taking inverse z-transform using Table 1, the general solutions of Eq. (2.1) is

$$y_k = \frac{1}{a(d - z_1)(d - z_2)}d^k$$

$$+\frac{1}{\sqrt{b^2-4ac}}\left(\frac{b+\sqrt{b^2-4ac}}{2}\alpha+a\beta+\frac{1}{d-z_1}\right)z_1^k$$
$$+\frac{1}{\sqrt{b^2-4ac}}\left(\frac{-b+\sqrt{b^2-4ac}}{2}-a\beta+\frac{1}{d-z_2}\right)z_2^k, d\neq z_1, z_2. \tag{2.6}$$

Case 3: If $\Delta<0$, it is similar to case 2, then the solution is

$$y_k=\frac{1}{a}\left(\frac{b+az_1}{z_1-z_2}z_1^k-\frac{b+az_2}{z_1-z_2}z_2^k\right)\alpha+\frac{1}{z_1-z_2}\left(z_1^k-z_2^k\right)\beta$$
$$+\frac{1}{a\left(d-z_1\right)\left(d-z_2\right)}\left(d^k+\frac{d-z_1}{z_1-z_2}z_2^k+\frac{z_2-d}{z_1-z_2}z_1^k\right),$$

where

$$z_{1,2}=\frac{-b}{2a}\pm i\frac{\sqrt{4ac-b^2}}{2a}, Re\left(z\right)=\frac{-b}{2a}, Im\left(z\right)=\pm\frac{\sqrt{4ac-b^2}}{2a},$$
$$z_1=r(\cos\varphi+i\sin\varphi), z_2=r(\cos\varphi-i\sin\varphi) \tag{2.7}$$

and

$$z_{1,2}^k=r^k\left(\cos\varphi\pm i\sin\varphi\right)^k=r^k\left(\cos k\varphi\pm i\sin k\varphi\right),$$
$$r=\sqrt{\frac{c}{a}}, \varphi=\arctan\left(\frac{Im(z)}{Re(z)}\right). \tag{2.8}$$

We attain the solution as follows

$$y_k=\left(\frac{d^k}{ad^2+bd+ac}+\frac{2ar^k\beta}{\sqrt{4ac-b^2}}\sin k\varphi\right)+\left(1-\frac{d^k}{ad^2+bd+ac}\right)r^k\cos k\varphi$$
$$+\left(\frac{b}{\sqrt{4ac-b^2}}+\frac{b+2ad}{ad^2+bd+ac}\right)r^k\sin k\varphi,$$
$$4ac-b^2>0, ad^2+bd+ac\neq 0. \tag{2.9}$$

2.2 The marginal probability density function computation

The distribution of input and output values using the z-transform is coherent with the Multivariate Random Variable Transform (R.V.T). We will determine the first probability density function (1-p.d.f) of the solutions of the random non-homogeneous linear second difference equation as follows

$$AZ_{k+2}+BZ_{k+1}+CZ_k=D^k, Z_0=E_0, Z_1=E_1. \tag{2.10}$$

by using the R.V.T construction as follows:
Proposition 1 (Multivariate Random Variable Transform)**.** *Let $V=(V_1,\cdots,V_m)$ be a random vector of dimension m with the joint p.d.f $f_V(V)$. We define $u:\mathbb{R}^m\to\mathbb{R}^m$*

to be a one-to-one deterministic map that is assumed to be continuous concerning each one of its arguments and with continuous partial derivatives. Then, the joint p.d.f $f_W(W)$ *of the random vector* $W = u(V)$ *is given by* $f_W(W) = f_V\left(u^{-1}(W)\right)|J_m|$, *is the joint p.d.f, and* $u^{-1}(W)$ *is inverse transformation of* $u(V) : V = u^{-1}(W)$, J_m *is the Jacobian of the transform satisfies the condition:*

$$J_m = \det\left(\frac{\partial V}{\partial W}\right) = \det\begin{vmatrix} \frac{\partial V_1}{\partial W_1} & \cdots & \frac{\partial V_m}{\partial W_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial V_1}{\partial W_m} & \cdots & \frac{\partial V_m}{\partial W_m} \end{vmatrix} \neq 0.$$

To provide more generalization to the study, we consider both initial conditions, $E_0 = E_0(\omega), E_1 = E_1(\omega)$, and coefficients, $A = A(\omega), B = B(\omega), C = C(\omega), D = D(\omega)$ are dependent continuous random variables defined on a same probability space, $(\Omega, \mathbb{F}, \mathbb{P})$, where Ω is a sample space, $\mathbb{F}$ is a sigma-algebra of events and $\mathbb{P}$ is a probability measure on and whose joint p.d.f $f_{E_0,E_1,A,B,C,D}(\alpha, \beta, a, b, c, d)$ on the domain is given, and we define the domains of the input parameters denoted by

$$\begin{aligned}
\mathbb{D}_{E_0} &= \{\alpha = E_0(\omega), \omega \in \Omega : \alpha_1 \leq \alpha \leq \alpha_2\}, \\
\mathbb{D}_{E_1} &= \{\beta = E_1(\omega), \omega \in \Omega : \beta_1 \leq \beta \leq \beta_2\}, \\
\mathbb{D}_A &= \{a = A(\omega), \omega \in \Omega : a_1 \leq a \leq a_2\}, \qquad (2.11) \\
\mathbb{D}_B &= \{b = B(\omega), \omega \in \Omega : b_1 \leq b \leq b_2\}, \\
\mathbb{D}_C &= \{c = C(\omega), \omega \in \Omega : c_1 \leq c \leq c_2\}, \qquad (2.12) \\
\mathbb{D}_D &= \{d = D(\omega), \omega \in \Omega : d_1 \leq d \leq d_2\},
\end{aligned}$$

associated with probabilities, $A = A(\omega), B = B(\omega), C = C(\omega), \omega \in \Omega$:

$$\begin{aligned}
p_1 &= P\left[\varepsilon_1 = \left\{\omega \in \Omega : B^2 - 4AC > 0\right\}\right], \qquad (2.13) \\
p_2 &= P\left[\varepsilon_2 = \left\{\omega \in \Omega : B^2 - 4AC < 0\right\}\right], \qquad (2.14) \\
p_3 &= P\left[\varepsilon_3 = \left\{\omega \in \Omega : B^2 - 4AC = 0\right\}\right],
\end{aligned}$$

The probability consistent with the sample is $p_1 + p_2 + p_3 = 1$, and $p_3 = 0$.

2.3 Real and distinct random roots

The general solution Z_k of Eq. (2.10) is called a discrete stochastic process is given by

$$\begin{aligned}
Z_k = \frac{1}{A}\left(\frac{B + Az_1}{z_1 - z_2}z_1^k - \frac{B + Az_2}{z_1 - z_2}z_2^k\right)E_0 + \frac{1}{z_1 - z_2}\left(z_1^k - z_2^k\right)E_1 \\
+ \frac{1}{A(D - z_1)(D - z_2)}\left(D^k + \frac{D - z_1}{z_1 - z_2}z_1^k + \frac{z_2 - D}{z_1 - C}z_2^k\right).
\end{aligned}$$

where $z_1 = z_1(\omega), z_2 = z_2(\omega), A = A(\omega), B = B(\omega), C = C(\omega), \omega \in \Omega$ are real and complex roots given by

$$z_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2}, z_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2}. \tag{2.15}$$

Applying R.V.T definition, the first term of the transform of $V = (E_0, E_1, A, B, C, D)$ and $W = (W_1, W_2, W_3, W_4, W_5, W_6)$ is established. The domain of the input parameters $V = (E_0, E_1, A, B, C, D)$ is the random vector with the joint p.d.f $f_V(V)$ setting $u : \mathbb{R}^6 \to \mathbb{R}^6$ is the 1-1 mapping as $u = (u_1(V), u_2(V), u_3(V), u_4(V), u_5(V), u_6(V)$, and $u(V) = (W_1, W_2, W_3, W_4, W_5, W_6)$ where

$$\begin{aligned} W_1 = u_1(V) = \frac{1}{A}\left(\frac{B + Az_1}{z_1 - z_2} z_1^k - \frac{B + Az_2}{z_1 - z_2} z_2^k\right) E_0 + \frac{1}{z_1 - z_2}\left(z_1^k - z_2^k\right) E_1 \\ + \frac{1}{A(D - z_1)(D - z_2)}\left(D^k + \frac{D - z_1}{z_1 - z_2} z_1^k + \frac{z_2 - D}{z_1 - C} z_2^k\right). \\ W_2 = u_2(V) = E_1, W_3 = u_3(V) = A, W_4 = u_4(V) = B, \\ W_5 = u_5(V) = C, W_6 = u_6(V) = D. \end{aligned}$$

and we can attain

$$\begin{aligned} \Rightarrow E_0 = u^{-1}(W) = \frac{A}{\frac{B + A\tilde{z}_1}{\tilde{z}_1 - \tilde{z}_2}\tilde{z}_1^k - \frac{B + A\tilde{z}_2}{\tilde{z}_1 - \tilde{z}_2}\tilde{z}_2^k}\Bigg(W_1 - \frac{1}{\tilde{z}_1 - \tilde{z}_2}\left(\tilde{z}_1^k - \tilde{z}_2^k\right) W_2 \\ - \frac{1}{A(D - \tilde{z}_1)(D - \tilde{z}_2)}\left(D^k + \frac{D - \tilde{z}_1}{\tilde{z}_1 - \tilde{z}_2}\tilde{z}_1^k + \frac{\tilde{z}_2 - D}{\tilde{z}_1 - C}\tilde{z}_2^k\right)\Bigg), \\ E_1 = u_2^{-1}(W) = W_2, A = u_3^{-1}(W) = W_3, B = u_4^{-1}(W) = W_4, \\ C = u_5^{-1}(W) = W_5, D = u_6^{-1}(W) = W_6. \end{aligned}$$

with the conditions $W_4^2 - 4W_3W_5 > 0$ or $W_5 < \frac{W_4^2}{4W_3}, W_3 < \frac{W_4^2}{4W_3}$, where

$$\tilde{z}_1 = \frac{-W_4 + \sqrt{W_4^2 - 4W_3W_5}}{2W_3}, \tilde{z}_2 = \frac{-W_4 - \sqrt{W_4^2 - 4W_3W_5}}{2W_3}.$$

and Jacobian of the transform given by

$$J_6 = \det\left(\frac{\partial V}{\partial W}\right) = \det\begin{vmatrix} \frac{\partial E_0}{\partial W_1} & \cdots & \frac{\partial D}{\partial W_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial E_0}{\partial W_6} & \cdots & \frac{\partial D}{\partial W_6} \end{vmatrix}$$

$$= \det \begin{vmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{12} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \frac{A}{\frac{B+A\tilde{z}_1}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_1^k - \frac{B+A\tilde{z}_2}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_2^k} \neq 0,$$

where

$$a_{11} = \frac{A}{\frac{B+A\tilde{z}_1}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_1^k - \frac{B+A\tilde{z}_2}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_2^k},$$
$$a_{12} = \frac{A}{\left(\frac{B+A\tilde{z}_1}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_1^k - \frac{B+A\tilde{z}_2}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_2^k\right)} \frac{-1}{(\tilde{z}_1-\tilde{z}_2)} \left(\tilde{z}_1^k - \tilde{z}_2^k\right).$$

and

$$f_W(W) = f_V(V) \times |J_6|$$
$$= f_V(e_0, e_1, a, b, c, d) \times \left| \frac{A(\tilde{z}_1 - \tilde{z}_2)}{(B + A\tilde{z}_1)\tilde{z}_1^k - (B + A\tilde{z}_2)\tilde{z}_2^k} \right|,$$

where e_0 is given by

$$e_0 = \frac{A}{\frac{B+A\tilde{z}_1}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_1^k - \frac{B+A\tilde{z}_2}{\tilde{z}_1-\tilde{z}_2}\tilde{z}_2^k} \left(w_1 - \frac{1}{\tilde{z}_1 - \tilde{z}_2} \left(\tilde{z}_1^k - \tilde{z}_2^k\right) w_2 \right.$$
$$\left. - \frac{1}{A\left(D - \tilde{z}_1\right)\left(D - \tilde{z}_2\right)} \left(D^k + \frac{D - \tilde{z}_1}{\tilde{z}_1 - \tilde{z}_2}\tilde{z}_1^k + \frac{\tilde{z}_2 - D}{\tilde{z}_1 - C}\tilde{z}_2^k \right) \right).$$

Combine the condition (2.11), and (2.12)) then

$$w_5 \leq min\{w_{51}, \frac{w_4^2}{4w_3}\}, w_3 \leq min\{w_{31}, \frac{w_4^2}{4w_3}\}$$

The marginal p.d.f $f_{1R}(z,k)$ concerning $Z = W_1$ with the constraints $w_5 \leq min\{w_{51}, \frac{w_4^2}{4w_3}\}$, $w_3 \leq min\{w_{32}, \frac{w_4^2}{4w_3}\}$:

$$f_{1R}(z,k) = \int_{w_{21}}^{w_{22}} \int_{min\{w_{31}, \frac{w_4^2}{4w_3}\}}^{min\{w_{32}, \frac{w_4^2}{4w_3}\}} \int_{w_{41}}^{w_{42}} \int_{\min\left[w_{51}; \frac{w_4^2}{4w_3}\right]}^{\min\left[w_{52}; \frac{w_4^2}{4w_3}\right]} \int_{w_{61}}^{w_{62}} f_w(w)\, dw_6 dw_5 dw_4 dw_3 dw_2$$
$$= \int_{w_{21}}^{w_{22}} \int_{min\{w_{31}, \frac{w_4^2}{4w_3}\}}^{min\{w_{32}, \frac{w_4^2}{4w_3}\}} \int_{w_{41}}^{w_{42}} \int_{\min\left[w_{51}; \frac{w_4^2}{4w_3}\right]}^{\min\left[w_{52}; \frac{w_4^2}{4w_3}\right]} \int_{w_{61}}^{w_{62}} f_V(v) \times |J_6|\, dw_6 dw_5 dw_4 dw_3 dw_2.$$

2.4 Complex and random roots

The general solution Z_k of Eq. (2.10) corresponding to complex zeroes given by

$$Z_k = \frac{1}{A}\left(Ar^k\cos k\varphi + \frac{ABr^k\sin k\varphi}{\sqrt{4AC-B^2}}\right)E_0 + \frac{2Ar^k\sin k\varphi}{\sqrt{4AC-B^2}}E_1 + \frac{1}{AD^2+BD+AC}\left(D^k - r^k\cos k\varphi + \frac{r^k\,(B+2AD)}{\sqrt{4AC-B^2}}\sin k\varphi\right).$$

We will determine the piece $f_{1I}(z,n)$ associated with the event ε_2 defined in Eq. (2.14) that contributes to determining the 1-p.d.f. $f_1(z,n)$. This will be done by applying again Proposition 1 with the same identification for the maps $u_1, u_2, u_3, \cdots$ as we did in the previous case. Taking

$$W_1 = u_1(V) = \frac{1}{A}\left(Ar^k\cos k\varphi + \frac{ABr^k\sin k\varphi}{\sqrt{4AC-B^2}}\right)E_0 + \frac{2Ar^k\sin k\varphi}{\sqrt{4AC-B^2}}E_1 + \frac{1}{AD^2+BD+AC}\left(D^k - r^k\cos k\varphi + \frac{r^k\,(B+2AD)}{\sqrt{4AC-B^2}}\sin k\varphi\right).$$

and we have

$$E_0 = u_1^{-1}(W) = \frac{A}{\left(Ar^k\cos k\varphi + \frac{ABr^k\sin k\varphi}{\sqrt{4AC-B^2}}\right)}\left(W_1 - \frac{2Ar^k\sin k\varphi}{\sqrt{4AC-B^2}}E_1 - \frac{1}{AD^2+BD+AC}\left(D^k - r^k\cos k\varphi + \frac{r^k\,(B+2AD)}{\sqrt{4AC-B^2}}\sin k\varphi\right)\right).$$

with

$$J_6 = \frac{A}{\left(Ar^k\cos k\varphi + \frac{ABr^k\sin k\varphi}{\sqrt{4AC-B^2}}\right)} \neq 0. \tag{2.16}$$

The joint p.d.f. of the random vector $W = (W_1, W_2, W_3, W_4)$ is computed as follows:

$$f_W(W) = f_V(e_0, e_1, a, b, c, d) \times |J_6| = f_V(e_0, e_1, a, b, c, d) \times \left|\frac{A}{\left(Ar^k\cos k\varphi + \frac{ABr^k\sin k\varphi}{\sqrt{4AC-B^2}}\right)}\right|.$$

where

$$e_0 = \frac{a}{\left(ar^k\cos k\varphi + \frac{abr^k\sin k\varphi}{\sqrt{4ac-b^2}}\right)}\left(w_1 - \frac{2ar^k\sin k\varphi}{\sqrt{4ac-b^2}}e_1 - \frac{1}{ad^2+bd+ac}\left(d^k - r^k\cos k\varphi + \frac{r^k\,(b+2ad)}{\sqrt{4ac-b^2}}\sin k\varphi\right)\right).$$

and the marginal p.d.f $f_{1I}(z,k)$ concerning $Z = W_1$ with the constraints $w_5 \leq min\{w_{51}, \frac{w_4^2}{4w_3}\}$, $w_3 \leq min\{w_{32}, \frac{w_4^2}{4w_3}\}$:

$$f_{1I}(z,k) = \int_{w_{21}}^{w_{22}} \int_{min\{w_{31},\frac{w_4^2}{4w_3}\}}^{min\{w_{32},\frac{w_4^2}{4w_3}\}} \int_{w_{41}}^{w_{42}} \int_{\min\left[w_{51};\frac{w_4^2}{4w_3}\right]}^{\min\left[w_{52};\frac{w_4^2}{4w_3}\right]} \int_{w_{61}}^{w_{62}} f_W(w)\, dw_6 dw_5 dw_4 dw_3 dw_2$$

$$= \int_{w_{21}}^{w_{22}} \int_{min\{w_{31},\frac{w_4^2}{4w_3}\}}^{min\{w_{32},\frac{w_4^2}{4w_3}\}} \int_{w_{41}}^{w_{42}} \int_{\min\left[w_{51};\frac{w_4^2}{4w_3}\right]}^{\min\left[w_{52};\frac{w_4^2}{4w_3}\right]} \int_{w_{61}}^{w_{62}} f_V(v) \times |J_6|\, dw_6 dw_5 dw_4 dw_3 dw_2.$$

Finally, the 1-p.d.f of the solution Eq. (2.10) corresponding to the values $f_{1R}(z,n)$ and $f_{1I}(z,n)$ calculated above is given by

$$f_1(z,n) = f_{1R}(z,n) + f_{1I}(z,n). \tag{2.17}$$

2.5 Numerical results

This section will illustrate the transform of a set of input values k and apply the R.V.T to compute the 1-p.d.f [18]. First, we consider the covariance Matrix related as follows: Let X_j be random variables, $1 \leq j \leq m$, where $X_j = x_{1j}, x_{2j}, x_{3j}, ..., x_{ij}, ..., x_{nj}, 1 \leq i \leq n$. We can write the matrix form as

$$M = \begin{bmatrix} x_{11} & x_{12} & ... & x_{1m} \\ x_{21} & x_{22} & ... & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & ... & x_{nm} \end{bmatrix}_{n \times m},$$

and so $\mu = [\mu_1, \mu_2, ..., \mu_n]$, where

$$\mu_j = \sum_{i=1}^{n} \frac{x_{ij}}{n}.$$

The covariance matrix of a random variable is defined by

$$M = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & ... & Cov(X_1, X_m) \\ Cov(X_2, X_1) & Var(X_{22}) & ... & Cov(X_2, X_m) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_m, X_1) & Cov(X_m, X_2) & ... & Var(X_m) \end{bmatrix},$$

where

$$Cov(X_i, X_j) = \sum_{k=1}^{n} \frac{(x_{ki} - \mu_i)(x_{kj} - \mu_j)}{n-1}, \; i, \; j = 1, ..., m,$$

and

$$Var\left(X_i\right) = \sum_{k=1}^{n} \frac{\left(x_{ki} - \mu_i\right)^2}{n-1}.$$

We consider that the four random inputs E_0, E_1, A, B, C, D follow a joint Gaussian distribution: $\eta_i = (E_0, E_1, A, B, C, D)^T \sim N(\mu_{\eta_i}, \Sigma_{\eta_i})$ with different vector $\mu_{\eta_i}, i = 1, 2, 3$, and a common covariance matrix $\Sigma = \Sigma_{\eta_i}$ given, respectively, by

$$\mu_{\eta_i} = \begin{cases} (1.27, -1.68, 1.26, -1.39, -1.56, 1.84) & \text{if } \Delta > 0, \\ (-1.65, 1.97, -1.56, 1.89, -1.87, 1.96) & \text{if } \Delta < 0, \\ (-1.98, -1.16, -1.95, -1.18, -0.71, -1.69) & \text{if } \Delta = 0 \end{cases}$$

The structure corresponds to the Gaussian distribution (Normal distribution), and the Gaussian distribution p.d.f is computed by the formula

$$f_V(X) = \frac{1}{\sqrt{(2\pi)^6 \left|\Sigma\right|}} \exp\left(-\left(X-\mu\right)^T \Sigma^{-1} \left(X-\mu\right)/2\right), X = (x_1, x_2, \cdots, x_n).$$

associated with standard deviation $\sigma = [0.15, 0.15, 2.77, 0.74, 1.82, 0.17]$ and

$$\Sigma = \begin{bmatrix} 3.19 & -1.89 & 0.81 & -1.54 & -0.10 & -1.73 \\ -1.88 & 3.90 & -3.49 & 3.62 & -0.52 & 4.23 \\ 0.81 & -3.49 & 3.46 & -3.31 & 0.61 & -3.90 \\ -1.54 & 3.62 & -3.31 & 3.37 & -0.52 & 3.95 \\ -0.10 & -0.52 & 0.61 & -0.52 & 0.13 & -0.62 \\ -1.73 & 4.23 & -3.90 & 3.95 & -0.62 & 4.63 \end{bmatrix}.$$

The determinant is 1.1246×10^{-8} and the inverse matrix (rounded values) are:

$$\Sigma^{-1} = \begin{bmatrix} 7.34 & -5.24 & 12.39 & -8.47 & 35.51 & 29.94 \\ -1.85 & -44.81 & -56.51 & -86.00 & 153.18 & 86.53 \\ 16.31 & -65.02 & 55.47 & -91.68 & 93.24 & 202.92 \\ -1.99 & -86.72 & -76.24 & -792.62 & 364.95 & 739.34 \\ 23.60 & 145.28 & 54.21 & 347.38 & -123.08 & -391.09 \\ 23.03 & 77.65 & 175.29 & 720.91 & -375.96 & -579.85 \end{bmatrix}.$$

Case 1 ($\Delta > 0$): Taking into account the mean: $\mu = (1.27, -1.68, 1.26, -1.39, -1.56, -1.84)$, we compute the value $\left(X-\mu\right)\Sigma^{-1}\left(X-\mu\right)^T \sim \chi_6^2 = \sum_{i=1}^{6}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2$, with standard deviation

$\sigma = [1.79, 1.97, 1.86, 1.84, 0.36, 2.15]$, and we obtain the marginal p.d.f as follows:

$$f_{VR}(X) = \frac{1}{\sqrt{(2\pi)^6 \times 1.1246 \times 10^{-8}}} \times \exp\left\{ -\frac{1}{2}\left[\left(\frac{x_1 - 1.27}{1.79}\right)^2 + \left(\frac{x_2 + 1.68}{1.97}\right)^2 + \left(\frac{x_3 - 1.26}{1.86}\right)^2 + \left(\frac{x_4 - 1.39}{1.84}\right)^2 + \left(\frac{x_5 + 1.56}{0.36}\right)^2 + \left(\frac{x_6 + 1.84}{2.15}\right)^2 \right] \right\} \times |J_6|,$$

The Jacobian value is calculated as follows:

$$J_6 = \frac{1.26}{0.90981(-0.76108)^k + 0.35019 \times 1.7935^k}.$$

Considering the 1-p.d.f for the event in equation (2.13), the existing probability is calculated as follows

$$f_1(Z_k) = \int_{2.0}^{2.1}\int_{1.9}^{2.1}\int_{1.6}^{6.4}\int_{1.8}^{3.2}\int_{-1.5}^{1.7}\int_{0.6.1}^{0.9} \frac{1.26}{0.90981(-0.76108)^k + 0.35019 \times 1.7935^k}$$

$$\times 38.015 \times \exp\left\{ -\frac{1}{2}\left[\left(\frac{x_1 - 1.27}{1.79}\right)^2 + \left(\frac{x_2 + 1.68}{1.97}\right)^2 + \left(\frac{x_3 - 1.26}{1.86}\right)^2 + \left(\frac{x_4 - 1.39}{1.84}\right)^2 + \left(\frac{x_5 + 1.56}{0.36}\right)^2 + \left(\frac{x_6 + 1.84}{2.15}\right)^2 \right] \right\} dx_6 dx_5 dx_4 dx_3 dx_2 dx_1$$

$$= \frac{0.40439}{0.90981(-0.76108)^k + 0.35019 \times 1.7935^k}.$$

The first probability density function of the solution of the random homogeneous linear second-order difference equation when $k = 0$ is $p_1 = 0.32094$.

Case 2 ($\Delta < 0$): Considering the p.d.f corresponding to the event (2.14), the solution given in Eq. (2.9), we compute $J_6 = \frac{A}{Ar^k \cos k\varphi + \frac{ABr^k \sin k\varphi}{\sqrt{4AC - B^2}}} \neq 0.$.

The joint p.d.f of random vector w is computed by

$$f_w(w) = f_v(v) \times \left| \frac{A}{Ar^k \cos k\varphi + \frac{ABr^k \sin k\varphi}{\sqrt{4AC - B^2}}} \right| \times$$

$$= f_v(v) \times \frac{-1.56}{0.75523 \times 1.0949^k \times \sin(0.98449k) - 1.56 \times 1.0949^k \cos(0.98449k)}.$$

and

$$f_{iw_1}(w) = \int_{w_{21}}^{w_{22}} \int_{min\left[w_{31}, \frac{w_4^2}{4w_3}\right]}^{min\{w_{32}, \frac{w_4^2}{4w_3}\}} \int_{w_{41}}^{w_{42}} \int_{\min\left[w_{51}; \frac{w_4^2}{4w_3}\right]}^{\min\left[w_{52}; \frac{w_4^2}{4w_3}\right]} \int_{w_{61}}^{w_{62}} f_v(v) \times |J_6| \, dw_6 dw_5 dw_4 dw_3 dw_2.$$

With the mean: $\mu_2 = (-1.65, 1.97, -1.56, 1.89, -1.87, 1.96)$, we compute the values $(X-\mu)\Sigma^{-1}(X-\mu)^T \sim \chi_6^2 = \sum_{i=1}^{6}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2$ with standard deviation $\sigma = [1.79, 1.97, 1.86, 1.84, 0.36, 2.15]$, and we obtain the marginal p.d.f as follows:

$$f_{VI}(X) = \frac{1}{\sqrt{(2\pi)^6 \times 1.1246 \times 10^{-8}}} \times \exp\left\{ -\frac{1}{2}\left[\left(\frac{x_1+1.65}{1.79}\right)^2 + \left(\frac{x_2-1.97}{1.97}\right)^2 + \left(\frac{x_3+1.56}{1.86}\right)^2 + \left(\frac{x_4-1.89}{1.84}\right)^2 + \left(\frac{x_5+1.87}{0.36}\right)^2 + \left(\frac{x_6-1.96}{2.15}\right)^2 \right] \right\} \times |J_6|,$$

and the marginal p.d.f of the transform p_2 as follows:

$$\begin{aligned} f_2(Z_k) =& \frac{-1.56}{0.75523 \times 1.0949^k \times \sin(0.98449k) - 1.56 \times 1.0949^k \cos(0.98449k)} \times \\ & \int_{2.0}^{2.1}\int_{1.9}^{2.1}\int_{1.6}^{6.4}\int_{1.8}^{3.2}\int_{-1.5}^{1.7}\int_{0.6}^{0.9} 1.3928 \times 10^{21} \times \exp\left\{ -\frac{1}{2}\left[\left(\frac{x_1+1.65}{1.79}\right)^2 \right.\right. \\ & + \left(\frac{x_2-1.97}{1.97}\right)^2 + \left(\frac{x_3+1.56}{1.86}\right)^2 + \left(\frac{x_4-1.89}{1.84}\right)^2 + \left(\frac{x_5+1.87}{0.36}\right)^2 \\ & \left.\left. + \left(\frac{x_6-1.96}{2.15}\right)^2 \right] \right\} dx_6 dx_5 dx_4 dx_3 dx_2 dx_1 \\ =& \frac{2.2973}{0.75523 \times 1.0949^k \times \sin(0.98449k) - 1.56 \times 1.0949^k \cos(0.98449k)}. \end{aligned}$$

The first probability density function of the solution of the random homogeneous linear second-order difference equation when $k = 0$ is $p_2 = 0.67906$.
The marginal p.d.f corresponding to the event (2.14) is given by

$$f_{W_1}(Z_k) = f_{W_1 R}(Z_k) + f_{W_1 I}(Z_k).$$

We can choose the specific one to calculate the first probability by setting $x_2 = x_3 = x_4 = x_5 = x_6 = 0, x = x_1$. Then the p.d.f function becomes

$$f_{VR}(x) = 38.015 \times \exp\left(-\frac{\left(\frac{x-1.27}{1.79}\right)^2 + 21.267}{2} \right) \times |J_6|,$$

$$f_{VI}(x) = 38.015 \times \exp\left(-\frac{\left(\frac{x+1.65}{1.79}\right)^2 + 30.572}{2} \right) \times |J_6|,$$

where $J_6 = \frac{A}{\frac{B+AZ_1}{Z_1-Z_2}Z_1^k - \frac{B+AZ_2}{Z_1-Z_2}Z_2^k} = \frac{1.26}{0.90981(-0.76108)^k + 0.35019 \times 1.7935^k}$,

and $J_6 = \frac{-1.56}{0.75523i \times 1.0949^k \times \sin(0.98449k) - 1.56 \times 1.0949^k \cos(0.98449k)}$.

The graph of marginal p.d.f $f_{VR}(V)$ is shown in Figure 1 when $x_2 = x_3 = x_4 = x_5 = x_6 = 0$, and Figure 3 when $x_3 = x_4 = x_5 = x_6 = 0$.

The graph of marginal p.d.f $f_{VI}(V)$ is shown in Figure 2 when $x_2 = x_3 = x_4 = x_5 = x_6 = 0$, and Figure 4 when $x_3 = x_4 = x_5 = x_6 = 0$. The 1-p.d.f corresponding to the real values and complex values could happen with the cases summarized in Table 2.

Table 2 The 1-p.d.f of the real values and complex values

Case	p_1	p_2
$p_1 < p_2$	0.3102	0.6898
$p_1 > p_2$	0.8631	0.1369
$p_1 = p_2$	0.5000	0.5000

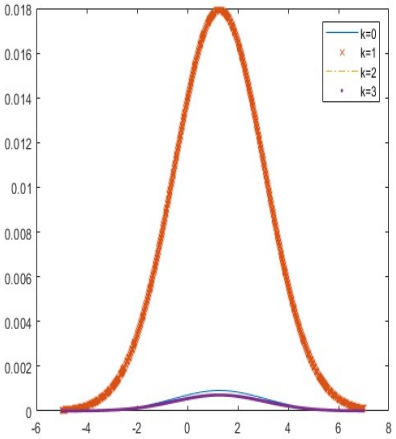


Fig. 1 Probability density function 3D graph for case I when n=6.

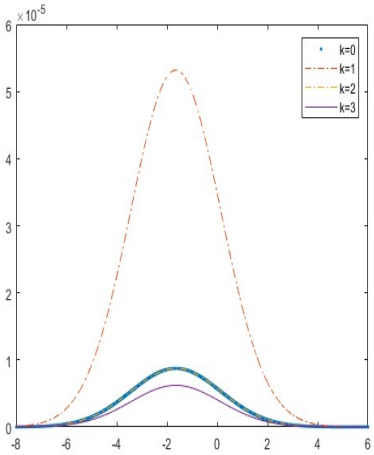


Fig. 2 Probability density function 2D graph for case II when n=6.

3 Discussion

The paper constructed a z-transform performance and demonstrated the application in calculating the marginal p.d.f using the R.V.T transform. Moreover, the z-transform is to establish the transform for the solutions, this purpose is consistent with discrete systems in different mappings. However, the z-transform is straightforward and effective in collecting the image by establishing the preliminary functions. By connecting R.V.T, the marginal p.d.f has been computed to show the distribution of input values

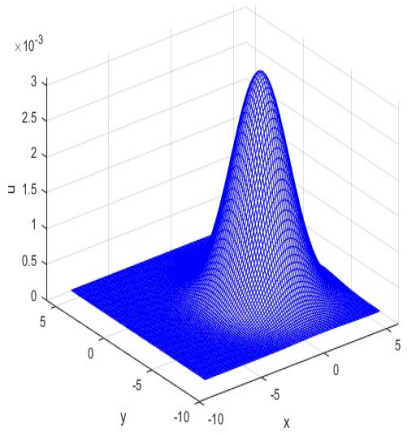


Fig. 3 3D probability density function graph for case I when n=6.

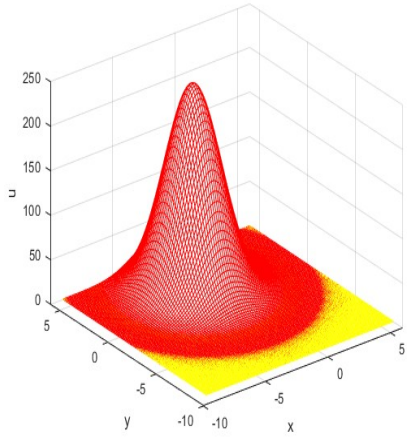


Fig. 4 3D probability density function graph for case II when n=6.

via output values. The R.V.T is effective and easy to establish the system based on one-to-one mapping. This paper calculates the case that the probability of an event happening in complex numbers more frequently than in real numbers. The complete cases are the probability of the real event equals the complex event, and the likelihood of the real event happening more frequently than the complex event [19–21]. The probability is completely attained by selecting the appropriate values of the solutions.

4 Conclusion

In this paper, a z-transform has been used and combined with the random variable transform to calculate the marginal probability density function. The z-transform effectively illustrates the solutions to a second-difference equation. In the application, R.V.T has been established to compute directly the first p.d.f by setting the values of non-homogeneous difference equations. The aim is to measure the distribution of the entries concerning the images via the R.V.T.

Supplementary information The authors confirm the supplementary data used in this paper are available with the agreement of the relevant authors.

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Declarations

- **Authors' contributions:** Nguyen Minh Tuan: Conceptualization, data curation, investigation, methodology, software, visualization, writing-original draft and writing-review and editing, validation, visualization, writing-original draft and

writing-review and editing. Natchaya Ratasukharom: methodology, resources, supervision, validation, visualization, and writing review and editing.

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