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Two effective new directions algorithms for solving a constrained optimization problem

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Abstract

This work introduces two new direction-type iterative algorithms for the numerical solution of large-scale monotone nonlinear equations. The proposed methods are designed to improve computational efficiency while maintaining stability in high-dimensional settings. Unlike classical approaches, the new strategies construct modified search directions that enhance convergence behavior and reduce computational cost per iteration. A detailed theoretical analysis is presented to justify the validity of the algorithms. In particular, it is shown that both methods are globally convergent under standard monotonicity assumptions. Moreover, their convergence properties are studied, and a convergence rate analysis is provided to better understand their performance. To assess the practical efficiency of the proposed schemes, several numerical experiments are carried out on benchmark problems. The obtained results indicate that the methods are robust and perform competitively when compared with existing techniques in the literature, especially for large-scale nonlinear systems.

Keywords: Constrained optimization, Picard-Mann, Iterative procedure, Numerical results.

MSC Classification: 65K05 , 90C52 , 90C56 , 94A08.

1 Introduction

A typical expression for a system of nonlinear equations is represented as follows:

$$F(X) = 0, \quad X \in \mathcal{D}, \tag{1}$$

where the set $\mathcal{D} \subseteq \mathbb{R}^n$ is a nonempty closed and convex set, and $F : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map. In this article, $\mathbb{R}^n$ represents the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$.

The Newton method and its type are highly valued for their straightforward implementation and quadratic convergence. Their general iterative technique can be described as follows:

$$X_{k+1} = X_k + \alpha_k d_k, \quad k = 0, 1, \ldots \tag{2}$$

where X_{k+1} denotes the current iterate, while X_k refers to the preceding iterate. The step size, $\alpha_k > 0$ is determined through a suitable line search, while the search direction d_k is defined using $d_k = -F(X_k) + \mathcal{P}_k$, where $\mathcal{P}_k = \left(I - \left(F(X_k)'\right)^{-1}\right) F(X_k)$, with I standing for the identity matrix and $F(X_k)'$ representing the Jacobian matrix at X_k. It is clear that when $\mathcal{P}_k = 0$, then d_k simplifies to the steepest descent direction. Newton's method is rarely used for large-scale problems because computing and storing the Jacobian and its inverse is costly, and some functions may lack derivatives.

Research focuses on improving the double-direction method, a matrix-free algorithm for large-scale optimization. Studies show that integrating it with the Picard-Mann iterative process significantly boosts its convergence and robustness [10, 11, 24]. Recent hybrid versions, which combine techniques like dynamic line search or Jacobian approximation with this iteration scheme, have demonstrated strong performance on complex problems [21].

The paper is structured as follows: Section 2 derives the proposed methods, Section 3 presents convergence analysis, Section 4 shows numerical experiments and, and Section 5 concludes the study.

1.1 Preliminaries

A Picard-Mann-based double-direction iterative algorithm is proposed for convex-constrained nonlinear equations, updating each iterate using two search directions as

$$X_{k+1} = X_k + \alpha_k \widehat{d_k^1} + \alpha_k^2 \widehat{d_k^2}, \tag{3}$$

where α_k is the corresponding line search, and $\widehat{d_k^1}$ and $\widehat{d_k^2}$ defined as follows:

$$\widehat{d_k^1} = -F(X_k), \tag{4}$$

and

$$\widehat{d_k^2} = -\delta_k^{-1} F(X_k), \tag{5}$$

where δ_k is the acceleration parameter.

Substituting (4) and (5) in (3), we obtain

$$X_{k+1} = X_k - \left(\alpha_k + \alpha_k^2 \delta_k\right) \delta_k^{-1} F(X_k), \tag{6}$$

where $\mu_k = \left(\alpha_k + \alpha_k^2 \delta_k\right)$. Next, we derive the acceleration parameter δ_k by using the Broyden update approximation.

Now we consider Broyden's matrix updating formula given as follows:

$$H_{k+1} = H_k + \frac{(y_k - B_k s_k)\, s_k^T}{s_k^T s_k}. \tag{7}$$

Approximating the Jacobian matrix H_k as $H_k \approx \delta_k I$, where δ_k is the acceleration parameter and I stand for identity matrix.

Therefore

$$\delta_{k+1} I = \delta_k I + \frac{(y_k - \delta_k s_k)\, s_k^T}{s_k^T s_k}. \tag{8}$$

We have two choices the proposed δ_{k+1} defined as:

$$\delta_{k+1} = \delta_k + \frac{s_k^T (y_k - \delta_k s_k)}{s_k^T s_k} = \frac{s_k^T y_k}{s_k^T s_k}, \tag{9}$$

and

$$\delta_{k+1} = \delta_k + \frac{y_k^T (y_k - \delta_k s_k)}{y_k^T s_k} = \frac{y_k^T y_k}{y_k^T s_k}. \tag{10}$$

2 The proposed mathods

The new proposed δ_{k+1} can further be simplified as follows:

$$\delta_{k+1} = (1 - \theta_k) \frac{s_k^T y_k}{s_k^T s_k} + \theta_k \frac{y_k^T y_k}{y_k^T s_k}, \tag{11}$$

where

$$\theta_k = \frac{\left(s_k^T y_k\right)^2}{\left(s_k^T s_k\right) y_k^T y_k} \text{ where } 0 < \theta_k < 1, \tag{12}$$

where $s_k = X_{k+1} - X_k$ and $y_k = F(X_{k+1}) - F(X_k)$.

The first new method is details are provided:

Algorithm 2.1 (AH1 algorithm)

Step 1: Initial iterate $X_0 \epsilon D$, initial search direction $\hat{d}_0^1 = -F(X_0)$; parameters ρ, σ, $0 < w < 2$, $\varepsilon > 0$, $\delta_0 = 0$, and $\xi > 0$.

Step 2: Solution $\hat{X}$ such that if $\left\|F(\hat{X})\right\| \leq \varepsilon$ then Stop and return X_k else go to Step 3

Step 3: Compute $F(X_k)$.

Step 4: Compute $\alpha_k = \xi \rho^{m_i}$, where m_i is the smallest nonnegative integer m such that:

$$-F(X_k + \left(\alpha_k + \alpha_k^2 \delta_k\right) \hat{d}_0^1)^T \hat{d}_k^1 \geq \sigma \left(\alpha_k + \alpha_k^2 \delta_k\right) \left\|F\left(X_k + \omega_k \hat{d}_k^1\right)\right\| \left\|\hat{d}_k^1\right\|^2. \tag{13}$$

Step 5: Set $z_k = X_k + \left(\alpha_k + \alpha_k^2 \delta_k\right) \hat{d}_k^1$, $z_k \epsilon D$ and $\|F(z_k)\| \leq \varepsilon$ then return z_k.

Step 6: Compute

$$\lambda_k = \frac{F(z_k)^T (X_k - z_k)}{\|F(z_k)\|^2}, \tag{14}$$

and

$$X_{k+1} = P_D \left[X_k - w_k F(z_k) \right]. \tag{15}$$

Step 7: Compute δ_{k+1} according to (11).
Step 8: Compute $\hat{d}_{k+1}^1$ according to (4).
Step 9: Set $k \leftarrow k+1$ and go to **Step 2**.

The study proposes a second algorithm that modifies the iterative scheme by incorporating a Picard-Mann-based correction parameter into the search direction, enhancing numerical performance. The resulting general iteration updates each step as:

$$X_{k+1} = X_k - (\lambda_k + 1) \left(\alpha_k + \alpha_k^2 \delta_k \right) \delta_k^{-1} F(X_k),$$

where $(\lambda_k + 1) \in (1, 2)$, and α_k is the corresponding step length.

The second search direction in the proposed algorithm is defined as:

$$d_k = \begin{cases} -F(X_k), & \text{for } k = 0, \\ -(\lambda_k + 1)\delta_k^{-1} F(X_k), & \text{for } k \geq 1. \end{cases} \tag{16}$$

Therefore, the algorithm steps for the proposed new second algorithm are defined as:

Algorithm 2.2 (AH2 algorithm)

Step 1: Initial iterate $X_0 \epsilon D$, initial search direction $\hat{d}_0^2 = -F(X_0)$; parameters ρ, σ, $0 < w < 2$, $\varepsilon > 0$, $\delta_0 = 0$, and $\xi > 0$.

Step 2: Solution $\hat{X}$ such that if $\left\| F(\hat{X}) \right\| \leq \varepsilon$ then Stop and return X_k else go to Step 3

Step 3: Compute $F(X_k)$.

Step 4: Compute $\alpha_k = \xi \rho^{m_i}$, where m_i is the smallest nonnegative integer m such that:

$$-F(X_k + \left(\alpha_k + \alpha_k^2 \delta_k \right) \hat{d}_0^2)^T \hat{d}_k^2 \geq \sigma \left(\alpha_k + \alpha_k^2 \delta_k \right) \left\| F\left(X_k + \omega_k \hat{d}_k^2 \right) \right\| \left\| \hat{d}_k^2 \right\|^2. \tag{17}$$

Step 5: Set $z_k = X_k + \left(\alpha_k + \alpha_k^2 \delta_k \right) \hat{d}_k^2$, $z_k \epsilon D$ and $\|F(z_k)\| \leq \varepsilon$ then return z_k.

Step 6: Compute

$$\lambda_k = \frac{F(z_k)^T (X_k - z_k)}{\|F(z_k)\|^2}, \tag{18}$$

and

$$X_{k+1} = P_D \left[X_k - w_k F(z_k) \right]. \tag{19}$$

Step 7: Compute δ_{k+1} according to (11).
Step 8: Compute $\hat{d}_{k+1}^2$ according to (5).
Step 9: Set $k \leftarrow k+1$ and go to **Step 2**.

Definition 1 Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set. For any vector $X \in \mathbb{R}^n$, the orthogonal projection of $X \epsilon \mathcal{D}$, denoted by $P_{\mathcal{D}}[X]$, is defined by

$$P_{\mathcal{D}}[X] := \arg\min\{\|X - Y\| : Y \in \mathcal{D}\}.$$

Definition 2 Suppose for any $X, Y \in \mathbb{R}^n$, the mapping $P_{\mathcal{D}} : \mathbb{R}^n \to \mathcal{D}$ is known as the projection operator, which has a non-expansive property. Then, it holds that

$$\|P_{\mathcal{D}}[X] - P_{\mathcal{D}}[Y]\| \leq \|X - Y\|, \forall X, Y \in \mathbb{R}^n.$$

Consequently, we have

$$\|P_{\mathcal{D}}[X] - Y\| \leq \|X - Y\|, \quad \forall Y \in \mathcal{D}.$$

Definition 3 Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set. Then for any $X \in \mathbb{R}^n$, its orthogonal projection onto $\mathcal{D}$, denoted by $P_{\mathcal{D}}[X]$, is defined by

$$P_{\mathcal{D}}[X] := \arg\min\{\|X - Y\| : Y \in \mathcal{D}\}. \tag{20}$$

Definition 4 Suppose for any $X, Y \in \mathbb{R}^n$, the mapping $P_{\mathcal{D}} : \mathbb{R}^n \to \mathcal{D}$ is known as the projection operator, which has a non-expansive property. Then, it holds that

$$\|P_{\mathcal{D}}[X] - P_{\mathcal{D}}[Y]\| \leq \|X - Y\|, \forall X, Y \in \mathbb{R}^n.$$

Consequently, we have

$$\|P_{\mathcal{D}}[X] - Y\| \leq \|X - Y\|, \quad \forall Y \in \mathcal{D}. \tag{21}$$

3 Convergence analysis

This section establishes the global convergence of the proposed Algorithm **2.1**. The analysis is based on the following assumptions and a supporting lemma.

Assumption 1. The operator F is Lipschitz continuous, meaning there exists a constant $\mathcal{L} > 0$ such that

$$\|F(X) - F(Y)\| \leq \mathcal{L}\|X - Y\|, \quad \forall X, Y \in \mathbb{R}^n.$$

Assumption 2. The solution set of (1) is nonempty and is denoted by $\mathcal{D}^{\nu}$.

Assumption 3. The mapping F is monotone; that is,

$$(F(X) - F(Y))^T (X - Y) \geq 0, \quad \forall X, Y \in \mathbb{R}^n.$$

Now, using Assumption **3** and the definitions of s_k and y_k, we obtain

$$s_k^T y_k = s_k^T \left(F\left(X_{k+1}\right) - F\left(X_k\right)\right) + \gamma s_k^T s_k \geq \gamma \left\|s_k\right\|^2 > 0. \tag{22}$$

Furthermore, from Assumption **1**, and the definition of y_k, we have

$$\|y_k\| = \left\|F\left(X_{k+1}\right) - F\left(X_k\right) + \gamma s_k\right\| \leq \left\|F\left(X_{k+1}\right) - F\left(X_k\right)\right\| + \gamma \left\|s_k\right\|,$$

and

$$\|y_k\| \leq \mathcal{L} \left\|s_k\right\| + \gamma \left\|s_k\right\| \leq (\mathcal{L} + \gamma) \left\|s_k\right\|. \tag{23}$$

Lemma 1 Assume that Assumption **1**, **2**, and **3** hold. If

$$\delta_{k+1} = (1-\theta_k)\,\frac{s_k^T y_k}{s_k^T s_k} + \theta_k \frac{y_k^T y_k}{y_k^T s_k},$$

then the sequence $\{\delta_{k+1}\}$ satisfies

$$\frac{1}{M'} \leq \frac{1}{\delta_{k+1}} \leq \frac{1}{\gamma},$$

and consequently,

$$\delta_{k+1}^{-1} \in \left[\frac{1}{M'}, \frac{1}{\gamma}\right].$$

Proof By the definition of δ_{k+1}, three possible cases:

Case 1: If $\theta_k = 0$ we get $\delta_{k+1} = \frac{s_k^T y_k}{s_k^T s_k}$, we have

$$\delta_{k+1} = \frac{s_k^T y_k}{s_k^T s_k} \geq \frac{\gamma \|s_k\|^2}{\|s_k\|^2} = \gamma$$

Since $\delta_{k+1} > 0$, this clearly shows that it's positive.

Case 2: If $\theta_k = 1$ we get $\delta_{k+1} = \frac{y_k^T y_k}{y_k^T s_k}$, from (22) and (23), we have

$$\delta_{k+1} = \frac{y_k^T y_k}{y_k^T s_k} \leq \frac{\|y_k\|^2}{\gamma \|s_k\|^2} \leq \frac{(\mathcal{L}+\gamma)^2 \|s_k\|^2}{\gamma \|s_k\|^2} = \frac{(\mathcal{L}+\gamma)^2}{\gamma} = M.$$

Case 3: If $0 < \theta_k < 1$ we get $\delta_{k+1} = (1-\theta_k)\,\frac{s_k^T y_k}{s_k^T s_k} + \theta_k \frac{y_k^T y_k}{y_k^T s_k}$, from (22) and (23), we have

$$\delta_{k+1} \leq \frac{s_k^T y_k}{s_k^T s_k} + \frac{y_k^T y_k}{y_k^T s_k} = \frac{2\|y_k\|^2 \|s_k\|^2}{\|s_k\|^2 \left(y_k^T s_k\right)},$$

and

$$\delta_{k+1} \leq \frac{2\|y_k\|^2 \|s_k\|^2}{\gamma \|s_k\|^4} = \frac{2\|y_k\|^2}{\gamma \|s_k\|^2},$$

and

$$\delta_{k+1} \leq \frac{2(\mathcal{L}+\gamma)^2 \|s_k\|^2}{\gamma \|s_k\|^2} = \frac{2(\mathcal{L}+\gamma)^2}{\gamma} = M'.$$

Thus, in both cases, the bounds for δ_{k+1}^{-1} are given by:

$$\gamma \leq \delta_{k+1} \leq M \leq M'.$$

Consequently, this implies

$$\frac{1}{M'} \leq \frac{1}{\delta_{k+1}} \leq \frac{1}{\gamma}.$$

Therefore, $\delta_{k+1}^{-1} \in \left[\frac{1}{M'}, \frac{1}{\gamma}\right]$. □

Lemma 2 Assume that Assumption **1**, **2**, and **3** are satisfied, and let the sequence $\{X_k\}$ is generated by Algorithm **2.2**, then the search direction d_k fulfills the following descent condition for all

$$F(X_k)^T d_k \leq -C \|F(X_k)\|^2, \quad \forall k > 0. \tag{24}$$

Proof For $k = 0$, from (16), we have $F(X_0)^T d_0 = -\|F(X_0)\|^2$.

For $k \geq 1$, applying (8) and (16)yields

$$\begin{aligned} F(X_k)^T d_k &= -\delta_{k+1}^{-1} \|F(X_k)\|^2, \\ &\leq -\frac{1}{M'} \|F(X_k)\|^2, \\ &= -C \|F(X_k)\|^2, \end{aligned}$$

where $C = \frac{1}{M'}$.

□

Next, we show the boundedness of the search direction.

Proof For $k = 0$. From (16) we get $\|d_0\| = \|F(v_0)\|$.

For $k > 0$, using (22), and (16), we obtain

$$\|d_k\| = \left\| -(1+\lambda_k)\delta_{k+1}^{-1} F(X_k) \right\| \leq (1+\lambda_k) \left| \delta_{k+1}^{-1} \right| \|F(X_k)\| \leq \frac{(1+\lambda_k)\|F(X_k)\|}{M'} = \Omega.$$

Thus, we have

$$\|d_k\| \leq \Omega, \quad \forall k \geq 0. \tag{25}$$

□

Lemma 3 Assume that Assumption **1**, **2**, and **3** are satisfied, and let the sequences $\{X_k\}$ and $\{z_k\}$ be produced by the Algorithm **2.2**. Then there exists a step size ω_k satisfies the following inequality:

$$\omega_k \geq \omega := \min \left\{ \xi, \frac{\rho C}{(\mathcal{L} + \sigma \|F(z_k)\|) \frac{(1+\lambda_k)^2}{M'^2}} \right\}. \tag{26}$$

Proof If $\omega_k \neq \xi$, then according to the definition $\omega_k = \left(\alpha_k + \alpha_k^2 \delta_k\right), \rho_k^{-1}\omega$ does not satisfies (17); in other words,

$$-F\left(X_k + \rho_k^{-1}\omega d_k\right)^T d_k < \sigma \rho_k^{-1} \omega \|F(z_k)\| \|d_k\|^2. \tag{27}$$

Now, using Assumption **1** and combining (24) with (27), we obtain

$$\begin{aligned} C\|F(X_k)\|^2 &\leq -F(X_k)^T d_k, \\ &= \left(F\left(X_k + \rho_k^{-1}\omega d_k\right) - F(X_k)\right)^T d_k - F\left(X_k + \rho_k^{-1}\omega d_k\right)^T d_k, \\ &\leq \mathcal{L}\rho_k^{-1}\omega \|d_k\|^2 + \sigma\rho_k^{-1}\omega \|F(z_k)\| \|d_k\|^2, \\ &= \omega_k (\mathcal{L} + \sigma \|F(z_k)\|) \rho^{-1} \|d_k\|^2. \end{aligned} \tag{28}$$

Consequently,

$$\omega_k > \frac{\rho C \|F(X_k)\|^2}{(\mathcal{L} + \sigma \|F(z_k)\|) \|d_k\|^2} \geq \frac{\rho C}{(\mathcal{L} + \sigma \|F(z_k)\|) \frac{(1+\lambda_k)^2}{M'^2}}.$$

□

Lemma 4 Assume that Assumptions **1**, **2**, and **3** are satisfied. Let the sequences $\{X_k\}$ and $\{z_k\}$ are generated by Algorithm **2.2**, and then $\{X_k\}$ and $\{z_k\}$ are bounded. Furthermore, the following hold:

$$\lim_{k\to\infty} \|X_k - z_k\| = 0, \tag{29}$$

and

$$\lim_{k\to\infty} \|X_{k+1} - X_k\| = 0. \tag{30}$$

Proof The initial part of the proof addresses the boundedness of the sequences $\{X_k\}$ and $\{z_k\}$. Let $\hat{X} \in \mathcal{D}^v$ be an arbitrary solution to equation (1). . Given the monotonicity of F, it follows that

$$F(z_k)^T \left(X_k - \hat{X}\right) \geq F(z_k)^T (X_k - z_k).$$

Applying the line search conditionch (17), and definition of z_k, we have

$$F(z_k)^T \left(X_k - \hat{X}\right) \geq F(z_k)^T (X_k - z_k) \geq \sigma\omega_k^2 \|F(z_k)\| \|d_k\|^2 > 0. \tag{31}$$

Then, using the nonexpansive projection property (21), (19), and (31), it is easy to see that

$$\begin{aligned}
\left\|X_{k+1} - \hat{X}\right\|^2 &= \left\|P_{\mathcal{D}}\left[X_k - w\lambda_k F(z_k)\right] - \hat{X}\right\|^2, \\
&\leq \left\|X_k - w\lambda_k F(z_k) - \hat{X}\right\|^2, \\
&= \left\|X_k - \hat{X}\right\|^2 - 2w\lambda_k \left\langle F(z_k), X_k - \hat{X}\right\rangle + w^2\lambda_k^2 \|F(z_k)\|^2, \\
&\leq \left\|X_k - \hat{X}\right\|^2 - 2w\lambda_k \langle F(z_k), X_k - z_k\rangle + w^2\lambda_k^2 \|F(z_k)\|^2,
\end{aligned}$$

and

$$\left\|X_{k+1} - \hat{X}\right\|^2 \leq \left\|X_k - \hat{X}\right\|^2 - w(2-w)\frac{\left(\langle F(z_k), X_k - z_k\rangle\right)^2}{\|F(z_k)\|^2}, \tag{32}$$

and

$$\left\|X_{k+1} - \hat{X}\right\|^2 \leq \left\|X_k - \hat{X}\right\|^2 - w(2-w)\sigma^2 \|X_k - z_k\|^4, \tag{33}$$

where $0 < w < 2$. From (32), it follows that

$$\left\|X_{k+1} - \hat{X}\right\|^2 \leq \left\|X_k - \hat{X}\right\|^2.$$

This shows that the sequence $\left\{\left\|X_k - \hat{X}\right\|\right\}$ is nonincreasing and convergent; consequently, $\{X_k\}$ is bounded. Moreover, using (32) together with Assumptions **1**, **2**, and **3**, we obtain

$$\|F(X_k)\| = \left\|F(X_k) - F(\hat{X})\right\| \leq \mathcal{L}\left\|X_k - \hat{X}\right\| \leq \mathcal{L}\left\|X_0 - \hat{X}\right\|. \tag{34}$$

Setting $\mathcal{L}\left\|X_0 - \hat{X}\right\| = S$, then we have

$$\|F(X_k)\| \leq S, \forall k \geq 0.$$

Given that the sequences $\{X_k\}$ and $\{d_k\}$ are bounded by the definition of $\{z_k\}$ in **Step 3** of Algorithm **2.2** implies that $\{z_k\}$ is also bounded. Following the same reasoning used in (34), we can assert the existence of a positive constant $\mathcal{P} > 0$ such that

$$\|F(z_k)\| \leq \mathcal{P}, \quad \forall k \geq 0.$$

From (31), we obtain

$$\left(F\left(z_k\right)^T\left(X_k - z_k\right)\right)^2 \leq \frac{\|F\left(z_k\right)\|^2}{w(2-w)}\left(\left\|X_k - \hat{X}\right\|^2 - \left\|X_{k+1} - \hat{X}\right\|^2\right). \tag{35}$$

Furthermore, using (31) and (35), it is not di cult to see that

$$\omega_k^4 \|F\left(z_k\right)\|^2 \|d_k\|^4 \leq \frac{\|F\left(z_k\right)\|^2}{\sigma^2 w(2-w)}\left(\left\|X_k - \hat{X}\right\|^2 - \left\|X_{k+1} - \hat{X}\right\|^2\right). \tag{36}$$

Because the sequence $\left\{\left\|X_k - \hat{X}\right\|^2\right\}$ converges, and given that $\sigma > 0$, $0 < w < 2$, and $\{F\left(z_k\right)\}$ is bounded, taking the limit on both sides of (36) yields

$$\lim_{k\to\infty} \omega_k^4 \|d_k\|^4 \leq 0.$$

Hence, it follows that

$$\lim_{k\to\infty} \omega_k \|d_k\| = 0. \tag{37}$$

Consequently, together with the definition of z_k □

Proof , this establishes (29). Moreover, from the definitions of λ_k and $P_D\left[.\right]$ in (21), we obtain

$$\begin{aligned}
\|X_{k+1} - X_k\| &= \|P_D\left[X_k - w\lambda_k F(z_k)\right] - X_k\|, \\
&\leq \|X_k - w\lambda_k F(z_k) - X_k\|, \\
&= \|w\lambda_k F(z_k)\|, \\
&= w\|X_k - z_k\|.
\end{aligned}$$

This gives

$$\lim_{k\to\infty} \|X_{k+1} - X_k\| = 0,$$

which confirms that (30) holds as well. Thus, the proof is concluded. □

Theorem 1 *Assume that Assumptions **1**, **2**, and **3** are satisfied, and let $\{X_k\}$ be the sequence of iterates produced by Algorithm **2.2**. Then $\{X_k\}$converges to a solution of (1).*

Proof First from (26) and (37) that $0 \leq \omega\|d_k\| \leq \omega_k\|d_k\| \to 0$. This directly implies that $\lim_{k\to\infty}\|d_k\| = 0$. Combining this result with (31) gives

$$0 \leq p\|F\left(X_k\right)\| \leq \|d_k\| \to 0,$$

which leads to $\lim_{k\to\infty}\|F\left(X_k\right)\| = 0$. Next, using (37) together with the boundedness of $\{X_k\}$, we conclude that the sequence has at least one cluster point.

Let $\{\widetilde{X}\}$ be an accumulation point of $\{X_k\} \subset \mathcal{D}$ and let $\mathcal{W} \subset \{0, 1, 2, \ldots\}$ be an infinite index set such that

$$\lim_{k\to\infty, k\in\mathcal{W}} X_k = \widetilde{X} \in \mathcal{D}.$$

Since $\mathcal{D}$ is closed under Assumptions **1**, **2** and **3**, it follows that

$$0 = \lim_{k\to\infty}\|F\left(X_k\right)\| = \lim_{k\to\infty, k\in\mathcal{W}}\|F\left(X_k\right)\| = \|F(\widetilde{X})\|,$$

which shows $\widetilde{X}$ satisfies (1). Moreover, Lemma **8** guarantees that the sequence $\left\{\left\|X_k - \hat{X}\right\|\right\}$ converges. Therefore, by setting $\widetilde{X} := \widehat{X}$, we have

$$\lim_{k\to\infty}\left\|X_k - \widetilde{X}\right\| = \lim_{k\to\infty, k\in\mathcal{K}}\left\|X_k - \widetilde{X}\right\| = 0.$$

Consequently, the sequence $\{X_k\}$ converges to $\widetilde{X} \in \mathcal{D}^X$. □

3.1 Convergence rate

Assumption **4**. For every point $\bar{X} \in \mathcal{D}'$, there exist a positive constants G and ε such that the inequality

$$G \operatorname{dist}(X, \mathcal{D}') \leq \|F(X)\|^2, \quad \forall X \in S(\bar{X}, \varepsilon),$$

where $\operatorname{dist}(X, \mathcal{D}')$ is the distance from X to the solution set $\mathcal{D}'$, and $S(\bar{X}, \varepsilon) = \left\{X \in \mathbb{R}^n : \|X - \bar{X}\| \leq \varepsilon\right\}$.

Theorem 2 *Let Assumptions **1, 2** and **3** hold, and assume the sequences $\{X_k\}$ and $\{z_k\}$ are generated by Algorithm **2.2**. It follows that $\left\{\operatorname{dist}\left(X_k, \mathcal{D}'\right)\right\}$ converges to zero at a Q-linear rate.*

Proof Let $K_k := \operatorname{argmin}\left\{\|X_k - K\| : K \in \mathcal{D}'\right\}$. This means that K_k is the nearest solution to X_k,

$$\|X_k - K_k\| = \operatorname{dist}\left(X_k, \mathcal{D}'\right), \quad \forall k \geq 0. \tag{38}$$

Applying the **Cauchy–Schwarz** inequality together with relation (24) yields

$$\|d_k\| \geq C \|F(X_k)\|. \tag{39}$$

We have for some $C > 0$. From (38), Assumption **4**, and (39). Finally, we assume that the given condition remains true for all iterations up to some $k \geq 2$

$$\begin{aligned}
\operatorname{dist}\left(x_k, \mathcal{D}'\right) &\leq \|X_{k+1} - K_k\|^2 \\
&\leq \|X_k - K_k\|^2 - w(2-w)\sigma^2 \|X_k - z_k\|^4 \\
&\leq \operatorname{dist}\left(x_k, \mathcal{D}'\right)^2 - w(2-w)\sigma^2 \mu_k^4 C^4 \|F(X_k)\|^4 \\
&\leq \left(1 - w(2-w)\sigma^2 C^4 G_k^2 \omega^4\right) \operatorname{dist}\left(x_k, \mathcal{D}'\right)^2.
\end{aligned}$$

We now proceed with the proof by examining two separate situations. Letting $G = \frac{1}{\sqrt{w(2-w)\sigma^2 C^2}}$, we have $w(2-w)\sigma^2 C^4 G_k^2 \omega^4 \in (0, 1)$.

Consequently, this shows that the sequence $\left\{\operatorname{dist}\left(x_k, \mathcal{D}'\right)\right\}$ tends to zero at a Q-linear rate. □

4 Numerical Experiments

The performance and stability of the proposed hybrid algorithms are assessed in this section using a series of established benchmark problems. These methods are referred to as "**AH1**" and "**AH2**," corresponding to Algorithm **2.1** and **2.2**. The search directions and associated parameters for the comparative algorithm are described below.

HDDM proposed in [10]

$$d_k = \begin{cases} -F(X_k), & \text{for } k = 0, \\ -\eta_k \delta_k^{-1} F(X_k), & \text{for } k \geq 1. \end{cases}$$

where $s_k = X_{k+1} - X_k$ and $y_k = F(X_{k+1}) - F(X_k)$, $\delta_{k+1} = \frac{\|s_k\|^2 \|y_k\|^2}{\left(\alpha_k + \alpha_k^2 \delta_k\right)^2 \left(y_k^T \delta_k^i\right)^2}$.

Algorithm **2.1** proposed in [3]

$$d_k = \begin{cases} -F(X_k), & \text{for } k = 0, \\ -\eta_k \delta_k^{-1} F(X_k), & \text{for } k \geq 1, \end{cases}$$

where $s_k = X_{k+1} - X_k$ and $y_k = F(X_{k+1}) - F(X_k) + \gamma s_k$, $\delta_{k+1} = \max\left\{\frac{s_k^T y_k}{s_k^T s_k}, \frac{y_k^T y_k}{y_k^T s_k}\right\}$.

For our numerical simulation, we use the benchmark problem described [6, 12, 19, 26], with the parameters: $\rho = 0.33$, and $\sigma = 10^{-5}$ and initial points: $X_0 = (1, ..., 1)^T$, $X_1 = (0.2, ..., 0.2)^T$, $X_2 = (2, ..., 2)^T$, and $X_3 = (20, ..., 20)^T$, where the dimensions used are $n = 1000$, 2000, 3000, and 4000.

The numerical information presented in Tables 1 is depicted graphically in Figures. **1**, **2**, and **3** we analyze the performance of the **AH1** and **AH2** methods with other methods present the performance profiles based on CPU time, number of iterations.

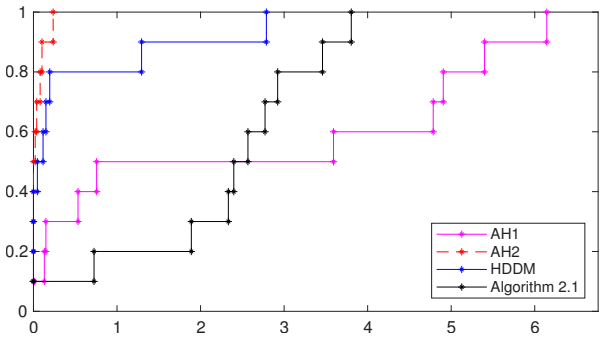


Fig. 1 Results using the number of CpuTime.

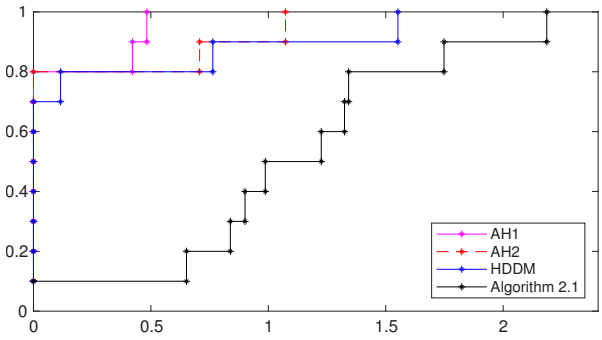


Fig. 2 Results using the number of iterations.

Table 1 Numerical results for AH 1, AH2, HDDM, and Algorithm 2.1 on Problem 1, 2, 3, and 4

	AH 1			AH 2			HDDM			Algorithm 2.1		
	NI	NF	CPU	NI	NF	CPU	NI	NF	CPU	NI	NF	CPU
Problem 1												
500	1	5	0.01	4	22	0.01	12	33	0.02	1	13	0.02
2000	3	12	0.02	6	24	0.01	12	25	0.02	5	20	0.04
3000	4	20	0.06	9	34	0.36	14	46	0.26	7	24	0.16
4000	7	23	0.12	16	44	0.43	22	55	0.33	20	73	0.22
Problem 2												
500	1	10	0.09	1	11	0.05	1	10	0.06	4	12	0.02
2000	15	34	0.04	12	22	0.04	21	45	0.03	11	33	0.02
3000	50	102	0.35	44	96	0.26	112	455	0.20	3	7	0.12
4000	300	230	0.49	200	900	0.42	400	86	0.21	21	56	0.11
Problem 3												
500	3	7	0.17	4	9	0.17	2	7	0.16	1	2	0.15
2000	2	8	0.14	2	5	0.12	4	33	0.11	12	50	0.22
3000	12	45	0.50	12	47	0.36	44	78	0.54	82	107	0.76
4000	20	55	0.66	21	66	0.56	45	98	0.53	56	101	0.88
Problem 4												
500	12	26	0.16	14	28	0.23	5		0.16	10	21	0.06
2000	22	56	0.22	15	30	0.22	34		0.27	23	98	0.06
3000	34	83	0.24	20	43	0.33	21		0.21	55	105	0.06
4000	54	201	0.34	55	202	0.33	67	206	0.35	87	206	0.06

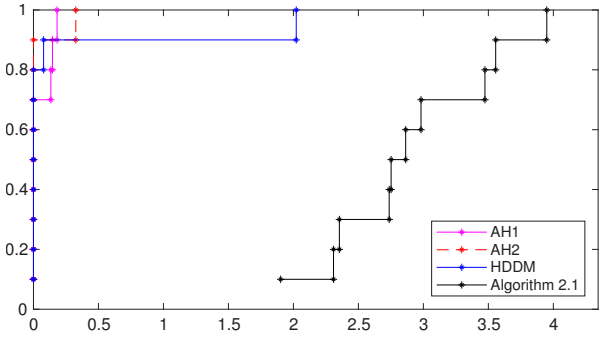


Fig. 3 Results using the number of functions.

Conclusions

This research formulates two novel computational methodologies for the resolution of constrained monotone nonlinear systems. The fundamental strategy employs a diagonal matrix approximation of the Jacobian operator, refined through an acceleration parameter.

A subsequent, enhanced algorithm is constructed via the incorporation of a Picard-Mann iterative schema, yielding superior convergence properties relative to established techniques. Both iterative frameworks satisfy the requisite descent criterion, thereby furnishing a proof of global convergence. Their convergence rate is formally established as Q-linear.

The efficacy and computational robustness of the proposed solvers are empirically substantiated through extensive numerical experiments on large-scale test problems.

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- **Consent for publication:** The authors give their consent for publication.
- **Data availability:** The data used to support the findings of this study are available from the corresponding author upon reasonable request.
- **Materials availability:** Not applicable.
- **Code availability:** The code used during the current study is available from the corresponding author upon reasonable request.
- **Author contribution:** All authors contributed equally to the conception, implementation, and writing of this manuscript.

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