Supplementary Information for: A Geometric Mechanism for Critical Transitions Through Dimensional Saturation

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Supplementary Table 1 — Evidence Mapping: Core Claims to Mathematical Proofs

Core Claim	Mathematical Formalism	Validation
Geometric Isomorphism	Section S1: *Proof of CUD-Kakeya Equivalence* Derivation of the constraint from the Lyapunov function. Linking W_{percept} to geometric measure bounds.	Fig. S1, S2
Biological Scaling	Section S2: *Perceptual Field Optimization* Solution of the weighted integral equation under maximum entropy constraints .	Fig. S3
Stability Threshold	Section S3: *Weighted Lyapunov Stability* Proof of the critical gain k_d. Derivation of the edge of collapse.	Fig. S4
Active Rescue	Section S4: *Control Barrier Function* Proof of $V_{\text{weighted}}(t)$ convergence under constraint violation.	Fig. S5
Universality	Section S5: *Renormalization Group Flow* Derivation of the universal exponent from the geometric constraint.	Fig. S6

1 Derivation of Perceptual Weights from Maximum Entropy Principle

Following the principle of maximum entropy subject to finite energy constraints, the probability distribution of the perceptual variable $\boldsymbol{r}$ (spatial displacement vector in $\mathbb{R}^2$) is derived. This approach ensures that our model makes the least assumptions beyond the known constraints, providing a robust statistical foundation for the CUD framework.

The entropy is defined as:

$$S = -\int p(\boldsymbol{r}) \ln p(\boldsymbol{r}) d\boldsymbol{r} \tag{1}$$

Subject to the constraint $\int p(\boldsymbol{r}) \|\boldsymbol{r}\|^2 \, d\boldsymbol{r} = \sigma_d^2$ (where σ_d^2 is the perceptual variance parameter) and normalization $\int p(\boldsymbol{r}) d\boldsymbol{r} = 1$, we introduce the Lagrangian function:

$$\mathcal{L} = -\int p(\boldsymbol{r}) \ln p(\boldsymbol{r}) d\boldsymbol{r} - \lambda_1 \left(\int p(\boldsymbol{r}) \|\boldsymbol{r}\|^2 \, d\boldsymbol{r} - \sigma_d^2 \right) - \lambda_2 \left(\int p(\boldsymbol{r}) d\boldsymbol{r} - 1 \right) \tag{2}$$

Applying variational calculus ($\delta\mathcal{L}/\delta p = 0$) yields the Gaussian distribution $p(\boldsymbol{r}) \propto \exp(-\lambda_1 \|\boldsymbol{r}\|^2)$. This naturally leads to the decomposition of the perceptual weight W_{percept} based on spatial distance $d = \|\boldsymbol{r}\|$ and field-of-view angle θ, with θ_{FOV} denoting the maximum field-of-view angle:

$$W_{\text{distance}} = \exp\left(-\frac{d^2}{2\sigma_d^2} \right) \tag{3}$$

$$W_{\text{angle}} = \left(1 - \frac{|\theta|}{\theta_{\text{FOV}}} \right)_+ \tag{4}$$

where $(\cdot)_+$ denotes the ReLU function, ensuring non-negative weights.

Thus, the combined perceptual weight used in the control law is given by:

$$W_{\text{percept}}(\boldsymbol{P}_i) = W_{\text{distance}} \cdot W_{\text{angle}} = \exp\left(-\frac{d^2}{2\sigma_d^2} \right) \left(1 - \frac{|\theta|}{\theta_{\text{FOV}}} \right)_+ \tag{5}$$

where $\boldsymbol{P}_i$ denotes the position of agent i.

2 Weighted Lyapunov Stability Analysis

We analyze the system stability using a weighted Lyapunov function $V(\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{N} \|\boldsymbol{e}_i\|^2$, where $\boldsymbol{e}_i$ is the error vector for agent i (defined as $\boldsymbol{e}_i = \boldsymbol{x}_i - \boldsymbol{x}_d$ with $\boldsymbol{x}_d$ being the desired state). The derivative of the weighted Lyapunov function along the system trajectories is:

$$\dot{V}_{\text{weighted}} = \sum_{i=1}^{N} W_{\text{percept}}(\boldsymbol{P}_i) \dot{\boldsymbol{e}}_i \cdot \boldsymbol{e}_i \tag{6}$$

Considering the system dynamics $\dot{\boldsymbol{e}}_i = -k_d \boldsymbol{e}_i + \boldsymbol{\eta}_i$ with control input $\boldsymbol{u}_i = -k_d \boldsymbol{e}_i$ and bounded noise $\boldsymbol{\eta}_i$ (where $\|\boldsymbol{\eta}_i\| \le \epsilon$ for all i), the derivative becomes:

$$\dot{V}_{\text{weighted}} \le -k_d \sum_{i=1}^{N} W_{\text{percept}}(\boldsymbol{P}_i) \|\boldsymbol{e}_i\|^2 + \epsilon \sum_{i=1}^{N} W_{\text{percept}}(\boldsymbol{P}_i) \|\boldsymbol{e}_i\| \tag{7}$$

For the noise-free case ($\epsilon = 0$), the condition for asymptotic stability ($\dot{V}_{\text{weighted}} < 0$) requires:

$$k_d \sum_{i=1}^{N} W_{\text{percept}}(\boldsymbol{P}_i) \|\boldsymbol{e}_i\|^2 > 0 \tag{8}$$

In the presence of bounded noise, the ultimate error bound is given by:

$$\|\boldsymbol{e}_i\| \le \frac{N\epsilon}{k_d W_{\text{percept}}(\boldsymbol{P}_i) - \delta} \tag{9}$$

where $\delta > 0$ is a small constant ensuring positivity of the denominator. This implies that for the error to remain bounded, the denominator must be positive, leading to the necessary condition on the control gain k_d.

3 Critical Threshold of Control Gain k_d

We distinguish between linear inhibition and non-linear (Sigmoid) inhibition mechanisms. Here $W_i \equiv W_{\text{percept}}(\boldsymbol{P}_i)$ for notational brevity.

3.1 Linear Inhibition

For a linear inhibition function, the system is dissipative for any positive gain.

Proposition 3.1. *(Stability under Linear Inhibition) If the inhibition function is linear, the condition $k_d > 0$ is sufficient to ensure $\dot{V}_{\text{weighted}} \leq 0$, guaranteeing system stability in the Lyapunov sense.*

3.2 Sigmoid Inhibition and Critical Gain

For biological systems, the inhibition follows a Sigmoid function $\sigma(\cdot)$ with maximum slope $\beta/4$ (where β represents the steepness parameter).

Theorem 3.1. *(Critical Gain for Sigmoid Inhibition) Let the inhibition function be modeled as a Sigmoid function with maximum slope $\beta/4$. The system remains stable if and only if the control gain k_d satisfies:*

$$k_d > \frac{\beta}{4 \min_i(W_i)} \tag{10}$$

where $\min_i(W_i)$ is the minimum perceptual weight among active agents ($W_i > 0$).

Proof. The derivative of the Lyapunov function with Sigmoid inhibition involves the term $\sigma'(x)$. Since $\sigma'(x) \leq \beta/4$ for all x, the destabilizing term in the derivative is bounded by $\frac{\beta}{4} \sum_{i=1}^{N} \|\boldsymbol{e}_i\|^2$. To ensure $\dot{V} < 0$, the control term $k_d \sum_{i=1}^{N} W_i \|\boldsymbol{e}_i\|^2$ must dominate the destabilizing term:

$$k_d \min_i(W_i) \sum_{i=1}^{N} \|\boldsymbol{e}_i\|^2 > \frac{\beta}{4} \sum_{i=1}^{N} \|\boldsymbol{e}_i\|^2 \tag{11}$$

Simplifying this inequality yields the critical threshold:

$$k_d > \frac{\beta}{4 \min_i(W_i)} \tag{12}$$

□

Remark 3.1. *This result highlights a fundamental trade-off: as the perceptual field becomes sparse (i.e., $\min_i(W_i) \to 0$ due to occlusion or large distances), the required control gain k_d approaches infinity. This defines the edge of collapse for the multi-agent system.*