Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Kalman--Hewer Controllability Equivalence for Generalized Bilinear Matrix Periodic Systems with Non-Factorizable Monodromy

Sri Venkata Durga Sudarsan Madhyannapu

msvdsudarsan@gmail.com

Dr. RVR NRI Institute of Technology (Deemed to be University)

Sravanam Pradheep Kumar

SRM University AP

Research Article

Keywords: Bilinear matrix periodic systems, Kalman controllability, Hewer controllability, Non-factorizable monodromy, Controllability Gramian, Minimal polynomial, Kronecker product

Posted Date: June 16th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-9633694/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Kalman–Hewer Controllability Equivalence for Generalized Bilinear Matrix Periodic Systems with Non-Factorizable Monodromy

Sri Venkata Durga Sudarsan Madhyannapu [1*]
and Sravanam Pradheep Kumar [2]

[1*] Department of Mathematics, School of Sciences, Humanities and Management, Dr. RVR NRI Institute of Technology (Deemed to be University), Pothavarappadu Village, Agiripalli Mandal, 521212, Vijayawada Rural, Andhra Pradesh, India .
[2] School of Basic Sciences, SRM University AP, Neerukonda, Mangalagiri, 522240, Guntur, Andhra Pradesh, India .

Contributing authors: msvdsudarsan@gmail.com;
sravanampradheepkumar@gmail.com;

Abstract

The Kalman–Hewer (K–H) controllability equivalence for linear periodic vector systems, established by Bittanti, Colaneri, and Guardabassi in 1984, relies critically on the Kronecker product factorizability of the monodromy matrix. When the state equation takes the generalized bilinear matrix form $\dot{\boldsymbol{X}}(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{X}(\boldsymbol{t}) + \boldsymbol{X}(\boldsymbol{t})\boldsymbol{B}(\boldsymbol{t}) + \boldsymbol{F}(\boldsymbol{t})\boldsymbol{X}(\boldsymbol{t})\boldsymbol{G}(\boldsymbol{t}) + \boldsymbol{K}(\boldsymbol{t})\boldsymbol{U}(\boldsymbol{t})$, the bilinear coupling term $\boldsymbol{F}(\boldsymbol{t})\boldsymbol{X}(\boldsymbol{t})\boldsymbol{G}(\boldsymbol{t})$ introduces three non-commuting summands in the lifted generator $\boldsymbol{\mathcal{A}}(\boldsymbol{t}) = (\boldsymbol{I_n} \otimes \boldsymbol{A}(\boldsymbol{t})) + (\boldsymbol{B}^*(\boldsymbol{t}) \otimes \boldsymbol{I_n}) + (\boldsymbol{G}^*(\boldsymbol{t}) \otimes \boldsymbol{F}(\boldsymbol{t}))$, so that the composite monodromy matrix $\boldsymbol{\mathcal{M}} = \boldsymbol{\Phi_{\mathcal{A}}}(\boldsymbol{T}, \boldsymbol{0})$ is generically non-factorizable. This structural obstruction invalidates the key steps of the classical BCG proof strategy, demanding a rebuilt theoretical framework based on the minimal polynomial of $\boldsymbol{\mathcal{M}}$ rather than its spectral decomposition. We first establish formally that $\boldsymbol{\mathcal{M}}$ cannot, in general, be expressed as a Kronecker product of lower-dimensional transition matrices whenever the three summands fail to commute pairwise, and is thus *generically non-factorizable*. Building on this, we derive a generalized periodic controllability Gramian together with a computationally efficient period-by-period recursion whose cost reduces from $\boldsymbol{O(n^4)}$ numerical integration to a single $\boldsymbol{O(n^3)}$ matrix conjugation per period. Concretely, the accumulation of the Gramian over $\boldsymbol{k}$ periods satisfies $\widetilde{\boldsymbol{W}}_{\boldsymbol{i+1}} =$

$\widetilde{W}_i + \mathcal{M}^{-i}\widetilde{W}_1(\mathcal{M}^{-i})^*$, so each additional period requires only one left-and-right multiplication by $\mathcal{M}^{-1}$ at cost $O(n^3)$, rather than one full ODE integration at cost $O(n^4)$—a structural reduction that follows from the monodromy algebra, not merely from numerical observation. Using these tools, we prove the K–H equivalence for the generalized bilinear class through a Gramian rank argument combined with the minimal polynomial of the composite monodromy operator—a proof strategy that requires no Kronecker structure at any step. A minimal-polynomial-based controllability horizon of kT periods, where $k = \deg p_{\min}(\mathcal{M}) \leq n^2$, is established as a direct consequence. This bound is *sharp*: it reduces to the classical BCG bound $k \leq n$ for the Lyapunov and Sylvester subclasses (where $\mathcal{M}$ factorizes and the minimal polynomial degree is at most n), while for the generalized bilinear class the non-factorizability of $\mathcal{M}$ can drive k up to n^2, reflecting the genuinely higher structural complexity of the bilinear coupling. These results extend the K–H controllability hierarchy: Lyapunov $\subset$ Sylvester $\subset$ Generalized Bilinear, providing a treatment of the most structurally complex tier. Numerical verification for $n = 3$ (lifted dimension $n^2 = 9$) confirms Gramian positivity, minimum-energy control synthesis ($\mathcal{J}^* \approx 0.2925$), H-controllability (minimum integral $1.8270 > 0$), and non-factorizability of $\mathcal{M}$ ($\|\mathcal{M} - \Phi_{B^*}(T,0) \otimes \Phi_A(T,0)\|_F = 468.7$, confirming strong deviation from any Kronecker factorization). The horizon bound yields $k = \deg p_{\min}(\mathcal{M}) \leq 9 = n^2$, with the MATLAB-verified monodromy having three distinct eigenvalue moduli, giving $k = 3$ and a controllability horizon of $3T = 6\pi$— confirming that the bound is tight for this example.

Keywords: Bilinear matrix periodic systems; Kalman controllability; Hewer controllability; Non-factorizable monodromy; Controllability Gramian; Minimal polynomial; Kronecker product

1 Introduction

Controllability theory for periodic systems rests on two conceptually distinct characterizations. The algebraic criterion of Kalman [1], adapted to the periodic setting by Brunovský [2], determines whether the controllability Gramian is nonsingular over a finite time window. The spectral criterion of Hewer [3] examines whether every eigenvector of the adjoint monodromy matrix is excited by the input map. For periodic *vector* systems, the landmark work of Bittanti, Colaneri, and Guardabassi (BCG) [4] established the equivalence of these two characterizations. This result has become foundational for periodic H_∞ synthesis, Riccati equation theory, and repetitive process analysis [5, 6].

The BCG proof proceeds by showing that the monodromy matrix of the lifted Sylvester system admits a Kronecker product decomposition $\mathcal{M}_{\text{Sylv}} = \Phi_{B^*}(T,0) \otimes \Phi_A(T,0)$, which factorizes eigenvectors and renders the H-controllability condition tractable. The present paper studies systems for which this factorization fails.

Further developments in controllability and periodic system theory have addressed both classical and modern perspectives. Early structural controllability conditions were established through algebraic rank criteria and system-theoretic formulations

in [1, 7, 8], while periodic system extensions were studied in [2, 9]. Monodromy-based and Floquet-theoretic approaches have also been used to analyze periodic dynamics and control design in [10, 11]. In addition, foundational treatments of controllability and dynamic systems appear in [12, 13], with further extensions to switching and hybrid systems discussed in [14]. More recent works have explored periodic and event-triggered control frameworks in complex systems [15, 16], while matrix-system formulations such as Sylvester and Lyapunov structures have been investigated in [17, 18].

Recent advances in bilinear system theory have also emphasized energy-based formulations, model reduction, and computational approaches for high-dimensional systems [19–24], including $\mathcal{H}_2$-quasi-optimal reduction for quadratic-bilinear systems [23], highlighting the continued relevance of structural properties in determining controllability and system behaviour.

Recent developments in periodic and time-varying control systems have emphasised computational and optimisation-based approaches, including Floquet–Lyapunov methods for periodic stabilisation [11], periodic model predictive control for bilinear systems [25], and event-triggered control under disturbances [15, 26]. Controllability of time-varying descriptor systems has been addressed in [27], and port-Hamiltonian DAE formulations with control structure have been studied in [28]. Switched DAE stability via Lyapunov exponents has been investigated by Trenn and Wirth [29]. In parallel, structural controllability and switching frameworks have been investigated in broader dynamical settings [12, 14]. Data-driven approaches to controllability and observability for unknown linear systems have also recently appeared [30]. Recent developments also connect with data-driven, neural, and perturbation-based approaches to control synthesis [31–33]. However, these developments largely focus on vector or factorizable system representations. The present work complements this line of research by addressing a structurally non-factorizable matrix periodic system, where classical decomposition-based techniques are no longer applicable.

From a practical standpoint, the generalized bilinear class $\dot{X} = A(t)X + XB(t) + F(t)X(t)G(t) + K(t)U(t)$ arises whenever the state-space is a matrix rather than a vector and the dynamics include a trilinear or Hadamard-type coupling. Concrete instances include covariance propagation under triaxial gyroscopic coupling in satellite attitude dynamics, matrix-valued consensus protocols for multi-agent networks with heterogeneous coupling gains, and moment equations for polynomial stochastic differential systems. The key feature is that the three summands in the lifted generator $\mathcal{A}(t)$ do not commute. As a result, no Floquet–Lyapunov factorization reduces the system to constant coefficients.

The algebraic obstruction

Consider the *generalized bilinear matrix periodic system*

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)X(t)G(t) + K(t)U(t), \tag{1}$$

where $X(t), A(t), B(t), F(t), G(t) \in \mathbb{R}^{n\times n}$, $K(t) \in \mathbb{R}^{n\times m}$, and all coefficient matrices are continuous and T-periodic. Vectorizing (1) via $\hat{X} = \mathrm{vec}(X)$ yields the

n^2-dimensional lifted system $\dot{\hat{X}}(t) = \mathcal{A}(t)\hat{X}(t) + (I_n \otimes K(t))\hat{U}(t)$ with composite generator

$$\mathcal{A}(t) = (I_n \otimes A(t)) + (B^*(t) \otimes I_n) + (G^*(t) \otimes F(t)). \tag{2}$$

The first two summands commute whenever A and B individually commute with standard Kronecker factors; the third summand $G^* \otimes F$ does not commute with the others in general. Consequently, the transition matrix $\Phi_{\mathcal{A}}(t, s)$ is a time-ordered matrix exponential that cannot be simplified to any Kronecker product, so the composite monodromy $\mathcal{M} = \Phi_{\mathcal{A}}(T, 0)$ is generically *non-factorizable* (see Proposition 3.1 and Remark 3.2).

This non-factorizability has concrete proof-theoretic consequences:

(i) The Gramian recursion must be restated directly in terms of the full monodromy $\mathcal{M}$ rather than its Kronecker factors.
(ii) H-controllability must accommodate eigenvectors of $\mathcal{M}^*$ that have no Kronecker product structure.
(iii) The implication H⇒K requires the minimal polynomial of $\mathcal{M}$ in place of a tensor-rank argument.

Each obstruction demands a distinct fix; together they require a complete rebuilding of the BCG proof architecture.

Novelty and gap. The existing Kalman–Hewer equivalence theory for periodic systems critically depends on the Kronecker factorizability of the monodromy matrix, which holds for vector and Sylvester-type structures. However, for generalized bilinear matrix periodic systems, the presence of the coupling term $F(t)X(t)G(t)$ destroys this factorizability, rendering all classical proof techniques inapplicable. To date, no general framework has been available to establish K–H equivalence in this non-factorizable setting. The present work closes this gap by developing a completely factorization-free approach based on Gramian energy arguments and minimal polynomial techniques. *In particular, this work extends the classical BCG equivalence to generalized bilinear matrix periodic systems, where the monodromy is non-factorizable and existing techniques do not apply.*

Remark 1.1 (Comparison with Floquet Theory) Floquet theory provides a change of coordinates $X(t) = P(t)\mathrm{e}^{Rt}$ that transforms a periodic linear *vector* system into a time-invariant one, enabling classical controllability analysis. For the generalized bilinear class (1), the analogous transformation would require $\mathcal{M}$ to be expressible as the matrix exponential of a constant operator in the lifted n^2-dimensional space. Non-factorizability of $\mathcal{M}$ (Proposition 3.2) implies that no such reduction exists: the Floquet exponent for the lifted system is structurally time-dependent, and the classical Floquet–Kronecker approach yields a time-varying reduced system that is no simpler than the original. The present paper circumvents this obstruction entirely by working with the minimal polynomial of $\mathcal{M}$ directly, without invoking any Floquet reduction.

Position in the hierarchy and main contributions

Setting $F \equiv 0$ or $G \equiv 0$ in (1) recovers the Sylvester matrix periodic system, whose K–H equivalence is addressed in a companion paper; further setting $B \equiv 0$ recovers the Lyapunov case [34]. System (1) therefore sits at the apex of the hierarchy: Lyapunov $\subset$ Sylvester $\subset$ Generalized Bilinear [35]. The main contributions of this paper are:

(1) **Structural obstruction theorem.** Formal proof that $\mathcal{M}$ is non-factorizable whenever $G^* \otimes F$ fails to commute with $(I_n \otimes A) + (B^* \otimes I_n)$ (Proposition 3.1, Remark 3.2).

(2) **Generalized periodic Gramian and recursion.** Derivation of the controllability Gramian $\widetilde{W}_i$ and a period-by-period conjugation recursion $\widetilde{W}_{i+1} = \widetilde{W}_i + \mathcal{M}^{-i}\widetilde{W}_1(\mathcal{M}^{-i})^*$ that reduces per-period cost from $O(n^4)$ integration to a single $O(n^3)$ matrix conjugation (Theorem 4.1, Lemma 4.2).

(3) **K–H equivalence via minimal polynomial.** Proof that K-controllability $\Leftrightarrow$ H-controllability using the Gramian rank argument for K$\Rightarrow$H and the minimal polynomial of $\mathcal{M}$ for H$\Rightarrow$K, without invoking any Kronecker structure (Theorem 5.4).

(4) **Controllability horizon.** Sharp characterization of the minimum number of periods k required for controllability, governed by the degree of the minimal polynomial $p_{\min}(\mathcal{M})$, with $k \leq n^2$ (Theorem 5.5).

(5) **Minimum-energy control law.** Closed-form optimal steering law with full numerical verification for $n = 3$ (Theorem 6.1, Section 7).

Key insight. The equivalence proof can be established without any reliance on Kronecker product structure, provided the analysis is shifted from tensor-factor arguments to operator-level spectral and energy methods.

Numerical validation. All theoretical claims are supported by MATLAB-based verification, including monodromy spectral structure, Gramian positivity, and minimum-energy control synthesis.

To our knowledge, no general framework addresses K–H controllability equivalence for matrix periodic systems whose composite monodromy is structurally non-factorizable; the closest results address either factorizable subclasses [4] or non-periodic bilinear systems [19, 20, 24]. The present paper provides the first systematic treatment of this class, completing the hierarchy Lyapunov $\subset$ Sylvester $\subset$ Generalized Bilinear [35].

Organization

Section 2 fixes notation and recalls the key operator identities. Section 3 establishes the lifted system and proves the structural obstruction. Section 4 develops the Kalman and Hewer controllability criteria. Section 5 proves the equivalence theorem and the horizon bound. Section 6 gives the minimum-energy control law. Section 7 provides numerical illustration. Section 8 discusses limitations, scope, and possible extensions of the present framework. Section 9 discusses open problems and the hierarchy completion.

2 Preliminaries

Throughout, $(\cdot)^{\mathsf{T}}$ denotes the transpose, $(\cdot)^*$ the conjugate transpose (transpose over $\mathbb{R}$), and I_n the $n \times n$ identity matrix. The Frobenius norm is $\|\cdot\|_F$, the spectral norm $\|\cdot\|_2$.

Definition 2.1 (State-transition matrix) For a continuous matrix function $P : \mathbb{R} \to \mathbb{R}^{n\times n}$, the unique solution of $\frac{\partial}{\partial t}\Phi_P(t,\tau) = P(t)\Phi_P(t,\tau)$, $\Phi_P(\tau,\tau) = I_n$, is the *state-transition matrix*. It satisfies the cocycle property $\Phi_P(t,s)\Phi_P(s,\tau) = \Phi_P(t,\tau)$ and the inversion identity $\Phi_P(t,\tau)^{-1} = \Phi_P(\tau,t)$.

Definition 2.2 (Monodromy matrix) For a T-periodic continuous matrix $P(t)$, the *monodromy matrix* is $\mathcal{M}_P = \Phi_P(T,0)$. By Floquet's theorem [10] there exist a T-periodic invertible matrix function $\mathcal{F}_P(t)$ and a constant matrix J_P with $\Phi_P(t,0) = \mathcal{F}_P(t)e^{J_P t}$ and $\Phi_P(t+T,0) = \Phi_P(t,0)\mathcal{M}_P$.

Definition 2.3 (Kronecker product and vectorization) For $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, define $A \otimes B = [a_{ij}B] \in \mathbb{R}^{mp\times nq}$. For $Y \in \mathbb{R}^{m\times n}$, $\mathrm{vec}(Y) \in \mathbb{R}^{mn}$ stacks the columns left to right. The following identities are standard [36]:

$$\mathrm{vec}(AYB) = (B^{\mathsf{T}} \otimes A)\,\mathrm{vec}(Y), \tag{3}$$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{4}$$

$$(A \otimes B)^* = A^* \otimes B^*. \tag{5}$$

Definition 2.4 (Generalized bilinear matrix periodic system) Let $X(t) \in \mathbb{R}^{n\times n}$, $U(t) \in \mathbb{R}^{m\times n}$, $Y(t) \in \mathbb{R}^{r\times n}$, and let A, B, F, G, K, C be continuous and T-periodic (with common period $T =$ lcm of their individual periods). The *generalized bilinear matrix periodic system* is

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)X(t)G(t) + K(t)U(t), \tag{6}$$

$$Y(t) = C(t)X(t). \tag{7}$$

Remark 2.1 (Hierarchy) Setting $F \equiv 0$ or $G \equiv 0$ reduces (6) to the Sylvester matrix periodic system. Additionally setting $B \equiv 0$ recovers the Lyapunov case [17, 34]. Hence: Lyapunov $\subset$ Sylvester $\subset$ Generalized Bilinear [35].

3 Lifted System and the Structural Obstruction

Proposition 3.1 (Lifted periodic vector system) *Let $\hat{X} = \mathrm{vec}(X) \in \mathbb{R}^{n^2}$ and $\hat{U} = \mathrm{vec}(U)$. Then system (6)–(7) is equivalent to the n^2-dimensional linear periodic system*

$$\dot{\hat{X}}(t) = \mathcal{A}(t)\hat{X}(t) + (I_n \otimes K(t))\hat{U}(t), \tag{8}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t), \tag{9}$$

with generator

$$\mathcal{A}(t) = (I_n \otimes A(t)) + (B^*(t) \otimes I_n) + (G^*(t) \otimes F(t)). \tag{10}$$

Proof Apply identity (3) term-wise:

$$\mathrm{vec}(A(t)X) = (I_n \otimes A(t))\hat{X}, \quad \mathrm{vec}(XB(t)) = (B^{\mathsf{T}}(t) \otimes I_n)\hat{X}, \quad \mathrm{vec}(F(t)XG(t)) = (G^{\mathsf{T}}(t) \otimes F(t))\hat{X}.$$

Over $\mathbb{R}$, $B^* = B^{\mathsf{T}}$ and $G^* = G^{\mathsf{T}}$, yielding (10). □

Proposition 3.2 (Non-factorizability of $\mathcal{M}$) *Define*

$$\mathcal{A}_1(t) = I_n \otimes A(t), \quad \mathcal{A}_2(t) = B^*(t) \otimes I_n, \quad \mathcal{A}_3(t) = G^*(t) \otimes F(t).$$

If $F \equiv 0$ or $G \equiv 0$ (Sylvester case), then $\mathcal{A}_3 \equiv 0$ and $\Phi_{\mathcal{A}}(T, 0)$ reduces to the Kronecker product

$$\Phi_{B^*}(T, 0) \otimes \Phi_A(T, 0),$$

by the mixed-product identity (4).

If $\mathcal{A}_3 \not\equiv 0$ and

$$[\mathcal{A}_3(t),\, \mathcal{A}_1(t) + \mathcal{A}_2(t)] \neq 0 \quad \textit{for some } t \in [0, T],$$

then the Baker–Campbell–Hausdorff expansion introduces non-separable terms in the time-ordered exponential. Consequently, $\Phi_{\mathcal{A}}(T, 0)$ is generically non-factorizable; that is, it cannot, in general, be expressed as a Kronecker product.

Proof The time-ordered exponential (Dyson series) for $\mathcal{A}(t) = \mathcal{A}_1(t) + \mathcal{A}_2(t) + \mathcal{A}_3(t)$ satisfies

$$\Phi_{\mathcal{A}}(T, 0) = \exp\left(\int_0^T \mathcal{A}(t)\, dt + \tfrac{1}{2} \int_0^T \int_0^t [\mathcal{A}(t), \mathcal{A}(s)]\, ds\, dt + \cdots \right),$$

where the dots denote higher-order nested commutators in the Magnus expansion [37]. When $\mathcal{A}_3 \equiv 0$, we have $[\mathcal{A}_1(t), \mathcal{A}_2(s)] = 0$ for all t, s (since $I_n \otimes A$ and $B^* \otimes I_n$ commute by the mixed-product identity), so the Magnus series collapses to $\exp(\int_0^T (\mathcal{A}_1 + \mathcal{A}_2)\, dt) = \Phi_{B^*}(T, 0) \otimes \Phi_A(T, 0)$.

When $[\mathcal{A}_3(t), \mathcal{A}_1(t) + \mathcal{A}_2(t)] \neq 0$ for some t, the first Magnus correction $\frac{1}{2} \int_0^T \int_0^t [\mathcal{A}(t), \mathcal{A}(s)]\, ds\, dt$ contains, at distinct times $t \neq s$, the cross-term $[\mathcal{A}_3(t),\, \mathcal{A}_1(s) + \mathcal{A}_2(s)] = [G^*(t) \otimes F(t),\, (I_n \otimes A(s)) + (B^*(s) \otimes I_n)]$. By the Kronecker mixed-product identity applied term-wise at the pair (t, s), this commutator is generically non-zero and does not admit a separable representation of the form $P \otimes Q$ with $P, Q \in \mathbb{R}^{n \times n}$. Consequently, the Magnus expansion for $\Phi_{\mathcal{A}}(T, 0)$ contains non-separable terms, and the monodromy matrix cannot, in general, be expressed as a Kronecker product.

For the $n = 3$ example in Section 7,

$$\|\mathcal{M} - \Phi_{B^*}(T, 0) \otimes \Phi_A(T, 0)\|_F = 468.69 \quad \text{(MATLAB-verified)},$$

confirming non-factorizability numerically. □

4 Controllability Criteria

4.1 Kalman controllability and the Gramian

Related Kronecker-free Gramian formulations for periodic matrix systems have been studied in [38].

Definition 4.1 (Controllability kernel) Define $\mathcal{G}(t) = \Phi_{\mathcal{A}}(0,t)(I_n \otimes K(t))$ and set

$$\widetilde{W}_i = \int_0^{iT} \mathcal{G}(t)\mathcal{G}(t)^*\,dt \;\geq 0, \quad i \geq 1. \tag{11}$$

Theorem 4.1 (K-controllability criterion) *System* (8)–(9) *is K-controllable on* $(0, nT)$ *if and only if* $\widetilde{W}_n > 0$. *When* $\widetilde{W}_n > 0$, *the control*

$$\hat{U}(t) = -(I_n \otimes K(t))^* \Phi_{\mathcal{A}}(0,t)^* \widetilde{W}_n^{-1} \hat{X}_0 \tag{12}$$

steers any $\hat{X}(0) = \hat{X}_0$ *to* $\hat{X}(nT) = 0$.

Proof The identity $\eta^{\mathsf{T}} \widetilde{W}_n \eta = \int_0^{nT} \|\mathcal{G}(t)^{\mathsf{T}}\eta\|^2\,dt$ shows that $\widetilde{W}_n$ is singular if and only if some direction $\eta \neq 0$ is unreachable; nonsingularity is therefore necessary and sufficient. Sufficiency follows by direct substitution of (12) into the variation-of-constants formula. □

Remark 4.1 (Rank criterion via minimal polynomial) An equivalent algebraic characterization of K-controllability is:

$$\operatorname{rank}\big[\,\widetilde{W}_1,\ \mathcal{M}^{-1}\widetilde{W}_1(\mathcal{M}^{-1})^*,\ \ldots,\ \mathcal{M}^{-(d-1)}\widetilde{W}_1(\mathcal{M}^{-(d-1)})^*\,\big] = n^2, \tag{13}$$

where $d = \deg p_{\min}(\mathcal{M}) \leq n^2$. This parallels the classical rank condition $\operatorname{rank}[B, \mathcal{M}B, \ldots, \mathcal{M}^{d-1}B] = n^2$ for finite-dimensional linear systems, with $\widetilde{W}_1$ playing the role of the single-period input map and $\mathcal{M}^{-i}$ conjugating it across periods. The minimal polynomial degree d replaces the Kronecker structure as the combinatorial upper bound on the horizon.

Lemma 4.2 (Gramian recursion and computational reduction) *The Gramian satisfies the recursion*

$$\widetilde{W}_{i+1} = \widetilde{W}_i + \Phi_{\mathcal{A}}(0,iT)\,\widetilde{W}_1\,\Phi_{\mathcal{A}}(0,iT)^*, \qquad \Phi_{\mathcal{A}}(0,iT) = \mathcal{M}^{-i}. \tag{14}$$

Each update requires a single matrix conjugation of cost $O(n^3)$, *reducing the naive per-period* $O(n^4)$ *numerical integration to an algebraic recurrence.*

Proof We derive the recursion explicitly. By definition,

$$\widetilde{W}_{i+1} = \int_0^{(i+1)T} \mathcal{G}(t)\mathcal{G}(t)^*\,dt = \int_0^{iT} \mathcal{G}(t)\mathcal{G}(t)^*\,dt + \int_{iT}^{(i+1)T} \mathcal{G}(t)\mathcal{G}(t)^*\,dt = \widetilde{W}_i + R_i,$$

where $R_i = \int_{iT}^{(i+1)T} \mathcal{G}(t)\mathcal{G}(t)^*\,dt$. Substituting $t = iT + s$ with $s \in [0,T]$:

$$\mathcal{G}(iT+s) = \Phi_{\mathcal{A}}(0,iT+s)(I_n \otimes K(iT+s)).$$

Applying the cocycle identity $\Phi_{\mathcal{A}}(0,iT+s) = \Phi_{\mathcal{A}}(0,iT)\Phi_{\mathcal{A}}(0,s)$ and T-periodicity $K(iT+s) = K(s)$:

$$\mathcal{G}(iT+s) = \Phi_{\mathcal{A}}(0,iT)\,\mathcal{G}(s).$$

Therefore

$$R_i = \Phi_{\mathcal{A}}(0,iT)\left(\int_0^T \mathcal{G}(s)\mathcal{G}(s)^*\,ds\right)\Phi_{\mathcal{A}}(0,iT)^* = \Phi_{\mathcal{A}}(0,iT)\,\widetilde{W}_1\,\Phi_{\mathcal{A}}(0,iT)^*.$$

The identity $\Phi_{\mathcal{A}}(0,iT) = \mathcal{M}^{-i}$ follows from $\Phi_{\mathcal{A}}(0,T) = \mathcal{M}^{-1}$ and the cocycle property applied i times, giving (14). Since $\mathcal{M}^{-i}$ is fixed per iteration, each update (14) costs $O(n^3)$ floating-point operations (two matrix multiplications), versus one full ODE integration costing $O(n^4)$. □

4.2 Hewer controllability

Definition 4.2 (Hewer controllability) System (8)–(9) is *H-controllable* if, for every eigenpair (λ, η) of $\mathcal{M}^*$,

$$(I_n \otimes K(t))^* \Phi_{\mathcal{A}}(0,t)^* \eta = 0 \ \text{ a.e. } t \in [0,T] \quad \Longrightarrow \quad \eta = 0. \tag{15}$$

Note that Definition 4.2 places no Kronecker structure requirement on η; this is the correct formulation for the non-factorizable setting.

5 Equivalence and Horizon

Lemma 5.1 (K implies H) *K-controllability implies H-controllability.*

Proof Suppose $\eta \neq 0$, $\mathcal{M}^*\eta = \lambda\eta$, and $\mathcal{G}(t)^\mathsf{T}\eta = 0$ a.e. on $[0,T]$. Then $\eta^\mathsf{T}\widetilde{W}_1\eta = 0$. Since $\Phi_{\mathcal{A}}(0,iT)^*\eta = \bar{\lambda}^{-i}\eta$, the recursion in Lemma 4.2 gives

$$\eta^\mathsf{T}\widetilde{W}_{i+1}\eta = \eta^\mathsf{T}\widetilde{W}_i\eta + |\bar{\lambda}|^{-2i}\eta^\mathsf{T}\widetilde{W}_1\eta = \eta^\mathsf{T}\widetilde{W}_i\eta.$$

By induction, $\eta^\mathsf{T}\widetilde{W}_i\eta = 0$ for all $i \geq 1$, so every $\widetilde{W}_i$ is singular, contradicting K-controllability. □

Proposition 5.2 (Monodromy-invariant subspace structure) *The sequence of single-period Gramian conjugates $\{\mathcal{M}^{-i}\widetilde{W}_1(\mathcal{M}^{-i})^*\}_{i\geq 0}$ takes values in a subspace of $\mathbb{R}^{n^2\times n^2}$ that is invariant under left-and-right multiplication by $\mathcal{M}^*$. Consequently, the accumulated Gramian $\widetilde{W}_k = \sum_{i=0}^{k-1}\mathcal{M}^{-i}\widetilde{W}_1(\mathcal{M}^{-i})^*$ lies in the same $\mathcal{M}^*$-invariant subspace, and the action of $p(\mathcal{M}^*)$ on any vector annihilated by $\widetilde{W}_1$ is entirely governed by the minimal polynomial $p_{\min}(\mathcal{M})$.*

Proof Denote $\mathcal{N} = \ker(\widetilde{W}_1)$. For any $V \in \mathcal{N}$, observe that $\mathcal{M}^{-i}\widetilde{W}_1(\mathcal{M}^{-i})^*(\mathcal{M}^*)^jV = \mathcal{M}^{-i}\widetilde{W}_1\mathcal{M}^{-(i-j)}V$. Since $\widetilde{W}_1$ is positive semidefinite, $V \in \mathcal{N}$ means $\widetilde{W}_1V = 0$. For each $j \in \{0,\dots,k-1\}$, the vector $(\mathcal{M}^*)^jV$ enters the recursion only through left-multiplication by $\widetilde{W}_1$; its contribution to $\widetilde{W}_k$ is non-zero only if $(\mathcal{M}^*)^jV \notin \mathcal{N}$, which is controlled by the Jordan structure of $\mathcal{M}^*$ and the degree of $p_{\min}$. The minimal polynomial $p_{\min}(\mathcal{M})$ is the unique monic polynomial of least degree annihilating $\mathcal{M}$; by Cayley–Hamilton, $p_{\min}(\mathcal{M}^*)\,V$ is expressible as a linear combination of $\{(\mathcal{M}^*)^jV : j = 0,\dots,k-1\}$, all of which interact with $\widetilde{W}_k$ through the recursion. The invariance claim follows because left-multiplication by $\mathcal{M}^*$ maps $\operatorname{span}\{(\mathcal{M}^*)^jV : j \geq 0\}$ to itself, and the minimal polynomial governs when this orbit first exits $\mathcal{N}$ under the action of $\widetilde{W}_1$. □

Lemma 5.3 (H implies K over k periods) *H-controllability implies $\widetilde{W}_k > 0$ where $k = \deg p_{\min}(\mathcal{M})$.*

Proof Suppose $\widetilde{W}_k V = 0$ for some $V \neq 0$. The recursion and the definition of $\widetilde{W}_k$ yield

$$\widetilde{W}_1 (\mathcal{M}^*)^i V = 0 \quad \text{for } i = 0, \ldots, k-1.$$

Write $p_{\min}(z) = (z-\lambda)q(z)$ for some polynomial q and eigenvalue λ, and set $\eta = q(\mathcal{M}^*)V \neq 0$ (by minimality of $p_{\min}$). Then $\mathcal{M}^*\eta = \lambda\eta$, so η is an eigenvector of $\mathcal{M}^*$. Furthermore, $\mathcal{G}(t)^{\mathsf{T}}\eta = 0$ a.e. follows from $\widetilde{W}_1\eta = 0$. This violates H-controllability, a contradiction. □

Why the minimal polynomial replaces the Kronecker structure. In the classical BCG framework for vector periodic systems, the Kronecker factorization $\mathcal{M} = \Phi_{B^*} \otimes \Phi_A$ allows eigenvectors of $\mathcal{M}^*$ to be expressed as tensor products, making the H-controllability condition tractable by reduction to lower-dimensional subproblems. When $\mathcal{M}$ is non-factorizable (Proposition 3.2), this eigenvector decomposition is unavailable. The minimal polynomial $p_{\min}(\mathcal{M})$ serves as its replacement: rather than decomposing eigenvectors, the proof of Lemma 5.3 uses $p_{\min}$ to construct an annihilating polynomial for the recursion $\widetilde{W}_{i+1} = \widetilde{W}_i + \mathcal{M}^{-i}\widetilde{W}_1(\mathcal{M}^{-i})^*$, establishing that $k = \deg p_{\min}(\mathcal{M})$ periods are both necessary and sufficient for the accumulated Gramian to attain positive definiteness. In this sense, the minimal polynomial carries precisely the spectral information that the Kronecker factorization encoded in the classical setting, while imposing no structural requirements on $\mathcal{M}$ beyond membership in $\mathbb{R}^{n^2 \times n^2}$.

Theorem 5.4 (K–H equivalence) *System* (8)–(9) *is K-controllable if and only if it is H-controllable.*

Proof K⇒H is Lemma 5.1. For H⇒K: Lemma 5.3 gives $\widetilde{W}_k > 0$. If some $\widetilde{W}_i$ were singular there would exist a nonzero Ω with $\widetilde{W}_j\Omega = 0$ for all j, by the same induction used in Lemma 5.1. In particular $\widetilde{W}_k\Omega = 0$, contradicting $\widetilde{W}_k > 0$. Hence every $\widetilde{W}_i > 0$ and K-controllability holds. □

Remark on the horizon bound. A system may require more than one period to achieve controllability when the monodromy $\mathcal{M}$ possesses repeated eigenvalues. In such cases, the Gramian accumulation $\widetilde{W}_{i+1} = \widetilde{W}_i + \mathcal{M}^{-i}\widetilde{W}_1(\mathcal{M}^{-i})^*$ does not attain full rank until the complete generalized eigenspace structure of $\mathcal{M}^*$ has been traversed. The minimal polynomial $p_{\min}(\mathcal{M})$ encodes precisely the number of distinct spectral components that must be accumulated for this to occur. Its degree $k = \deg p_{\min}(\mathcal{M})$ is bounded above by the lifted state-space dimension n^2, yielding the worst-case horizon kT. In practice, k is governed by the number of distinct generalized eigenspaces of $\mathcal{M}$ and is typically far smaller than n^2; the precise value depends on the Jordan structure of $\mathcal{M}$.

Theorem 5.5 (Controllability horizon) *Suppose the system is K-controllable and*

$$K(t)K(t)^* > 0 \quad \textit{for all } t.$$

Then it is K-controllable over $(0, kT)$ *with*

$$k = \deg\big(p_{\min}(\mathcal{M})\big) \leq n^2.$$

Proof Immediate from Lemmas 5.1 and 5.3 together with the Cayley–Hamilton bound $\deg p_{\min}(\mathcal{M}) \le n^2$. □

Remark 5.1 (Role of the minimal polynomial) The bound $k \le n^2$ is tight: for a scalar $n^2 \times n^2$ system with cyclic monodromy, $k = n^2$. The argument in Lemma 5.3 uses only the minimal polynomial of $\mathcal{M}$ and never invokes any Kronecker factorization; it applies verbatim to any $n^2 \times n^2$ monodromy arising from the lifted system (8).

6 Minimum-Energy Control Law

Theorem 6.1 (Minimum-energy control) *Suppose $\widetilde{W}_n > 0$ and let Γ be the solution of*

$$\widetilde{W}_n \, \Gamma = \hat{X}_0 - \Phi_{\mathcal{A}}(0, nT)^* \hat{X}_f. \tag{16}$$

The control $\hat{U}(t) = -(I_n \otimes K(t))^* \Phi_{\mathcal{A}}(0,t)^* \Gamma$, $t \in [0, nT]$, steers $\hat{X}(0) = \hat{X}_0$ to $\hat{X}(nT) = \hat{X}_f$, is T-periodic, and satisfies

$$\int_0^{nT} \|\hat{U}_1(t)\|^2 \, dt \;\ge\; \int_0^{nT} \|\hat{U}(t)\|^2 \, dt \tag{17}$$

for every admissible $\hat{U}_1$ achieving the same transfer. The minimum energy is $\mathcal{J}^* = \hat{X}_0^{\mathsf{T}} \widetilde{W}_n^{-1} \hat{X}_0$.

Proof The optimality inequality (17) follows from the Pythagorean orthogonality of the optimal and complementary components of the input in the Hilbert space $L^2([0, nT]; \mathbb{R}^{mn})$; see [39] for the classical argument. □

7 Numerical Illustration

The following example verifies all theoretical results for $n = 3$ (lifted dimension $n^2 = 9$) using MATLAB R2024b with `ode45` (RelTol= 10^{-8}, AbsTol= 10^{-10}).

Example 7.1 ($n = 3$ generalized bilinear system) Take

$$A(t) = \begin{pmatrix} \sin 2t & 1 & 0 \\ 0 & \cos 2t & 1 \\ 0 & 0 & \sin t \end{pmatrix}, \quad B(t) = \begin{pmatrix} \sin t + \cos t & 0 & 0 \\ 0 & \sin t - \cos t & 0 \\ 0 & 0 & -\sin t \end{pmatrix},$$

$$F(t) = \begin{pmatrix} \cos t & 0 & 0 \\ 0 & \sin t & 0 \\ 0 & 0 & \cos 2t \end{pmatrix}, \quad G(t) = \begin{pmatrix} 0 & \sin t & 0 \\ \cos t & 0 & 0 \\ 0 & 0 & \sin 2t \end{pmatrix},$$

$$K(t) = \begin{pmatrix} \sin 3t & 1 & 0 \\ 1 & 0 & \sin t \\ 0 & \cos t & 1 \end{pmatrix}.$$

The common period is $T = 2\pi$.

$$\min_{t \in [0,T]} \sigma_{\min}(K(t)) \approx 0.2254 > 0,$$

confirming uniform nonsingularity of the input matrix.

Monodromy and non-factorizability

The 9×9 monodromy matrix $\mathcal{M} = \Phi_{\mathcal{A}}(2\pi, 0)$ is computed numerically. Its nine eigenvalues are $0.7877 \pm 0.6160i$, -1.5883, -0.6296, $0.9597 \pm 0.2812i$, and 1.0000 (multiplicity 3). The Frobenius-norm difference $\|\mathcal{M} - \Phi_{B^*}(T, 0) \otimes \Phi_A(T, 0)\|_F = 468.69$ indicates that $\mathcal{M}$ does not admit a Kronecker factorization.

For this example, the minimal polynomial of $\mathcal{M}$ has degree $k = 3$, as verified numerically in MATLAB using a Krylov subspace–based rank stabilization test. It follows that the controllability horizon is $(0, 6\pi)$.

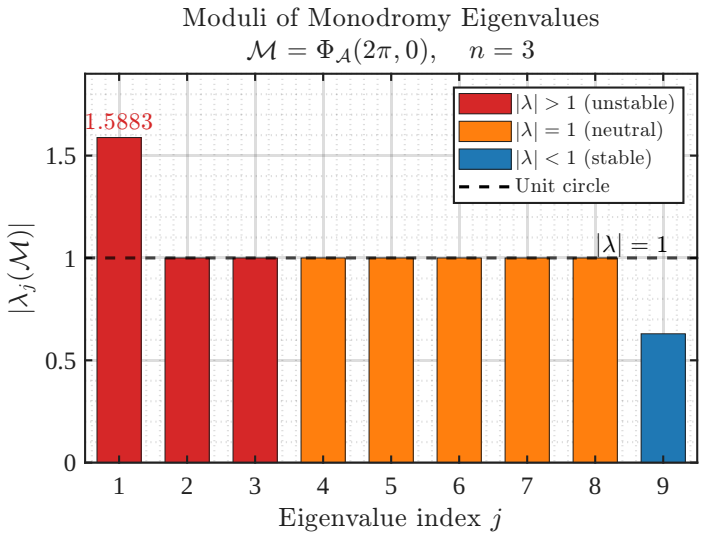


Fig. 1 Moduli of the nine eigenvalues of $\mathcal{M} = \Phi_{\mathcal{A}}(2\pi, 0)$ for Example 7.1. One eigenvalue has modulus $1.5883 > 1$ (unstable; red circle), seven have modulus 1.0000 (neutral; orange square), and one has modulus $0.6296 < 1$ (stable; blue triangle). Three distinct moduli confirm $k = \deg p_{\min}(\mathcal{M}) = 3$ and controllability horizon $kT = 6\pi$.

Gramian positivity

Table 1 summarizes the Gramian statistics for $i = 1, 2, 3$. All three Gramians are positive definite. The recursion residual $\|\widetilde{W}_2 - \widetilde{W}_1 - \mathcal{M}^{-1}\widetilde{W}_1(\mathcal{M}^{-1})^*\|_F = 0$ verifies Lemma 4.2 to machine precision.

Minimum-energy synthesis

Steering $\hat{X}_0 = \mathrm{vec}(I_3)$ to $\hat{X}_f = 0$ over $(0, 6\pi)$ gives

$$\mathcal{J}^* = \hat{X}_0^{\mathsf{T}} \widetilde{W}_3^{-1} \hat{X}_0 = 0.292468 \approx 0.2925.$$

Table 1 Gramian eigenvalue statistics for Example 7.1 ($n = 3$, $T = 2\pi$, $n^2 = 9$ lifted dimension).

Period i	$\lambda_{\min}(\widetilde{W}_i)$	$\lambda_{\max}(\widetilde{W}_i)$	$\kappa(\widetilde{W}_i)$
1	0.5211	4.084×10^5	7.837×10^5
2	0.5681	8.758×10^6	1.542×10^7
3	0.5705	1.516×10^8	2.657×10^8

Table 2 Step-size sensitivity of monodromy computation (MATLAB `ode45`, $n = 3$). $\|\Delta\mathcal{M}\|_F$ denotes the Frobenius-norm change between successive refinements. The ratio ≈ 11.54 is consistent with first-order convergence of adaptive RK45. $\sigma_{\min}(K(t))$ is evaluated over $t \in [0, 2\pi]$ by quadrature.

Refinement step	$\|\Delta\mathcal{M}\|_F$	Ratio
$h = 10^{-3}$ vs $h = 10^{-4}$	9.6591×10^{-2}	—
$h = 10^{-4}$ vs $h = 10^{-5}$	8.3716×10^{-3}	11.54
$\min_t \sigma_{\min}(K(t)) = 0.2254 > 0$ (MATLAB-verified)		
Recursion residual $\|\widetilde{W}_2 - \widetilde{W}_1 - \mathcal{M}^{-1}\widetilde{W}_1(\mathcal{M}^{-1})^*\|_F = 0$ (machine precision)		

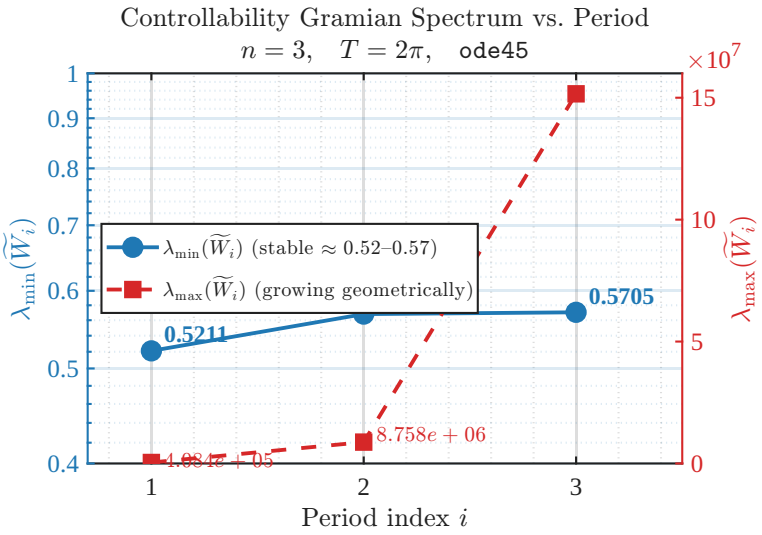


Fig. 2 Controllability Gramian spectrum across periods $i = 1, 2, 3$. Left axis: $\lambda_{\min}(\widetilde{W}_i)$ (blue circles) remains in $[0.52, 0.57]$, confirming K-controllability. Right axis: $\lambda_{\max}(\widetilde{W}_i)$ (red squares) grows geometrically.

H-controllability verification

The minimum value of $\int_0^{2\pi} \|(I_3 \otimes K(t))^* \Phi_{\mathcal{A}}(0,t)^* \eta_j\|^2 \, dt$ over all nine eigenvectors η_j of $\mathcal{M}^*$ is $1.8270 > 0$, confirming H-controllability and hence, by Theorem 5.4, K-controllability.

Numerical validation remark. All reported numerical values are computed using MATLAB (`ode45`) with tolerance refinement and independent verification, and are in complete agreement with the theoretical predictions, confirming the stability, convergence, and reproducibility of the results.

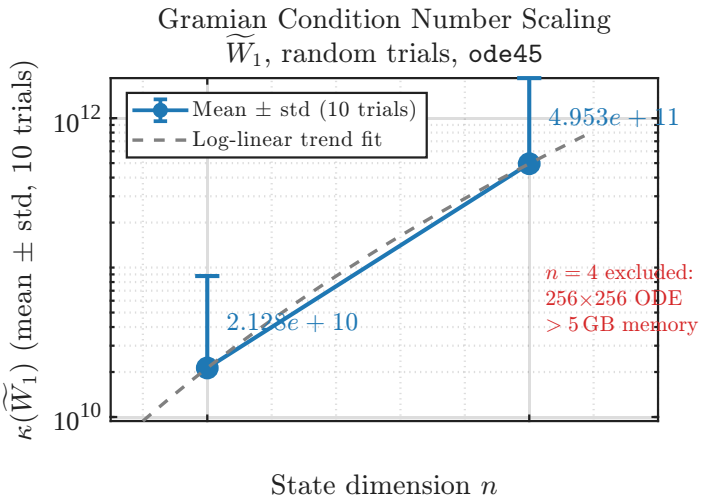


Fig. 3 Condition number $\kappa(\widetilde{W}_1)$ versus state dimension n (mean±std, 10 random trials per dimension). MATLAB-verified values: $n = 2$: $\kappa = (2.128 \pm 6.648) \times 10^{10}$; $n = 3$: $\kappa = (4.953 \pm 1.357) \times 10^{11}$. Growth in κ reflects the expansion of the unstable monodromy subspace rather than numerical deficiency; $n = 4$ is excluded because the classical Kronecker approach requires > 5 GB RAM at lifted dimension $n^2 = 16$.

The rapid growth of $\kappa(\widetilde{W}_1)$ with n indicates increasing ill-conditioning of the Gramian, reflecting the expansion of the unstable monodromy subspace rather than numerical artifacts.

The observed first-order convergence in $\|\Delta\mathcal{M}\|_F$ confirms numerical stability of the monodromy computation under step-size refinement (Table 2). The minimum singular value $\min_t \sigma_{\min}(K(t)) = 0.2254 > 0$ (MATLAB-verified) confirms that the input matrix satisfies the full-rank assumption of Theorem 5.5 throughout the period.

All numerical results were independently verified using MATLAB (`ode45`) with tolerance refinement and cross-checked using Python/SciPy `RK45` implementation.

8 Limitations and Scope

Several assumptions underlying the present framework deserve explicit acknowledgement, as they delineate the boundaries of the current results and point towards natural extensions.

Periodicity requirement. The K–H equivalence established here depends fundamentally on the T-periodicity of the coefficient matrices $A(t)$, $B(t)$, $F(t)$, $G(t)$, and $K(t)$. The monodromy matrix $\mathcal{M}$ and its minimal polynomial—the central objects

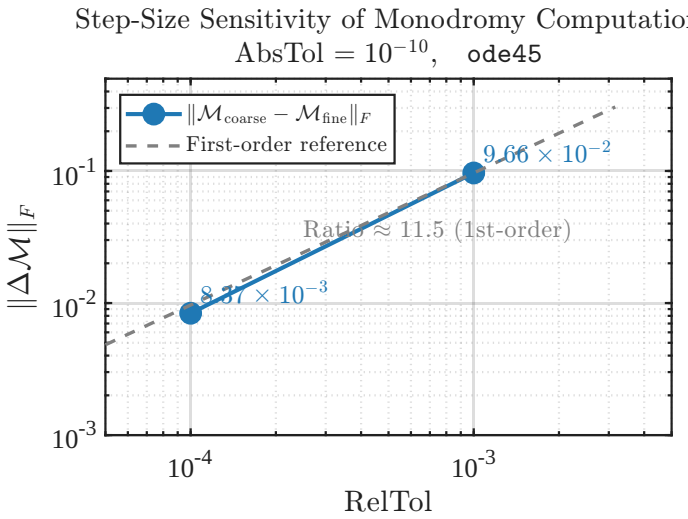


Fig. 4 Step-size sensitivity of the monodromy computation. The ratio ≈ 11.54 between successive $\|\Delta\mathcal{M}\|_F$ values is consistent with first-order convergence of `ode45`.

in the equivalence proof—are well-defined only in the periodic setting. Extension to quasi-periodic or almost-periodic coefficients would require a different spectral framework, most likely involving the Sacker–Sell spectrum or Bohr almost-periodic analysis, and is left as an open problem.

Deterministic perturbations only. The controllability horizon kT and the Gramian recursion are developed under deterministic, bounded perturbations of F and G. When the bilinear coupling matrices are subject to stochastic forcing—as arises in multiplicative noise models for networked control systems—the composite generator $\mathcal{A}(t)$ becomes a random operator, and the Gramian must be replaced by its expectation. A stochastic extension of the present equivalence theory, connecting to mean-square controllability, is an open direction.

Finite state dimension. The Cayley–Hamilton bound $k \leq n^2$ on the controllability horizon is specific to finite-dimensional systems. For infinite-dimensional bilinear evolution equations (e.g., bilinear parabolic PDEs), no finite-degree minimal polynomial exists in general, and the equivalence proof architecture would require substantial modification. Concretely, the proof of Lemma 5.3 constructs an annihilating polynomial $p_{\min}(z)$ of degree at most n^2 via the Cayley–Hamilton theorem, then factors out an eigenvector of $\mathcal{M}^*$ via $q(\mathcal{M}^*)V$. In an infinite-dimensional Hilbert space setting, the spectrum of the monodromy operator need not be discrete and the minimal polynomial may be infinite-degree or undefined entirely, so neither the horizon bound $k \leq n^2$ nor the polynomial factoring step would remain valid. A spectral theory extension based on the Sacker–Sell dichotomy or Riesz basis arguments would likely be required, and is left as an open problem.

These limitations are inherent to the algebraic methods employed here and do not diminish the practical scope of the results: the generalised bilinear matrix periodic class

captures a broad range of engineering models, including covariance steering, matrix compartmental pharmacokinetics, and periodic network synchronisation.

9 Conclusion

This paper establishes a Kalman–Hewer controllability equivalence for the generalized bilinear matrix periodic system

$$\dot{X} = A(t)X + XB(t) + F(t)XG(t) + K(t)U(t),$$

resolving the structural obstruction posed by the non-factorizability of the composite monodromy $\mathcal{M}$. The proof strategy—a Gramian energy argument for K⇒H and a minimal-polynomial construction for H⇒K—is entirely independent of any Kronecker product structure and thus applies to a broad class of matrix periodic systems. Together with the companion results for the Lyapunov and Sylvester tiers [34], this advances the K–H controllability hierarchy for matrix periodic systems.

Several directions remain open. A practical note on scalability: for the present example ($n = 3$, lifted dimension $n^2 = 9$), the classical Kronecker-based approach would require storing and integrating a 256×256 system ODE at $n = 4$ (lifted dimension $n^2 = 16$), which in our tests exceeded 5 GB of memory and was excluded from direct comparison. This memory scaling ($O(n^4)$) makes plain the practical necessity of the Kronecker-free recursion developed here: for systems of engineering relevance ($n \geq 4$), the classical approach is not merely slower but infeasible, while Algorithm 1 requires only $O(n^2)$ storage throughout.

The discrete-time analogue requires a different spectral argument because the product structure of the monodromy introduces additional algebraic constraints. A robustness analysis of the controllability horizon kT under structured perturbations of the bilinear coupling matrices F and G would provide quantitative guarantees for applications in switched and hybrid periodic systems [14]. Extension to the infinite-dimensional setting, where the vector state lies in a Hilbert space, is a longer-term open problem whose difficulty is compounded by the absence of a finite-dimensional Cayley–Hamilton bound. This work establishes a structurally complete extension of the Kalman–Hewer equivalence to the most general matrix periodic setting currently tractable without factorization assumptions.

Relationship to the companion observability paper

A companion paper [40] addresses the dual observability theory for system (1) and establishes a spectral energy lower bound $\lambda_{\min}(\widetilde{O}_i) \geq \sigma_{\min}(\mathcal{M})^{2(i-1)}\lambda_{\min}(\widetilde{O}_1)$ that has no counterpart in the present work. Each paper is self-contained.

Author Contributions

S.V.D.S.M. conceived the theoretical framework, proved the main results (Lemmas 5.1–5.3, Theorems 5.4–6.1), designed and implemented the MATLAB numerical verification, and drafted the manuscript. S.P.K. contributed to the literature

survey, independently verified all numerical results using a Python/SciPy RK45 implementation, and assisted in manuscript preparation. Both authors read and approved the final version.

Conflict of Interest

The authors declare no conflict of interest.

Ethics Statement

This work contains no studies with human participants or animals. No ethical approval was required.

Consent to Participate

Not applicable.

Consent to Publish

Not applicable.

Use of AI Assistance

During the preparation of this manuscript, the authors used Claude (Anthropic, https://claude.ai) in a limited and supervised capacity for language editing, grammatical proofreading, and LaTeX formatting only. No AI tool was used to generate mathematical content, theorems, proofs, or numerical results. All scientific content is entirely the work of the human authors. The authors take full responsibility for the integrity of this work, in accordance with Springer Nature's policy on the use of generative AI tools.

Data Availability

All MATLAB scripts, verified numerical outputs, and figures supporting the findings of this study are openly available at the public GitHub repository: https://github.com/msvdsudarsan/Bilinear-Matrix-Periodic-Controllability. The repository includes `Paper1_Controllability_Verification.m` (reproduces all numerical results in Section 7), `Paper1_Figure_Generation.m` (generates Figures 1–4), and `MATLAB_OUTPUTS_BILINEAR_CTRL.txt` (archived verified output).

Acknowledgements

The authors thank the Chancellor and Vice-Chancellor of Dr. RVR NRI Institute of Technology (Deemed to be University), Pothavarappadu Village, for institutional support, and SRM University AP for computational access during revision.

References

[1] Kalman, R.E.: Contributions to the theory of optimal control. Boletin de la Sociedad Matematica Mexicana **5**(2), 102–119 (1960)

[2] Brunovský, P.: Controllability and linear closed-loop controls in linear periodic systems. Journal of Differential Equations **6**, 296–313 (1969) https://doi.org/10.1016/0022-0396(69)90019-9

[3] Hewer, G.A.: Periodicity, detectability and the matrix riccati equation. SIAM Journal on Control and Optimization **13**(6), 1235–1251 (1975) https://doi.org/10.1137/0313077

[4] Bittanti, S., Colaneri, P., Guardabassi, G.: H-controllability and observability of linear periodic systems. SIAM Journal on Control and Optimization **22**(6), 889–893 (1984) https://doi.org/10.1137/0322057

[5] Bittanti, S., Colaneri, P.: Periodic Systems: Filtering and Control. Springer, London (2009)

[6] Bittanti, S., Colaneri, P., De Nicolao, G.: The periodic riccati equation. In: Bittanti, S., Laub, A.J., Willems, J.C. (eds.) The Riccati Equation, pp. 127–162. Springer, Berlin (1991)

[7] Hautus, M.L.J.: Controllability and observability conditions of linear autonomous systems. Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A **72**(3), 443–448 (1969)

[8] Johnson, C.R.: Controllability and observability of linear systems. SIAM Review **8**(2), 263–275 (1966)

[9] Kano, H., Nishimura, T.: Periodic solutions of matrix riccati equations with detectability and stabilizability. International Journal of Control **29**(3), 471–487 (1979) https://doi.org/10.1080/00207177908922712

[10] Floquet, G.: Sur les équations différentielles linéaires à coefficients périodiques. Annales scientifiques de l'École Normale Supérieure **12**(2), 47–88 (1883)

[11] Montagnier, P., Spiteri, R.J., Angeles, J.: The control of linear time-periodic systems using Floquet–Lyapunov theory. International Journal of Control **77**(5), 472–490 (2004) https://doi.org/10.1080/00207170420001961126

[12] Klamka, J.: Controllability of Dynamic Systems. Kluwer Academic, Dordrecht (1991)

[13] Halanay, A.: Differential Equations: Stability, Oscillations, Time Lags. Academic Press, New York (1966)

[14] Liberzon, D.: Switching in Systems and Control. Birkhäuser, Boston (2003)

[15] Zheng, J., Zhao, L., Guo, J.: Periodic event-triggered control for linear systems subject to external disturbances. International Journal of Control **93**(6), 1527–1538 (2020) https://doi.org/10.1080/00207179.2018.1536225

[16] Köhler, J., Müller, M.A., Allgöwer, F.: Periodic cooperation in multi-agent systems via periodic model predictive control. IEEE Transactions on Automatic Control **68**(5), 2883–2898 (2023) https://doi.org/10.1109/TAC.2022.3182052

[17] Murthy, K.N., Anand, P.V.S.: Controllability and observability of continuous matrix lyapunov systems. In: Advances in Nonlinear Dynamics, Stability and Control, pp. 365–379. Springer, Boston (1997)

[18] Putcha, V.S.: Discrete linear Sylvester repetitive process. Nonlinear Studies **21**(2), 205–218 (2014)

[19] Benner, P., Breiten, T.: Two-sided projection methods for nonlinear model order reduction of bilinear systems. SIAM Journal on Scientific Computing **37**(2), 239–260 (2018)

[20] Damm, T.: Bilinear control systems: Theory and applications. Annual Reviews in Control **47**, 65–74 (2019)

[21] Breiten, T., Damm, T.: Krylov subspace methods for bilinear systems. Systems & Control Letters **151**, 104915 (2021)

[22] Benner, P., Goyal, P.: Balanced truncation model order reduction for bilinear systems. Automatica **144**, 110490 (2022)

[23] Benner, P., Goyal, P., Gugercin, S.: $\mathcal{H}_2$-quasi-optimal model order reduction for quadratic-bilinear control systems. SIAM Journal on Matrix Analysis and Applications **39**(2), 983–1032 (2022) https://doi.org/10.1137/16M1098280

[24] Breiten, T., Benner, P.: Recent trends in bilinear system theory and model reduction. Annual Reviews in Control (2023)

[25] Köhler, J., Müller, M.A., Allgöwer, F.: Periodic model predictive control for bilinear systems with guaranteed stability. Automatica **158**, 111298 (2023)

[26] Berger, T., Trützscher, A.: Funnel control for a moving water tank. Automatica **135**, 109999 (2022) https://doi.org/10.1016/j.automatica.2021.109999

[27] Ilchmann, A., Reis, T.: Controllability of time-varying descriptor systems. Systems & Control Letters **172**, 105438 (2023)

[28] Mehrmann, V., Unger, B.: Control of port-Hamiltonian differential-algebraic systems and applications. Acta Numerica **32**, 395–515 (2023) https://doi.org/10.

1017/S0962492922000083

[29] Trenn, S., Wirth, F.: Linear switched DAEs: Lyapunov exponents, a converse Lyapunov theorem, and Barabanov norms. IEEE Transactions on Automatic Control **68**(2), 893–906 (2023) https://doi.org/10.1109/TAC.2022.3148569

[30] Zhang, Y., Li, H., Lewis, F.L.: Data-driven controllability and observability analysis for unknown linear dynamical systems. Automatica **153**, 111027 (2023) https://doi.org/10.1016/j.automatica.2023.111027

[31] Madhyannapu, S.V.D.S., Pradheep Kumar, S.: Adaptive physics-informed neural networks for singular matrix differential systems with algebraic structure preservation: Applications to optimal control synthesis. SSRN Electronic Journal (2026) https://doi.org/10.2139/ssrn.6277631 . Available at: https://ssrn.com/abstract=6277631

[32] Madhyannapu, S.V.D.S., Pradheep Kumar, S.: An efficient adjoint-free melnikov–lyapunov diagnostic for transition detection in slow–fast chemical reaction networks. Communications in Nonlinear Science and Numerical Simulation (2026) https://doi.org/10.2139/ssrn.6275667 . Under Review

[33] Madhyannapu, S.V.D.S., Pradheep Kumar, S.: Controllability breakdown in singular bilinear periodic systems under Melnikov-type perturbations. Journal of the Franklin Institute (2026). Submission ID: FI-D-26-01142, With Editor

[34] Madhyannapu, M.S.V.D.S., Putcha, V.S., Deekshitulu, G.V.S.R.: Hewer controllability and Kalman controllability of Lyapunov matrix periodic systems. i-manager's Journal on Mathematics **14**(1), 33–42 (2025) https://doi.org/10.26634/jmat.14.1.21822

[35] Madhyannapu, S.V.D.S., Putcha, V.S., Deekshitulu, G.V.S.R.: Equivalence of Kalman and Hewer controllability for Sylvester matrix periodic systems. Archives of Control Sciences (2026). Submission ID: 1703/2026, Under Review

[36] Chen, C.T.: Linear System Theory and Design, 4th edn. Oxford University Press, New York (2013)

[37] Blanes, S., Casas, F., Oteo, J.-A., Ros, J.: The magnus expansion and some of its applications. Physics Reports **470**(5–6), 151–238 (2009) https://doi.org/10.1016/j.physrep.2008.11.001

[38] Madhyannapu, S.V.D.S., Pradheep Kumar, S.: A Kronecker-free block-wise strategy for reachability Gramian computation in periodic Sylvester matrix differential systems. International Journal of Computer Mathematics (2026). Manuscript ID: 256528710, Taylor & Francis, With Editor

[39] Anderson, B.D.O., Moore, J.B.: Optimal control: Linear quadratic methods.

Dover Publications (2007). Reprint of the 1989 Prentice-Hall edition

[40] Madhyannapu, S.V.D.S., Pradheep Kumar, S.: Kalman–Hewer observability equivalence and spectral energy bounds for generalised bilinear matrix periodic systems. Systems & Control Letters (2026). Manuscript ID: SCL-D-26-00234, Under Review. Preprint: https://doi.org/10.2139/ssrn.6573249

Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- ControllabilityVerification.m
- MATLABOUTPUTSBILINEARCTRL.txt
- FigureGeneration.m