

A consistent projection method for the Micropolar Navier-Stokes equations

Zhiyong Si

sizhiyong@hpu.edu.cn

Henan Polytechnic University

Ziyi Li

Henan Polytechnic University

Yunxia Wang

Henan Polytechnic University

Research Article

Keywords: Micropolar Navier-Stokes equations; Consistent projection method; Energy estimates; Optimal error estimates

Posted Date: May 7th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-9630157/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

A consistent projection method for the Micropolar Navier-Stokes equations*

Zhiyong Si[†], Ziyi Li[†], Yunxia Wang[†]

May 6, 2026

Abstract

In this paper, we introduce a consistent projection method, namely the Gauge-Uzawa method, for the numerical solution of incompressible Micropolar Navier-Stokes Equations (MNSE). This method integrates the advantages of both gauge and Uzawa methodologies into a unified variational framework. By adopting a fully discrete projection strategy, the scheme efficiently decouples the computations of velocity, angular velocity, and pressure fields—thus enhancing computational efficiency while maintaining numerical accuracy. We conduct a rigorous theoretical analysis of the proposed method, establishing its unconditional stability and deriving optimal error estimates for all primary variables (velocity, angular velocity, and pressure) under appropriate norms. Numerical experiments verify that the method achieves the expected convergence order across a series of test cases. The results are in excellent agreement with theoretical predictions, fully demonstrating the effectiveness and reliability of the proposed approach.

Keywords: Micropolar Navier-Stokes equations; Consistent projection method; Energy estimates; Optimal error estimates

1 Introduction

In this paper, we investigate the Micropolar Navier-Stokes Equations (MNSE)—a time-dependent system of partial differential equations that couples linear velocity and pressure with angular velocity. This model describes the dynamics of continuous media where material particles exhibit both translational and rotational degrees of freedom. It plays a crucial role in modeling polarizable media and has broad applications in fields including smart fluids, ferrofluid pumping, liquid crystals, and collisional granular flows. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) denote a bounded, simply-connected domain that is either convex or has a $C^{1,1}$ -smooth boundary $\partial\Omega$. We propose and analyze a consistent projection finite element method for the numerical solution of the MNSE. This method integrates the advantages of both gauge and Uzawa methodologies into a unified variational framework and adopts a fully discrete projection strategy, which efficiently decouples the computations of velocity, angular velocity, and pressure fields, thereby enhancing computational efficiency while ensuring numerical accuracy and stability. The MNSE are given as follows,

$$\begin{cases} \mathbf{u}_t - (\nu + \nu_r)\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 2\nu_r \operatorname{curl} \mathbf{w} + \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ j\mathbf{w}_t - (c_a + c_d)\Delta \mathbf{w} + j(\mathbf{u} \cdot \nabla)\mathbf{w} - (c_0 + c_d - c_a)\nabla \operatorname{div} \mathbf{w} + 4\nu_r \mathbf{w} = 2\nu_r \operatorname{curl} \mathbf{u} + \mathbf{g}, \end{cases} \quad (1.1)$$

for $(x, t) \in Q_T$, where $Q_T = \Omega \times (0, T]$ with $T \in (0, \infty)$ fixed; here, $\mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}$ and $\mathbf{w}_t = \frac{\partial \mathbf{w}}{\partial t}$. Specifically, $\mathbf{u}$ denotes the velocity field, p the pressure, and $\mathbf{w}$ the angular velocity—all of which

*This work is supported by the Natural Science Foundation of Henan Province (No. 252300421989) and the Innovative Research Team of Henan Polytechnic University, China (No. T2024-4).

[†]School of Mathematics and Information Science, Henan Polytechnic University, 454003, Jiaozuo, P.R. China. sizhiyong@hpu.edu.cn

are the unknowns. The vector $\mathbf{f}$ represents the external body force density per unit mass, $\mathbf{g}$ denotes the body moment source, j is the inertia density, and the constants ν, ν_r, c_a, c_d , and c_0 are material parameters (among which ν and ν_r are kinematic viscosities). For further discussions and derivations of these equations, refer to [29]. The initial conditions for the velocity $\mathbf{u}$ and angular velocity $\mathbf{w}$ are specified as follows:

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \mathbf{w}(x, 0) = \mathbf{w}_0(x), \quad (1.2)$$

and the boundary conditions are given for any $x \in \partial\Omega$ and $t \in (0, T)$,

$$\mathbf{u} = \mathbf{0}, \mathbf{w} = \mathbf{0}. \quad (1.3)$$

To simplify notation, we set

$$\nu_0 = \nu + \nu_r, \quad c_1 = c_a + c_d, \quad c_2 = c_0 + c_d - c_a, \quad (1.4)$$

and assume $c_1, c_2 > 0$ (see, e.g., [25]).

In recent years, remarkable advancements have been achieved in both theoretical analysis and numerical applications pertaining to the MNSE. Ortega and Rojas [30] derived optimal error estimates for the penalty finite element method for the MNSE. Mitarai and Hayakawa [24] studied particle-laden flows in micropolar fluids, a topic with wide-ranging applications across diverse disciplines. Eringen laid the foundation for microcontinuum field theories of fluids and solids in [9, 10, 11]. Additionally, Papautsky and Brazzle [32] explored the application of micropolar fluid theory to model laminar flow in microchannels. Franchi and Straughan [12] analyzed the nonlinear stability of thermal convection in micropolar fluids with temperature-dependent viscosity. Ciarletta [5] established a theoretical framework for heat-conducting micropolar fluids and derived fundamental theorems for unsteady thermal flows. Nochetto et. al. [29] proposed and analyzed a first-order semi-implicit fully discrete scheme for the MNSE, and examined several of its key properties. Hills [22] established uniqueness and continuous dependence results for linear micropolar fluid flows, whereas Forste [13] investigated the existence of time-independent solutions for incompressible micropolar fluids under specific transformations. Rosensweig [38] utilized the MNSE to examine the effect of ferrofluids on magnetic fields and derived the corresponding magnetic field equation. Ren and Liu [36] proposed three fractional-step finite element schemes for two- and three-dimensional unsteady incompressible thermal micropolar fluid flows. Rinaldi and Zahn [37] investigated the role of spin viscosity in determining ferrofluid flow profiles under alternating and rotating magnetic fields. Xin and Liu [48] compared three iterative methods for finite element solutions of steady-state 2D/3D micropolar fluid equations. Furthermore, the analysis and numerical modeling of the MNSE are motivated by experimental studies, such as those on ferromagnetic fluid pumping. Zahn and Greer [50] performed numerical simulations under spatially uniform and time-sinusoidal magnetizing fields, demonstrating that angular momentum effects in the MNSE can effectively induce fluid motion. In [43, 42], we presented rotational velocity-correction projection method and the modified characteristics FEMs for the Micropolar Navier-Stokes Equations.

In the late 1960s, the projection method for decoupling velocity and pressure was pioneered by Chorin [4] and Temam [45]. Projection methods fall broadly into three categories: velocity-correction methods (see [15, 16]), pressure-correction methods (see [46, 35]), and consistent splitting methods (see [17, 23, 40]). E and Liu [31, 8] first proposed the gauge method, introducing an auxiliary vector field $\mathbf{a}$ and a gauge variable ϕ to solve the incompressible Navier-Stokes equations. Wang and Liu [47] carried out theoretical analyses of E and Liu's formulation, validating its computational merits. They further examined the implicit gauge method, which adopts backward Euler or Crank-Nicolson schemes for time discretization. However, the gauge method has certain limitations, such as the non-variational nature of the boundary term involving ∂_t . To mitigate these limitations while retaining the merits of the gauge method, Nochetto and Pyo [26] introduced a new vector variable $\tilde{\mathbf{u}}^{n+1}$ subject to the zero boundary condition $\tilde{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^n$. Through a sequence of technical manipulations and innovations, they developed the gauge-Uzawa method—also termed the consistent projection method. In recent years, a growing number of researchers have applied the consistent projection method to various fluid dynamic equations. For example, Nochetto and

Pyo [27] derived error estimates for the semi-discrete gauge method for the Navier-Stokes equations. For time-dependent viscoelastic Oldroyd flows, Si et al. [44] proposed a gauge-Uzawa finite element method and established optimal error estimates. Yang and Si [49] studied a consistent projection finite element method for incompressible magnetohydrodynamic (MHD) equations, while Si and Hou [41] extended its use to non-stationary incompressible thermally coupled MHD equations.

In this paper, we propose and analyze a fully discrete consistent projection method (Gauge-Uzawa method) for the MNSE. Error estimation for this method is quite challenging; to overcome these challenges, we first derive semi-discrete error estimates for the velocity and angular velocity, and establish the bound $\|\mathbf{U}\|_2 \leq C$, which together lay the foundation for deriving the fully discrete error estimates. The remainder of the paper is organized as follows: In Section 2, we introduce the consistent projection scheme and elaborate on the necessary functional framework for the MNSE. Section 3 is dedicated to analyzing the stability of the numerical solutions. In Section 4, we derive error estimates for the proposed consistent projection finite element method for the MNSE. Finally, in Section 5, we report numerical experiments to verify the theoretical results.

2 Notation and preliminaries

In this section, we first introduce the necessary notations for subsequent analysis. We adopt Sobolev spaces $L^2(\Omega)^d$ and $H^1(\Omega)^d$, which are endowed with the norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. The standard L^2 -inner product, which applies to both scalar and vector-valued functions on Ω , is denoted by the pairing $(\cdot, \cdot)$. Additionally, we further define the following Sobolev spaces:

$$\begin{aligned} X &= H_0^1(\Omega)^d, \quad d = \{2, 3\}, \quad H_0^1(\Omega) = \{\mathbf{v} \in H^1(\Omega), \mathbf{v}|_{\partial\Omega} = 0\}, \\ M &= L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q dx = 0 \right\}, \\ V &= \{\mathbf{v} \in X, \operatorname{div} \mathbf{v} = 0\}, \\ W &= H_n^1(\Omega) = \{\mathbf{v} \in H^1(\Omega)^d, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}. \end{aligned}$$

The Sobolev space $H_0^1(\Omega)^d$ is equipped with the following norm:

$$\forall \alpha \in H_0^1(\Omega)^d, \|\alpha\|_1 = \sqrt{(\|\nabla \alpha\|_0^2 + \|\alpha\|_0^2)}. \quad (2.1)$$

Furthermore, let T_h be a family of regular and quasi-uniform triangulations of Ω , consisting of triangles (for $d = 2$) or tetrahedrons (for $d = 3$) K_j with $j = 1, 2, \dots, N$. Here, $h = \max_{K \in T_h} h_K$ denotes the mesh size, where h_K is the diameter of element K [3, 2, 14]. The scalar and vector-valued finite element spaces are:

$$\begin{aligned} X_h &:= \{\mathbf{v}_h \in X : \mathbf{v}_h|_K \in P_k(K)^d \ (d = 2, 3)\}, \\ V_h &:= \{\mathbf{v}_h \in X_h : \nabla \cdot \mathbf{v}_h = 0\}, \\ P_h &:= \{q_h \in L_0^2(\Omega) \cap C^0(\Omega) : q_h|_K \in P_l(K), \ \forall K \in T_h\}, \\ Y_h &:= \{\mathbf{v}_h \in C^0(\Omega)^d : \mathbf{v}_h|_K \in P_k(K)^d \ (d = 2, 3)\}, \\ W_h &:= Y_h \cap H_0^1(\Omega), \end{aligned}$$

Here, $P_k(K)$, $P_q(K)$, and $P_l(K)$ are polynomial spaces whose degrees are uniformly bounded with respect to $K \in T_h$ [3, 2, 14]. Let $Z(\Omega) = \{\mathbf{z} \in H_0^1(\Omega)^d : \nabla \cdot \mathbf{z} = 0\}$ and $Z(\Omega)^*$ denote the dual space of $Z(\Omega)$, endowed with the norm $\|\cdot\|_*$. Let $0 < t_1 < \dots < t_{N+1} = T$ be a uniform partition of the time interval $[0, T]$ with $t_n = n\tau$, where $\tau = T/(N+1)$ denotes the time step size and N is a positive integer. We define the backward difference quotient operator as

$$D_{\tau} f^n = \frac{f^n - f^{n-1}}{\tau},$$

where $\{f^n\}_{n=0}^{N+1}$ is a sequence of functions.

For a nonnegative integer m and $r \in [1, \infty]$, we consider the Sobolev space $W^{m,r}(\Omega)$, which is equipped with the norm (see [3]) as follows:

$$\|f\|_{m,r} = \begin{cases} (\sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha} f|^r dx)^{\frac{1}{r}}, & 1 \leq r < \infty, \\ \sum_{|\alpha|} \text{esssup}_{x \in \Omega} |D^{\alpha} f(x)|, & r = \infty, \end{cases}$$

where $\partial^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$ for the multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $\alpha_i \geq 0$ for $i = 1, \dots, d$. We further denote ∂_t as the partial derivative with respect to time. Let $\tau > 0$ be the time step, and set $t_k = k\tau$ for $0 \leq k \leq K$, where $K = \lceil T/\tau \rceil$ denotes the integer part of T/τ .

Next, we define the following trilinear form as follows, for all $\mathbf{u} \in V$, $\mathbf{v}, \mathbf{w} \in X$,

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} dx + \frac{1}{2} \int_{\Omega} (\text{div } \mathbf{u}) \mathbf{v} \cdot \mathbf{w} dx,$$

and the properties of the trilinear form [1, 18, 19] are helpful in our analysis

$$B(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -B(\mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \mathbf{u} \in V, \mathbf{v}, \mathbf{w} \in X,$$

$$|B(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C_0^2 \|\nabla \mathbf{u}\|_0 \|\nabla \mathbf{v}\|_0 \|\nabla \mathbf{w}\|_0, \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in X,$$

$$|B(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{N}{2} \|\mathbf{u}\|_0 (\|\nabla \mathbf{v}\|_0 \|\mathbf{w}\|_{L^\infty} + \|\mathbf{v}\|_{L^6} \|\nabla \mathbf{w}\|_{L^3}), \quad \mathbf{u} \in L^2(\Omega)^d, \mathbf{v} \in X, \mathbf{w} \in L^\infty(\Omega)^d \cap X,$$

$$|B(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \frac{N}{2} (\|\mathbf{u}\|_{L^\infty} \|\nabla \mathbf{v}\|_0 + \|\nabla \mathbf{u}\|_{L^3} \|\mathbf{v}\|_{L^6}) \|\mathbf{w}\|_0, \quad \mathbf{u} \in L^\infty(\Omega)^d \cap X, \mathbf{v} \in X, \mathbf{w} \in L^2(\Omega)^d,$$

$$\|\mathbf{v}\|_0 \leq \gamma_0 \|\nabla \mathbf{v}\|_0, \quad \|\mathbf{v}\|_{L^3} \leq C \|\mathbf{v}\|_0^{\frac{1}{2}} \|\nabla \mathbf{v}\|_0^{\frac{1}{2}}, \quad \|\mathbf{v}\|_{L^6} \leq C \|\nabla \mathbf{v}\|_0, \quad \forall \mathbf{v} \in X,$$

$$\|\mathbf{v}\|_{L^\infty} \leq C \|\mathbf{v}\|_1^{\frac{1}{2}} \|\mathbf{v}\|_2^{\frac{1}{2}}, \quad \forall \mathbf{v} \in H^2(\Omega)^d,$$

$$\|\nabla \mathbf{v}\|_0^2 = \|\nabla \cdot \mathbf{v}\|_0^2 + \|\nabla \times \mathbf{v}\|_0^2, \quad \forall \mathbf{v} \in H_0^1(\Omega)^d,$$

$$\|\mathbf{u}\| = \|\mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{u}\|_{L^3}, \quad \forall \mathbf{u} \in X.$$

Here, $N > 0$ is a constant, γ_0 is a positive constant depending only on Ω , and C_0 is an embedding constant also depending only on Ω . We denote C as a generic positive constant, which may depend on the data and the solution and whose value may differ in different contexts. We assume that the pair $X_h \times P_h$ satisfies the discrete inf-sup condition.

$$\exists \beta > 0, \quad \forall q_h \in P_h, \quad \beta \|q_h\|_0 \leq \sup_{\mathbf{v}_h \in X_h \cap H_0^1(\Omega)} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_0}.$$

The following properties of $B(\cdot, \cdot, \cdot)$ for all $\mathbf{u}_h \in V_h$, $\mathbf{v}_h, \mathbf{w}_h \in X_h$ are as follow

$$B(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) = -B(\mathbf{u}_h, \mathbf{w}_h, \mathbf{v}_h), \quad B(\mathbf{u}_h, \mathbf{v}_h, \mathbf{v}_h) = 0.$$

Lemma 2.1 (Grad-Div inequality [33, 34]) If $\mathbf{v} \in H_0^1(\Omega)$ we can get

$$\|\nabla \cdot \mathbf{v}\|_0 \leq \|\nabla \mathbf{v}\|_0.$$

We recall a discrete version of Grönwall's inequality established in [1].

Lemma 2.2 [1] Let a_n, b_n, c_n and γ_n , for integer $n \geq 0$, be the nonnegative numbers such that

$$a_m + \tau \sum_{n=0}^m b_n \leq \tau \sum_{n=0}^m \gamma_n a_n + \tau \sum_{n=0}^m c_n + B, \quad \forall m \geq 0. \quad (2.2)$$

Suppose that $\tau\gamma_n < 1$, for all n , and set $\sigma_n = (1 - \tau\gamma_n)^{-1}$. Then

$$a_m + \tau \sum_{n=0}^m b_n \leq \exp(\tau \sum_{n=0}^m \gamma_n \sigma_n) (\tau \sum_{n=0}^m c_n + B), \quad \forall m \geq 0. \quad (2.3)$$

Assumption (A1): The initial data $\mathbf{u}_0, \mathbf{w}_0$, and the body forces $\mathbf{f}, \mathbf{g}$ satisfy

$$\|\mathcal{A}\mathbf{u}_0\|_0 + \|\mathcal{A}\mathbf{w}_0\|_0 + \sup_{0 \leq t \leq T} \{\|\mathbf{f}(t)\|_0 + \|\mathbf{g}(t)\|_0\} \leq C. \quad (2.4)$$

Assumption (A2): We assume that the MNSE problem has a unique strong solution $(\mathbf{u}, \mathbf{w}, p)$ on $[0, T]$ such that

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\mathcal{A}\mathbf{u}(t)\|_0 + \|\mathbf{w}(t)\|_0 + \|p(t)\|_1 + \|\mathbf{u}_t(t)\|_0 + \|\mathbf{w}_t(t)\|_0) \\ & + \int_0^T (\|\mathbf{u}_t\|_1^2 + \|\mathbf{w}_t\|_1^2 + \|p_t\|_1^2 + \|\mathbf{u}_{tt}\|_0^2 + \|\mathbf{w}_{tt}\|_0^2) dt \leq C. \end{aligned} \quad (2.5)$$

Assumption (A3): In order to get the optimal approximation order of the velocity, we need further regularity assumptions as follows

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\nabla \mathbf{u}_t(t)\|_0 + \|\nabla \times \nabla \times \mathbf{u}_t(t)\|_0 + \|\mathbf{u}_{tt}(t)\|_0 + \|\mathbf{w}_{tt}(t)\|_0 + \|p_t(t)\|_1) \\ & + \int_0^T (\|\mathcal{A}\mathbf{u}_t\|_0^2 + \|\mathbf{w}_t\|_2^2 + \|\mathbf{u}_{tt}\|_0^2 + \|p_{tt}\|_1^2) dt \leq C. \end{aligned} \quad (2.6)$$

Assumption (A4): [26, 28, 44, 41] Assume that the boundary of Ω is smooth enough that the unique solution θ of the steady Poisson problem

$$-\Delta \theta = \varphi, \quad \text{in } \Omega, \quad \theta|_{\Gamma_d} = 0, \quad \frac{\partial \theta}{\partial \mathbf{n}} \Big|_{\Gamma_n} = 0, \quad (2.7)$$

for prescribed $\varphi \in L^2(\Omega)$ satisfies

$$\|\theta\|_2 \leq c\|\varphi\|_0; \quad (2.8)$$

the steady Stokes problem

$$-\Delta \mathbf{v} + \nabla q = \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0, \quad \text{in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = \mathbf{0}, \quad (2.9)$$

for prescribed $\mathbf{f} \in L^2(\Omega)^3$ has a unique solution $(\mathbf{v}, q)$ which satisfies

$$\|\mathbf{v}\|_2 + \|q\|_1 \leq c\|\mathbf{f}\|_0, \quad (2.10)$$

and Maxwell's equations

$$\operatorname{curl} \operatorname{curl} \mathbf{B} = \mathbf{s}, \quad \operatorname{div} \mathbf{B} = 0, \quad \text{in } \Omega, \quad \mathbf{n} \times \operatorname{curl} \mathbf{B} = \mathbf{0}, \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (2.11)$$

for the prescribed $\mathbf{s} \in L^2(\Omega)^3$ admit a unique solution $\mathbf{B} \in V_n$ which satisfies

$$\|\mathbf{B}\|_2 \leq c\|\mathbf{s}\|_0. \quad (2.12)$$

In [6], we can see the validity of Assumption (A4) if $\partial\Omega$ is of C^2 or if Ω is a convex polyhedron.

Lemma 2.3 ([26]) Now let $(\mathbf{v}_h, q_h) \in V_h \times P_h$ indicate the finite element solution of (2.9) namely

$$\begin{aligned} (\nabla \mathbf{v}_h, \nabla v_h) - (q_h, \nabla \cdot v_h) &= (f, v_h), & \forall v_h \in V_h, \\ (r_h, \nabla \cdot \mathbf{v}_h) &= 0, & \forall r_h \in P_h. \end{aligned}$$

Let $(\mathbf{v}, q) \in X \times M$ be the solution of (2.9), there holds that

$$\begin{aligned} \|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 + h\|q - q_h\|_0 &\leq Ch^{k+1}(\|\mathbf{v}\|_{k+1} + \|q\|_k), \\ \|f\|_* &\leq C\|\nabla \mathbf{v}\|_0 \leq \|f\|_0 + C\|\nabla \mathbf{v}_h\|_0. \end{aligned}$$

Furthermore, we have

$$|||\mathbf{v} - \mathbf{v}_h||| = \|\mathbf{v} - \mathbf{v}_h\|_{L^\infty(\Omega)} + \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L^3(\Omega)} \leq \|f\|_0.$$

Lemma 2.4 ([28]) Let $\varpi \in H_0^1(\Omega)$ be the solution of the Poission's equations (2.11) and $\varpi_h \in \mathcal{T}_h(\Omega)$ indicate the finite element solution of (2.11) given as follows

$$(\nabla \varpi_h, \nabla \chi_h) + (\nabla \cdot \varpi_h, \nabla \cdot \chi_h) = (g, \chi_h), \quad \forall \chi_h \in W_h.$$

Then, there holds that

$$\begin{aligned} \|\varpi - \varpi_h\|_0 + h\|\varpi - \varpi_h\|_1 &\leq Ch^2\|g\|_0, \\ |||\varpi - \varpi_h||| &\leq \|g\|_0. \end{aligned}$$

In addition, we define the Stokes projection $(R_h(\mathbf{u}, p), Q_h(\mathbf{u}, p)) : (X, M) \rightarrow (X_h, P_h)$ by

$$\begin{aligned} \nu_0(\nabla(R_h(\mathbf{u}, p) - \mathbf{u}), \nabla \mathbf{v}_h) - (Q_h(\mathbf{u}, p) - p, \operatorname{div} \mathbf{v}_h) &= 0, \quad \forall \mathbf{v}_h \in X_h, \\ (\operatorname{div}(R_h(\mathbf{u}, p) - \mathbf{u}), q_h) &= 0, \quad \forall q_h \in P_h. \end{aligned}$$

Lemma 2.5 ([7]) Assume that $\mathbf{u} \in H_0^2(\Omega)^d \cap H^{r+1}(\Omega)^d$ and $p \in L_0^2(\Omega) \cap H^r(\Omega)$ ($d=2,3$). Then,

$$\|R_h(\mathbf{u}, p) - \mathbf{u}\|_0 + h(\|\nabla(R_h(\mathbf{u}, p) - \mathbf{u})\|_0 + \|Q_h(\mathbf{u}, p) - p\|_0) \leq Ch^{r+1}(\|\mathbf{u}\|_{r+1} + \|p\|_r),$$

and

$$\|R_h(\mathbf{u}, p)\|_{L^\infty} \leq C(\|\mathbf{u}\|_2 + \|p\|_1). \quad (2.13)$$

Lemma 2.6 ([7]) If $(\mathbf{u}, p) \in W^{2,k}(\Omega)^d \times W^{1,k}(\Omega)$ for $k > d$ ($d=2,3$),

$$\|\nabla R_h(\mathbf{u}, p)\|_{L^\infty} \leq C(\|\mathbf{u}\|_{W^{1,\infty}} + \|p\|_{L^\infty}). \quad (2.14)$$

For simplicity, we denote [7] $R_h^n = R_h(\mathbf{U}^n, P^n)$, $Q_h^n = Q_h(\mathbf{U}^n, P^n)$, $R_{1h}^n = R_{1h}(\mathbf{W}^n)$, $n = 1, 2, \dots, \mathcal{M}$, where $R_{1h} : W \rightarrow W_h$ are the H^1 -orthogonal projection, then

$$(\nabla R_{1h}(\mathbf{W}) - \nabla \mathbf{W}, \nabla \chi_h) + (\nabla \cdot R_{1h}(\mathbf{W}), \nabla \cdot \chi_h) = 0, \quad \forall \chi_h \in W_h.$$

If $\mathbf{W} \in \mathcal{T} \cap H^{r+1}(\Omega)$, there holds that

$$\|R_{1h}(\mathbf{W}) - \mathbf{W}\|_0 + h\|\nabla(R_{1h}(\mathbf{W}) - \mathbf{W})\|_0 \leq Ch^{r+1}\|\mathbf{W}\|_{r+1}, \quad (2.15)$$

$$\|R_{1h}(\mathbf{W})\|_{L^\infty} \leq C\|\mathbf{W}\|_2, \quad \|\nabla R_{1h}(\mathbf{W})\|_{L^\infty} \leq C\|\mathbf{W}\|_{W^{1,\infty}}. \quad (2.16)$$

Next, we give the consistent projection method as follows.

Algorithm 2.1 (The Consistent Projection Scheme) The initial data are given as $s_h^0, (\mathbf{u}_h^0, \mathbf{w}_h^0)$ is given as follows

$$\tilde{\mathbf{u}}_h^0 = \mathbf{u}_h^0 = \mathbf{u}_0, \quad \tilde{\mathbf{w}}_h^0 = \mathbf{w}_h^0 = \mathbf{w}_0, \quad s_h^0 = 0, \quad (2.17)$$

Step 1. Compute $\tilde{\mathbf{u}}_h^{n+1} \in X_h$ as the solution of

$$\begin{aligned} & \left(\frac{\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n}{\tau}, v_h \right) + B(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, v_h) + \nu_0(\nabla \tilde{\mathbf{u}}_h^{n+1}, \nabla v_h) - \nu_0(s_h^n, \operatorname{div} v_h) \\ &= 2\nu_r(\operatorname{curl} \mathbf{w}_h^{n+1}, v_h) + (f(t^{n+1}), v_h), \quad \forall v_h \in X_h. \end{aligned} \quad (2.18)$$

Step 2. Compute $\rho_h^{n+1} \in P_h$ as the solution of

$$(\nabla \rho_h^{n+1}, \nabla \psi_h) = (\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}, \psi_h), \quad \forall \psi_h \in P_h. \quad (2.19)$$

Step 3. Update $s_h^{n+1} \in P_h$ according to

$$(s_h^{n+1}, q_h) = (s_h^n, q_h) - (\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}, q_h), \quad \forall q_h \in P_h. \quad (2.20)$$

Step 4. Update $\mathbf{u}_h^{n+1} \in X_h \oplus \nabla P_h$ according to

$$\mathbf{u}_h^{n+1} = \tilde{\mathbf{u}}_h^{n+1} + \nabla \rho_h^{n+1}. \quad (2.21)$$

Step 5. Compute $p_h^{n+1} \in P_h$ via

$$p_h^{n+1} = \nu_0 s_h^{n+1} - \frac{1}{\tau} \rho_h^{n+1}. \quad (2.22)$$

Step 6. Compute $\mathbf{w}_h^{n+1} \in W_h$ via

$$\begin{aligned} & j \left(\frac{\mathbf{w}_h^{n+1} - \mathbf{w}_h^n}{\tau}, \chi_h \right) + c_1(\nabla \mathbf{w}_h^{n+1}, \nabla \chi_h) + jB(\mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1}, \chi_h) - c_2(\nabla \operatorname{div} \mathbf{w}_h^{n+1}, \chi_h) \\ & + 4\nu_r(\mathbf{w}_h^{n+1}, \chi_h) = 2\nu_r(\operatorname{curl} \mathbf{u}_h^{n+1}, \chi_h) + (g(t^{n+1}), \chi_h), \quad \forall \chi_h \in W_h. \end{aligned} \quad (2.23)$$

3 Stability analysis

Theorem 3.1 Suppose that Assumptions (A1-A3) are satisfied, then the consistent projection FEM is unconditional stable in the sense that, for all $\tau > 0$, there exists a priori bound such that

$$\begin{aligned} & \|\mathbf{u}_h^{N+1}\|_0^2 + j\|\mathbf{w}_h^{N+1}\|_0^2 + \tau\nu_0\|s_h^{N+1}\|_0^2 + \sum_{n=0}^N (\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_0^2 + \|\mathbf{w}_h^{n+1} - \mathbf{w}_h^n\|_0^2) \\ & + \frac{\tau\nu_0}{2}\|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0^2 + \tau c_1\|\nabla \mathbf{w}_h^{n+1}\|_0^2 + \tau\nu_r\|\mathbf{w}_h^{n+1}\|_0^2 + 2\tau c_2\|\operatorname{div} \mathbf{w}_h^{n+1}\|_0^2 + 2\|\nabla \rho_h^{n+1}\|_0^2 \leq C. \end{aligned} \quad (3.1)$$

Proof. Setting $v_h = 2\tau\tilde{\mathbf{u}}_h^{n+1}$ in (2.18), we deduce that

$$\begin{aligned} & 2(\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}) + 2\tau\nu_0\|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0^2 \\ &= 4\tau\nu_r(\operatorname{curl} \mathbf{w}_h^{n+1}, \tilde{\mathbf{u}}_h^{n+1}) + 2\tau\nu_0(s_h^n, \operatorname{div} \tilde{\mathbf{u}}_h^{n+1}) + 2\tau(f(t^{n+1}), \tilde{\mathbf{u}}_h^{n+1}). \end{aligned} \quad (3.2)$$

Thanks to (2.21), we derive that

$$\begin{aligned} (\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}) &= (\mathbf{u}_h^{n+1} - \nabla \rho_h^{n+1} - \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \nabla \rho_h^{n+1}) \\ &= (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{u}_h^{n+1}) + (\nabla \rho_h^{n+1}, \nabla \rho_h^{n+1}). \end{aligned} \quad (3.3)$$

Using (2.20), we deduce that

$$\begin{aligned} 2(s_h^n, \operatorname{div} \tilde{\mathbf{u}}_h^{n+1}) &= 2(s_h^n - s_h^{n+1}, s_h^n) \\ &= \|s_h^n\|_0^2 - \|s_h^{n+1}\|_0^2 + \|s_h^n - s_h^{n+1}\|_0^2. \end{aligned} \quad (3.4)$$

Thanks to (2.20), it infers that

$$\|s_h^n - s_h^{n+1}\|_0 \leq \|\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}\|_0 \leq \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0. \quad (3.5)$$

Setting $\chi_h = 2\tau \mathbf{w}_h^{n+1}$ in (2.23), we derive that

$$\begin{aligned} & 2j(\mathbf{w}_h^{n+1} - \mathbf{w}_h^n, \mathbf{w}_h^{n+1}) + 2\tau c_1 \|\nabla \mathbf{w}_h^{n+1}\|_0^2 + 8\tau \nu_r \|\mathbf{w}_h^{n+1}\|_0^2 + 2\tau c_2 \|\operatorname{div} \mathbf{w}_h^{n+1}\|_0^2 \\ & = 4\tau \nu_r (\operatorname{curl} \mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1}) + 2\tau (g(t^{n+1}), \mathbf{w}_h^{n+1}). \end{aligned} \quad (3.6)$$

Using Hölder's inequality and Young's inequality, we deduce that

$$\begin{aligned} |4\tau \nu_r (\operatorname{curl} \mathbf{w}_h^{n+1}, \tilde{\mathbf{u}}_h^{n+1})| & \leq \frac{\tau c_1}{2} \|\operatorname{curl} \mathbf{w}_h^{n+1}\|_0^2 + C\tau \|\mathbf{u}_h^{n+1}\|_0^2 + C\tau \|\nabla \rho_h^{n+1}\|_0^2, \\ |2\tau (f(t^{n+1}), \tilde{\mathbf{u}}_h^{n+1})| & \leq C\tau \|f(t^{n+1})\|_0^2 + \frac{\tau c_1}{2} \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0^2, \\ |4\tau \nu_r (\operatorname{curl} \mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1})| & \leq \frac{\tau \nu_0}{2} \|\operatorname{curl} \mathbf{w}_h^{n+1}\|_0^2 + C\tau \|\mathbf{u}_h^{n+1}\|_0^2, \\ |2\tau (g(t^{n+1}), \mathbf{w}_h^{n+1})| & \leq C\tau \|g(t^{n+1})\|_0^2 + 7\tau \nu_r \|\mathbf{w}_h^{n+1}\|_0^2. \end{aligned}$$

Combining above inequalities with (3.2), (3.6)

$$\begin{aligned} & \|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2 + \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_0^2 + 2\|\nabla \rho_h^{n+1}\|_0^2 + \frac{\tau \nu_0}{2} \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|_0^2 \\ & + j(\|\mathbf{w}_h^{n+1}\|_0^2 - \|\mathbf{w}_h^n\|_0^2 + \|\mathbf{w}_h^{n+1} - \mathbf{w}_h^n\|_0^2) + \tau c_1 \|\nabla \mathbf{w}_h^{n+1}\|_0^2 \\ & + \tau \nu_r \|\mathbf{w}_h^{n+1}\|_0^2 + 2\tau c_2 \|\operatorname{div} \mathbf{w}_h^{n+1}\|_0^2 + \tau \nu_0 (\|s_h^{n+1}\|_0^2 - \|s_h^n\|_0^2) \\ & \leq C\tau (\|\mathbf{u}_h^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2 + \|f(t^{n+1})\|_0^2 + \|g(t^{n+1})\|_0^2). \end{aligned} \quad (3.7)$$

Summing (3.7) with respect to n from 0 to N and using Gronwall's Lemma, we complete the proof. $\square$

4 Error estimates

To derive the error estimates, we set the initial values as $\mathbf{U}^0 = \mathbf{u}_0$ and $\mathbf{W}^0 = \mathbf{w}_0$, and then compute $\mathbf{U}^{n+1} \in X \cap H^2(\Omega)^d$ and $\mathbf{W}^{n+1} \in W \cap H^2(\Omega)^d$ for spatial dimensions $d = 2, 3$.

$$\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\tau} - \nu_0 \Delta \mathbf{U}^{n+1} + (\mathbf{U}^n \cdot \nabla) \mathbf{U}^{n+1} + \nabla P^{n+1} = 2\nu_r \operatorname{curl} \mathbf{W}^n + f(t^{n+1}), \quad (4.1)$$

$$\nabla \cdot \mathbf{U}^{n+1} = 0, \quad (4.2)$$

$$\begin{aligned} & j \frac{\mathbf{W}^{n+1} - \mathbf{W}^n}{\tau} - c_1 \Delta \mathbf{W}^{n+1} + j(\mathbf{U}^n \cdot \nabla) \mathbf{W}^{n+1} - c_2 \nabla \operatorname{div} \mathbf{W}^{n+1} + 4\nu_r \mathbf{W}^{n+1} \\ & = 2\nu_r \operatorname{curl} \mathbf{U}^{n+1} + g(t^{n+1}). \end{aligned} \quad (4.3)$$

To get Theorem 4.1, we derive the following Lemma.

Lemma 4.1 Suppose Assumptions (A1-A2) are satisfied, we deduce that

$$\begin{aligned} & \|\mathbf{U}^{N+1}\|_0^2 + j\|\mathbf{W}^{N+1}\|_0^2 + \tau \sum_{n=0}^N \left(\nu_0 \|\nabla \mathbf{U}^{n+1}\|_0^2 + c_1 \|\nabla \mathbf{W}^{n+1}\|_0^2 \right. \\ & \quad \left. + 2c_2 \|\operatorname{div} \mathbf{W}^{n+1}\|_0^2 + 8\nu_r \|\mathbf{W}^{n+1}\|_0^2 \right) \leq C. \end{aligned} \quad (4.4)$$

$$\|\nabla \mathbf{U}^{N+1}\|_0^2 + \nu_0 \tau \sum_{n=0}^N \|\mathcal{A} \mathbf{U}^{n+1}\|_0^2 \leq C. \quad (4.5)$$

Proof. Testing (4.1) by $2\tau\mathbf{U}^{n+1}$, (4.3) by $2\tau\mathbf{W}^{n+1}$, we can get

$$2(\mathbf{U}^{n+1} - \mathbf{U}^n, \mathbf{U}^{n+1}) + 2\tau\nu_0\|\nabla\mathbf{U}^{n+1}\|_0^2 = 4\tau\nu_r(\text{curl } \mathbf{W}^n, \mathbf{U}^{n+1}) + 2\tau(f(t^{n+1}), \mathbf{U}^{n+1}), \quad (4.6)$$

$$\begin{aligned} & 2j(\mathbf{W}^{n+1} - \mathbf{W}^n, \mathbf{W}^{n+1}) + 2\tau c_1\|\nabla\mathbf{W}^{n+1}\|_0^2 + 2\tau c_2\|\text{div } \mathbf{W}^{n+1}\|_0^2 + 8\tau\nu_r\|\mathbf{W}^{n+1}\|_0^2 \\ & = 4\tau\nu_r(\text{curl } \mathbf{U}^{n+1}, \mathbf{W}^{n+1}) + 2\tau(g(t^{n+1}), \mathbf{W}^{n+1}). \end{aligned} \quad (4.7)$$

Using Hölder's inequality and Young's inequality, we deduce that

$$\begin{aligned} |4\tau\nu_r(\text{curl } \mathbf{W}^n, \mathbf{U}^{n+1})| & \leq 4\tau\nu_r\|\mathbf{W}^n\|_0\|\text{curl } \mathbf{U}^{n+1}\|_0 \leq \frac{\tau\nu_0}{2}\|\nabla\mathbf{U}^{n+1}\|_0^2 + C\tau\|\mathbf{W}^n\|_0^2, \\ |2\tau(f(t^{n+1}), \mathbf{U}^{n+1})| & \leq 2\tau\|f(t^{n+1})\|_0\|\mathbf{U}^{n+1}\|_0 \leq \frac{\tau\nu_0}{2}\|\nabla\mathbf{U}^{n+1}\|_0^2 + C\tau\|f(t^{n+1})\|_0^2, \\ |4\tau\nu_r(\text{curl } \mathbf{U}^{n+1}, \mathbf{W}^{n+1})| & \leq 4\tau\nu_r\|\mathbf{U}^{n+1}\|_0\|\text{curl } \mathbf{W}^{n+1}\|_0 \leq \frac{\tau\nu_0}{2}\|\nabla\mathbf{W}^{n+1}\|_0^2 + C\tau\|\mathbf{U}^{n+1}\|_0^2, \\ |2\tau(g(t^{n+1}), \mathbf{W}^{n+1})| & \leq 2\tau\|g(t^{n+1})\|_0\|\mathbf{W}^{n+1}\|_0 \leq \frac{\tau c_1}{2}\|\nabla\mathbf{W}^{n+1}\|_0^2 + C\tau\|g(t^{n+1})\|_0^2. \end{aligned}$$

Combining above inequalities with (4.6) and (4.7), it yields that

$$\begin{aligned} & \|\mathbf{U}^{n+1}\|_0^2 - \|\mathbf{U}^n\|_0^2 + \|\mathbf{U}^{n+1} - \mathbf{U}^n\|_0^2 + j(\|\mathbf{W}^{n+1}\|_0^2 - \|\mathbf{W}^n\|_0^2 + \|\mathbf{W}^{n+1} - \mathbf{W}^n\|_0^2) \\ & + \tau\nu_0\|\nabla\mathbf{U}^{n+1}\|_0^2 + \tau c_1\|\nabla\mathbf{W}^{n+1}\|_0^2 + 2\tau c_2\|\text{div } \mathbf{W}^{n+1}\|_0^2 + 8\tau\nu_r\|\mathbf{W}^{n+1}\|_0^2 \\ & \leq C\tau(\|f(t^{n+1})\|_0^2 + \|g(t^{n+1})\|_0^2 + \|\mathbf{W}^n\|_0^2 + \|\mathbf{U}^{n+1}\|_0^2). \end{aligned} \quad (4.8)$$

Adding it with respect to n from $n = 0$ to $n = N$ and using Gronwall's Lemma, we derive that

$$\begin{aligned} & \|\mathbf{U}^{N+1}\|_0^2 + j\|\mathbf{W}^{N+1}\|_0^2 + \tau \sum_{n=0}^N (\nu_0\|\nabla\mathbf{U}^{n+1}\|_0^2 + c_1\|\nabla\mathbf{W}^{n+1}\|_0^2 \\ & + 2c_2\|\text{div } \mathbf{W}^{n+1}\|_0^2 + 8\nu_r\|\mathbf{W}^{n+1}\|_0^2) \leq C. \end{aligned} \quad (4.9)$$

Testing (4.1) by $2\tau\mathcal{A}\mathbf{U}^{n+1}$, we deduce that

$$\begin{aligned} & \|\nabla\mathbf{U}^{n+1}\|_0^2 - \|\nabla\mathbf{U}^n\|_0^2 + \|\nabla(\mathbf{U}^{n+1} - \mathbf{U}^n)\|_0^2 + 2\nu_0\tau\|\mathcal{A}\mathbf{U}^{n+1}\|_0^2 + 2\tau(\mathbf{U}^n \cdot \nabla\mathbf{U}^{n+1}, \mathcal{A}\mathbf{U}^{n+1}) \\ & = 4\nu_r\tau(\text{curl } \mathbf{W}^n, \mathcal{A}\mathbf{U}^{n+1}) + 2\tau(f(t^{n+1}), \mathcal{A}\mathbf{U}^{n+1}). \end{aligned}$$

Using Young's and Cauchy's inequalities, there holds that

$$2\tau(\mathbf{U}^n \cdot \nabla\mathbf{U}^{n+1}, \mathcal{A}\mathbf{U}^{n+1}) \leq \frac{\nu_0\tau}{4}\|\mathcal{A}\mathbf{U}^{n+1}\|_0^2 + C\tau\|\nabla\mathbf{U}^n\|_0^2\|\mathcal{A}\mathbf{U}^n\|_0^2.$$

$$4\nu_r\tau(\text{curl } \mathbf{W}^n, \mathcal{A}\mathbf{U}^{n+1}) \leq \frac{\nu_0\tau}{4}\|\mathcal{A}\mathbf{U}^{n+1}\|_0^2 + C\tau\|\nabla\mathbf{W}^n\|_0^2.$$

$$2\tau(f(t^{n+1}), \mathcal{A}\mathbf{U}^{n+1}) \leq \frac{\nu_0\tau}{4}\|\mathcal{A}\mathbf{U}^{n+1}\|_0^2 + C\tau\|f(t^{n+1})\|_0^2.$$

Then, we arrive at

$$\begin{aligned} & \|\nabla\mathbf{U}^{n+1}\|_0^2 - \|\nabla\mathbf{U}^n\|_0^2 + \|\nabla(\mathbf{U}^{n+1} - \mathbf{U}^n)\|_0^2 + \nu_0\tau\|\mathcal{A}\mathbf{U}^{n+1}\|_0^2 \\ & \leq C\tau\|\nabla\mathbf{U}^n\|_0^2\|\mathcal{A}\mathbf{U}^n\|_0^2 + C\tau\|\nabla\mathbf{W}^n\|_0^2 + C\tau\|f(t^{n+1})\|_0^2. \end{aligned} \quad (4.10)$$

Taking sum of (4.10) over all n and using Gronwall's lemma, it yields that

$$\|\nabla\mathbf{U}^{N+1}\|_0^2 + \nu_0\tau \sum_{n=0}^N \|\mathcal{A}\mathbf{U}^{n+1}\|_0^2 \leq C.$$

The proof is completed. $\square$

Lemma 4.2 Suppose the assumptions (A1-A2) are satisfied, we deduce that

$$\begin{aligned} & \nu_0 \|\nabla \mathbf{U}^{N+1}\|_0^2 + c_1 \|\nabla \mathbf{W}^{N+1}\|_0^2 + c_2 \|\operatorname{div} \mathbf{W}^{N+1}\|_0^2 \\ & + \nu_0 \sum_{n=0}^N \|\nabla \mathbf{U}^{n+1} - \nabla \mathbf{U}^n\|_0^2 + \tau \sum_{n=0}^N \left(\nu_0 \|D_\tau \mathbf{U}^{n+1}\|_0^2 + j \|D_\tau \mathbf{W}^{n+1}\|_0^2 \right) \leq C. \end{aligned} \quad (4.11)$$

Proof. Testing (4.1) by $2\tau D_\tau \mathbf{U}^{n+1}$, (4.3) by $2\tau D_\tau \mathbf{W}^{n+1}$, it yields that

$$\begin{aligned} & 2\tau \|D_\tau \mathbf{U}^{n+1}\|_0^2 + 2\tau j \|D_\tau \mathbf{W}^{n+1}\|_0^2 + \nu_0 (\|\nabla \mathbf{U}^{n+1}\|_0^2 - \|\nabla \mathbf{U}^n\|_0^2 + \|\nabla \mathbf{U}^{n+1} - \nabla \mathbf{U}^n\|_0^2) \\ & + c_1 (\|\nabla \mathbf{W}^{n+1}\|_0^2 - \|\nabla \mathbf{W}^n\|_0^2 + \|\nabla \mathbf{W}^{n+1} - \nabla \mathbf{W}^n\|_0^2) \\ & + c_2 (\|\operatorname{div} \mathbf{W}^{n+1}\|_0^2 - \|\operatorname{div} \mathbf{W}^n\|_0^2 + \|\operatorname{div} \mathbf{W}^{n+1} - \operatorname{div} \mathbf{W}^n\|_0^2) \\ & \leq 4\tau \nu_r (\operatorname{curl} \mathbf{W}^n, D_\tau \mathbf{U}^{n+1}) + 2\tau (f(t^{n+1}), D_\tau \mathbf{U}^{n+1}) - 2\tau ((\mathbf{U}^n \cdot \nabla) \mathbf{U}^{n+1}, D_\tau \mathbf{U}^{n+1}) \\ & \quad - 2\tau j ((\mathbf{U}^n \cdot \nabla) \mathbf{W}^{n+1}, D_\tau \mathbf{W}^{n+1}) + 4\tau \nu_r (\operatorname{curl} \mathbf{U}^{n+1}, D_\tau \mathbf{W}^{n+1}) + 2\tau (g(t^{n+1}), D_\tau \mathbf{W}^{n+1}) \\ & \leq \frac{\tau}{4} \|D_\tau \mathbf{U}^{n+1}\|_0^2 + C\tau \|\nabla \mathbf{W}^n\|_0^2 + \frac{\tau}{4} \|D_\tau \mathbf{U}^{n+1}\|_0^2 + C\tau \|f(t^{n+1})\|_0^2 + \frac{\tau}{4} \|D_\tau \mathbf{U}^{n+1}\|_0^2 \\ & \quad + C\tau \|\mathcal{A} \mathbf{U}^n\|_0^2 \|\nabla \mathbf{U}^{n+1}\|_0^2 + \frac{\tau j}{4} \|D_\tau \mathbf{W}^{n+1}\|_0^2 + C\tau \|\mathcal{A} \mathbf{U}^n\|_0^2 \|\nabla \mathbf{W}^{n+1}\|_0^2 \\ & \quad + \frac{\tau j}{4} \|D_\tau \mathbf{W}^{n+1}\|_0^2 + C\tau \|\nabla \mathbf{U}^{n+1}\|_0^2 + \frac{\tau j}{4} \|D_\tau \mathbf{W}^{n+1}\|_0^2 + C\tau \|g(t^{n+1})\|_0^2. \end{aligned} \quad (4.12)$$

Summing it over all n and using (4.9), (4.32), Gronwall's Lemma, we derive that

$$\begin{aligned} & \nu_0 \|\nabla \mathbf{U}^{N+1}\|_0^2 + c_1 \|\nabla \mathbf{W}^{N+1}\|_0^2 + c_2 \|\operatorname{div} \mathbf{W}^{N+1}\|_0^2 \\ & + \nu_0 \sum_{n=0}^N \|\nabla \mathbf{U}^{n+1} - \nabla \mathbf{U}^n\|_0^2 + \tau \sum_{n=1}^N (\|D_\tau \mathbf{U}^{n+1}\|_0^2 + j \|D_\tau \mathbf{W}^{n+1}\|_0^2) \leq C. \end{aligned} \quad (4.13)$$

Therefore, the Lemma 4.1 is completed. $\square$

Theorem 4.1 Suppose that the assumptions (A1-A3) are satisfied, and denote the error as follows

$$\varepsilon^{n+1} = \mathbf{u}(t^{n+1}) - \mathbf{U}^{n+1}, \zeta^{n+1} = \mathbf{w}(t^{n+1}) - \mathbf{W}^{n+1}, \varrho^{n+1} = p(t^{n+1}) - P^{n+1}.$$

Then, there holds that

$$\|\varepsilon^{N+1}\|_0^2 + j \|\zeta^{N+1}\|_0^2 + \tau \sum_{n=0}^N (\nu_0 \|\nabla \varepsilon^{n+1}\|_0^2 + c_1 \|\nabla \zeta^{n+1}\|_0^2 + 2c_2 \|\operatorname{div} \zeta^{n+1}\|_0^2 + 8\nu_r \|\zeta^{n+1}\|_0^2) \leq C\tau^2, \quad (4.14)$$

and

$$\tau \sum_{n=0}^N \|\varrho^{n+1}\|_0^2 \leq C\tau^2. \quad (4.15)$$

Proof. Taking $t = t^{n+1}$ in (1.1), we obtain

$$\begin{aligned} & \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\tau} - \nu_0 \Delta \mathbf{u}(t^{n+1}) + (\mathbf{u}(t^{n+1}) \cdot \nabla) \mathbf{u}(t^{n+1}) + \nabla p(t^{n+1}) \\ & = 2\nu_r \operatorname{curl} \mathbf{w}(t^{n+1}) + f(t^{n+1}) - R_u^{n+1}, \end{aligned} \quad (4.16)$$

$$\operatorname{div} \mathbf{u}(t^{n+1}) = 0, \quad (4.17)$$

$$\frac{j(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))}{\tau} - c_1 \Delta \mathbf{w}(t^{n+1}) + j(\mathbf{u}(t^{n+1}) \cdot \nabla) \mathbf{w}(t^{n+1}) - c_2 \nabla \operatorname{div} \mathbf{w}(t^{n+1}) + 4\nu_r \mathbf{w}(t^{n+1})$$

$$= 2\nu_r \operatorname{curl} \mathbf{u}(t^{n+1}) + g(t^{n+1}) - R_w^{n+1}, \quad (4.18)$$

where

$$R_u^{n+1} = \frac{1}{\tau} \int_{t^n}^{t^{n+1}} (t - t^n) \mathbf{u}_{tt} dt, R_w^{n+1} = \frac{j}{\tau} \int_{t^n}^{t^{n+1}} (t - t^n) \mathbf{w}_{tt} dt. \quad (4.19)$$

Subtracting (4.1), (4.2), (4.3) from (4.16), (4.17), (4.18), respectively, we derive that

$$\begin{aligned} & \frac{\varepsilon^{n+1} - \varepsilon^n}{\tau} - \nu_0 \Delta \varepsilon^{n+1} + \nabla \varrho^{n+1} + ((\mathbf{u}(t^{n+1}) \cdot \nabla) \mathbf{u}(t^{n+1}) - (\mathbf{U}^n \cdot \nabla) \mathbf{U}^{n+1}) \\ &= 2\nu_r \operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{W}^n) - R_u^{n+1}, \end{aligned} \quad (4.20)$$

$$\operatorname{div} \varepsilon^{n+1} = 0, \quad (4.21)$$

$$\begin{aligned} & \frac{j(\zeta^{n+1} - \zeta^n)}{\tau} - c_1 \Delta \zeta^{n+1} + j[(\mathbf{u}(t^{n+1}) \cdot \nabla) \mathbf{w}(t^{n+1}) - (\mathbf{U}^n \cdot \nabla) \mathbf{W}^{n+1}] - c_2 \nabla \operatorname{div} \zeta^{n+1} + 4\nu_r \zeta^{n+1} \\ &= 2\nu_r \operatorname{curl} \varepsilon^{n+1} - R_w^{n+1}. \end{aligned} \quad (4.22)$$

Testing (4.20) by $2\tau \varepsilon^{n+1}$, we get

$$\begin{aligned} & \|\varepsilon^{n+1}\|_0^2 - \|\varepsilon^n\|_0^2 + \|\varepsilon^{n+1} - \varepsilon^n\|_0^2 + 2\tau \nu_0 \|\nabla \varepsilon^{n+1}\|_0^2 = 4\tau \nu_r (\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{W}^n), \varepsilon^{n+1}) \\ & - 2\tau (R_u^{n+1}, \varepsilon^{n+1}) - 2\tau (((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \cdot \nabla) \mathbf{u}(t^{n+1}), \varepsilon^{n+1}) - 2\tau ((\varepsilon^n \cdot \nabla) \mathbf{u}(t^{n+1}), \varepsilon^{n+1}). \end{aligned} \quad (4.23)$$

Using Hölder's inequality and Young's inequality, we deduce that

$$\begin{aligned} |4\tau \nu_r (\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{W}^n), \varepsilon^{n+1})| &\leq \frac{\tau \nu_0}{5} \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0^2 \\ &+ \frac{\tau c_1}{2} \|\nabla \zeta^n\|_0^2 + C\tau \|\varepsilon^{n+1}\|_0^2, \\ |2\tau (R_u^{n+1}, \varepsilon^{n+1})| &\leq \frac{\tau \nu_0}{5} \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|R_u^{n+1}\|_0^2, \\ |-2\tau (((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \cdot \nabla) \mathbf{u}(t^{n+1}), \varepsilon^{n+1})| &\leq 2\tau \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|_0 \|\nabla \mathbf{u}(t^{n+1})\|_{L^3} \|\varepsilon^{n+1}\|_{L^6} \\ &\leq \frac{\tau \nu_0}{5} \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|_0^2, \\ |-2\tau ((\varepsilon^n \cdot \nabla) \mathbf{u}(t^{n+1}), \varepsilon^{n+1})| &\leq 2\tau \|\varepsilon^n\|_0 \|\nabla \mathbf{u}(t^{n+1})\|_{L^3} \|\varepsilon^{n+1}\|_{L^6} \\ &\leq \frac{\tau \nu_0}{5} \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|\varepsilon^n\|_0^2. \end{aligned}$$

Taking sum of above inequalities, it follows that

$$\begin{aligned} & \|\varepsilon^{n+1}\|_0^2 - \|\varepsilon^n\|_0^2 + \|\varepsilon^{n+1} - \varepsilon^n\|_0^2 + \tau \nu_0 \|\nabla \varepsilon^{n+1}\|_0^2 \\ & \leq C\tau (\|R_u^{n+1}\|_0^2 + \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|_0^2 + \|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0^2 + \|\varepsilon^n\|_0^2 + \|\varepsilon^{n+1}\|_0^2 + \|\nabla \zeta^n\|_0^2). \end{aligned} \quad (4.24)$$

Testing (4.22) by $2\tau \zeta^{n+1}$, we obtain

$$\begin{aligned} & j(\|\zeta^{n+1}\|_0^2 - \|\zeta^n\|_0^2 + \|\zeta^{n+1} - \zeta^n\|_0^2) + 2\tau c_1 \|\nabla \zeta^{n+1}\|_0^2 + 2\tau c_2 \|\operatorname{div} \zeta^{n+1}\|_0^2 + 8\tau \nu_r \|\zeta^{n+1}\|_0^2 \\ & \leq 4\tau \nu_r (\operatorname{curl} \varepsilon^{n+1}, \zeta^{n+1}) - 2\tau (R_w^{n+1}, \zeta^{n+1}) - 2\tau j(((\mathbf{u}(t^{n+1}) \cdot \nabla) \mathbf{w}(t^{n+1}) - (\mathbf{U}^n \cdot \nabla) \mathbf{W}^{n+1}), \zeta^{n+1}) \\ & \leq 4\tau \nu_r (\operatorname{curl} \varepsilon^{n+1}, \zeta^{n+1}) - 2\tau (R_w^{n+1}, \zeta^{n+1}) - 2\tau j(((\mathbf{u}(t^{n+1}) \cdot \nabla) \mathbf{w}(t^{n+1}), \zeta^{n+1}) + ((\varepsilon^n \cdot \nabla) \mathbf{w}^{n+1}, \zeta^{n+1}) \\ & \quad - ((\mathbf{u}(t^n) \cdot \nabla) \mathbf{w}(t^{n+1}), \zeta^{n+1})) \\ & \leq 4\tau \nu_r (\operatorname{curl} \varepsilon^{n+1}, \zeta^{n+1}) - 2\tau (R_w^{n+1}, \zeta^{n+1}) - 2\tau j(((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \cdot \nabla) \mathbf{w}(t^{n+1}), \zeta^{n+1}) \end{aligned}$$

$$\begin{aligned}
& + ((\varepsilon^n \cdot \nabla) \mathbf{w}^{n+1}, \zeta^{n+1})] \\
& \leq \frac{\tau \nu_0}{5} \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|\zeta^{n+1}\|_0^2 + C\tau \|R_w^{n+1}\|_0^2 + C\tau \|\nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))\|_0^2 \\
& \quad + \|\varepsilon^n\|_{L^6} \|\nabla \mathbf{w}(t^{n+1})\|_{L^3} \|\zeta^{n+1}\|_0 \\
& \leq \frac{\tau \nu_0}{5} \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|\zeta^{n+1}\|_0^2 + C\tau \|R_w^{n+1}\|_0^2 + \frac{\tau c_1}{2} \|\nabla \zeta^{n+1}\|_0^2 + C\tau \|\nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))\|_0^2. \quad (4.25)
\end{aligned}$$

Taking sum of (4.24) and (4.25), we derive that

$$\begin{aligned}
& \|\varepsilon^{n+1}\|_0^2 - \|\varepsilon^n\|_0^2 + \|\varepsilon^{n+1} - \varepsilon^n\|_0^2 + \tau \nu_0 \|\nabla \varepsilon^{n+1}\|_0^2 + j(\|\zeta^{n+1}\|_0^2 - \|\zeta^n\|_0^2 + \|\zeta^{n+1} - \zeta^n\|_0^2) \\
& \quad + \tau c_1 \|\nabla \zeta^{n+1}\|_0^2 + 2\tau c_2 \|\operatorname{div} \zeta^{n+1}\|_0^2 + 8\tau \nu_r \|\zeta^{n+1}\|_0^2 \\
& \leq C\tau (\|R_u^{n+1}\|_0^2 + \|R_w^{n+1}\|_0^2 + \|\varepsilon^n\|_0^2 + \|\varepsilon^{n+1}\|_0^2 + \|\zeta^{n+1}\|_0^2 \\
& \quad + \|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0^2 + \|\nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))\|_0^2). \quad (4.26)
\end{aligned}$$

Here, we have used

$$\begin{aligned}
& \sum_{n=0}^N (\|\nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))\|_0^2 + \|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0^2) \leq C\tau, \\
& \sum_{n=0}^N (\|R_u^{n+1}\|_0^2 + \|R_w^{n+1}\|_0^2) \leq C\tau.
\end{aligned}$$

Taking sum of (4.26) over n from 0 to N and using the Gronwall's Lemma, there holds that

$$\|\varepsilon^{N+1}\|_0^2 + j\|\zeta^{N+1}\|_0^2 + \tau \sum_{n=0}^N (\nu_0 \|\nabla \varepsilon^{n+1}\|_0^2 + c_1 \|\nabla \zeta^{n+1}\|_0^2 + 2c_2 \|\operatorname{div} \zeta^{n+1}\|_0^2 + 8\nu_r \|\zeta^{n+1}\|_0^2) \leq C\tau^2. \quad (4.27)$$

We are now starting to prove (4.15). Testing (4.20) by $2\tau D_\tau \varepsilon^{n+1}$, it yields that

$$\begin{aligned}
& 2\tau (D_\tau \varepsilon^{n+1}, D_\tau \varepsilon^{n+1}) - 2\tau \nu_0 (\Delta \varepsilon^{n+1}, D_\tau \varepsilon^{n+1}) + 2\tau (((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \cdot \nabla) \mathbf{u}(t^{n+1}), D_\tau \varepsilon^{n+1}) \\
& \quad + 2\tau ((\varepsilon^n \cdot \nabla) \mathbf{u}(t^{n+1}), D_\tau \varepsilon^{n+1}) + 2\tau ((\mathbf{U}^n \cdot \nabla) \varepsilon^{n+1}, D_\tau \varepsilon^{n+1}) \\
& = 4\tau \nu_r (\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{W}^n), D_\tau \varepsilon^{n+1}) - 2\tau (R_u^{n+1}, D_\tau \varepsilon^{n+1}). \quad (4.28)
\end{aligned}$$

Using Hölder's inequality and Young's inequality, we deduce that

$$\begin{aligned}
|2\tau (((\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)) \cdot \nabla) \mathbf{u}(t^{n+1}), D_\tau \varepsilon^{n+1})| & \leq C\tau \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|_{L^3} \|\nabla \mathbf{u}(t^{n+1})\|_{L^6} \|D_\tau \varepsilon^{n+1}\|_0 \\
& \leq \frac{\tau}{5} \|D_\tau \varepsilon^{n+1}\|_0^2 + C\tau \|\nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))\|_0^2, \\
|2\tau ((\varepsilon^n \cdot \nabla) \mathbf{u}(t^{n+1}), D_\tau \varepsilon^{n+1})| & \leq C\tau \|\varepsilon^n\|_{L^3} \|\nabla \mathbf{u}(t^{n+1})\|_{L^6} \|D_\tau \varepsilon^{n+1}\|_0 \\
& \leq \frac{\tau}{5} \|D_\tau \varepsilon^{n+1}\|_0^2 + C\tau \|\nabla \varepsilon^n\|_0^2, \\
|2\tau ((\mathbf{U}^n \cdot \nabla) \varepsilon^{n+1}, D_\tau \varepsilon^{n+1})| & \leq C\tau \|\mathbf{U}^n\|_2 \|\nabla \varepsilon^{n+1}\|_0 \|D_\tau \varepsilon^{n+1}\|_0 \\
& \leq \frac{\tau}{5} \|D_\tau \varepsilon^{n+1}\|_0^2 + C\tau \|\mathbf{U}^n\|_2^2 \|\nabla \varepsilon^{n+1}\|_0^2, \\
|4\tau \nu_r (\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{W}^n), D_\tau \varepsilon^{n+1})| & \leq 4\tau \nu_r (\|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0 + \|\nabla \zeta^n\|_0) \|D_\tau \varepsilon^{n+1}\|_0 \\
& \leq \frac{\tau}{5} \|D_\tau \varepsilon^{n+1}\|_0^2 + C\tau (\|\nabla \zeta^n\|_0^2 + \|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0^2),
\end{aligned}$$

Using Young's inequality, there holds that

$$|-2\tau (R_u^{n+1}, D_\tau \varepsilon^{n+1})| \leq \frac{\tau}{5} \|D_\tau \varepsilon^{n+1}\|_0^2 + C\tau \|R_u^{n+1}\|_0^2.$$

Taking sum of the above inequalities with (4.28), we have

$$\begin{aligned}
& \tau \|D_\tau \varepsilon^{n+1}\|_0^2 + \nu_0 (\|\nabla \varepsilon^{n+1}\|_0^2 - \|\nabla \varepsilon^n\|_0^2 + \|\nabla \varepsilon^{n+1} - \nabla \varepsilon^n\|_0^2) \\
& \leq C\tau (\|\nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))\|_0^2 + \|\nabla \varepsilon^n\|_0^2 + \|\mathbf{U}^n\|_2^2 \|\nabla \varepsilon^{n+1}\|_0^2 + \|\nabla \zeta^n\|_0^2 \\
& \quad + \|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0^2 + \|R_u^{n+1}\|_0^2).
\end{aligned} \tag{4.29}$$

Summing it with respect to n from 0 to N and using (4.32), we arrive at

$$\nu_0 \|\nabla \varepsilon^{N+1}\|_0^2 + \tau \sum_{n=0}^N \|D_\tau \varepsilon^{n+1}\|_0^2 \leq C\tau^2. \tag{4.30}$$

By using the inf-sup condition, from (4.20), we derive that

$$\begin{aligned}
\tau \beta \|\varrho^{n+1}\|_0^2 & \leq C\tau \|D_\tau \varepsilon^{n+1}\|_0^2 + C\tau \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|R_u^{n+1}\|_0^2 + C\tau \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|_0^2 \|\nabla \mathbf{u}(t^{n+1})\|_{L^3}^2 \\
& \quad + C\tau \|\varepsilon^n\|_0^2 \|\nabla \mathbf{u}(t^{n+1})\|_{L^3}^2 + C\tau \|\nabla \mathbf{U}^n\|_0^2 \|\nabla \varepsilon^{n+1}\|_0^2 + C\tau \|\nabla \zeta^{n+1}\|_0^2 + C\tau \|\operatorname{curl}(\mathbf{w}(t^{n+1}) - \mathbf{w}(t^n))\|_0^2.
\end{aligned} \tag{4.31}$$

By summing the above inequalities from 0 to N and applying Theorem 4.1 and (4.30), we obtain that

$$\tau \sum_{n=0}^N \|\varrho^{n+1}\|_0^2 \leq C\tau^2.$$

Therefore, the proof is completed. $\square$

Lemma 4.3 Suppose the assumptions (A4) and $\tau \leq \min\{1, \sqrt{\frac{1}{2C'}}\}$ are satisfied, we deduce that

$$\tau \sum_{n=0}^N (\|\mathbf{U}^{n+1}\|_2^2 + \|P^{n+1}\|_1^2) \leq C. \tag{4.32}$$

Proof. From the assumptions (A4) and (4.1), using Hölder's inequality, we can acquire that

$$\begin{aligned}
& \|\mathbf{U}^{n+1}\|_2 + \|P^{n+1}\|_1 \\
& \leq C(\|f(t^{n+1})\|_0 + \|(\mathbf{U}^n \cdot \nabla) \mathbf{U}^{n+1}\|_0 + \|\operatorname{curl} \mathbf{W}^n\|_0 + \|D_\tau \mathbf{U}^{n+1}\|_0) \\
& \leq C(\|f(t^{n+1})\|_0 + \|(\varepsilon^n \cdot \nabla) \mathbf{U}^{n+1}\|_0 + \|(\mathbf{u}(t^n) \cdot \nabla) \mathbf{U}^{n+1}\|_0 + \|\operatorname{curl} \mathbf{W}^n\|_0 + \|D_\tau \mathbf{U}^{n+1}\|_0) \\
& \leq C(\|f(t^{n+1})\|_0 + \|\nabla \varepsilon^n\|_0 \|\mathcal{A} \mathbf{U}^{n+1}\|_0 + \|\nabla \mathbf{U}^{n+1}\|_0 + \|\operatorname{curl} \mathbf{W}^n\|_0 + \|D_\tau \mathbf{U}^{n+1}\|_0).
\end{aligned}$$

Taking sum of the above inequalities yields that

$$\begin{aligned}
& \tau \sum_{n=0}^N (\|\mathbf{U}^{n+1}\|_2^2 + \|P^{n+1}\|_1^2) \\
& \leq C\tau \sum_{n=0}^N (\|f(t^{n+1})\|_0^2 + \|\nabla \varepsilon^n\|_0^2 \|\mathbf{U}^{n+1}\|_0^2 + \|\nabla \mathbf{U}^{n+1}\|_0^2 + \|\operatorname{curl} \mathbf{W}^n\|_0^2 + \|D_\tau \mathbf{U}^{n+1}\|_0^2) \\
& \leq C + C_1 \tau^2 \sum_{n=0}^N \|\mathcal{A} \mathbf{U}^{n+1}\|_0^2.
\end{aligned}$$

Making use of (4.14), Lemma 2.2 and $\tau \leq \min\{1, \sqrt{\frac{1}{2C'}}\}$, we find

$$\tau \sum_{n=0}^N (\|\mathbf{U}^{n+1}\|_2^2 + \|P^{n+1}\|_1^2) \leq C,$$

where C and C' denote different positive constants. □

For the derivation of error estimates, we first introduce the following definitions.

$$\begin{aligned}
G^{n+1} &= \mathbf{u}(t^{n+1}) - R_h^{n+1}, & H^{n+1} &= p(t^{n+1}) - Q_h^{n+1}, \\
\epsilon^{n+1} &= \mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, & \tilde{\epsilon}^{n+1} &= \mathbf{U}^{n+1} - \tilde{\mathbf{u}}_h^{n+1}, \\
l^{n+1} &= \mathbf{w}(t^{n+1}) - R_{1h}^{n+1}, & L^{n+1} &= \mathbf{W}^{n+1} - \mathbf{w}_h^{n+1}, \\
L_h^{n+1} &= R_{1h}^{n+1} - \mathbf{w}_h^{n+1}, & \epsilon_h^{n+1} &= R_h^{n+1} - \mathbf{u}_h^{n+1}, \\
\tilde{\epsilon}_h^{n+1} &= R_h^{n+1} - \tilde{\mathbf{u}}_h^{n+1}, & F^{n+1} &= \mathbf{U}^{n+1} - R_h^{n+1}, \\
M^{n+1} &= \mathbf{W}^{n+1} - R_{1h}^{n+1}, & e_h^{n+1} &= Q_h^{n+1} - p_h^{n+1}, \\
\mathfrak{g}^{n+1} &= P^{n+1} - Q_h^{n+1}.
\end{aligned}$$

Then, there holds that

$$\begin{aligned}
\tilde{\epsilon}^{n+1} &= \epsilon^{n+1} + \nabla \rho_h^{n+1}, & \tilde{\epsilon}_h^{n+1} &= \epsilon_h^{n+1} + \nabla \rho_h^{n+1}, \\
\tilde{\epsilon}^{n+1} &= F^{n+1} + \tilde{\epsilon}_h^{n+1}, & \epsilon^{n+1} &= F^{n+1} + \epsilon_h^{n+1}, \\
L^{n+1} &= M^{n+1} + L_h^{n+1}.
\end{aligned}$$

Furthermore, we have

$$(\epsilon^{n+1}, \nabla q_h) = (\epsilon_h^{n+1}, \nabla q_h) = (F^{n+1}, \nabla q_h) = 0, \quad \forall q_h \in P_h.$$

Therefore, we have the crucial orthogonality properties

$$\|\tilde{\epsilon}^{n+1}\|_0^2 = \|\epsilon^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2, \quad \|\tilde{\epsilon}_h^{n+1}\|_0^2 = \|\epsilon_h^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2. \quad (4.33)$$

Noting $F^{n+1} = G^{n+1} - \epsilon^{n+1}$, $\varrho^{n+1} = \tilde{\vartheta}^{n+1} - \mathfrak{g}^{n+1}$, $M^{n+1} = l^{n+1} - \zeta^{n+1}$, the following Lemma holds.

Lemma 4.4 ([26, 28]) Suppose the assumptions (A1-A3) are satisfied, then

$$\begin{aligned}
\|F^{n+1}\|_0^2 &\leq Ch^4, \\
\tau \sum_{n=1}^N \|\nabla F^{n+1}\|_0^2 &\leq Ch^2, \\
\tau \sum_{n=1}^N \|\mathfrak{g}^{n+1}\|_0^2 &\leq Ch^2, \\
\|M^{n+1}\|_0^2 &\leq Ch^4, \\
\tau \sum_{n=1}^N \|\nabla M^{n+1}\|_0^2 &\leq Ch^2, \\
\|G^{n+1}\|_0 + h\|G^{n+1}\|_1 + h\|H^{n+1}\|_0 &\leq Ch^2.
\end{aligned}$$

Then, the following theorem presents the error estimate.

Theorem 4.2 Suppose that the assumptions (A1-A3) are satisfied and $h^2 \leq C\tau$, then the errors satisfy that

$$\begin{aligned}
&\|\epsilon^{N+1}\|_0^2 + \tau \nu_0 \|s_h^{N+1}\|_0^2 + j\|L^{N+1}\|_0^2 + \sum_{n=0}^N (\tau \nu_0 \|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2 \\
&+ 2c_1 \|\nabla L^{n+1}\|_0^2 + 2c_2 \|\operatorname{div} L^{n+1}\|_0^2 + 8\nu_r \|L^{n+1}\|_0^2) \leq C(\tau + h^2).
\end{aligned} \quad (4.34)$$

Proof. Subtracting (2.18), (2.23) from (4.1), (4.3), respectively, it follows that

$$\begin{aligned} & \left(\frac{\tilde{\epsilon}^{n+1} - \epsilon^n}{\tau}, v_h \right) + \nu_0 (\nabla \tilde{\epsilon}^{n+1}, \nabla v_h) \\ = & (P^{n+1}, \nabla \cdot v_h) - \nu_0 (s_h^n, \nabla \cdot v_h) + 2\nu_r (\text{curl}(\mathbf{W}^n - \mathbf{w}_h^{n+1}), v_h) - B(\mathbf{U}^n, \mathbf{U}^{n+1}, v_h) + B(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, v_h), \end{aligned} \quad (4.35)$$

$$\begin{aligned} & j \left(\frac{L^{n+1} - L^n}{\tau}, \chi_h \right) + c_1 (\nabla L^{n+1}, \nabla \chi_h) + c_2 (\text{div } L^{n+1}, \text{div } \chi_h) \\ = & -4\nu_r (L^{n+1}, \chi_h) + 2\nu_r (\text{curl } \epsilon^{n+1}, \chi_h) - jB(\mathbf{U}^n, \mathbf{W}^{n+1}, \chi_h) + jB(\mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1}, \chi_h). \end{aligned} \quad (4.36)$$

Choosing $v_h = 2\tau \tilde{\epsilon}_h^{n+1} = 2\tau(\tilde{\epsilon}^{n+1} - F^{n+1})$ in (4.35), we derive that

$$\begin{aligned} & \|\epsilon^{n+1}\|_0^2 - \|\epsilon^n\|_0^2 + \|\epsilon^{n+1} - \epsilon^n\|_0^2 + 2\tau\nu_0 \|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + 2\|\nabla \rho_h^{n+1}\|_0^2 \\ = & 2(\epsilon^{n+1} - \epsilon^n, F^{n+1}) + 2\tau\nu_0 (\nabla \tilde{\epsilon}^{n+1}, \nabla F^{n+1}) + 2\tau(P^{n+1}, \nabla \cdot \tilde{\epsilon}_h^{n+1}) - 2\tau\nu_0 (s_h^n, \nabla \cdot \tilde{\epsilon}_h^{n+1}) \\ & + 4\tau\nu_r (\text{curl}(\mathbf{W}^n - \mathbf{w}_h^{n+1}), \tilde{\epsilon}_h^{n+1}) - 2\tau B(\mathbf{U}^n, \mathbf{U}^{n+1}, \tilde{\epsilon}_h^{n+1}) + 2\tau B(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \tilde{\epsilon}_h^{n+1}). \end{aligned} \quad (4.37)$$

Using Lemma 4.4, (2.20), Hölder's inequality and Young's inequality, we deduce that

$$\begin{aligned} & |2(\epsilon^{n+1} - \epsilon^n, F^{n+1}) + 2\tau\nu_0 (\nabla \tilde{\epsilon}^{n+1}, \nabla F^{n+1})| \\ \leq & 2\|\epsilon^{n+1} - \epsilon^n\|_0 \|F^{n+1}\|_0 + 2\tau\nu_0 \|\nabla \tilde{\epsilon}^{n+1}\|_0 \|\nabla F^{n+1}\|_0 \\ \leq & \frac{1}{2} \|\epsilon^{n+1} - \epsilon^n\|_0^2 + C\|F^{n+1}\|_0^2 + \frac{\tau\nu_0}{4} \|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + C\tau \|\nabla F^{n+1}\|_0^2, \\ & |2\tau(P^{n+1}, \nabla \cdot \tilde{\epsilon}_h^{n+1})| \\ = & |2\tau(P^{n+1} - Q_h^{n+1}, \nabla \cdot \tilde{\epsilon}_h^{n+1}) - 2\tau((p(t^{n+1}) - Q_h^{n+1}), \nabla \cdot \tilde{\epsilon}_h^{n+1}) + 2\tau(p(t^{n+1}), \nabla \cdot \tilde{\epsilon}_h^{n+1})| \\ = & |2\tau(\mathbf{g}^{n+1}, \nabla \cdot \tilde{\epsilon}_h^{n+1}) - 2\tau(H^{n+1}, \nabla \cdot \tilde{\epsilon}_h^{n+1}) + 2\tau(\nabla p(t^{n+1}), \nabla \rho_h^{n+1})| \\ \leq & 2\tau\|\mathbf{g}^{n+1}\|_0 \|\nabla \tilde{\epsilon}_h^{n+1}\|_0 + 2\tau\|H^{n+1}\|_0 \|\nabla \tilde{\epsilon}_h^{n+1}\|_0 + 2\tau\|\nabla p(t^{n+1})\|_0 \|\nabla \rho_h^{n+1}\|_0 \\ \leq & C\tau\|\mathbf{g}^{n+1}\|_0^2 + C\tau\|H^{n+1}\|_0^2 + C\tau^2 \|\nabla p(t^{n+1})\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2 + \frac{\tau\nu_0}{4} \|\nabla \tilde{\epsilon}_h^{n+1}\|_0^2 \\ \leq & C\tau\|\mathbf{g}^{n+1}\|_0^2 + C\tau\|H^{n+1}\|_0^2 + C\tau^2 \|\nabla p(t^{n+1})\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2 + \frac{\tau\nu_0}{4} (\|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + \|\nabla F^{n+1}\|_0^2) \\ \leq & C\tau^2 + C\tau h^2 + C\tau\|\mathbf{g}^{n+1}\|_0^2 + \frac{\tau\nu_0}{4} (\|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + \|\nabla F^{n+1}\|_0^2) + \|\nabla \rho_h^{n+1}\|_0^2, \\ & |-2\tau\nu_0(s_h^n, \nabla \cdot \tilde{\epsilon}_h^{n+1})| = |2\tau\nu_0(s_h^n, \nabla \cdot \tilde{\mathbf{u}}_h^{n+1})| = |2\tau\nu_0(s_h^n - s_h^{n+1}, s_h^n)| \\ \leq & \tau\nu_0 (\|s_h^n\|_0^2 - \|s_h^{n+1}\|_0^2) + \frac{\tau\nu_0}{4} \|\nabla \tilde{\epsilon}^{n+1}\|_0^2, \\ & |4\tau\nu_r (\text{curl}(\mathbf{W}^n - \mathbf{w}_h^{n+1}), \tilde{\epsilon}_h^{n+1})| = |4\tau\nu_r(\mathbf{W}^n - \mathbf{w}_h^{n+1}, \text{curl } \tilde{\epsilon}_h^{n+1})| \\ \leq & 4\tau\nu_r(\mathbf{W}^n + \mathbf{W}^{n+1} - \mathbf{W}^{n+1} - \mathbf{w}_h^{n+1}, \text{curl } \tilde{\epsilon}_h^{n+1}) \\ \leq & 4\tau\nu_r(\mathbf{W}^{n+1} - \mathbf{w}_h^{n+1}, \text{curl } \tilde{\epsilon}_h^{n+1}) + 4\tau\nu_r(\mathbf{W}^n - \mathbf{W}^{n+1}, \text{curl } \tilde{\epsilon}_h^{n+1}) \\ \leq & C\tau\|L^{n+1}\|_0^2 + \frac{\tau\nu_0}{4} (\|\nabla F^{n+1}\|_0^2 + \|\nabla \tilde{\epsilon}^{n+1}\|_0^2) \\ & + 4\tau\nu_r(\mathbf{w}(t^n) - \mathbf{w}(t^n) + \mathbf{w}(t^{n+1}) - \mathbf{w}(t^{n+1}) + \mathbf{W}^n - \mathbf{W}^{n+1}, \text{curl } \tilde{\epsilon}_h^{n+1}) \\ \leq & C\tau\|L^{n+1}\|_0^2 + \frac{\tau\nu_0}{4} (\|\nabla F^{n+1}\|_0^2 + \|\nabla \tilde{\epsilon}^{n+1}\|_0^2) + C\tau(\|\zeta^{n+1}\|_0^2 + \|\zeta^n\|_0^2) \\ & + C\tau\|\text{curl}(\mathbf{w}(t^n) - \mathbf{w}(t^{n+1}))\|_0^2, \\ & |-2\tau B(\mathbf{U}^n, \mathbf{U}^{n+1}, \tilde{\epsilon}_h^{n+1}) + 2\tau B(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \tilde{\epsilon}_h^{n+1})| \\ = & |-2\tau B(\mathbf{U}^n - \mathbf{u}_h^n, \mathbf{U}^{n+1}, \tilde{\epsilon}_h^{n+1}) - 2\tau B(\mathbf{u}_h^n, \mathbf{U}^{n+1} - R_h^{n+1}, \tilde{\epsilon}_h^{n+1}) - 2\tau B(\mathbf{u}_h^n, \tilde{\epsilon}_h^{n+1}, \tilde{\epsilon}_h^{n+1})| \\ = & |-2\tau B(\mathbf{U}^n - \mathbf{u}_h^n, \mathbf{U}^{n+1}, \tilde{\epsilon}_h^{n+1}) - 2\tau B(\mathbf{u}_h^n, \mathbf{U}^{n+1} - R_h^{n+1}, \tilde{\epsilon}_h^{n+1})| \\ \leq & \frac{\tau\nu_0}{4} \|\nabla F^{n+1}\|_0^2 + \frac{\tau\nu_0}{4} \|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + C\tau(\|\epsilon^n\|_0^2 + h^2). \end{aligned}$$

Plugging these estimates with (4.37), we deduce that

$$\begin{aligned} & \|\epsilon^{n+1}\|_0^2 - \|\epsilon^n\|_0^2 + \frac{1}{2}\|\epsilon^{n+1} - \epsilon^n\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2 + \tau\nu_0(\|s_h^{n+1}\|_0^2 - \|s_h^n\|_0^2) + \tau\nu_0\|\nabla \tilde{\epsilon}^{n+1}\|_0^2 \\ & \leq C\tau^2 + C\tau h^2 + C\tau(\|\epsilon^n\|_0^2 + \|\mathbf{g}^{n+1}\|_0^2 + \|\nabla F^{n+1}\|_0^2 + \|\zeta^{n+1}\|_0^2 + \|\zeta^n\|_0^2 + \|\operatorname{curl}(\mathbf{w}(t^n) - \mathbf{w}(t^{n+1}))\|_0^2 \\ & \quad + \|L^{n+1}\|_0^2) + C\|F^{n+1}\|_0^2. \end{aligned} \quad (4.38)$$

From (4.36), choosing $\chi_h = 2\tau L_h^{n+1} = 2\tau(L^{n+1} - M^{n+1})$, we get

$$\begin{aligned} & j(\|L^{n+1}\|_0^2 - \|L^n\|_0^2 + \|L^{n+1} - L^n\|_0^2) + 2\tau c_1\|\nabla L^{n+1}\|_0^2 + 2\tau c_2\|\operatorname{div} L^{n+1}\|_0^2 + 8\tau\nu_r\|L^{n+1}\|_0^2 \\ & = (L^{n+1} - L^n, M^{n+1}) + 2c_1\tau(\nabla L^{n+1}, \nabla M^{n+1}) + 2c_2\tau(\operatorname{div} L^{n+1}, \operatorname{div} M^{n+1}) \\ & \quad + 4\tau\nu_r(\operatorname{curl} \epsilon^{n+1}, L^{n+1} - M^{n+1}) - 2\tau jB(\mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1} - \mathbf{W}^{n+1} + \mathbf{W}^{n+1}, L^{n+1} - M^{n+1}) \\ & \quad - 2\tau jB(\mathbf{U}^{n+1}, L^{n+1}, L^{n+1} - M^{n+1}) + 2\tau jB(\mathbf{U}^n - \mathbf{U}^{n+1}, \mathbf{W}^{n+1}, L^{n+1} - M^{n+1}) \\ & \leq \frac{j}{2}\|L^{n+1} - L^n\|_0^2 + \frac{\tau\nu_0}{4}\|\nabla \epsilon^{n+1}\|_0^2 + C\tau\|L^{n+1}\|_0^2 + 2\tau j\|\epsilon^{n+1}\|_0\|\mathbf{W}^{n+1}\|_2\|L^{n+1}\|_1 \\ & \quad + 2\tau j\|\mathbf{U}^n - \mathbf{U}^{n+1}\|_1\|\mathbf{W}^{n+1}\|_1\|\nabla L^{n+1}\|_0 + C\|M^{n+1}\|_0^2 + C\tau\|\nabla M^{n+1}\|_0^2. \end{aligned} \quad (4.39)$$

Combining (4.38) with (4.39), there holds that

$$\begin{aligned} & \|\epsilon^{n+1}\|_0^2 - \|\epsilon^n\|_0^2 + \frac{1}{2}\|\epsilon^{n+1} - \epsilon^n\|_0^2 + j(\|L^{n+1}\|_0^2 - \|L^n\|_0^2 + \|L^{n+1} - L^n\|_0^2) + \tau\nu_0\|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2 \\ & \quad + \tau\nu_0(\|s_h^{n+1}\|_0^2 - \|s_h^n\|_0^2) + \tau c_1\|\nabla L^{n+1}\|_0^2 + 2\tau c_2\|\operatorname{div} L^{n+1}\|_0^2 + 8\tau\nu_r\|L^{n+1}\|_0^2 \\ & \leq C\tau^2 + C\tau h^2 + C\tau(\|\epsilon^n\|_0^2 + \|\epsilon^{n+1}\|_0^2 + \|\mathbf{g}^{n+1}\|_0^2 + \|L^n\|_0^2 + \|L^{n+1}\|_0^2 + \|\mathbf{U}^n - \mathbf{U}^{n+1}\|_1^2 + \|\nabla F^{n+1}\|_0^2) \\ & \quad + C\|F^{n+1}\|_0^2 + C\|M^{n+1}\|_0^2 + C\tau\|\zeta^{n+1}\|_0^2 + C\tau\|\zeta^n\|_0^2 + C\tau\|\operatorname{curl}(\mathbf{w}(t^n) - \mathbf{w}(t^{n+1}))\|_0^2. \end{aligned} \quad (4.40)$$

Taking sum of the above inequalities for n from 0 to N and applying Gronwall's Lemma, Theorem 4.1, Lemma 4.1, Lemma 4.4, noting $\|F^{n+1}\|_0 \leq Ch^2$ and $\|M^{n+1}\|_0 \leq Ch^2$, using $h^2 \leq C\tau$, we deduce that

$$\begin{aligned} & \|\epsilon^{N+1}\|_0^2 + \tau\nu_0\|s_h^{N+1}\|_0^2 + j\|L^{N+1}\|_0^2 + \sum_{n=0}^N(\tau\nu_0\|\nabla \tilde{\epsilon}^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2 \\ & \quad + 2c_1\|\nabla L^{n+1}\|_0^2 + 2c_2\|\operatorname{div} L^{n+1}\|_0^2 + 8\tau\nu_r\|L^{n+1}\|_0^2) \leq C(\tau + h^2). \end{aligned} \quad (4.41)$$

Therefore, the proof is completed. $\square$

Theorem 4.3 Suppose that the assumptions (A1-A3) are satisfied and $h^2 \leq C\tau$, then the errors satisfy that

$$\|\epsilon^{N+1}\|_*^2 + j\|L^{N+1}\|_*^2 + \sum_{n=0}^N(\|\epsilon^{n+1} - \epsilon^n\|_*^2 + \tau\nu_0\|\epsilon^{n+1}\|_0^2 + \|\nabla L^{n+1} - \nabla L^n\|_*^2) \leq C(\tau^2 + h^4). \quad (4.42)$$

Proof. We assume that $(\mathbf{v}^n, q^n)$ and $(\mathbf{v}_h^n, q_h^n)$ are solutions of the Stokes Equations with $f = \epsilon^n$. Then, from Lemma 2.3, we can get a crucial inequality,

$$\|\mathbf{v}^n - \mathbf{v}_h^n\|_0 + h\|\nabla(\mathbf{v}^n - \mathbf{v}_h^n)\|_0 \leq Ch^2\|\epsilon^n\|_0. \quad (4.43)$$

Noting $(\rho_h^{n+1}, \nabla \cdot \mathbf{v}_h^{n+1}) = 0$ and

$$(\tilde{\epsilon}^{n+1} - \epsilon^n, \mathbf{v}_h^{n+1}) = (\epsilon^{n+1} - \epsilon^n, \mathbf{v}_h^{n+1}) = (\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n), \nabla \mathbf{v}_h^{n+1}),$$

choosing $v_h = 2\tau \mathbf{v}_h^{n+1}$ in (4.35), we can derive that

$$\|\nabla \mathbf{v}_h^{n+1}\|_0^2 - \|\nabla \mathbf{v}_h^n\|_0^2 + \|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\|_0^2 + 2\tau\nu_0\|\epsilon^{n+1}\|_0^2$$

$$\begin{aligned}
&= -2\tau\nu_0(\nabla F^{n+1}, \nabla \mathbf{v}_h^{n+1}) + 2\tau\nu_0(F^{n+1}, \epsilon^{n+1}) + 2\tau\nu_0(\nabla \rho_h^{n+1}, \nabla q_h^{n+1}) + 2\tau(P^{n+1}, \nabla \cdot \mathbf{v}_h^{n+1}) \\
&\quad - 2\tau B(\mathbf{U}^n, \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1}) + 2\tau B(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1}) + 2\nu_r(\text{curl}(\mathbf{W}^n - \mathbf{w}_h^{n+1}), \mathbf{v}_h^{n+1}). \tag{4.44}
\end{aligned}$$

Using Lemma 4.3, Hölder's inequality and Young's inequality, we arrive at

$$\begin{aligned}
&| -2\tau\nu_0(\nabla F^{n+1}, \nabla \mathbf{v}_h^{n+1}) | \\
&= |2\tau\nu_0(\nabla F^{n+1}, \nabla(\mathbf{v}^{n+1} - \mathbf{v}_h^{n+1}) - \nabla \mathbf{v}^{n+1})| \\
&\leq C\tau(h^2\|\nabla F^{n+1}\|_0^2 + \|F^{n+1}\|_0^2) + \frac{\tau\nu_0}{5}\|\epsilon^{n+1}\|_0^2, \\
&\quad |2\tau\nu_0(F^{n+1}, \epsilon^{n+1}) + 2\tau\nu_0(\nabla \rho_h^{n+1}, \nabla q_h^{n+1})| \\
&\leq C\tau\|F^{n+1}\|_0\|\nabla \epsilon^{n+1}\|_0 + 2\tau\nu_0\|\nabla \rho_h^{n+1}\|_0\|\epsilon^{n+1}\|_0 \\
&\leq C\tau(\|F^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2) + \frac{\tau\nu_0}{5}\|\nabla \epsilon^{n+1}\|_0^2, \\
&\quad |2\tau(P^{n+1}, \nabla \cdot \mathbf{v}_h^{n+1})| \\
&\leq 2\tau(P^{n+1} - Q_h^{n+1}, \nabla \cdot (\mathbf{v}_h^{n+1} - \mathbf{v}^{n+1})) \\
&\leq C\tau h\|\mathbf{g}^{n+1}\|_0\|\epsilon^{n+1}\|_0 \\
&\leq C\tau h^2\|\mathbf{g}^{n+1}\|_0^2 + \frac{\tau\nu_0}{5}\|\epsilon^{n+1}\|_0^2, \\
&\quad | -2\tau B(\mathbf{U}^n, \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1}) + 2\tau B(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h^{n+1}) | \\
&= | -2\tau B(\mathbf{U}^n - \mathbf{u}_h^n, \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1}) + 2\tau B(\mathbf{u}_h^n - \mathbf{U}^n, \tilde{\mathbf{u}}_h^{n+1} - \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1}) \\
&\quad + 2\tau B(\mathbf{U}^n, \tilde{\mathbf{u}}_h^{n+1} - \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1}) | \\
&= | -2\tau B(\mathbf{U}^n - \mathbf{u}_h^n, \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1}) + 2\tau B(\mathbf{u}_h^n - \mathbf{U}^n, \tilde{\mathbf{u}}_h^{n+1} - \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1} - \mathbf{v}^{n+1} + \mathbf{v}^{n+1}) \\
&\quad + 2\tau B(\mathbf{U}^n, \tilde{\mathbf{u}}_h^{n+1} - \mathbf{U}^{n+1}, \mathbf{v}_h^{n+1}) | \\
&\leq 2\tau\|\epsilon^n\|_0\|\mathbf{U}^{n+1}\|_2\|\nabla \mathbf{v}_h^{n+1}\|_0 + 2\tau\|\epsilon^n\|_0\|\nabla \tilde{\epsilon}^{n+1}\|_0(\|\mathbf{v}_h^{n+1} - \mathbf{v}^{n+1}\| + \|\mathbf{v}^{n+1}\|_2) \\
&\quad + 2\tau\|\mathbf{U}^n\|_2\|\tilde{\epsilon}^{n+1}\|_0\|\nabla \mathbf{v}_h^{n+1}\|_0 \\
&\leq C\tau\|\nabla \mathbf{v}_h^{n+1}\|_0^2 + \frac{\tau\nu_0}{5}\|\epsilon^n\|_0^2 + \frac{\tau\nu_0}{5}(\|\epsilon^{n+1}\|_0^2 + \|\nabla \rho_h^{n+1}\|_0^2) + C\tau(\tau + h^2)\|\nabla \tilde{\epsilon}^{n+1}\|_0^2, \\
&\quad |2\nu_r(\text{curl}(\mathbf{W}^n - \mathbf{w}_h^{n+1}), \mathbf{v}_h^{n+1})| \\
&\leq 2\nu_r(\mathbf{W}^n - \mathbf{w}_h^{n+1}, \nabla \mathbf{v}_h^{n+1}) \\
&\leq 2\tau\nu_r(\mathbf{w}(t^n) - \mathbf{w}(t^n) + \mathbf{w}(t^{n+1}) - \mathbf{w}(t^{n+1}) + \mathbf{W}^n + \mathbf{W}^{n+1} - \mathbf{W}^{n+1} - \mathbf{w}_h^{n+1}, \nabla \mathbf{v}_h^{n+1}) \\
&\leq C\tau(\|\mathbf{w}(t^n) - \mathbf{w}(t^{n+1})\|_0^2 + \|\zeta^n\|_0^2 + \|\zeta^{n+1}\|_0^2) + \tau c_1\|L^{n+1}\|_0^2 + C\tau\|\nabla \mathbf{v}_h^{n+1}\|_0^2.
\end{aligned}$$

Taking sum of the above inequalities for n from 0 to N and invoking Gronwall's Lemma, Lemma 4.4, (4.34), we deduce that

$$\|\nabla \mathbf{v}_h^{N+1}\|_0^2 + \sum_{n=0}^N (\tau\nu_0\|\epsilon^{n+1}\|_0^2 + \|\nabla(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\|_0^2) \leq C(\tau^2 + h^4) + \tau c_1 \sum_{n=0}^N \|L^{n+1}\|_0^2. \tag{4.45}$$

Set ϖ^n , ϖ_h^n are solutions of the Poisson's Equations with $g = L^n$, respectively. Then Lemma 2.4 yields

$$\|\varpi^n - \varpi_h^n\|_0 + h\|\nabla(\varpi^n - \varpi_h^n)\|_0 \leq Ch^2\|\varpi^n\|_2 \leq Ch^2\|L^n\|_0. \tag{4.46}$$

From Lemma 2.4, it yields that

$$\|\varpi^{n+1}\|_2 + \|\varpi^{n+1} - \varpi_h^{n+1}\|_0 \leq C\|L^{n+1}\|_0, \quad \|\nabla \varpi^{n+1}\|_0 \simeq \|L^{n+1}\|_{-1}. \tag{4.47}$$

Taking $\chi_h = 2\tau\varpi_h^{n+1}$ in (4.36), we have

$$2j(L^{n+1} - L^n, \varpi_h^{n+1}) + 2\tau c_1(\nabla L^{n+1}, \nabla \varpi_h^{n+1}) + 2\tau c_2(\text{div } L^{n+1}, \text{div } \varpi_h^{n+1})$$

$$= -8\tau\nu_r(L^{n+1}, \varpi_h^{n+1}) + 4\tau\nu_r(\operatorname{curl} \epsilon^{n+1}, \varpi_h^{n+1}) - 2\tau jB(\mathbf{U}^n, \mathbf{W}^{n+1}, \varpi_h^{n+1}) - 2\tau jB(\mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1}, \varpi_h^{n+1}). \quad (4.48)$$

Noting

$$\begin{aligned} & 2j(L^{n+1} - L^n, \varpi_h^{n+1}) \\ &= 2j(\nabla(\varpi_h^{n+1} - \varpi_h^n), \nabla \varpi_h^{n+1}) \\ &= j\|\nabla \varpi_h^{n+1}\|_0^2 - j\|\nabla \varpi_h^n\|_0^2 + j\|\nabla(\varpi_h^{n+1} - \varpi_h^n)\|_0^2. \end{aligned}$$

Using (4.32), (4.34), Hölder's inequality and Young's inequality, there holds that

$$\begin{aligned} & |2\tau c_1(\nabla L^{n+1}, \nabla \varpi_h^{n+1})| \\ &= 2\tau c_1(\nabla L_h^{n+1} + \nabla M^{n+1}, \nabla \varpi_h^{n+1}) \\ &= 2\tau c_1(\nabla M^{n+1}, \nabla \varpi_h^{n+1}) + 2\tau c_1(L_h^{n+1}, L^{n+1}) \\ &= 2\tau c_1\|L^{n+1}\|_0^2 + 2\tau c_1(\nabla M^{n+1}, \nabla \varpi_h^{n+1}) - 2\tau c_1(M^{n+1}, L^{n+1}), \\ & \quad |2\tau c_1(\nabla M^{n+1}, \nabla \varpi_h^{n+1}) - 2\tau c_1(M^{n+1}, L^{n+1})| \\ &\leq 2\tau c_1(\nabla M^{n+1}, \nabla(\varpi_h^{n+1} - \varpi_h^n) + \nabla \varpi_h^{n+1}) + C\tau\|M^{n+1}\|_0^2 + \frac{\tau c_1}{7}\|L^{n+1}\|_0^2 \\ &\leq 2\tau c_1(h\|\nabla M^{n+1}\|_0\|L^{n+1}\|_0 + \|M^{n+1}\|_0\|\varpi_h^{n+1}\|_2) + C\tau\|M^{n+1}\|_0^2 + \frac{\tau c_1}{7}\|L^{n+1}\|_0^2 \\ &\leq C\tau h^2\|\nabla M^{n+1}\|_0^2 + C\tau\|M^{n+1}\|_0^2 + \frac{\tau c_1}{7}\|L^{n+1}\|_0^2 + C\tau\|M^{n+1}\|_0^2 + \frac{\tau c_1}{7}\|L^{n+1}\|_0^2, \\ & \quad |2\tau c_2(\operatorname{div} L^{n+1}, \operatorname{div} \varpi_h^{n+1})| \\ &\leq 2\tau c_2(\nabla L^{n+1}, \nabla \varpi_h^{n+1}) \\ &\leq \frac{\tau c_1}{7}\|\nabla L^{n+1}\|_0^2 + C\tau\|\nabla \varpi_h^{n+1}\|_0^2, \\ & \quad |-8\tau\nu_r(L^{n+1}, \varpi_h^{n+1})| \\ &\leq \frac{\tau c_1}{7}\|L^{n+1}\|_0^2 + C\tau\|\nabla \varpi_h^{n+1}\|_0^2, \\ & \quad |4\tau\nu_r(\operatorname{curl} \epsilon^{n+1}, \varpi_h^{n+1})| \\ &\leq C\tau\|\epsilon^{n+1}\|_0^2 + C\tau\|\nabla \varpi_h^{n+1}\|_0^2, \\ & \quad |-2\tau jB(\mathbf{U}^n, \mathbf{W}^{n+1}, \varpi_h^{n+1}) - 2\tau jB(\mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1}, \varpi_h^{n+1})| \\ &= |-2\tau jB(\mathbf{U}^{n+1}, \mathbf{W}^{n+1}, \varpi_h^{n+1}) + 2\tau jB(\mathbf{u}_h^{n+1}, \mathbf{w}_h^{n+1}, \varpi_h^{n+1}) - 2\tau jB(\mathbf{U}^n - \mathbf{U}^{n+1}, \mathbf{W}^{n+1}, \varpi_h^{n+1})| \\ &= |-2\tau jB(\mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, \mathbf{W}^{n+1}, \varpi_h^{n+1}) + 2\tau jB(\mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, L^{n+1}, \varpi_h^{n+1}) - 2\tau jB(\mathbf{U}^{n+1}, L^{n+1}, \varpi_h^{n+1}) \\ & \quad - 2\tau jB(\mathbf{U}^n - \mathbf{U}^{n+1}, \mathbf{W}^{n+1}, \varpi_h^{n+1})|, \\ & \quad |-2\tau jB(\mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, \mathbf{W}^{n+1}, \varpi_h^{n+1})| \\ &\leq C\tau\|\epsilon^{n+1}\|_0\|\mathbf{W}^{n+1}\|_2\|\nabla \varpi_h^{n+1}\|_0 \\ &\leq C\tau\|\nabla \varpi_h^{n+1}\|_0^2 + \frac{\tau\nu_0}{5}\|\epsilon^{n+1}\|_0^2, \\ & \quad |2\tau jB(\mathbf{U}^{n+1} - \mathbf{u}_h^{n+1}, L^{n+1}, \varpi_h^{n+1})| \\ &\leq C\tau\|\epsilon^{n+1}\|_0\|\nabla L^{n+1}\|_0(\|\varpi_h^{n+1} - \varpi_h^n\|_2 + \|\varpi_h^{n+1}\|_2) \\ &\leq C\tau(\tau + h^2)\|\nabla L^{n+1}\|_0^2 + \frac{\tau c_1}{7}\|L^{n+1}\|_0^2, \\ & \quad |-2\tau jB(\mathbf{U}^{n+1}, L^{n+1}, \varpi_h^{n+1})| \\ &\leq C\tau\|\mathbf{U}^{n+1}\|_2\|L^{n+1}\|_0\|\nabla \varpi_h^{n+1}\|_0 \\ &\leq C\tau\|\nabla \varpi_h^{n+1}\|_0^2 + \frac{\tau c_1}{7}\|L^{n+1}\|_0^2, \\ & \quad |-2\tau jB(\mathbf{U}^n - \mathbf{U}^{n+1}, \mathbf{W}^{n+1}, \varpi_h^{n+1})| \\ &= |2\tau jB(\epsilon^n, \mathbf{W}^{n+1}, \varpi_h^{n+1}) - 2\tau jB(\epsilon^{n+1}, \mathbf{W}^{n+1}, \varpi_h^{n+1}) - 2\tau jB(\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n), \mathbf{W}^{n+1}, \varpi_h^{n+1})| \end{aligned}$$

$$\begin{aligned}
&\leq C\tau(\|\varepsilon^n\|_0\|\mathbf{W}^{n+1}\|_0 + \|\varepsilon^{n+1}\|_0\|\mathbf{W}^{n+1}\|_0 + \|\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)\|_0\|\mathbf{W}^{n+1}\|_0)(\|\varpi_h^{n+1} - \varpi^{n+1}\| + \|\varpi^{n+1}\|_2) \\
&\leq C\tau^3 + \frac{\tau c_1}{7}\|L^{n+1}\|_0^2.
\end{aligned}$$

Summing it over all n from 0 to N and using discrete Gronwall's Lemma, it yields that

$$j\|\nabla\varpi_h^{N+1}\|_0^2 + \sum_{n=0}^N j\|\nabla\varpi_h^{n+1} - \nabla\varpi_h^n\|_0^2 + \tau c_1 \sum_{n=0}^N \|L^{n+1}\|_0^2 \leq C(\tau^2 + h^4). \quad (4.49)$$

Thanks to Lemma 2.3, we get

$$\|L^{N+1}\|_*^2 + j \sum_{n=0}^N \|\nabla L^{n+1} - \nabla L^n\|_*^2 + \tau c_1 \sum_{n=0}^N \|L^{n+1}\|_0^2 \leq C(\tau^2 + h^4). \quad (4.50)$$

Combining (4.45) with (4.50), the proof is completed. $\square$

5 Numeical results

In this section, numerical experiments are carried out to validate the efficacy of the proposed numerical algorithm. All simulations are implemented via FreeFEM++ [20], an open-source software framework tailored for solving partial differential equations using the finite element method. Specifically, we consider the two-dimensional time-dependent micropolar Navier-Stokes equations defined on the unit square domain $\Omega = [0, 1] \times [0, 1]$, whose exact solutions are derived as follows:

$$\begin{aligned}
u_1 &= \sin(2.0\pi x + t) \sin(2.0\pi y + t), \\
u_2 &= \cos(2.0\pi x + t) \cos(2.0\pi y + t), \\
p &= \sin(2.0\pi(x - y) + t), \\
w &= \sin(2.0\pi x + t) \sin(2.0\pi y + t).
\end{aligned}$$

The boundary conditions and body force terms are derived from the exact solutions. For spatial discretization, we adopt the P1b-P1-P1b finite element spaces, which satisfy the Ladyzhenskaya-Babuska-Brezzi (LBB) condition, thus ensuring stability and optimal convergence. The temporal and spatial step sizes are set as $\tau = h^2$ and $h = 1/n_h$ with $n_h = 8, 16, 24, 32, 40, 48, 56$ to attain optimal convergence rates. The numerical results presented in Tables 1–3 demonstrate optimal convergence orders for different values of the material parameters ν and ν_r .

References

- [1] Adams R., Sobolev Spaces, Academic press, New York, 1975.
- [2] Brenner S., Scott L., The mathematical theory of finite element methods, Springer, New York, 1994.
- [3] Brezzi F., Fortin M., Mixed and hybrid finite element methods, Springer, New York, 2012.
- [4] Chorin A., Numerical solution of the Navier-Stokes equations, Mathematics of computation, 1968. 22(104): 745-762.
- [5] Ciarletta M., On the theory of heat-conducting micropolar fluids, International Journal of Engineering Science, 1995. 33(10): 1403-1417.
- [6] Diening L., Prohl A., Ruzicka M. Semi-implicit Euler scheme for generalized Newtonian fluids, SIAM Journal on Numerical Analysis, 2006. 44(3): 1172-1190.
- [7] Dong X., He Y., Two-level Newton iterative method for the 2D/3D stationary incompressible magnetohydrodynamics, Journal of Scientific Computing, 2015. 63(2): 426-451.

Table 1: The numerical results of $\nu = 1.0$, $\nu_r = 1.0$ and $T = 1$ for $\tau = h^2$.

n_h	$\ \mathbf{u}_h^N - \mathbf{u}\ _0$	$\ \nabla(\mathbf{u}_h^N - \mathbf{u})\ _0$	$\ p_h^{N-1} - p\ _0$	$\ w_h^N - w^N\ _0$	$\ \nabla(w_h^N - w^N)\ _0$
8	0.162009	0.372432	3.85906	0.166341	0.358006
16	0.0429448	0.187683	1.33623	0.0460381	0.184439
24	0.0192728	0.124975	0.684293	0.020865	0.123638
32	0.0108722	0.093624	0.427185	0.0118179	0.0929071
40	0.00696452	0.0748366	0.297856	0.00758754	0.0743918
48	0.00483922	0.0623251	0.222347	0.00527855	0.0620231
56	0.00355589	0.0533964	0.174049	0.00388223	0.0531781
64	0.00272258	0.0467047	0.140967	0.00297438	0.0465397
n_h	$\mathbf{u}_{\text{order}L^2}$	$\mathbf{u}_{\text{order}H^1}$	$p_{\text{order}L^2}$	$w_{\text{order}L^2}$	$w_{\text{order}H^1}$
8	—	—	—	—	—
16	1.9155	0.9887	1.5301	1.8532	0.9568
24	1.9761	1.0029	1.6505	1.9518	0.9864
32	1.9900	1.0040	1.6378	1.9760	0.9933
40	1.9959	1.0037	1.6160	1.9858	0.9960
48	1.9969	1.0034	1.6036	1.9902	0.9974
56	1.9990	1.0031	1.5887	1.9931	0.9981
64	1.9997	1.0027	1.5787	1.9948	0.9986

Table 2: The numerical results of $\nu = 0.5$, $\nu_r = 0.5$ and $T = 1$ for $\tau = h^2$.

n_h	$\ \mathbf{u}_h^N - \mathbf{u}\ _0$	$\ \nabla(\mathbf{u}_h^N - \mathbf{u})\ _0$	$\ p_h^{N-1} - p\ _0$	$\ w_h^N - w^N\ _0$	$\ \nabla(w_h^N - w^N)\ _0$
8	0.161721	0.372421	1.93612	0.167036	0.357724
16	0.0428582	0.187684	0.668725	0.0462801	0.184401
24	0.0192311	0.124975	0.342351	0.0209801	0.123627
32	0.0108479	0.0936241	0.213689	0.0118843	0.0929022
40	0.00694883	0.0748366	0.148982	0.00763066	0.0743893
48	0.00482822	0.0623252	0.111208	0.00530858	0.0620217
56	0.00354778	0.0533965	0.0870477	0.00390438	0.0531772
64	0.00271635	0.0467047	0.0704997	0.00299138	0.0465391
n_h	$\mathbf{u}_{\text{order}L^2}$	$\mathbf{u}_{\text{order}H^1}$	$p_{\text{order}L^2}$	$w_{\text{order}L^2}$	$w_{\text{order}H^1}$
8	—	—	—	—	—
16	1.9159	0.9886	1.5337	1.8517	0.9560
24	1.9764	1.0029	1.6513	1.9512	0.9861
32	1.9902	1.0040	1.6383	1.9756	0.9932
40	1.9960	1.0038	1.6164	1.9855	0.9959
48	1.9970	1.0034	1.6039	1.9902	0.9973
56	1.9991	1.0030	1.5890	1.9930	0.9981
64	1.9998	1.0028	1.5790	1.9948	0.9985

- [8] E W., Liu J., Gauge method for viscous incompressible flows, Communications in Mathematical Sciences, 2003. 1(2): 317-332.
- [9] Eringen A., Theory of micropolar fluids, Journal of mathematics and Mechanics, 1966. 16(1): 1-18.
- [10] Eringen A., Microcontinuum Field Theories. I. Foundations and Solids, Springer-Verlag, New York, 1999.
- [11] Eringen A., Microcontinuum Field Theories. II. Fluent Media, Springer-Verlag, New York, 2001.

Table 3: The numerical results of $\nu = 0.05$, $\nu_r = 0.05$ and $T = 1$ for $\tau = h^2$.

n_h	$\ \mathbf{u}_h^N - \mathbf{u}\ _0$	$\ \nabla(\mathbf{u}_h^N - \mathbf{u})\ _0$	$\ p_h^{N-1} - p\ _0$	$\ w_h^N - w^N\ _0$	$\ \nabla(w_h^N - w^N)\ _0$
8	0.164167	0.37432	0.219828	0.158609	0.364777
16	0.0438202	0.187991	0.0714397	0.0431627	0.185556
24	0.019671	0.125068	0.0359876	0.0194748	0.123985
32	0.0110962	0.0936635	0.0222599	0.0110113	0.0930556
40	0.00710815	0.0748568	0.0154231	0.00706356	0.0744683
48	0.00493824	0.0623368	0.0114608	0.00491153	0.0620675
56	0.00362847	0.0534038	0.00893967	0.00361116	0.0532062
64	0.00277804	0.0467096	0.00722026	0.0027661	0.0465585
n_h	$\mathbf{u}_{\text{order}L^2}$	$\mathbf{u}_{\text{order}H^1}$	$p_{\text{order}L^2}$	$w_{\text{order}L^2}$	$w_{\text{order}H^1}$
8	—	—	—	—	—
16	1.9055	0.9936	1.6216	1.8776	0.9752
24	1.9754	1.0051	1.6911	1.9628	0.9944
32	1.9902	1.0051	1.6699	1.9820	0.9975
40	1.9958	1.0044	1.6443	1.9896	0.9986
48	1.9978	1.0039	1.6286	1.9930	0.9991
56	1.9993	1.0034	1.6116	1.9952	0.9993
64	2.0000	1.0030	1.5997	1.9965	0.9995

- [12] Franchi F., Straughan B., Nonlinear stability for thermal convection in a micropolar fluid with temperature dependent viscosity, International journal of engineering science, 1992. 30(10): 1349-1360.
- [13] Forste J., Existence of time independent flows of incompressible micropolar fluids, Fluid Dynamics Trans, 1971(6): 193-197.
- [14] Girault V., Raviart P., Finite element methods for the Navier-Stokes equations, Springer, New York, 1986.
- [15] Guermond J., Shen J., Quelques resultats nouveaux sur les methodes de projection, Comptes Rendus de l'Academie des Sciences-Series I-Mathematics, 2001. 333 (12): 1111-1116.
- [16] Guermond J., Shen J., Velocity-correction projection methods for incompressible flows, SIAM Journal on Numerical Analysis, 2003. 41(1): 112-134.
- [17] Guermond J., Shen J., A new class of truly consistent splitting schemes for incompressible flows, Journal of computational physics, 2003. 192 (1): 262-276.
- [18] He Y., Two-level method based on finite element and crank-nicolson extrapolation for the time-dependent Navier-Stokes equations, SIAM Journal on Numerical Analysis, 2003. 41(4): 1263-1285.
- [19] He Y., Liu K., A multilevel finite element method in space-time for the Navier-Stokes problem, Numerical Methods for Partial Differential Equations: An International Journal, 2005. 21(6): 1052-1078.
- [20] Hecht F., New development in FreeFem++, Journal of numerical mathematics, 2012. 20(3-4): 251-265.
- [21] Heywood J., Rannacher R., Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization, SIAM Journal on Numerical Analysis, 1990. 27(2): 353-384.

- [22] Hills R., On uniqueness and continuous dependence for linear micropolar fluid, *International Journal of Engineering Science*, 1973. 11(3): 369-376.
- [23] Liu Q., Hou Y., Two-level consistent splitting methods based on three corrections for the time-dependent Navier-Stokes equations, *International Journal for Numerical Methods in Fluids*, 2016. 80 (7): 429-450.
- [24] Mitarai N., Hayakawa H., Collisional granular flow as a micropolar fluid, *Physical review letters*, 2002. 88(17): 174-301.
- [25] Lukaszewicz G., Micropolar fluids, in *Modeling and Simulation in Science, Engineering and Technology*, 1999.
- [26] Nocketto R., Pyo J. The Gauge-Uzawa finite element method, Part I: The Navier-Stokes equations, *SIAM journal on numerical analysis*, 2005. 43(3): 1043-1086.
- [27] Nocketto R., Pyo J., Error estimates for semi-discrete gauge methods for the Navier-Stokes equations, *Mathematics of computation*, 2005. 74(250): 521-542.
- [28] Nocketto R., Pyo J., The gauge-uzawa finite element method - Part II: The Boussinesq equations, *Mathematical Models and Methods in Applied Sciences*, 2006. 16(10): 1599-626.
- [29] Nocketto R., Salgado A., Tomas I. The Micropolar Navier-Stokes equations: A priori error analysis, *Mathematical Models and Methods in Applied Sciences*, 2014. 24(7): 1237-1264.
- [30] Ortega-Torres E., Rojas-Medar M. Optimal error estimate of the penalty finite element method for the micropolar fluid equations, *Numerical Functional Analysis and Optimization*, 2008. 29(5-6): 612-637.
- [31] Oseledets V., On a new way of writing the Navier-Stokes equation, *The Hamiltonian Formalism, Russian Mathematical Surveys*, 1989. 44: 169-170.
- [32] Papautsky I., Brazzle J., Laminar fluid behavior in microchannels using micropolar fluid theory, *Sensors and actuators A: Physical*, 1999. 73(1-2): 101-108.
- [33] Pyo J., Optimal error estimate for semi-discrete Gauge-Uzawa method for the Navier-Stokes equations, *Bulletin of the Korean Mathematical Society*, 2009. 46(4): 627-644.
- [34] Pyo J., Error estimate for the second order semi-discrete stabilized Gauge-Uzawa method for the Navier-Stokes equations, *International Journal of Numerical Analysis*, 2013. 10(1): 24-41.
- [35] Ren Y., Liu D., Pressure correction projection finite element method for the 2D/3D time-dependent thermomicropolar fluid problem, *Computers and Mathematics with Applications*, 2023. 136: 136-150.
- [36] Ren Y., Liu D., Research on three kinds of splitting finite element schemes for 2D/3D unsteady incompressible thermomicropolar fluid equations, *International Journal for Numerical Methods in Fluids*, 2023: 1-26.
- [37] Rinaldi C., Zahn M., Effects of spin viscosity on ferrofluid flow profiles in alternating and rotating magnetic fields, *Physics of Fluids*, 2002. 14(8): 2847-2870.
- [38] Rosensweig R., *Ferrohydrodynamics*, Cambridge University Press, Oxford, 1997.
- [39] Shen J.. Remarks on the pressure error estimates for the projection methods, *Numerische Mathematik*, 1994. 67(4): 513-520.
- [40] Shen X., Wang Y., Si Z., A rotational pressure-correction projection methods for unsteady incompressible magnetohydrodynamics equations, *Applied Mathematics and Computation*, 2020. 387: 124488.

- [41] Si Z., Hou A., Wang Y., A consistent projection finite element method for the non-stationary incompressible thermally coupled MHD equations, *Communications in Nonlinear Science and Numerical Simulation*, 2023. 11(12): 107496.
- [42] Si Z., Ji Y., Wang Y., Unconditional stability and error estimates of the modified characteristics FEMs for the Micropolar Navier-Stokes Equations, *Commun Nonlinear Sci*, 2024, 134: 108008.
- [43] Si Z., Li Z., Wei L., A rotational velocity-correction projection method for the Micropolar Navier-Stokes equations, *Appl Numer Math*, 2024. 205: 267-280.
- [44] Si Z., Li W., Wang Y., A Gauge-Uzawa finite element method for the time-dependent viscoelastic Oldroyd flows, *Journal of Mathematical Analysis and Applications*, 2015. 425(1): 96-110.
- [45] Temam R. Sur, l'approximation de la solution des equations de Navier-Stokes par la methode des pas fractionnaires (II), *Archive for rational mechanics and analysis*, 1969. 33(5): 377-385.
- [46] Van J., A second-order accurate pressure-correction scheme for viscous incompressible flow, *SIAM journal on scientific and statistical computing*, 1986. 7(3): 870-891.
- [47] Wang C., Liu J., Convergence of gauge method for incompressible flow, *Mathematics of computation*, 2000. 69(232): 1385-1407.
- [48] Xin X., Liu D., Numerical analysis and comparison of three iterative methods based on finite element for the 2D/3D stationary micropolar fluid equations, *Entropy*, 2022. 24(5): 628.
- [49] Yang Y., Si Z., A consistent projection finite element method for the incompressible MHD equations, *Applicable Analysis*, 2021. 100(12): 2606-2626.
- [50] Zahn M., Greer D., Ferrohydrodynamic pumping in spatially uniform sinusoidally time-varying magnetic fields, *J. Magnetism and Magnetic Mater*, 1995. 149(1-2): 165-173.