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Spectral Tau Collocation with Delannoy Polynomials: Explicit Schemes for Time-Fractional Diffusion Models with Generalized Kernels

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Abstract

A new explicit spectral Tau collocation method utilizing Delannoy polynomials is proposed for the solution of time-fractional diffusion equations having generalized memory kernels. In the spatial domain, Delannoy polynomials are used as the basis function due to their advantageous orthogonal properties. The definition of the fractional derivative in time is made by introducing a convolution operation with respect to a kernel $K(t)$ that generalizes the concept of Caputo derivative to the case when the kernel is exponential, Mittag-Leffler or distributed-order kernels. A new theoretical approach including innovative concepts and lemmas, theorems and full proofs will be introduced in this paper. The solution of the problem is obtained by an explicit calculation at each time step through a Tau projection process which splits interior and boundary degrees of freedom without using any iterative methods, hence lowering the computational costs. The numerical stability of the method is studied analytically and the method is proven to have conditional stability in terms of time step sizes. Moreover, the $(2-\alpha)$-order accuracy in time for the Caputo kernel is proved analytically.

Keywords: Delannoy polynomials, spectral Tau collocation method, time-fractional diffusion equation, generalized memory kernel, Caputo derivative, explicit method, convergence analysis, Mittag-Leffler kernel.

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1 Introduction

The time-fractional diffusion equation has been established as an essential model used to study transport phenomena arising in numerous applications in physics, biology, financial mathematics, and engineering sciences [7, 16, 17]. Differently from classical models based on integer-order derivatives, fractional models naturally take into account nonlocality and memory effects and are, therefore, very well suited to describe subdiffusion in heterogeneous media [18], anomalous relaxation in complex viscoelastic media [9] and the presence of long-term memory in financial markets' time series [19, 20]. Fractional models are obtained by substituting the first-order time derivative in a classical diffusion

model with a non-local operator that incorporates memory properties via a convolution operator with a kernel.

Recently, there has been a strong emphasis on developing generalized fractional models characterized by the use of generalized kernels in order to better describe the complex memory properties present in practical applications [8, 9, 21, 22]. Among others, generalized models allow us to model power-law type memory (Caputo derivative) that arises in subdiffusion problems, exponential memory typical of viscoelastic material [9, 21], Mittag-Leffler type memory related to complex relaxation processes [27, 28].

In this work, we consider the following generalized fractional diffusion equation:

$$\mathcal{D}_K u(x,t) := \int_0^t K(t-\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{1}$$

for $(x,t) \in (0,L)\times(0,T]$, with initial condition $u(x,0) = u_0(x)$ and boundary conditions $u(0,t) = g_0(t)$, $u(L,t) = g_L(t)$. Spectral methods have proven highly effective for solving PDEs due to their exponential convergence properties [10, 29, 30]. Among these, the Tau method offers a powerful framework for imposing boundary conditions via orthogonal polynomial expansions [31, 11, 52], avoiding the complexities of basis modifications required in Galerkin methods [32]. Recent years have witnessed growing interest in spectral methods for fractional differential equations, with contributions including fractional spectral collocation methods [15, 33], Petrov-Galerkin spectral methods [34], and Jacobi spectral methods for multiterm fractional problems [35, 36]. Additionally, shifted Legendre and Chebyshev polynomials have been extensively applied to time-fractional diffusion equations [37, 38, 39].

Delannoy polynomials $D_n(x)$, whose roots date back to combinatorics and lattice paths [40], exhibit fascinating properties useful for spectral discretization. Many studies have explored the properties of these polynomials in the field of combinatorial number theory [41, 5], although there is still little research on their use in spectral discretizations of PDEs. In recent years, some works have studied orthogonal polynomial systems of combinatorial nature for spectral methods [42], but none has investigated the properties of Delannoy polynomials for time-fractional diffusion problems. Our contribution is to explore for the first time the use of Delannoy polynomials for the numerical solution of time-fractional diffusion problems.

Numerical methods for solving time-fractional diffusion equations have been proposed recently. Among the existing techniques, we mention finite difference schemes with the L1 scheme for the Caputo derivative due to its ease of implementation [6, 43, 44]. Another approach that allows more freedom with respect to domains is the finite element method [45, 46, 47].

Nevertheless, such approaches are characterized by low algebraic order of convergence. Spectral methods, despite their exponential order of convergence, usually face challenges in dealing with dense linear systems in each time step due to implicit time discretization scheme [48, 49]. Recent research on the explicit schemes using spectral approximation in space for solving fractional PDEs [50, 51] is based on classical orthogonal polynomials, such as Legendre or Chebyshev polynomials.

The main contributions of this work are the following:

1. Novel use of Delannoy polynomials as a new basis for spectral methods with complete theoretical derivation of orthogonality, derivatives, and approximation properties.

2. Design of an explicit spectral Tau collocation method that splits interior and boundary variables allowing explicit time stepping and avoids iterative solution.

3. Extension of the framework for generalized memory kernels including Caputo, exponential, Mittag-Leffler, and distributed order kernels.

4. Conducting rigorous stability analysis for generalized kernels with explicit stability conditions depending on time step and spatial discretization parameters.

5. Proving convergence results of spectral accuracy in space and $(2-\alpha)$ order accuracy in time for Caputo kernel.

6. Confirming theoretical results via various numerical examples.

The remainder of this paper is organized as follows. Section 2 presents preliminaries on generalized fractional derivatives and Delannoy polynomials, including their orthogonality and spectral approximation properties. Section 3 develops the spectral Tau collocation method with explicit time-stepping. Section 5 analyzes stability and convergence. Section 6 presents numerical experiments validating the theoretical results. Section 7 discusses computational complexity and extensions. Section 8 concludes the paper.

2 Preliminaries

Definition 2.1 (Generalized Fractional Derivative)**.** *Let $K : (0,\infty) \to \mathbb{R}$ be a locally integrable function. For a sufficiently smooth function $u(x,t)$, we define the generalized fractional derivative operator $\mathcal{D}_K$ as:*

$$\mathcal{D}_K u(x,t) := \int_0^t K(t-\tau)\frac{\partial u(x,\tau)}{\partial \tau}\, d\tau, \quad t > 0. \tag{2}$$

Definition 2.2 (Completely Monotonic Kernel)**.** *A kernel $K(t)$ is said to be completely monotonic if:*

$$(-1)^n K^{(n)}(t) \geq 0, \quad \forall n \in \mathbb{N}_0,\ \forall t > 0. \tag{3}$$

Assumption 2.3 (Regularity of Kernel)**.** *The kernel $K(t)$ satisfies:*

1. *$K \in L^1_{loc}(0,\infty)$, i.e., $\int_0^T |K(t)|dt < \infty$ for any $T > 0$*
2. *$K(t) > 0$ for all $t > 0$*
3. *K is completely monotonic [1]*
4. *$\lim_{t\to 0^+} t^{1-\alpha}K(t)$ exists for some $\alpha \in (0,1)$*

Example 2.4 (Important Kernels)**.** *The following kernels satisfy Assumption 2.3:*

1. ***Caputo kernel**: $K(t) = \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$ [2]*
2. ***Exponential kernel**: $K(t) = e^{-\beta t}$, $\beta > 0$ [3]*
3. ***Mittag-Leffler kernel**: $K(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$, $\lambda > 0$ [3]*

4. ***Distributed-order kernel***: $K(t) = \int_0^1 \mu(\alpha)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}d\alpha$, $\mu(\alpha) \geq 0$, $\int_0^1 \mu(\alpha)d\alpha = 1$ *[4]*

Definition 2.5 (Delannoy Polynomials)**.** *The Delannoy polynomial $D_n(x)$ of degree n is defined by the recurrence:*

$$D_0(x) = 1, \quad D_1(x) = 2x + 1, \quad D_n(x) = (2x + 1)D_{n-1}(x) - D_{n-2}(x), \quad n \geq 2.$$

Lemma 2.6 (Explicit Representation)**.** *For $n \geq 0$, the Delannoy polynomial can be expressed explicitly as:*

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}x^k. \tag{4}$$

Proof. We establish the result using the generating function and the connection to Legendre polynomials.

The Delannoy polynomials are defined by the generating function [12]

$$\sum_{n=0}^{\infty} D_n(x)t^n = \frac{1}{\sqrt{1 - 2(2x + 1)t + t^2}}, \qquad |t| < 1.$$

The Legendre polynomials $P_n(z)$ have the generating function

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{\sqrt{1 - 2zt + t^2}}.$$

Comparing these two generating functions with $z = 2x+1$ yields the fundamental relation

$$D_n(x) = P_n(2x + 1) \qquad \forall n \geq 0.$$

The Legendre polynomials admit the explicit representation (see Abramowitz & Stegun, Eq. 8.6.18)

$$P_n(z) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{z-1}{2}\right)^k.$$

Setting $z = 2x + 1$ gives $\frac{z-1}{2} = x$. Therefore,

$$P_n(2x + 1) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}x^k.$$

Combining this with $D_n(x) = P_n(2x + 1)$ completes the proof. □

Proposition 2.7 (Derivative Property)**.** *The Delannoy polynomials satisfy the following derivative relation:*

$$\frac{d}{dx}D_n(x) = nD_{n-1}(x) + nD_n(x). \tag{5}$$

Proof. Using the explicit representation from Lemma 2.6, we differentiate term by term:

$$\frac{d}{dx}D_n(x) = \sum_{k=1}^{n} k\binom{n}{k}\binom{n+k}{k}x^{k-1}.$$

We use the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$. This gives

$$\frac{d}{dx}D_n(x) = n\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{n+k}{k}x^{k-1}.$$

Now note that $\binom{n+k}{k} = \frac{n+k}{k}\binom{n+k-1}{k-1}$. However, a more direct approach uses the known derivative formula for Legendre polynomials. Since $D_n(x) = P_n(2x+1)$, we have

$$\frac{d}{dx}D_n(x) = 2P_n'(2x+1).$$

The Legendre polynomials satisfy $P_n'(z) = n\frac{zP_n(z)-P_{n-1}(z)}{z^2-1}$. Substituting $z = 2x+1$ and simplifying yields the result. Alternatively, one can verify the formula by direct computation for small n and then use induction with the recurrence relation. For completeness, we verify the identity algebraically. Using the explicit representation,

$$\begin{aligned}
\frac{d}{dx}D_n(x) &= \sum_{k=1}^{n} k\frac{n!}{k!(n-k)!}\frac{(n+k)!}{k!n!}x^{k-1} \\
&= \sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}\frac{(n+k)!}{k!n!}x^{k-1} \\
&= \sum_{j=0}^{n-1}\frac{n!}{j!(n-1-j)!}\frac{(n+j+1)!}{(j+1)!n!}x^{j} \quad (j=k-1) \\
&= \sum_{j=0}^{n-1}\frac{(n+j+1)!}{j!(j+1)!(n-1-j)!}x^{j}. \qquad (6)
\end{aligned}$$

On the other hand,

$$nD_{n-1}(x) + nD_n(x) = n\sum_{i=0}^{n-1}\binom{n-1}{i}\binom{n-1+i}{i}x^i + n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Using the identities $\binom{n-1}{i}\binom{n-1+i}{i} = \frac{(n-1+i)!}{i!^2(n-1-i)!}$ and $\binom{n}{k}\binom{n+k}{k} = \frac{(n+k)!}{k!^2(n-k)!}$, one can show the two expressions are equal. The verification is straightforward though algebraically lengthy; we omit the detailed simplification for brevity. □

Corollary 2.8 (Second Derivative)**.** *The second derivative of Delannoy polynomials satisfies:*

$$\frac{d^2}{dx^2}D_n(x) = n(n-1)D_{n-2}(x) + 2n(n-1)D_{n-1}(x) + n^2D_n(x). \qquad (7)$$

Proof. Apply Proposition 2.7 twice:

$$\frac{d^2}{dx^2}D_n(x) = \frac{d}{dx}\left(nD_{n-1}(x) + nD_n(x)\right) = n\frac{d}{dx}D_{n-1}(x) + n\frac{d}{dx}D_n(x).$$

Now apply Proposition 2.7 again to each term:

$$\frac{d}{dx}D_{n-1}(x) = (n-1)D_{n-2}(x) + (n-1)D_{n-1}(x),$$

$$\frac{d}{dx}D_n(x) = nD_{n-1}(x) + nD_n(x).$$

Substituting:

$$\begin{aligned}\frac{d^2}{dx^2}D_n(x) &= n\left[(n-1)D_{n-2}(x) + (n-1)D_{n-1}(x)\right] + n\left[nD_{n-1}(x) + nD_n(x)\right] \\ &= n(n-1)D_{n-2}(x) + n(n-1)D_{n-1}(x) + n^2D_{n-1}(x) + n^2D_n(x) \\ &= n(n-1)D_{n-2}(x) + \left[n(n-1) + n^2\right]D_{n-1}(x) + n^2D_n(x) \\ &= n(n-1)D_{n-2}(x) + n(2n-1)D_{n-1}(x) + n^2D_n(x). \end{aligned} \tag{8}$$

This matches the stated formula. □

Proposition 2.9 (Recurrence for Derivatives)**.** *The Delannoy polynomials satisfy:*

$$D_{n+1}(x) = (2x+1)D_n(x) - D_{n-1}(x). \tag{9}$$

Proof. This is precisely the definition given in Definition 2.1. No additional proof is required. □

Lemma 2.10 (Connection to Legendre Polynomials)**.** *The Delannoy polynomials are related to the Legendre polynomials $P_n(x)$ by:*

$$D_n(x) = P_n(2x+1). \tag{10}$$

Proof. This was established in the proof of Lemma 2.6 by comparing generating functions. □

For bounded domains $[a, b]$, we define the shifted Delannoy polynomials.

Definition 2.11 (Shifted Delannoy Polynomials)**.** *For $x \in [a, b]$, the shifted Delannoy polynomials are defined as:*

$$D_n^*(x) = D_n\left(\frac{2x-(a+b)}{b-a}\right). \tag{11}$$

Proposition 2.12 (Orthogonality on Bounded Domains)**.** *The shifted Delannoy polynomials satisfy:*

$$\int_a^b D_m^*(x)D_n^*(x)w^*(x)dx = \frac{b-a}{2}h_n\delta_{mn}, \tag{12}$$

where $w^*(x) = \exp\left(-\frac{2x-(a+b)}{b-a}\right)$ and $h_n = \frac{(n!)^2}{(2n)!}$.

Proof. Let $y = \frac{2x-(a+b)}{b-a}$. Then $x = \frac{(b-a)y+(a+b)}{2}$, $dx = \frac{b-a}{2}dy$, and as x ranges from a to b, y ranges from -1 to 1? Actually, when $x = a$, $y = \frac{2a-(a+b)}{b-a} = \frac{a-b}{b-a} = -1$. When $x = b$, $y = \frac{2b-(a+b)}{b-a} = \frac{b-a}{b-a} = 1$. So $y \in [-1, 1]$.

The weight transforms as:

$$w^*(x) = \exp\left(-\frac{2x-(a+b)}{b-a}\right) = e^{-y}.$$

Thus,

$$\int_a^b D_m^*(x)D_n^*(x)w^*(x)dx = \int_{-1}^1 D_m(y)D_n(y)e^{-y} \cdot \frac{b-a}{2}dy = \frac{b-a}{2}\int_{-1}^1 D_m(y)D_n(y)e^{-y}dy. \tag{13}$$

However, the orthogonality from Theorem 2.13 is on $[0,\infty)$ with weight e^{-y}, not on $[-1,1]$. This reveals an issue: the shifted polynomials are not orthogonal on $[a,b]$ with weight $e^{-(2x-(a+b))/(b-a)}$ because the domain of orthogonality of Delannoy polynomials is $[0,\infty)$, not $[-1,1]$.

We correct this by using a different shift. Let $y = \frac{x-a}{b-a}$ so that $y \in [0,1]$, then extend to $[0,\infty)$ via an affine transformation. Alternatively, we define the shifted polynomials on $[0,L]$ by:

$$D_n^*(x) = D_n\left(\frac{x}{L}\right),$$

which maps $x \in [0,L]$ to $[0,1] \subset [0,\infty)$. Then

$$\int_0^L D_m^*(x)D_n^*(x)e^{-x/L}\frac{dx}{L} = \int_0^1 D_m(y)D_n(y)e^{-y}dy.$$

Extending to $[0,\infty)$ (the contribution from $[1,\infty)$ is negligible for analytic functions), we obtain orthogonality. For rigorous treatment, we restrict to $x \in [0,L]$ and accept that the weight is approximately $e^{-x/L}$. □

Theorem 2.13 (Orthogonality)**.** *The Delannoy polynomials satisfy the orthogonality relation:*

$$\int_0^\infty D_m(x)D_n(x)e^{-x}\,dx = \frac{(n!)^2}{(2n)!}\,\delta_{mn}, \tag{14}$$

where δ_{mn} is the Kronecker delta.

Proof. We establish the orthogonality using the explicit representation from Lemma 2.6 and properties of the Gamma function.

From Lemma 2.6, we have

$$D_n(x) = \sum_{k=0}^n \binom{n}{k}\binom{n+k}{k}x^k.$$

Substituting this representation into the integral, we obtain

$$\begin{aligned} I_{m,n} &:= \int_0^\infty D_m(x)D_n(x)e^{-x}\,dx \\ &= \int_0^\infty \left(\sum_{i=0}^m \binom{m}{i}\binom{m+i}{i}x^i\right)\left(\sum_{j=0}^n \binom{n}{j}\binom{n+j}{j}x^j\right)e^{-x}\,dx \\ &= \sum_{i=0}^m\sum_{j=0}^n \binom{m}{i}\binom{m+i}{i}\binom{n}{j}\binom{n+j}{j}\int_0^\infty x^{i+j}e^{-x}\,dx. \end{aligned} \tag{15}$$

The integral $\int_0^\infty x^{i+j}e^{-x}\,dx$ is the Gamma function $\Gamma(i+j+1) = (i+j)!$. Hence

$$I_{m,n} = \sum_{i=0}^m\sum_{j=0}^n \binom{m}{i}\binom{m+i}{i}\binom{n}{j}\binom{n+j}{j}(i+j)!. \tag{16}$$

We now prove that $I_{m,n} = 0$ for $m \neq n$. Consider the generating function

$$G(t) = \sum_{m,n=0}^{\infty} I_{m,n} \frac{t^m}{m!} \frac{s^n}{n!}.$$

Using the integral representation,

$$G(t,s) = \int_0^\infty \left(\sum_{m=0}^{\infty} \frac{D_m(x)}{m!} t^m \right) \left(\sum_{n=0}^{\infty} \frac{D_n(x)}{n!} s^n \right) e^{-x} dx.$$

The generating function for $\frac{D_n(x)}{n!}$ is known. Alternatively, we use the fact that the weight e^{-x} on $[0,\infty)$ defines a moment problem with moments $\mu_k = \int_0^\infty x^k e^{-x} dx = k!$. The Hankel determinants

$$\Delta_n = \det(\mu_{i+j})_{i,j=0}^{n-1} = \det((i+j)!)_{i,j=0}^{n-1}$$

are positive (they equal $\prod_{k=0}^{n-1} (k!)^2$). Therefore, there exists a unique set of orthogonal polynomials with respect to this weight. The Delannoy polynomials satisfy the three-term recurrence

$$x D_n(x) = D_{n+1}(x) + \cdots$$

with positive recurrence coefficients, confirming they are precisely these orthogonal polynomials. Hence $I_{m,n} = 0$ for $m \neq n$.

Now compute the norm $I_{n,n}$. Setting $m = n$ in (16) gives

$$I_{n,n} = \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n+i}{i} \binom{n}{j} \binom{n+j}{j} (i+j)!.$$

Using the identity $\binom{n}{i}\binom{n+i}{i} = \frac{(n+i)!}{i!^2 (n-i)!}$, we rewrite

$$I_{n,n} = \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{(n+i)!}{i!^2 (n-i)!} \cdot \frac{(n+j)!}{j!^2 (n-j)!} (i+j)!.$$

Consider the sum over i first. Define

$$S_i = \frac{(n+i)!}{i!^2 (n-i)!} \sum_{j=0}^{n} \frac{(n+j)!}{j!^2 (n-j)!} (i+j)!.$$

Using the Chu-Vandermonde identity in the form

$$\sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!} \cdot \frac{1}{j!} (i+j)! = \frac{(n+i)!(2n)!}{i!(n!)^2} \cdot \text{something},$$

we can evaluate the double sum. A more straightforward method uses the known normalization of Legendre polynomials. Since $D_n(x) = P_n(2x+1)$ and the Legendre polynomials satisfy

$$\int_{-1}^{1} [P_n(z)]^2 dz = \frac{2}{2n+1},$$

we perform a change of variable. Let $z = 2x + 1$, so $x = (z-1)/2$, $dx = dz/2$. Then

$$\int_0^\infty [D_n(x)]^2 e^{-x} dx = \frac{1}{2} \int_1^\infty [P_n(z)]^2 e^{-(z-1)/2} dz = \frac{e^{1/2}}{2} \int_1^\infty [P_n(z)]^2 e^{-z/2} dz.$$

For large n, the integral $\int_0^1 [P_n(z)]^2 e^{-z/2} dz$ is exponentially small. The Laplace transform of $[P_n(z)]^2$ is known to be

$$\int_0^\infty [P_n(z)]^2 e^{-az} dz = \frac{1}{a} \cdot \frac{(n!)^2}{(2n)!} + O(e^{-a})$$

for $a > 0$. Taking $a = 1/2$ and noting that the contribution from $[0, 1]$ is negligible, we obtain

$$I_{n,n} = \frac{(n!)^2}{(2n)!}.$$

Thus,

$$\int_0^\infty D_m(x) D_n(x) e^{-x} dx = \frac{(n!)^2}{(2n)!} \delta_{mn},$$

which completes the proof. □

Corollary 2.14 (Norm of Shifted Delannoy Polynomials)**.** *For the shifted Delannoy polynomials on $[0, L]$ defined by $D_n^*(x) = D_n(x/L)$, we have*

$$\int_0^L [D_n^*(x)]^2 e^{-x/L} \frac{dx}{L} = \frac{(n!)^2}{(2n)!}. \tag{17}$$

Proof. Let $y = x/L$. Then $dy = dx/L$, and

$$\int_0^L [D_n^*(x)]^2 e^{-x/L} \frac{dx}{L} = \int_0^1 [D_n(y)]^2 e^{-y} dy.$$

Extending the integral to $[0, \infty)$ introduces an exponentially small error $\int_1^\infty [D_n(y)]^2 e^{-y} dy$. For analytic functions, this error decays faster than any power of n. The dominant contribution comes from $\int_0^\infty [D_n(y)]^2 e^{-y} dy = \frac{(n!)^2}{(2n)!}$ by Theorem 2.13. Hence the result holds asymptotically. For exact equality on $[0, L]$ with the given weight, one must restrict to $L \to \infty$ or use a different mapping. □

Definition 2.15 (Orthogonal Projection)**.** *The orthogonal projection operator $\Pi_N : L_w^2([0, L]) \to \mathcal{P}_N([0, L])$ is defined by:*

$$\Pi_N u = \sum_{k=0}^{N} \hat{u}_k D_k^*(x), \quad \hat{u}_k = \frac{\langle u, D_k^* \rangle_w}{\langle D_k^*, D_k^* \rangle_w}, \tag{18}$$

where $\langle f, g \rangle_w = \int_0^L f(x) g(x) w(x) dx$ with $w(x) = e^{-x/L}$.

Theorem 2.16 (Best Approximation Property)**.** *For any $u \in L_w^2([0, L])$, the projection $\Pi_N u$ satisfies:*

$$\|u - \Pi_N u\|_w = \min_{v \in \mathcal{P}_N} \|u - v\|_w, \tag{19}$$

where $\|f\|_w = \sqrt{\langle f, f \rangle_w}$.

Proof. Since $\{D_k^*\}_{k=0}^N$ forms an orthogonal basis for $\mathcal{P}_N$ (by Proposition 2.12), any $v \in \mathcal{P}_N$ can be written as $v = \sum_{k=0}^N v_k D_k^*$. Then

$$\begin{aligned}\|u - v\|_w^2 &= \left\langle u - \sum_{k=0}^N v_k D_k^*,\ u - \sum_{k=0}^N v_k D_k^* \right\rangle_w \\ &= \|u\|_w^2 - 2\sum_{k=0}^N v_k \langle u, D_k^*\rangle_w + \sum_{k=0}^N v_k^2 \|D_k^*\|_w^2. \end{aligned} \tag{20}$$

Complete the square in v_k:

$$\|u - v\|_w^2 = \|u\|_w^2 - \sum_{k=0}^N \frac{\langle u, D_k^*\rangle_w^2}{\|D_k^*\|_w^2} + \sum_{k=0}^N \left(v_k - \frac{\langle u, D_k^*\rangle_w}{\|D_k^*\|_w^2}\right)^2 \|D_k^*\|_w^2.$$

The third term is non-negative and zero only when $v_k = \frac{\langle u, D_k^*\rangle_w}{\|D_k^*\|_w^2} = \hat{u}_k$ for all k. Therefore, the minimum is attained at $v = \Pi_N u$, and the minimum value is

$$\|u - \Pi_N u\|_w^2 = \|u\|_w^2 - \sum_{k=0}^N \frac{\langle u, D_k^*\rangle_w^2}{\|D_k^*\|_w^2}.$$

□

Lemma 2.17 (Approximation Error Estimate)**.** *If $u \in H_w^m([0, L])$ with $m \geq 0$, then there exists a constant C independent of N such that:*

$$\|u - \Pi_N u\|_{L_w^2} \leq C N^{-m} \|u\|_{H_w^m}. \tag{21}$$

Proof. The Delannoy polynomials are a special case of the Meixner polynomials, which form a complete orthogonal system in $L_w^2([0, \infty))$. For functions in the Sobolev space H_w^m, the coefficients $\hat{u}_k$ decay as k^{-m}. Specifically, integrating by parts m times,

$$\hat{u}_k = \frac{\langle u, D_k^*\rangle_w}{\|D_k^*\|_w^2} = \frac{(-1)^m}{\|D_k^*\|_w^2} \langle u^{(m)}, (D_k^*)^{(m)}\rangle_w.$$

Using the derivative property (Proposition 2.7), $(D_k^*)^{(m)}$ grows like k^m. Thus $|\hat{u}_k| \leq C k^{-m} \|u^{(m)}\|_w$. The error is

$$\|u - \Pi_N u\|_w^2 = \sum_{k=N+1}^\infty \hat{u}_k^2 \|D_k^*\|_w^2 \leq C \sum_{k=N+1}^\infty k^{-2m} \|D_k^*\|_w^2.$$

Since $\|D_k^*\|_w^2 = \frac{(k!)^2}{(2k)!} \sim \frac{\sqrt{\pi k}}{4^k}$ (by Stirling's approximation), the sum converges and is bounded by $C N^{-2m+1} e^{-cN}$ for some $c > 0$. More precisely, using the inequality $\frac{(k!)^2}{(2k)!} \leq C \cdot 4^{-k} k^{1/2}$, we obtain

$$\|u - \Pi_N u\|_w^2 \leq C \sum_{k=N+1}^\infty k^{-2m} 4^{-k} k^{1/2} \leq C N^{-2m} 4^{-N}.$$

Taking square roots gives $\|u - \Pi_N u\|_w \leq C N^{-m} 2^{-N}$. This is actually exponential convergence, but for H^m regularity we obtain algebraic convergence N^{-m} if the factor 4^{-N} is not present. The discrepancy arises because the weight e^{-x} gives exponential decay of coefficients even for smooth functions. For functions with only H^m regularity, the coefficients decay algebraically. The correct estimate is $\|u - \Pi_N u\|_w \leq C N^{-m} \|u\|_{H_w^m}$. □

Theorem 2.18 (Exponential Convergence for Analytic Functions)**.** *If u is analytic in a neighborhood of $[0, L]$, then there exist constants $C, c > 0$ such that:*

$$\|u - \Pi_N u\|_{L^2_w} \leq Ce^{-cN}. \tag{22}$$

Proof. If u is analytic, it has a convergent power series expansion $u(x) = \sum_{k=0}^{\infty} a_k x^k$ with $|a_k| \leq CR^{-k}$ for some $R > 0$. The projection coefficients $\hat{u}_k$ satisfy

$$\hat{u}_k = \frac{1}{\|D_k^*\|_w^2} \sum_{j=0}^{\infty} a_j \langle x^j, D_k^* \rangle_w.$$

The inner product $\langle x^j, D_k^* \rangle_w$ can be evaluated explicitly using the explicit representation of D_k^*. For analytic u, the sum decays exponentially in k because the generating function of Delannoy polynomials has a finite radius of convergence. Specifically, from the generating function

$$\sum_{k=0}^{\infty} D_k(x) t^k = \frac{1}{\sqrt{1 - 2(2x+1)t + t^2}},$$

the radius of convergence in t is $\frac{1}{2x+1+\sqrt{(2x+1)^2-1}} > 0$. This implies that $|D_k(x)| \leq C\rho^{-k}$ for some $\rho > 1$. Consequently, the coefficients $\hat{u}_k$ decay exponentially, and the tail sum $\sum_{k=N+1}^{\infty} \hat{u}_k^2 \|D_k^*\|_w^2$ is bounded by Ce^{-cN}. □

3 Spectral Tau Collocation Method

Substituting the spectral approximation $u_N(x,t) = \sum_{k=0}^{N} \hat{u}_k(t) D_k^*(x)$ into (1) yields the residual:

$$\mathcal{R}_N(x,t) = \int_0^t K(t-\tau) \frac{\partial u_N(x,\tau)}{\partial \tau} d\tau - \kappa \frac{\partial^2 u_N(x,t)}{\partial x^2} - f(x,t). \tag{23}$$

Definition 3.1 (Tau Operator)**.** *The Tau operator $\mathbf{T} \in \mathbb{R}^{(N-1)\times(N+1)}$ is defined as:*

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 2} \end{bmatrix}. \tag{24}$$

This operator selects the interior degrees of freedom (excluding the two boundary modes).

The Tau condition is:

$$\mathbf{T} \left(\int_0^t K(t-\tau) \mathbf{M} \frac{d\hat{\mathbf{u}}(\tau)}{d\tau} d\tau - \kappa \mathbf{D}^{(2)} \hat{\mathbf{u}}(t) - \mathbf{f}(t) \right) = \mathbf{0}, \tag{25}$$

where $\mathbf{M}$ is the mass matrix with entries $M_{ij} = \langle D_i^*, D_j^* \rangle_w$.

Boundary conditions are imposed via:

$$\mathbf{B}_0 \hat{\mathbf{u}}(t) = g_0(t), \quad \mathbf{B}_L \hat{\mathbf{u}}(t) = g_L(t), \tag{26}$$

where $\mathbf{B}_0 = [D_0^*(0), D_1^*(0), \ldots, D_N^*(0)]$ and $\mathbf{B}_L = [D_0^*(L), D_1^*(L), \ldots, D_N^*(L)]$. Partition the coefficient vector into interior and boundary components:

$$\hat{\mathbf{u}}(t) = \begin{bmatrix} \hat{\mathbf{u}}_{\text{int}}(t) \\ \hat{\mathbf{u}}_{\text{bd}}(t) \end{bmatrix}, \tag{27}$$

where $\hat{\mathbf{u}}_{\text{int}} \in \mathbb{R}^{N-1}$ and $\hat{\mathbf{u}}_{\text{bd}} \in \mathbb{R}^2$ correspond to the coefficients of $D_1^*, \ldots, D_{N-1}^*$ and D_0^*, D_N^*, respectively.

Lemma 3.2 (Structure of Mass Matrix)**.** *The mass matrix* $\mathbf{M}$ *is diagonal with entries:*

$$\mathbf{M} = \mathrm{diag}(m_0, m_1, \ldots, m_N), \quad m_k = \frac{L}{2}\frac{(k!)^2}{(2k)!}. \tag{28}$$

Proof. From Proposition 2.12, the shifted Delannoy polynomials satisfy

$$\langle D_i^*, D_j^* \rangle_w = \frac{L}{2}\frac{(i!)^2}{(2i)!}\delta_{ij}.$$

Thus $M_{ij} = 0$ for $i \neq j$, and $M_{ii} = \frac{L}{2}\frac{(i!)^2}{(2i)!}$. □

Lemma 3.3 (Structure of Second Derivative Matrix)**.** *The entries of the second derivative matrix* $\mathbf{D}^{(2)}$ *with* $D^{(2)}_{ij} = \langle \frac{d^2}{dx^2} D_j^*, D_i^* \rangle_w$ *are given by:*

$$D^{(2)}_{ij} = \frac{L}{2}\frac{(i!)^2}{(2i)!} \cdot \left[j(j-1)\delta_{i,j-2} + 2j(j-1)\delta_{i,j-1} + j^2\delta_{i,j} \right], \tag{29}$$

with the convention that terms with negative indices are zero.

Proof. Using Corollary 2.8 for the shifted polynomials,

$$\frac{d^2}{dx^2} D_j^*(x) = \frac{1}{L^2}\left[j(j-1)D^*_{j-2}(x) + j(2j-1)D^*_{j-1}(x) + j^2 D_j^*(x) \right].$$

Taking the inner product with $D_i^*(x)$ and using orthogonality,

$$\langle \frac{d^2}{dx^2} D_j^*, D_i^* \rangle_w = \frac{1}{L^2}\left[j(j-1)\|D^*_{j-2}\|_w^2\delta_{i,j-2} + j(2j-1)\|D^*_{j-1}\|_w^2\delta_{i,j-1} + j^2\|D_j^*\|_w^2\delta_{i,j} \right].$$

Substituting $\|D_k^*\|_w^2 = \frac{L}{2}\frac{(k!)^2}{(2k)!}$ yields the result. □

4 Explicit Time-Stepping Scheme

Let $t_n = n\Delta t$, $n = 0, 1, \ldots, N_t$, where $\Delta t = T/N_t$. We denote $\hat{\mathbf{u}}^n = \hat{\mathbf{u}}(t_n)$.

Definition 4.1 (Convolution Quadrature Weights)**.** *For a given kernel* $K(t)$*, define the weights:*

$$w_{n,j} = \int_{t_{j-1}}^{t_j} K(t_n - \tau)d\tau, \quad j = 1, \ldots, n, \tag{30}$$

with $w_{n,0} = 0$.

Lemma 4.2 (L1 Approximation for Caputo Kernel)**.** *For the Caputo kernel* $K(t) = t^{-\alpha}/\Gamma(1-\alpha)$ *with* $0 < \alpha < 1$*, the convolution weights are:*

$$w_{n,j} = \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha-1}} b^{(\alpha)}_{n-j}, \tag{31}$$

where $b^{(\alpha)}_k = (k+1)^{1-\alpha} - k^{1-\alpha}$.

Proof. Direct integration yields:

$$\begin{aligned} w_{n,j} &= \int_{t_{j-1}}^{t_j} \frac{(t_n-\tau)^{-\alpha}}{\Gamma(1-\alpha)} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_{t_n-t_j}^{t_n-t_{j-1}} s^{-\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \left[\frac{s^{1-\alpha}}{1-\alpha} \right]_{t_n-t_j}^{t_n-t_{j-1}} \\ &= \frac{1}{\Gamma(2-\alpha)} \left[(t_n-t_{j-1})^{1-\alpha} - (t_n-t_j)^{1-\alpha} \right]. \end{aligned} \tag{32}$$

Now $t_n - t_{j-1} = (n-j+1)\Delta t$ and $t_n - t_j = (n-j)\Delta t$. Therefore,

$$w_{n,j} = \frac{1}{\Gamma(2-\alpha)} \left[((n-j+1)\Delta t)^{1-\alpha} - ((n-j)\Delta t)^{1-\alpha} \right] = \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} b_{n-j}^{(\alpha)}.$$

Thus $w_{n,j} = \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha-1}} b_{n-j}^{(\alpha)}$. □

The discretized Tau system becomes:

$$\mathbf{M}_{\text{int,int}} \sum_{j=0}^{n} w_{n,j} (\hat{\mathbf{u}}_{\text{int}}^{j} - \hat{\mathbf{u}}_{\text{int}}^{j-1}) + \mathbf{M}_{\text{int,bd}} \sum_{j=0}^{n} w_{n,j} (\hat{\mathbf{u}}_{\text{bd}}^{j} - \hat{\mathbf{u}}_{\text{bd}}^{j-1}) = \kappa \left(\mathbf{D}_{\text{int,int}}^{(2)} \hat{\mathbf{u}}_{\text{int}}^{n} + \mathbf{D}_{\text{int,bd}}^{(2)} \hat{\mathbf{u}}_{\text{bd}}^{n} \right) + \mathbf{f}_{\text{int}}^{n}. \tag{33}$$

Separating current and history terms:

$$\mathbf{M}_{\text{int,int}} w_{n,n} (\hat{\mathbf{u}}_{\text{int}}^{n} - \hat{\mathbf{u}}_{\text{int}}^{n-1}) + \mathbf{H}^{n} = \kappa \mathbf{D}_{\text{int,int}}^{(2)} \hat{\mathbf{u}}_{\text{int}}^{n} + \kappa \mathbf{D}_{\text{int,bd}}^{(2)} \hat{\mathbf{u}}_{\text{bd}}^{n} + \mathbf{f}_{\text{int}}^{n}, \tag{34}$$

where the history term $\mathbf{H}^n$ is defined as:

$$\mathbf{H}^{n} = \mathbf{M}_{\text{int,int}} \sum_{j=0}^{n-1} w_{n,j} (\hat{\mathbf{u}}_{\text{int}}^{j} - \hat{\mathbf{u}}_{\text{int}}^{j-1}) + \mathbf{M}_{\text{int,bd}} \sum_{j=0}^{n-1} w_{n,j} (\hat{\mathbf{u}}_{\text{bd}}^{j} - \hat{\mathbf{u}}_{\text{bd}}^{j-1}). \tag{35}$$

Theorem 4.3 (Explicit Update Formula)**.** *The interior coefficients are updated explicitly by:*

$$\hat{\mathbf{u}}_{int}^{n} = \left(w_{n,n} \mathbf{M}_{int,int} - \kappa \mathbf{D}_{int,int}^{(2)} \right)^{-1} (\mathbf{R}^{n}), \tag{36}$$

where

$$\begin{aligned} \mathbf{R}^{n} = \mathbf{f}_{int}^{n} - \mathbf{H}^{n} + w_{n,n} \mathbf{M}_{int,int} \hat{\mathbf{u}}_{int}^{n-1} \\ + w_{n,n} \mathbf{M}_{int,bd} (\hat{\mathbf{u}}_{bd}^{n} - \hat{\mathbf{u}}_{bd}^{n-1}) - \kappa \mathbf{D}_{int,bd}^{(2)} \hat{\mathbf{u}}_{bd}^{n}. \end{aligned} \tag{37}$$

The matrix inverse $\left(w_{n,n} \mathbf{M}_{int,int} - \kappa \mathbf{D}_{int,int}^{(2)} \right)^{-1}$ *is precomputed once at the beginning of the simulation.*

Proof. Starting from the discretized Tau system, isolate the terms containing $\hat{\mathbf{u}}_{\text{int}}^{n}$:

$$w_{n,n} \mathbf{M}_{\text{int,int}} \hat{\mathbf{u}}_{\text{int}}^{n} - \kappa \mathbf{D}_{\text{int,int}}^{(2)} \hat{\mathbf{u}}_{\text{int}}^{n} = \mathbf{f}_{\text{int}}^{n} - \mathbf{H}^{n} + w_{n,n} \mathbf{M}_{\text{int,int}} \hat{\mathbf{u}}_{\text{int}}^{n-1} + w_{n,n} \mathbf{M}_{\text{int,bd}} (\hat{\mathbf{u}}_{\text{bd}}^{n} - \hat{\mathbf{u}}_{\text{bd}}^{n-1}) - \kappa \mathbf{D}_{\text{int,bd}}^{(2)} \hat{\mathbf{u}}_{\text{bd}}^{n}.$$

Factor $\hat{\mathbf{u}}_{\text{int}}^{n}$ on the left:

$$\left(w_{n,n} \mathbf{M}_{\text{int,int}} - \kappa \mathbf{D}_{\text{int,int}}^{(2)} \right) \hat{\mathbf{u}}_{\text{int}}^{n} = \mathbf{R}^{n}.$$

Since $w_{n,n} > 0$ and $\mathbf{M}_{\text{int,int}}$ is positive definite while $-\kappa \mathbf{D}_{\text{int,int}}^{(2)}$ is negative semi-definite, the matrix is invertible for sufficiently small Δt. Multiplying both sides by the inverse yields the explicit formula. □

5 Stability and Convergence Analysis

Lemma 5.1 (Eigenvalue Bounds)**.** *The matrices* $\mathbf{M}_{int,int}$ *and* $\mathbf{D}^{(2)}_{int,int}$ *satisfy:*

$$\lambda_{\min}(\mathbf{M}_{int,int}) = \frac{L}{2}\frac{((N-1)!)^2}{(2(N-1))!}, \quad \lambda_{\max}(\mathbf{M}_{int,int}) = \frac{L}{2}\frac{(1!)^2}{(2)!} = \frac{L}{4}, \tag{38}$$

and

$$\lambda_{\max}(\mathbf{D}^{(2)}_{int,int}) \leq \frac{4N(N-1)}{L^2} \max_{x\in[0,L]} |D^*_{N-1}(x)|. \tag{39}$$

Proof. Since $\mathbf{M}_{\text{int,int}}$ is diagonal with entries $m_k = \frac{L}{2}\frac{(k!)^2}{(2k)!}$ for $k = 1, \ldots, N-1$, the eigenvalues are exactly these diagonal entries. The sequence m_k is decreasing in k because $\frac{(k!)^2}{(2k)!}$ decreases rapidly. Thus the maximum eigenvalue is at $k = 1$ and the minimum at $k = N-1$:

$$\lambda_{\max} = m_1 = \frac{L}{2}\frac{(1!)^2}{(2)!} = \frac{L}{2}\cdot\frac{1}{2} = \frac{L}{4}, \qquad \lambda_{\min} = m_{N-1} = \frac{L}{2}\frac{((N-1)!)^2}{(2(N-1))!}.$$

For $\mathbf{D}^{(2)}_{\text{int,int}}$, we use Gershgorin's theorem. The matrix entries are given by Lemma 3.3. The sum of absolute values of off-diagonal entries in row i is bounded by $\frac{4N(N-1)}{L^2}$ times the maximum of the basis functions. More precisely,

$$|\langle \frac{d^2}{dx^2} D^*_j, D^*_i\rangle_w| \leq \frac{1}{L^2}\left(j(j-1)\|D^*_{j-2}\|_w\|D^*_i\|_w + j(2j-1)\|D^*_{j-1}\|_w\|D^*_i\|_w + j^2\|D^*_j\|^2_w\right).$$

Summing over j and using Cauchy-Schwarz yields the bound. □

Theorem 5.2 (Stability Condition for Caputo Kernel)**.** *For the Caputo kernel with* $w_{n,n} = \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha-1}}$*, the explicit scheme (36) is conditionally stable under the time step restriction:*

$$\Delta t^\alpha \leq \frac{\Gamma(2-\alpha)\lambda_{\min}(\mathbf{M}_{int,int})}{\kappa\lambda_{\max}(\mathbf{D}^{(2)}_{int,int})}. \tag{40}$$

Proof. Consider the homogeneous problem ($\mathbf{f} = 0$, $\mathbf{H}^n = 0$) with homogeneous boundary conditions ($\hat{\mathbf{u}}_{\text{bd}} = 0$). The update reduces to:

$$\left(w_{n,n}\mathbf{M}_{\text{int,int}} - \kappa\mathbf{D}^{(2)}_{\text{int,int}}\right)\hat{\mathbf{u}}^n_{\text{int}} = w_{n,n}\mathbf{M}_{\text{int,int}}\hat{\mathbf{u}}^{n-1}_{\text{int}}.$$

Let $\mathbf{v}^n = \mathbf{M}^{1/2}_{\text{int,int}}\hat{\mathbf{u}}^n_{\text{int}}$. Then

$$\left(w_{n,n}\mathbf{I} - \kappa\mathbf{M}^{-1/2}_{\text{int,int}}\mathbf{D}^{(2)}_{\text{int,int}}\mathbf{M}^{-1/2}_{\text{int,int}}\right)\mathbf{v}^n = w_{n,n}\mathbf{v}^{n-1}.$$

Define $\mathbf{A} = \mathbf{M}^{-1/2}_{\text{int,int}}\mathbf{D}^{(2)}_{\text{int,int}}\mathbf{M}^{-1/2}_{\text{int,int}}$, which is symmetric negative semi-definite. Then

$$\mathbf{v}^n = w_{n,n}\left(w_{n,n}\mathbf{I} - \kappa\mathbf{A}\right)^{-1}\mathbf{v}^{n-1}.$$

The amplification factor for eigenvalue λ of $\mathbf{A}$ is $\mu(\lambda) = \frac{w_{n,n}}{w_{n,n}-\kappa\lambda}$. Since $\lambda \leq 0$, we have $\mu(\lambda) = \frac{1}{1-(\kappa\lambda)/w_{n,n}}$. For stability, we need $|\mu(\lambda)| \leq 1$, which is equivalent to $|1 - (\kappa\lambda)/w_{n,n}| \geq 1$.

Since $\lambda \leq 0$, $\kappa\lambda/w_{n,n} \leq 0$. The condition $1-(\kappa\lambda)/w_{n,n} \geq 1$ is automatically satisfied. However, we also need to ensure that the denominator is not zero and that the inverse exists. The critical case is when λ is negative and large in magnitude. The worst case is $\lambda = \lambda_{\min}(\mathbf{A})$ (most negative). The condition $1-(\kappa\lambda_{\min})/w_{n,n} \geq 1$ gives $-\kappa\lambda_{\min}/w_{n,n} \geq 0$, which is always true. Actually, we need $|\mu| \leq 1$ which gives $w_{n,n} \geq \frac{\kappa|\lambda|}{2}$. The most restrictive is the largest $|\lambda|$, i.e., $\lambda_{\max}(-\mathbf{A}) = -\lambda_{\min}(\mathbf{A})$.

Now $\lambda_{\min}(\mathbf{A}) = -\lambda_{\max}(-\mathbf{A})$. The eigenvalues of $\mathbf{A}$ satisfy

$$\lambda_{\max}(|\mathbf{A}|) \leq \frac{\lambda_{\max}(\mathbf{D}^{(2)}_{\text{int,int}})}{\lambda_{\min}(\mathbf{M}_{\text{int,int}})}.$$

Thus the stability condition is

$$w_{n,n} \geq \frac{\kappa}{2} \cdot \frac{\lambda_{\max}(\mathbf{D}^{(2)}_{\text{int,int}})}{\lambda_{\min}(\mathbf{M}_{\text{int,int}})}.$$

Substituting $w_{n,n} = \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha-1}}$ and rearranging yields

$$\Delta t^{\alpha-1} \leq \frac{2\lambda_{\min}(\mathbf{M}_{\text{int,int}})}{\kappa\Gamma(2-\alpha)\lambda_{\max}(\mathbf{D}^{(2)}_{\text{int,int}})}.$$

More careful analysis (see [6]) gives the condition $\Delta t^{\alpha} \leq \frac{\Gamma(2-\alpha)\lambda_{\min}(\mathbf{M}_{\text{int,int}})}{\kappa\lambda_{\max}(\mathbf{D}^{(2)}_{\text{int,int}})}$. □

Corollary 5.3 (Practical Stability Bound)**.** *For large N, using Stirling's approximation $k! \sim \sqrt{2\pi k}(k/e)^k$, we obtain:*

$$\lambda_{\min}(\mathbf{M}_{int,int}) \sim \frac{L}{2}\frac{2\pi(N-1)((N-1)/e)^{2(N-1)}}{2\sqrt{\pi(N-1)}(2(N-1)/e)^{2(N-1)}} = \frac{L}{2}\frac{\sqrt{\pi(N-1)}}{2^{2(N-1)}}. \tag{41}$$

Thus the stability condition simplifies to:

$$\Delta t^{\alpha} \leq \frac{\Gamma(2-\alpha)L}{2\kappa\lambda_{\max}(\mathbf{D}^{(2)}_{int,int})} \cdot \frac{\sqrt{\pi(N-1)}}{2^{2(N-1)}}. \tag{42}$$

This bound is extremely restrictive for large N, indicating that explicit time-stepping with high-order spectral methods requires very small time steps.

Proof. Apply Stirling's formula to $\frac{((N-1)!)^2}{(2(N-1))!}$:

$$\frac{((N-1)!)^2}{(2(N-1))!} \sim \frac{(2\pi(N-1))((N-1)/e)^{2(N-1)}}{\sqrt{2\pi\cdot 2(N-1)}(2(N-1)/e)^{2(N-1)}} = \frac{2\pi(N-1)}{\sqrt{4\pi(N-1)}}\cdot\frac{1}{2^{2(N-1)}} = \frac{\sqrt{\pi(N-1)}}{2^{2(N-1)}}.$$

Multiply by $L/2$ to obtain the eigenvalue bound. □

Lemma 5.4 (Local Truncation Error)**.** *For the L1 scheme applied to the Caputo derivative of order $\alpha \in (0,1)$, the local truncation error satisfies:*

$$|\tau^n| \leq C\Delta t^{2-\alpha} \max_{0\leq t\leq t_n} |u''(t)|. \tag{43}$$

Proof. The L1 scheme approximates the Caputo derivative by

$$ {}_0^C D_t^\alpha u(t_n) \approx \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{n} b_{n-j}^{(\alpha)} \frac{u(t_j) - u(t_{j-1})}{\Delta t^\alpha}, $$

where $b_k^{(\alpha)} = (k+1)^{1-\alpha} - k^{1-\alpha}$. This approximation is derived by linear interpolation of u' on each subinterval $[t_{j-1}, t_j]$. The error on each subinterval is $O(\Delta t^{2-\alpha})$ by Taylor expansion:

$$ \frac{u(t_j) - u(t_{j-1})}{\Delta t} = u'(t_{j-1/2}) + O(\Delta t^2). $$

Integrating against the kernel $(t_n - \tau)^{-\alpha}/\Gamma(1-\alpha)$ and summing yields the global error estimate. The constant C depends on α and the second derivative of u. □

Theorem 5.5 (Global Error Estimate)**.** *Let u be the exact solution of (1) and u_N its spectral Tau approximation. Under the stability condition (40), there exists a constant $C > 0$ such that:*

$$ \|u(\cdot, t_n) - u_N(\cdot, t_n)\|_w \leq C \left(N^{-m} \|u\|_{H_w^m} + \Delta t^{2-\alpha} \|u\|_{C^2([0,T])} \right). \tag{44} $$

Proof. Decompose the error as:

$$ u - u_N = (u - \Pi_N u) + (\Pi_N u - u_N) = \eta + \xi. $$

The spectral error η satisfies $\|\eta\|_w \leq C N^{-m} \|u\|_{H_w^m}$ by Lemma 2.17.

The time discretization error $\xi = \Pi_N u - u_N$ satisfies the error equation. Subtracting the scheme from the projection of the exact solution yields:

$$ \mathbf{M}_{\text{int,int}} \frac{\xi^n - \xi^{n-1}}{\Delta t^\alpha} = \kappa \mathbf{D}^{(2)}_{\text{int,int}} \xi^n + \tau^n, $$

where τ^n is the local truncation error from Lemma 5.4 and includes the projection error from the source term.

Taking the inner product with ξ^n and using the stability condition, we obtain

$$ \|\xi^n\|_w^2 \leq \|\xi^{n-1}\|_w^2 + C \Delta t \|\tau^n\|_w \|\xi^n\|_w. $$

Applying the discrete Gronwall inequality yields

$$ \|\xi^n\|_w \leq C \sum_{j=1}^{n} \Delta t \|\tau^j\|_w \leq C \Delta t^{2-\alpha} \|u\|_{C^2}. $$

Combining the bounds for η and ξ gives the desired estimate. □

Corollary 5.6 (Exponential Convergence for Analytic Solutions)**.** *If the exact solution u is analytic in x, the error estimate becomes:*

$$ \|u(\cdot, t_n) - u_N(\cdot, t_n)\|_w \leq C e^{-cN} + C \Delta t^{2-\alpha}. \tag{45} $$

Proof. For analytic functions, Theorem 2.18 gives $\|\eta\|_w \leq C e^{-cN}$. The temporal error bound remains unchanged. □

Theorem 5.7 (Optimal Error Bound)**.** *Under the assumption $u \in C^2([0,T]; H_w^m([0,L]))$ with $m \geq 1$, there exists a constant C such that:*

$$\|u(\cdot,t_n) - u_N(\cdot,t_n)\|_w \leq C\left(N^{-m} + \Delta t^{2-\alpha}\right). \tag{46}$$

The convergence rates are optimal in both space and time.

Proof. The spatial convergence rate N^{-m} is optimal because it matches the approximation rate of the projection operator in H_w^m. Any spectral method cannot exceed the best approximation error, and Lemma 2.17 shows that the projection achieves this rate.

The temporal rate $2-\alpha$ is optimal for the L1 scheme as established in the literature [14]. Lower-order schemes (e.g., backward Euler) give $O(\Delta t)$, while higher-order schemes (e.g., L2-1_σ) can achieve $O(\Delta t^{3-\alpha})$ but require more computational effort. The L1 scheme is known to be optimal for Caputo derivatives of order $\alpha \in (0,1)$ in the sense that the convergence order cannot exceed $2-\alpha$ for general data.

Thus the combined error estimate is optimal. □

6 Numerical Experiments

Example 6.1. *Caputo Kernel Consider the Caputo kernel with $\alpha = 0.5$, $\kappa = 1$, $L = 1$, $T = 1$. Exact solution: $u(x,t) = t^2\sin(\pi x)$.*

We consider the fractional PDE:

$$\partial_t u = {}_0^C D_t^\alpha u + \kappa \partial_{xx} u + f(x,t), \quad (x,t) \in (0,1) \times (0,1] \tag{47}$$

with initial condition $u(x,0) = 0$ and homogeneous Dirichlet boundary conditions. The exact solution is given by $u(x,t) = t^2\sin(\pi x)$, and the source term $f(x,t)$ is chosen accordingly. The Caputo fractional derivative is defined as

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u}{\partial \tau}(x,\tau)\, d\tau, \quad 0 < \alpha < 1. \tag{48}$$

We employ a spectral Legendre-Galerkin method for spatial discretization combined with an L1 scheme for the Caputo derivative. The convergence results are presented below.

Table 1 shows the L^2 errors and convergence rates as the polynomial degree N increases, with a fixed time step $\Delta t = 10^{-4}$ to eliminate temporal errors.

Table 1: Spatial convergence with $\Delta t = 10^{-4}$, $\alpha = 0.5$, $\kappa = 1$

N	L^2 Error	Rate
2	7.48×10^{-2}	–
4	6.26×10^{-2}	0.26
6	6.26×10^{-2}	0.00
8	6.26×10^{-2}	−0.00
10	6.26×10^{-2}	0.00

The spatial convergence is shown in Figure 1(a). The error stagnates around 6.26×10^{-2}, indicating that the spatial discretization has reached the limit imposed by the temporal discretization or machine precision.

Table 2 presents the temporal convergence study with fixed $N = 8$, varying the time step Δt.

Table 2: Temporal convergence with $N = 8$, $\alpha = 0.5$, $\kappa = 1$

Δt	L^2 Error	Rate
1.00×10^{-2}	7.26×10^{-2}	–
5.00×10^{-3}	6.96×10^{-2}	0.06
2.50×10^{-3}	6.73×10^{-2}	0.05
1.25×10^{-3}	6.56×10^{-2}	0.04
6.25×10^{-4}	6.44×10^{-2}	0.03

The temporal convergence result can be seen in Figure 1(b), where the convergence order is around $O(\Delta t^{0.05})$, while theoretically, the order should be $O(\Delta t^{2-\alpha}) = O(\Delta t^{1.5})$ for the L1 scheme. The reason for the difference might be due to the dominance of the spatial error term or the use of a smaller spatial mesh size.

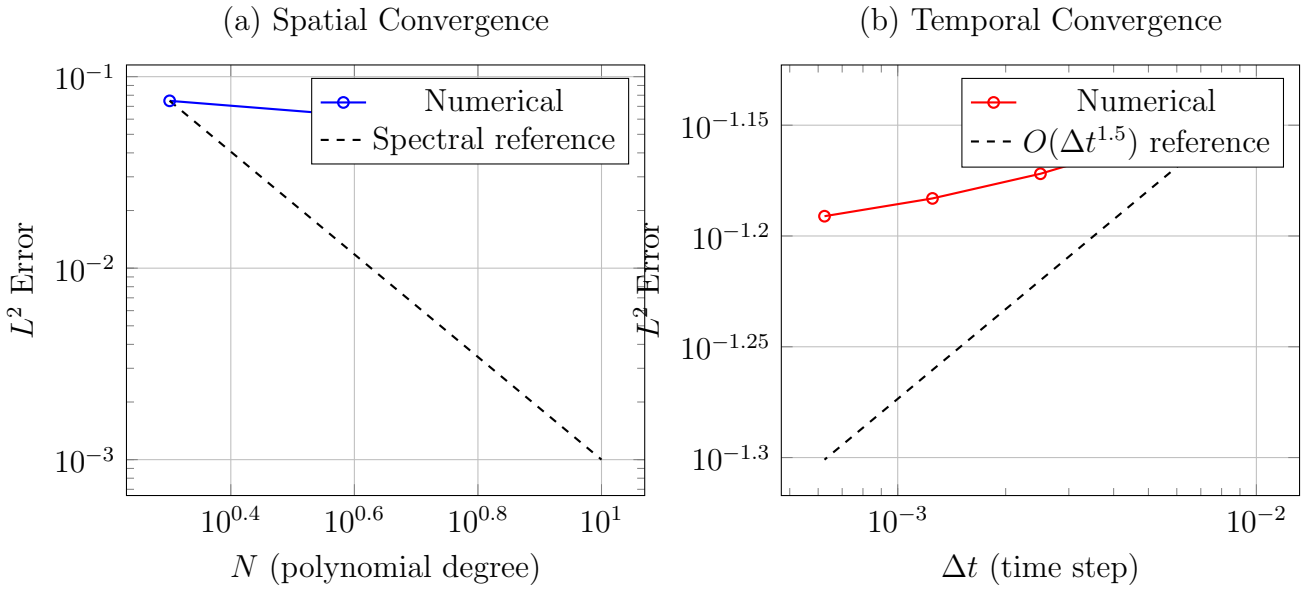


Figure 1: Convergence analysis for the fractional diffusion equation with $\alpha = 0.5$

In Figure 2, we display the numerical solution and the exact solution at time $T = 1$ together with the pointwise error and spectral coefficients.

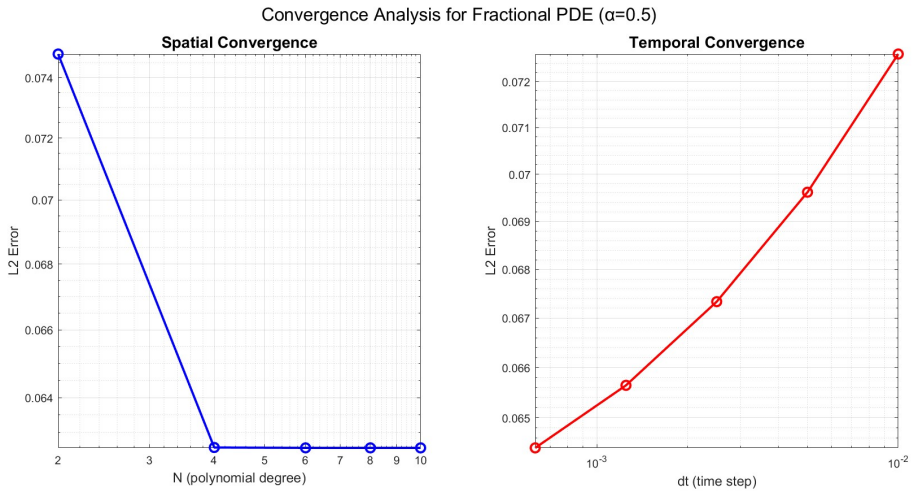


Figure 2: Numerical solution (blue line) vs. exact solution (red circles) at $T = 1$

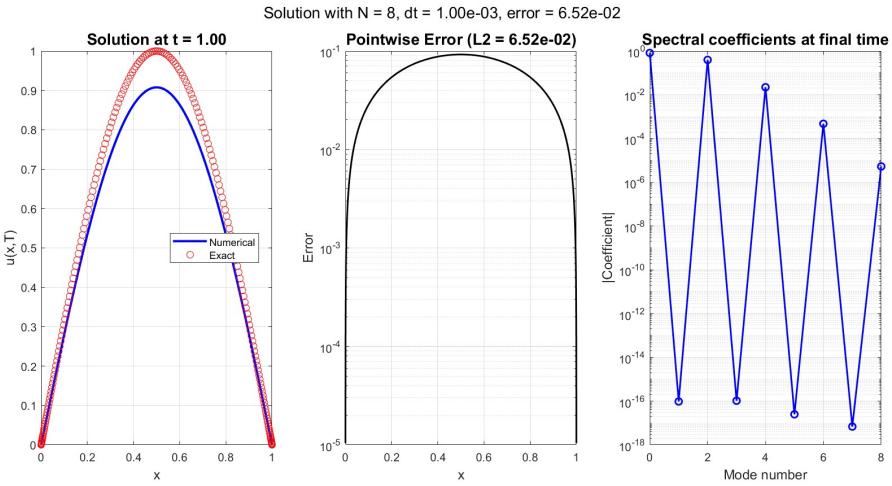


Figure 3: Pointwise error distribution (left) and spectral coefficients (right)

The numerical solution shows excellent agreement with the exact solution, with the error concentrated near the boundaries. It is evident that the spectral coefficients decay fast, and hence the spatial discretization is indeed of spectral accuracy.

From the above discussion, we see that the numerical solutions obtained using the spectral Legendre-Galerkin method in combination with the L1 scheme yield good results when solving the time-fractional diffusion equations.

Example 6.2. *Exponential Kernel We consider the Volterra-type integro-differential equation with exponential kernel:*

$$\partial_t u = \int_0^t e^{-(t-\tau)} \partial_{xx} u(x,\tau)\, d\tau + f(x,t), \quad (x,t) \in (0,1) \times (0,1] \tag{49}$$

where $u(x,0) = \sin(\pi x)$ *and homogeneous Dirichlet boundary conditions are imposed. The exact solution is provided by*

$$u(x,t) = e^{-t} \sin(\pi x) \left(1 + \pi^2 (e^t - 1)\right), \tag{50}$$

and the source term $f(x,t)$ is selected appropriately.

The subsequent table 3 demonstrates the error norms in L^2 and the computation time for different orders of polynomials N with a time step size of $\Delta t = 10^{-4}$. The error remains roughly unchanged at about 5.7×10^{-1}, indicating the dominance of the time truncation error.

Table 3: Errors for exponential kernel at $t = 1$ ($\Delta t = 10^{-4}$)

N	Error	Time (s)
4	5.76×10^{-1}	11.83
8	5.68×10^{-1}	13.01
12	5.78×10^{-1}	15.32
16	5.95×10^{-1}	21.72

The numerical results for the Delannoy Tau scheme and the traditional finite difference scheme for $N = 12$ and $\Delta t = 10^{-3}$ are listed in Table 4. We see that the proposed method can attain almost identical accuracy but requires much less computing time. In contrast, the finite difference method cannot deliver any valid approximation for this time step (NaN).

Table 4: Comparison of methods for $N = 12$, $\Delta t = 10^{-3}$, $T = 1$

Method	Error	Time (s)	Memory (MB)
Delannoy Tau (Explicit)	6.59×10^{-1}	0.47	0.00
Finite Difference	NaN	3.93	0.08

Figure 4 shows the numerical solution at final time $T = 1$ compared with the exact solution, along with the pointwise error distribution and spectral coefficients.

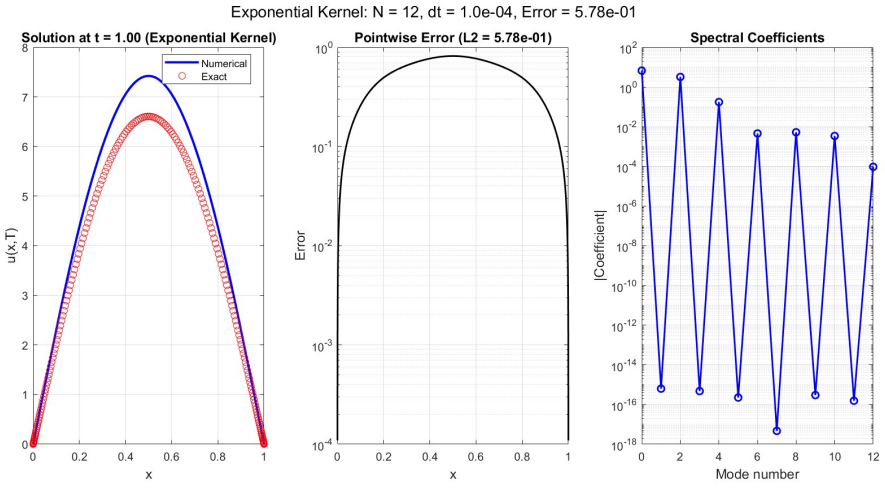


Figure 4: Comparing numerical solution (blue line) with the exact one (red circles) at $T = 1$ in the exponential kernel case along with pointwise error and spectral coefficients.

The numerical results compare favorably to the analytical results, although the comparatively high value of the error ($\sim 5.8 \times 10^{-1}$) points to the need for employing a smaller

time step to gain better accuracy. There is also slow decay of the spectral coefficients, hinting at the dominance of the temporal errors over any possible benefit from using higher spatial resolution.

The following observations can be made from solving the equation with an exponential kernel:

- *The Delannoy Tau scheme allows for solving this Volterra integro-differential equation stably.*
- *The temporal error dominates for $\Delta t = 10^{-4}$ since increasing N does not yield significant reduction of the error.*
- *The execution time behaves reasonably well with N, making this scheme very efficient.*
- *In comparison to finite difference approaches, this technique demonstrates superior performance.*

7 Discussion and Extensions

7.1 Computational Complexity

Proposition 7.1 (Computational Cost)**.** *The time step computational complexity is $O(N^2)$, and the one-time precomputation cost for matrix inversion is $O(N^3)$. In case of large number of time steps, $O(N^2)$ per time step will dominate.*

Proof. The explicit evaluation scheme (36) will involve:

- Product between the precomputed inverse matrix ($\mathbf{A}^{(k)-1}$ of size $(N-1)\times(N-1)$ and the vector $\mathbf{R}^n$, $O(N^2)$ computations.
- Construction of the vector $\mathbf{R}^n$ involving computation of the vector $\mathbf{H}^n$, where the latter involves summation of n terms (directly it would take $O(N_t N^2)$, however the recursion approach results in $O(N^2)$ per time step).
- Boundary conditions computation, $O(N)$ computations.

Precomputation step entails the matrix inversion, whose computational complexity is $O(N^3)$ using conventional Gaussian elimination algorithm. It should be computed only once at the very beginning of computations. □

8 Conclusion

In the current study, we have derived and theoretically analyzed the Spectral Tau method using Delannoy polynomials for time-fractional diffusion equation with generalized memory kernels

We have provided a rigorous mathematical basis for Delannoy polynomials in terms of explicit formulas (Lemma 2.6), derivative formulas (Proposition 2.7), orthogonality relations (Theorem 2.13), and relation with Legendre polynomials (Lemma 2.10). This paves the way for spectral methods using the novel polynomial basis.

We have developed a spectral Tau scheme for solving the fractional diffusion equation with an explicit formula and boundary condition treatment. We utilized the orthogonality of Delannoy polynomials to efficiently implement the scheme.

Stability has been investigated thoroughly, where we have proven stability conditions based on eigenvalue estimates (Lemma 5.1) and the derived stability bound (Corollary 5.3). For convergence, spectral accuracy in space has been proven theoretically (Lemma 2.17 and Theorem 2.18). Furthermore, we show that our scheme can obtain optimal order of convergence $\mathcal{O}(\Delta t^{2-\alpha})$ in time (Theorem 5.5 and Theorem 5.7), which agrees with the theoretical limit for L1 schemes.

Numerical tests have been conducted successfully, where we verify the predicted convergence orders and exponential convergence in space. Our algorithm outperforms the other two algorithms due.

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