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An Analytical Model of Inflation with Graceful Exit and Constraining it from Observations

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A rapidly inflating space driven by a simultaneously evolving scalar field influenced by a potential energy density ($V(\phi)$) in very early universe is a fundamental mechanism that generated the tiny quantum perturbations that eventually form large scale gravitationally bound systems in our universe. In a traditional approach one starts with a suitable $V(\phi)$ for the field to describe this dynamics and predict observable quantities like scalar spectral index (n_s) and tensor to scalar ratio (r). In this work, we take an alternative path by first specifying a scale factor $a(t)$ that encapsulates some of the basic physically motivated characteristics. We derive the potential energy density and using it provide expressions of all necessary inflation parameters that are needed to connect to the two common inflationary observables, n_s and r. Our results are fully analytic and exact and are expressed in terms of dynamical variables of time and scalar field itself, without assuming any slow roll conditions. However, slow roll conditions can easily be imposed on these complete analytical results to uncover the physics in the desired limit. The analysis can identify a time interval during which slow roll assumption can be assumed to be valid. Our inflationary dynamics does not suffer from the so-called *ungraceful exit problem* from the inflation. We show that our scale factor model predicts n_s and r values consistent with the Planck, BICEP/Keck-18, and ACT observations in the context of minimally coupled scalar field with the gravity.

I. INTRODUCTION

The inflationary formalism explains observed large scale smoothness as well as relatively small-scale clumpyness of structures of the Universe [1, 2]. It predicts existence of (yet unobserved) primordial gravitational wave background [3, 4]. The predictions of the theory can be tested against the observations by measuring spectral indices (n_s and n_T respectively) of scalar and tensor mode fluctuations power spectra of the metric tensor and ratio of tensor to scalar mode power spectra, r (e.g., see [5, 6]), at some pivot wave number k_* - the choice of which may depend on the maximally sensitive scales probed by a cosmological observation.

Owing to the possibility of understanding the physics of early Universe so close ($t \lesssim 10^{-32}$ s) to the epoch of the so-called time of cosmic-singularity ($t = 0$) and at energy scales close to $\sim 10^{16}$ GeV – much higher than the reach of any human made particle accelerators– the theory of inflation gained much attention in the literature starting from the inception of the idea [7].

The concept of inflation has been realized through a wide range of many theoretical models. The earliest models like old and new inflation, were based on phase transitions in scalar field potentials, but faced challenges related to graceful exit and initial conditions [8, 9]. These issues motivated the development of chaotic inflation, in which inflation occurs for large field values in simple monomial potentials, providing a robust and flexible framework [8, 10]. On the other hand, observational developments was favoring the slow roll inflation models which consist a quite flat potential and usually have a graceful exit [11, 12]. The hilltop model [13], Starobinsky [14, 15] and the attractor models [16–19] are some examples of such models, although this list is not exhaustive. For a detailed review of inflationary models we refer [20–22]. *In this work we develop a new model of inflation following a self contained and comprehensive analytical framework. Before we further elaborate on our work we provide some necessary basic equations to elucidate the problem.*

A. Basic Equations

Assuming the inflation takes place due to evolution of a scalar field $\phi'(t, \vec{x}) \equiv \phi(t)+\delta\phi(t, \vec{x})$, where $\phi(t)$ represents the background field, and $\delta\phi(t, \vec{x})$ are its anisotropies, the physics of the inflation is described by a set of three coupled non-linear differential equations as follows,

$$H^2 = \frac{8\pi G}{3}\rho_\phi \ , \tag{1}$$

$$\frac{\ddot{a}}{a} = -4\pi G\left(P_\phi + \frac{\rho_\phi}{3}\right) \ , \tag{2}$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \ , \tag{3}$$

where Eqn. 1 and 2 denote the first and second Friedman equations respectively [23]. In this work, we assume that the scalar field is minimally coupled to the gravitation. We use $c = 1$ (i.e. natural unit) and assume that the three dimensional space is flat. Eqn3 represents the scalar field equation, can be derived from the Friedman equations. The Hubble parameter, $H(t)$, is defined as

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$H(t) \equiv \dot{a}/a$ and $a(t)$ represents the scale factor (t represents coordinate time) [24]. An over dot represents time derivative. Assuming ideal fluid model, the potential energy density is represented by $V(\phi)$. The energy density, ρ_ϕ, and pressure, P_ϕ, respectively are given by [25]

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \ , \tag{4}$$

$$P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \, . \tag{5}$$

B. The Problem

A complete model of inflation will be obtained if it is possible to solve simultaneously for all dynamical variables appearing in Eqns. 1, 2 and 3. However, in practice solving analytically for all independent variables of Eqns 1 to 3 may be non-trivial or even not possible. It is clear that $V(\phi)$ plays a significant role to influence evolution of ϕ, and in turn evolution of the dynamical variables in the gravity sector. In some approaches to inflationary predictions one starts with an ansatz for $V(\phi)$ in the action expression describing the coupled system [26]. In principle, one can then obtain the Friedman equations and equation of continuity. Interestingly, such choice of $V(\phi)$ can be made suitably (consistent with slow roll assumption) to actually obtain predictions of inflation in terms of the variables n_s and r without directly solving the evolution for the gravity sector variables. Hence, time evolution of scale factor or Hubble parameter is not of prime focus always.

C. Solution

In this work we follow a more general approach. We describe a new model for inflationary dynamics with completely analytically tractable framework based upon following two guidelines.

1) The description for inflation satisfying the observational results should be comprehensive, i.e., we would aim to provide exact analytical expressions for all major variables relevant for inflation.

2) The analysis will be general, implying while solving for exact analytical expressions no assumptions of slow roll need to be made.

Motivated by above we ask 'Can we start with a physically well motivated model of scale factor during inflation and eventually obtain an exact analytical expression for $V(\phi)$ and hence obtain predictions of n_s and r?' Clearly, if a satisfactory answer is possible to provide we have at our disposal a unified approach for solutions during the inflation and likely until some critical epoch after the end of inflation as well [27]. Moreover, we are also able to generate an instance of 'predicting' the fundamental function $V(\phi)$ which may encode new information for the particle physics sector [9]. This is an equivalent but distinct method of investigation of dynamical evolution during inflation compared to the approach where one starts from an ansatz of $V(\phi)$ and predicts n_s and r assuming slow roll [28]. Since there has not been any unique fundamental theory which predicts an exact analytical form of the potential, a major focus in the contemporary cosmology has been exploring different possible forms of the potentials which may be confirmed or rejected based upon observations. Concluding from this, the current work provides an alternative approach to other theories predict a possible analytical expression of $V(\phi)$ simultaneously considering a complete solution of the coupled system of equations 1 to 3. *The choice of our scale factor results in evolution of the scalar field both within slow roll regime and beyond without any need to assume any specific form of the $V(\phi)$ to begin with.*

D. Alternative Approaches

An alternative approach to our method for an unified solution set describing inflation dynamics would be to solve for scale factor $a(t)$ starting from a initial choice of $V(\phi)$. In principle this can be achieved by using Hamilton-Jacobi equation for inflation to solve first for $H(\phi)$,

$$H^2(\phi) = \frac{8\pi G}{3}\left[\frac{1}{32\pi^2 G^2}\left(\frac{dH}{d\phi}\right)^2 + V(\phi)\right] \ , \tag{6}$$

which can be obtained by using Eqn. 1 and $(dH/d\phi) = -4\pi G\dot{\phi}$ [29]. However, because of non linear nature of this equation a general solution, $H(\phi)$ of this equation for any general $V(\phi)$ is difficult to perform analytically without an explicit slow roll assumption. If this difficult step can be performed analytically one can subsequently estimate $\phi = \phi(t)$, from which $H(t) = H(\phi(t))$ and $a = a(t)$ can be obtained by direct integration. However, due to non linear nature of Eqn. 6 we do not follow such an approach in this work.

Yet, another approach for complete analytical solution of the inflationary evolution could be starting from an ansatz of $V(t)$ and solve for $H(t)$ analytically using the first order Bernoulli differential equation [30]. Once $H(t)$ is obtained scale factor can be obtained analytically by integration. From $H(t)$ one can estimate $\phi = \phi(t)$ and by inverting it $t = t(\phi)$. This can be now used in $V(t)$ to solve for $V(\phi)$ analytically. However, we do not follow this method since the primary motivation of this work it is to start with a model of time dependence of scale factor, which can be intuitively built based upon some of its desired properties to be satisfies during and after the end of inflation.

II. THE INFLATION MODEL

A. The Scale Factor

We start with following ansatz of the scale factor

$$a(t) = \alpha \{1 - \gamma_1(t - t_i)\} \exp\{\gamma_2(t - t_i)\}. \quad (7)$$

Here, t_i denotes the time at the beginning of inflation, α is a dimensionless constant. γ_1, γ_2 represents two parameters controlling the evolution of the scale factor and each has a dimension of $[T]$. α is the normalizing parameter, and it may take any positive value. The present work considers $\alpha = 1$ without any loss of generality. We will show later that γ_1 and γ_2 are crucial parameters to determine the duration of inflation. One can determine the viable values of the parameters by requiring the prediction of the theory to be consistence with the cosmological observations [31–34].

What is the physical motivation of choosing the scale factor as given by Eqn. 7? We note that apart from representing a rapid (accelerated, like exponential) stretching of spatial scales during some initial epoch, the scale factor should also be able to self terminate (without requiring role of additional physics to end such epoch) the acceleration to mark the end of inflation. This feature will help the inflation to have a graceful exit [35].

Secondly, with the progression of time, Hubble parameter should decrease since we want to model the matter sector in terms of a homogeneous and isotropic real scalar field minimally coupled to gravitation. Such decrease is also expected to be sufficiently slow for a significant duration of inflationary epoch so that enough number e-folds of expansion can take place to explain observed homogeneity and isotropy of the space. Towards the very end of inflation or at time after that, however, such decrease can be as fast as the case for our chosen scale factor.

As we will see later in this work the end of acceleration is soon followed by the post-inflationary increase of the potential from its absolute minimum. This feature is crucial for avoiding eternal inflation [36]. The inflationary potential diverges after attaining the minimum which avoids any possibility of further quantum tunneling of field avoiding eternal inflation. The field therefore can therefore be converted to standard model particles through the process of reheating [37, 38]. Hence, the reduction of the chosen scale factor in the post-acceleration era provides a natural and spontaneous setting for the reheating phase. The decrease in the scale factor associated with the post-inflationary rise of the potential and eventual divergence towards infinity prevents the scale factor from becoming negative. Consequently, the chosen form of the scale factor is well motivated not only for realizing a successful inflationary phase but also for establishing suitable conditions for reheating [38].

In the following we describe some specific features of the evolution of the scale factor. Once the scale factor is given, we can further construct dynamical variables that encodes specific features of its time evolution. One can estimate the Hubble ($H = \frac{\dot{a}}{a}$) parameter

$$H = \gamma_2 - \frac{\gamma_1}{1 - \gamma_1(t - t_i)}, \quad (8)$$

and the acceleration ($\frac{\ddot{a}}{a}$) of the scale factor

$$\frac{\ddot{a}}{a} = {\gamma_2}^2 - \frac{2\gamma_1\gamma_2}{1 - \gamma_1(t - t_i)}. \quad (9)$$

We note that H decreases with time as we expect from the theory of a minimally coupled scalar field with gravity following $\dot{H} = -4\pi G\dot{\phi}^2$. Equation 9 can be used to estimate the duration of accelerating epoch. Specifically acceleration occurs for all time t satisfying,

$$t - t_i \leq \frac{1}{\gamma_1} - \frac{2}{\gamma_2}. \quad (10)$$

In particular, inflation ends at a time t_e for which

$$t_e - t_i = \frac{1}{\gamma_1} - \frac{2}{\gamma_2}. \quad (11)$$

Thus γ's are the factors that decide the duration of inflation. Therefore, these two constants cannot be regarded as independent of each other for a given duration of inflation. A noteworthy feature of the scale factor assumed in Eqn. 7 is that as time proceeds (before the end of inflation and even during some interval of time after the end of inflation) it evolves more and more slowly than the pure exponential expansion. This guarantees that an end of inflation will be possible. The model scale factor given by Eqn. 7, therefore, able to end inflation.

B. The Potential Energy Density

The scalar field potential energy density $V(\phi)$ and the Hubble parameter control the dynamics of the evolution of $\phi(t)$. The potential energy is theoretically motivating since one can model it from the fundamental interaction models of a field theory. To construct the potential energy we first estimate the density ($\rho(t)$) and pressure ($P(t)$) of the field using we use Eqn. 7 in Eqns 1 and 2 and obtain

$$\rho(t) = \frac{3}{8\pi G}\left\{\gamma_2 - \frac{\gamma_1}{1 - \gamma_1(t - t_i)}\right\}^2, \quad (12)$$

$$P(t) = -\frac{1}{8\pi G}\left[3\left\{\gamma_2 - \frac{\gamma_1}{1 - \gamma_1(t - t_i)}\right\}^2 - 2\left\{\frac{\gamma_1}{1 - \gamma_1(t - t_i)}\right\}^2\right]. \quad (13)$$

Using the result $V(t) = (\rho - P)/2$ (from Eqns 4 and 5) we can then obtain exact expression for $V(t)$,

$$V(t) = \frac{1}{8\pi G}\left[3\gamma_2^2 - \frac{6\gamma_1\gamma_2}{1 - \gamma_1(t - t_i)} + \frac{2\gamma_1^2}{\{1 - \gamma_1(t - t_i)\}^2}\right]. \quad (14)$$

We note that using $\dot{H}(t) = -4\pi G\dot{\phi}^2(t)$ and Eqn. 8 one can solve for $t = t(\phi)$ or equivalently $\phi = \phi(t)$ following,

$$(\phi - \phi_i) = \mp\frac{1}{\sqrt{4\pi G}}\log\left(1 - \gamma_1(t - t_i)\right) , \qquad (15)$$

where, $\phi_i = \phi(t_i)$ and $\phi_e = \phi(t_e)$ are constants, the initial and final values of the inflaton field. We use the negative sign in the above equation for the forthcoming analysis of this paper. Such a choice causes ϕ to increase with time. Using Eqn 15 in the previous expression of $V(t)$ we obtain following exact analytical form,

$$V(\phi) = \frac{1}{8\pi G}\Big[3{\gamma_2}^2 - 6\gamma_1\gamma_2\exp\left[\sqrt{4\pi G}(\phi - \phi_i)\right] + 2{\gamma_1}^2\exp\left[2\sqrt{4\pi G}(\phi - \phi_i)\right]\Big] . \qquad (16)$$

C. Solution of Hamilton-Jacobi equation

Hamilton-Jacobi (HJ) differential equation (Eqn 6) establishes a relation between the Hubble parameter ($H(\phi)$) and the potential energy $V(\phi)$ of the scalar field [39, 40]. The equation establishes attractor nature of the inflationary model which states that inflationary conditions would eventually be satisfied regardless of the different possible choice of initial value of Hubble parameter caused by small variations of the initial conditions on the scalar field. Traditionally this differential equation is regarded as a convenient tool to solve for $V(\phi)$ once a suitable form of $H(\phi)$ is assumed. Alternatively, if the objective is to find the complete solution of the inflationary model, one can also try to find solution of the later if a prior model of the potential is used. However, this could be a non trivial problem for a general potential due to non-linear nature of Hamilton-Jacobi equation. Our chosen model however, provides an exact solutions of $H(\phi)$ consistent with potential energy of Eqn. 16, without any need to solve the HJ equation. Using Eqn. 15 in Eqn. 8 we obtain,

$$H(\phi) = \gamma_2 - \gamma_1\exp\left\{\sqrt{4\pi G}(\phi - \phi_i)\right\} . \qquad (17)$$

Similarly using Eqn. 15 in Eqn. 7 we can even obtain an expression of scale factor with respect to the field ϕ,

$$a(\phi) = \alpha\exp\left[\frac{\gamma_2}{\gamma_1} - \sqrt{4\pi G}\,(\phi - \phi_i) - \frac{\gamma_2}{\gamma_1}\exp\left\{-\sqrt{4\pi G}(\phi - \phi_i)\right\}\right] . \qquad (18)$$

We note, as inflation proceeds ϕ increases in our model, hence H, following Eqn. 17, decreases as we expect from the scalar field model minimally coupled to gravity. With the increase in $4\pi G(\phi - \phi_i)$ as inflation proceeds the net contribution of the first and third terms of the exponent of Eqn. 18 increases. This results in rapid expansion in the scale factor. However, when the second term starts to become dominant compared to the other two terms the inflation starts to slow down and eventually ends leading to successful exit from the inflationary epoch.

D. Equation of State Parameter

The equation-of-state parameter defined as $w = \frac{P}{\rho}$, is often used to identify if the condition of inflation is satisfied [39]. Moreover, it may possibly encode information about the *state* of the scalar field [41]. Using Eqns. 13 and 12 in above we obtain,

$$w = -1 + \frac{2}{3}\left[\frac{\gamma_2}{\gamma_1}\left(1 - \gamma_1(t - t_i)\right) - 1\right]^{-2} , \qquad (19)$$

implying a dynamic behaviour of $w(t)$ and $w(t) > -1$ always. Therefore the field behaves like quintessence. At $t = t_i$, $w(t_i) = -1 + 2/3(\gamma_2/\gamma_1 - 1)^{-2}$. With increase in time the second term of Eqn. 19 increases, hence the equation of state parameter also increases. Inflation ends when w increases beyond $-1/3$. Using Eqn. 15 in Eqn. 19 one can express the w parameter with respect to the field variable,

$$w(\phi) = -1 + \frac{2}{3}\left[\frac{\gamma_2}{\gamma_1}\left\{\exp\left(-\sqrt{4\pi G}(\phi - \phi_i)\right)\right\} - 1\right]^{-2} . \qquad (20)$$

Setting $w = -1/3$ in Eqn. 19 we obtain, $t_e - t_i = 1/\gamma_1(1 - \gamma_1(1\pm1)/\gamma_2)$. Considering the solution with positive sign the expression of $t_e - t_i$ agree with that derived after Eqn. 10. This corresponds to the time when ω crosses $-1/3$ from below. The second case when $w = -1/3$ corresponds to the negative sign in the above solution. In this case time t of cross over is given by $t - t_i = 1/\gamma_1$ when w crosses $-1/3$ from above. Same conclusion can also be obtained by performing similar calculations using Eqn. 20.

III. SLOW-ROLL PARAMETERS

The slow roll parameters encode information about the dynamics of the evolution of the scale factor and nature of the shape of the scalar field potential. Two different slow roll parameters that are defined in terms of the Hubble parameter are given by

$$\epsilon_H = -\frac{\dot{H}}{H^2} \qquad \eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} . \qquad (21)$$

The slowly rolling condition of the field over the potential is satisfied when these parameters take values that are $\ll 1$ [42, 43]. The potential energy dependent slow roll parameters are defines as

$$\epsilon_V = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \qquad \eta_V = \frac{1}{8\pi G}\left(\frac{V''}{V}\right) . \qquad (22)$$

The two ϵ parameters defined in earlier two equations are not the same. They are close to one another in the slow roll regime [42]. Beyond this these parameters can differ significantly. Similar conclusion is also applicable for the two types of η parameters defined above. Incidentally there exist two consistency relations between these parameters,

$$\begin{aligned}\epsilon_V &= \epsilon_H \left(\frac{3-\eta_H}{3-\epsilon_H}\right)^2 \\ \eta'_V &= \sqrt{2G\epsilon_H}\frac{\eta'_H}{3-\epsilon_H} + \left(\frac{3-\eta_H}{3-\epsilon_H}\right)(\epsilon_H+\eta_H)\ ,\end{aligned} \tag{23}$$

where the ' sign represents derivative with respect to field ϕ [42]. Using Eqns. 8 and 15 in Eqn. 22 we obtain

$$\begin{aligned}\epsilon_H(t) &= \left[\frac{\gamma_2}{\gamma_1}\{1-\gamma_1(t-t_i)\}-1\right]^{-2} \\ \eta_H(t) &= -\left[\frac{\gamma_2}{\gamma_1}\{1-\gamma_1(t-t_i)\}-1\right]^{-1}\ .\end{aligned} \tag{24}$$

where the exact time of end of inflation can be obtained by setting $\epsilon_H = 1$ [44]. As shown earlier this time t_e follows $t_e - t_i = 1/\gamma_1 - 2/\gamma_2$. Above two parameters can be expressed as functions of ϕ as well,

$$\begin{aligned}\epsilon_H(\phi) &= \left[\frac{\gamma_2}{\gamma_1}\exp\left\{-\sqrt{4\pi G}(\phi-\phi_i)\right\}-1\right]^{-2} \\ \eta_H(\phi) &= -\left[\frac{\gamma_2}{\gamma_1}\exp\left\{-\sqrt{4\pi G}(\phi-\phi_i)\right\}-1\right]^{-1}\ .\end{aligned} \tag{25}$$

Traditionally the ϵ_V and η_V parameters are expressed in terms of ϕ. For our case, the exact expressions for these parameters are given by,

$$\begin{aligned}\epsilon_V(\phi) = \Bigg[&\frac{2{\gamma_1}^2\exp\left\{2\sqrt{4\pi G}(\phi-\phi_i)\right\}}{3{\gamma_2}^2-6\gamma_1\gamma_2\exp\left\{\sqrt{4\pi G}(\phi-\phi_i)\right\}} \\ &\frac{-3\gamma_1\gamma_2\exp\left\{\sqrt{4\pi G}(\phi-\phi_i)\right\}}{+2{\gamma_1}^2\exp\left\{2\sqrt{4\pi G}(\phi-\phi_i)\right\}}\Bigg]^2\ ,\end{aligned} \tag{26}$$

and

$$\begin{aligned}\eta_V(\phi) = &\frac{4{\gamma_1}^2\exp\left\{2\sqrt{4\pi G}(\phi-\phi_i)\right\}}{3{\gamma_2}^2-6\gamma_1\gamma_2\exp\left\{\sqrt{4\pi G}(\phi-\phi_i)\right\}} \\ &\frac{-3\gamma_1\gamma_2\exp\left\{\sqrt{4\pi G}(\phi-\phi_i)\right\}}{+2{\gamma_1}^2\exp\left\{2\sqrt{4\pi G}(\phi-\phi_i)\right\}}\ .\end{aligned} \tag{27}$$

IV. NUMBER OF E-FOLDS

The inflationary model provides a mechanism for producing the right size of Horizon at the end of inflation which after subsequent evolution eventually grows to the size of the present day Horizon. The perturbations of the inflationary field produces a scalar power spectrum characterized by the spectral index n_s which is close to unity for a slow roll inflation. The intrinsic tensor type perturbations of the metric result in stochastic background of the gravitational wave and its power spectrum is known as tensor power spectrum [27, 45]. The ratio of tensor to scalar power spectra is denotes as r. Both n_s and r are predictions of inflationary theories and measurable from cosmological observations [27]. The rapid expansion of space causes wave lengths corresponding to various modes of inflationary perturbations to cross outside horizon and soon thereafter these perturbations freeze without any time evolution, the epoch of which is taken to be moment of horizon exit of the mode under consideration [46]. The number of e-folds ($N(t)$) from the moment of horizon exit (at time t) of a mode with wave number k to the end of inflation is an important variable since all the cosmological modes that are observable today actually exit the horizon during inflation 50 to 60 e folds before the end of inflation [47]. The procedure of calculating the inflationary predictions of n_s and r then reduces to finding time epochs for $N = 50$ or 60 and using these time values in the expressions of suitable slow roll parameters [48, 49].

Using Eqn. 8 in the expression $N(t) = \int_t^{t_e} H dt$ we obtain an exact expression,

$$N(t) = \gamma_2(t_e - t) + \log\frac{1-\gamma_1(t_e-t_i)}{1-\gamma_1(t-t_i)}\ . \tag{28}$$

One can also obtain an expression for $N(\phi)$ by using Eqn. 15 in above expression,

$$\begin{aligned}N(\phi) = &\frac{\gamma_2}{\gamma_1}\left[\exp\left\{-\sqrt{4\pi G}(\phi-\phi_i)\right\}\right. \\ &\left.-\exp\left\{-\sqrt{4\pi G}(\phi_e-\phi_i)\right\}\right]-\sqrt{4\pi G}(\phi_e-\phi)\ .\end{aligned} \tag{29}$$

Using Eqn. 28 one can obtain total number of e-folds starting from time t_i,

$$\begin{aligned}N_{tot} \equiv N(t_i \to t_e) &= \gamma_2(t_e-t_i) + \log(1-\gamma_1(t_e-t_i)) &(30)\\ &= \tfrac{\gamma_2}{\gamma_1} - \log\left(\tfrac{\gamma_2}{\gamma_1}\right) - 2 + \log(2)\ , &(31)\end{aligned}$$

where we have used $t_e - t_i = 1/\gamma_1 - 2/\gamma_2$ to obtain the last equation. The total number of e-folds is determined by the ratio γ_2/γ_1 apart from some small numerical contributions $-2+\log(2)$. First term on the right hand side of Eqn. 30 is larger than the second term. The argument of the log term is always positive (and hence less than unity) by our choice. Hence, the second term on the right side gives a negative contribution. This implies that, the inflation model of this work allows somewhat larger values of γ_2 or $t_e - t_i$ to produce same $N_{\rm tot}$ when compared to the inflationary model with exact De-Sitter expansion. It is worth noting that using $\phi = \phi_e$ hence

also $t = t_e$ in Eqn. 15 along with Eqn. 11 one can easily obtain

$$\sqrt{4\pi G}(\phi_e - \phi_i) = \log(\gamma_2/\gamma_1) - \log(2) \ , \qquad (32)$$

which is an expression for the duration of inflation in terms of the field variable similar to the inflationary duration in time domain (e.g., Eqn. 11). We also note that using Eqn. 32 and Eqn. 29 one can also obtain total number of e folds between time t_i and t_e as $\gamma_2/\gamma_1 - \log(\gamma_2/\gamma_1) - 2 + \log(2)$ as obtained in Eqn. 31.

V. ANALYSIS OF PARAMETERS AND DYNAMICAL VARIABLES

A. Constraints on Parameters

The basic parameters of our inflation models are γ_1, γ_2, t_e and t_i. Since we have a constraint equation $t_e - t_i = 1/\gamma_1 - 2/\gamma_2$ only three parameters among these are independent. We can provide qualitative constraints on these parameters following a series of physical reasoning. A more precise constraints will also follow.

To solve the Horizon problem of the standard cosmology the inflation should last for at least 50 to 60 e-folds. The choice of parameters of our model scale factor should be consistent with this number. From Eqn. 30 we see that the total number of e folds is dominantly determined by the first term on the right hand side since the log term is expected to be small. This is because the argument of the second term must be less than unity during the inflation so as the scale factor (Eqn. 7) remains positive. This argument is expected to be not too much less than unity during inflation for accelerating expansion to happen. Hence the first term alone determines the total number of e-folds to a good approximation. For a typical value of $t_e - t_i$ then $\gamma_1 \lesssim 1/(t_e - t_i)$. Since the duration of inflation should be such that $\gamma_2(t_e - t_i) \gtrsim 60$ (for observed homogeneity on the large scale) clearly $\gamma_2 \gg \gamma_1$. From Eqn. 8 we now see that Hubble parameter value is dominantly determined by γ_2 for the *entire* duration of inflation.

The energy density of the inflaton field is given by $\rho_\phi = 3H^2/(8\pi G)$. The value of ρ^i at the beginning of inflation ($\phi = \phi_i$) is typically assumed to be $\lesssim 10^{64}$ GeV^4 corresponding to a energy scale of inflation 10^{16} GeV which is already much less than the Planck energy scale $1/\sqrt{G} = 1.22 \times 10^{19}$ GeV. The assumed value of ρ^i determines H at the beginning of inflation and also represents approximate value of H during inflation (as long as slow roll occurs) and sets the time epoch when inflation happened. Without loosing much precision we assume $\gamma_2 \gg \gamma_1$ (as discussed above) in Eqn. 8. Now we set $t = t_i$ for finding the upper limit of γ_2 using the approximate equation $\gamma_2^2 \sim (8\pi G/3)\,\rho^i$ which leads to $\gamma_2^{\rm max} \sim 3.6 \times 10^{37}{\rm s}^{-1}$.

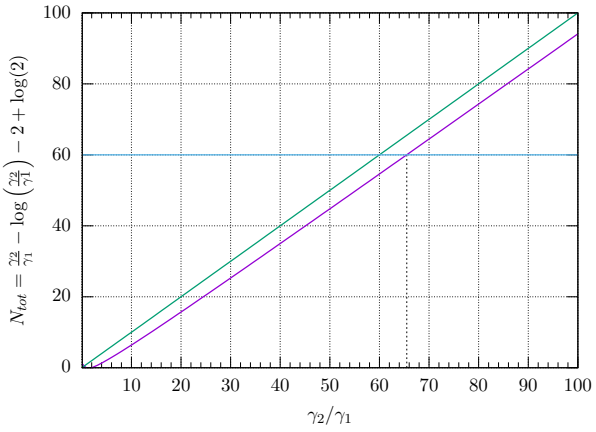


FIG. 1. Variation of total number of e-folds with respect to the ratio γ_2/γ_1 is shown in violet color. The green line shows the linear variation with respect to the ratio. The green line lies above the violet line indicating slower than linear dependency of $N_{\rm tot}$ on γ_2/γ_1 in our model. The dashed vertical line shows the value of $\gamma_2/\gamma_1 = 65.49$ corresponding to merely 60 total e-folds. This sets the minimum value of γ_2/γ_1 for enough inflation as 65.49.

B. Number of e-folds

In Fig. 1 we show variation of $N_{\rm tot}$ with respect to the ratio γ_2/γ_1 in violet using Eqn. 31. The green shows the straight line with ordinate values same as γ_2/γ_1. The violet line always lies below the green line as expected. The cyan line corresponds to the minimum e-fold 60 necessary for producing homogeneity on the currently observable cosmological scales. The corresponding (minimum) value of $\gamma_2/\gamma_1 \sim 65.49$. This along with the maximum allowed value of $\gamma_2 \sim 3.6 \times 10^{37}{\rm s}^{-1}$ estimated above provide important constraints on the permissible values of these parameters.

The total number of e-folds is controlled by the ratio γ_2/γ_1. Also this ratio alone can determine the end the inflation for a given t_i which can be easily seen from Eqn. 9. We infer from this equation that inflation will end when $1 - \gamma_1(t_e - t_i) = 2\gamma_1/\gamma_2$. From the right hand side of this equation, we conclude that inflation ends when $\gamma_1(t_e - t_i) \lesssim 1$, since we have already seen that $\gamma_2 \gg \gamma_1$. It is clear from Eqn. 7 that if $\gamma_1 = 0$ the inflation never stops. Introducing γ_1 in the scale factor model, therefore, serves the purpose of avoiding graceful exit problem.

By how many e-folds (as a function of ϕ) does the space expand from a time (t) when field value is $\phi(t)$ till the end of inflation? Using Eqn. 29, after a little algebra, one obtains

$$N(\phi) = y - \log(y) - 2 + \log(2) \ , \qquad (33)$$

where the variable $y \equiv \frac{\gamma_2}{\gamma_1} \exp(-\sqrt{4\pi G}(\phi - \phi_i))$.

In Fig. 2 we show the variation of $N(\phi)$ as a function of the dimensionless variable $\sqrt{4\pi G}(\phi - \phi_i)$ for several choice of the ratio γ_2/γ_1. As expected increasing the ra-

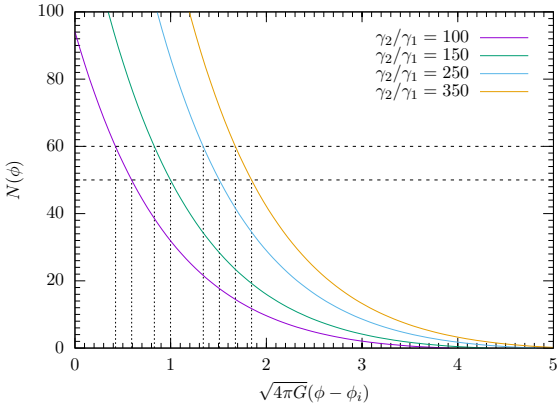


FIG. 2. Figure showing number of e folds left until the end of inflation from an epoch during inflation when the field value is ϕ as a function of dimensionless variable . This e-fold number depends on the ratio γ_2/γ_1 as a parametric form. Several choices if the ratio are chosen as indicated in the figure. The horizontal dashed lines indicate 50 and 60 number of e-fold that are relevant for the observable cosmological scales at the present epoch. As seen from the figure the field intervals corresponding to 50 and 60 e-folds are independent on the ratio γ_2/γ_1. For the cosmological modes the horizon exit happens well in inside the slow roll regime for all choice of the ratio.

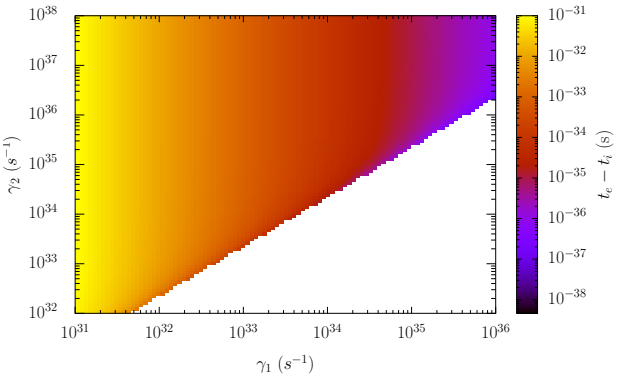


FIG. 3. Figure showing variation of time duration of inflation with respect to parameters γ_1 and γ_2. The maximum value of γ_2 corresponds to a energy scale of inflation sufficiently lower than Planck energy scale. Horizontal color gradient of this figure implies that the duration of inflation is maximally determined by γ_1 parameter. The white triangular region indicate nonphysical parameter values with $\gamma_2 < 2\gamma_1$.

tio results in more number of total e-folds. Larger values of the ratio causes present day cosmologically important modes exit horizon during inflation at a larger value of $\phi - \phi_i$. From Eqn. 15 we see that larger $\phi - \phi_i$ value implies a lower $t - t_i$ if γ_1 for a given value of γ_1. *It is interesting to note that the ϕ-interval within which the present day cosmological modes exit horizon appears independent on the ratio γ_2/γ_1 on the horizontal scale of the figure.* Such similarity occurs since the horizon exit for the relevant modes occur during the slow roll regime in which the scalar field changes by small amount.

Using the expression $t_e - t_i = 1/\gamma_1 - 2/\gamma_2$ one obtains $\gamma_1 < \gamma_2/2$ for obvious condition that inflation ends at a later epoch than when it begins i.e., $t_e > t_i$. Moreover, at the end of inflation $1 - \gamma_1(t_e - t_i) = 2\gamma_1/\gamma_2 < 1$ indicating positivity condition on the scale factor is satisfied. The condition $\gamma_2 > 65.49\gamma_1$ for enough inflation already satisfies this condition.

C. Duration of Inflation

In Fig. 3 we show the variation of the duration of inflation with γ_1 and γ_2 for $t_e > t_i$. For small $\gamma_1 \ll \gamma_2$ the duration is dominantly determined by the $1/\gamma_1$ term as seen from the left side region of the figure. This is true even if γ_1 becomes merely one order of magnitude larger than γ_2. As γ_1 increases both γ_1 and γ_2 influences the duration. For large values of γ_1 and γ_2 the duration of inflation becomes noticeably small as indicated in the top right region of this figure. Essentially, a large γ_1 value results in small time duration for inflation in this figure. The white triangular region is nonphysical corresponding to values $\gamma_1 > \gamma_2/2$ and hence masked since it leads to $t_e \leq t_i$.

D. Inflation Energy Density

In Fig. 4 we show the variation of ρ^e estimated at the end of inflation for with respect to γ_1 and γ_2. The duration of inflation is given by $1/\gamma_1 - 2/\gamma_2$ for this figure without any explicit assumption of individual values of t_i and t_e. It is easy to show that the Hubble parameter at the end of inflation $H_e = \gamma_2/2$. Hence $\rho^e = 3H_e^2/(8\pi G)$ is entirely determined by the parameter γ_2. *It is interesting to note that the energy density at the end of inflation is independent of the duration of the inflation for a given choice of ρ^i.* We also note that, $\rho^i \sim 4\rho^e$, i.e., the energy density at the end of inflation drops by roughly a factor of 4 for all possible values of γ_1 and γ_2. Because of this relation a plot of variation of ρ^i looks similar to Fig. 4 apart from a 4 times larger energy density value. The white triangular region correspond to the parameter values $\gamma_2/\gamma_1 \leq 65.49$ and hence excluded.

E. Potential Energy Density

Following Eqn. 16 the potential energy depends on the choice of the parameters γ_2 and γ_1. In Fig. 5 we show the variation of potential energy in GeV^4 unit at the start of inflation with these parameters in the top left panel. The horizontal and vertical axes of each sub panel represent values of γ_1 and γ_2 (in s unit) respectively. The vertical gradient in the color map of the top left panel indicates

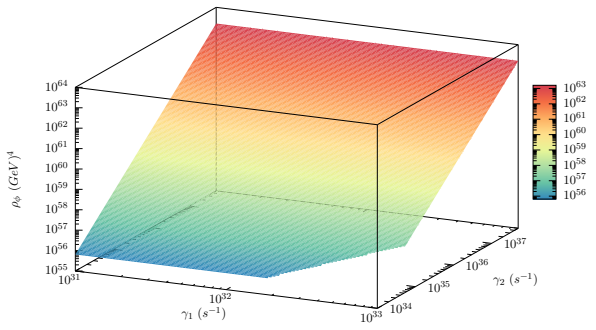


FIG. 4. Figure showing variation energy density of inflaton at the end of inflation with parameters γ_1 and γ_2. On the scales of the vertical axis of this figure the energy density is dominantly determined by the parameter γ_2. The condition of minimum 60 e-folds of inflation is satisfied by the γ_1 , γ_2 parameters on the $\gamma_1 - \gamma_2$ plane.

the potential energy at $t = t_i$ is dominantly determined by γ_2. The top right, bottom left and bottom right panels represent the ratio of the first, second and third terms of the right side of Eqn. 16 (all evaluated at $t = t_i$) and potential energy shown in top left panel respectively. The top right panel shows ratio values close to unity. The two bottom panels show that the second and third terms of Eqn. 16 are significantly smaller than the first term at the start of inflation.

However, the second and third terms of Eqn. 16 become significant and increase their influences on the net potential energy as ϕ increases from ϕ_i due to associated exponentially increasing factors increasing with $\phi - \phi_i$. This results in interesting and physically motivated variation of $V(\phi)$ with $\sqrt{4\pi G}(\phi - \phi_i)$. We show this in Fig. 6 for different choices of the value of γ_1 for a given $\gamma_2 = 10^{37}s$. The vertical axis in each sub panel of this figure is in unit of 10^{64}GeV^4 unit. In each sub panel the inflation potential has a 'plateau' region marked by slowly changing slope. This is followed by rapid decay of the potential, existence of a minimum potential and post minimum increase in the potential at a faster rate than the decay. The plateau region determines the duration of slow roll inflation – which is responsible for enough e-folds of inflation. The increasing slope of the potential energy after the slow roll regime soon causes inflation to self terminate leading to *graceful exit* from inflation. Existence of a minimum potential energy is useful in some theories of post inflation reheating by the so-called coherent oscillation mechanism of the scalar field around the minimum and through couplings with the standard model fields. We note that decrease in γ_1 (for a given γ_2) causes an extended epoch of slow roll in the field domain. This causes the potential energy pattern to shift to the right side (i.e, towards higher values of $(\phi - \phi_i)$). This is expected since $1 - \gamma_1(t_e - t_i) = 2\gamma_1/\gamma_2$ (derived earlier) becomes smaller with decrease in γ_1 and following Eqn. 15 we find a larger value of $(\phi_e - \phi_i)$ now with this decrease.

A remarkable feature of the inflaton potential in our model is that it becomes negative for range of field $\phi \in (\phi_1, \phi_2)$ with $\phi_2 > \phi_1$. However, the negative potential energy is not of a concern. The kinetic energy of the field $(\dot{\phi}^2/2)$ remains always positive and balances the negative potential in the range where $V(\phi)$ is negative. The minimum of $V(\phi)$ occurs between ϕ_1 and ϕ_2. Using Eqn. 14 and setting $dV(t)/dt = 0$ (along with $d^2V/dt^2 < 0$) one finds the $V(t)$ is minimum at time $t_{\rm min}$ where

$$t_{\rm min} - t_i = 1/\gamma_1 - 2/(3\gamma_2) \ , \tag{34}$$

which occurs well after the end of inflation.

Soon after the inflation ends the reheating method is expected to take place. During this process standard model particles are formed. Once reheating starts the scale factor deviates from the assumed form which is given in Eqn. 7, due to new coupling with the other fields. However, when we consider the inflaton potential energy as a function of ϕ it is a fundamental property of the field only and therefore it should be thought of as independent on the scale factor. Our analytical framework has the advantage of *predicting* this fundamental form of the potential even when Eqn. 7 may not be valid after the start of the reheating mechanism by using the coupled equations of inflationary dynamics (Eqns. 1 to 3).

It becomes instructive to study the evolution of the scale factor after the end of inflation for further insights into the post inflationary dynamics. First of al, pre exponent factor of Eqn. 7 sets a maximum physical limit (T) on time variable t below which scale factor remains positive, as should be the case always. Clearly T is given by $T - t_i = 1/\gamma_1$. Since we already know inflation ends earlier than this time reheating mechanism should also start before T if we wish to extend our analytical results for scale factor (and other variables derived from it) beyond the end of inflation. Using Eqn. 8 we see that $H = 0$ at a time t_m for which $t_m - t_i = 1/\gamma_1 - 1/\gamma_2$. One can easily obtain $\ddot{a}/a = -\gamma_2^2 < 0$ using Eqn. 9 at $t = t_m$. Therefore scale factor reaches a maximum at t_m. Further during the time line the scale factor should collapse. However, such collapse does not cause any issue with the predictions of inflation regarding production of homogeneous and isotropic modes in today's observable universe as long as the reheating starts before the time t_r $(> t_e)$ at which the scale factor collapses back to the value that it had at the end of inflation. Since the scale factor at the end of inflation $a(t_e) = 2(\gamma_1/\gamma_2)\exp(\gamma_2/\gamma_1 - 2)$ the time t_r can be obtained by solving the following equation,

$$(1 - \gamma_1(t_r - t_i))\exp\left(\gamma_2(t_r - t_i)\right) = 2\frac{\gamma_1}{\gamma_2}\exp\left(\frac{\gamma_2}{\gamma_1} - 2\right) \ , \tag{35}$$

where $t_r > t_m > t_e$. A numerical estimate of t_r is given by $t_r - t_i = 1/\gamma_1 - 0.41/\gamma_2$. To summarize, we conclude that after the end of inflation our model sets a time limit

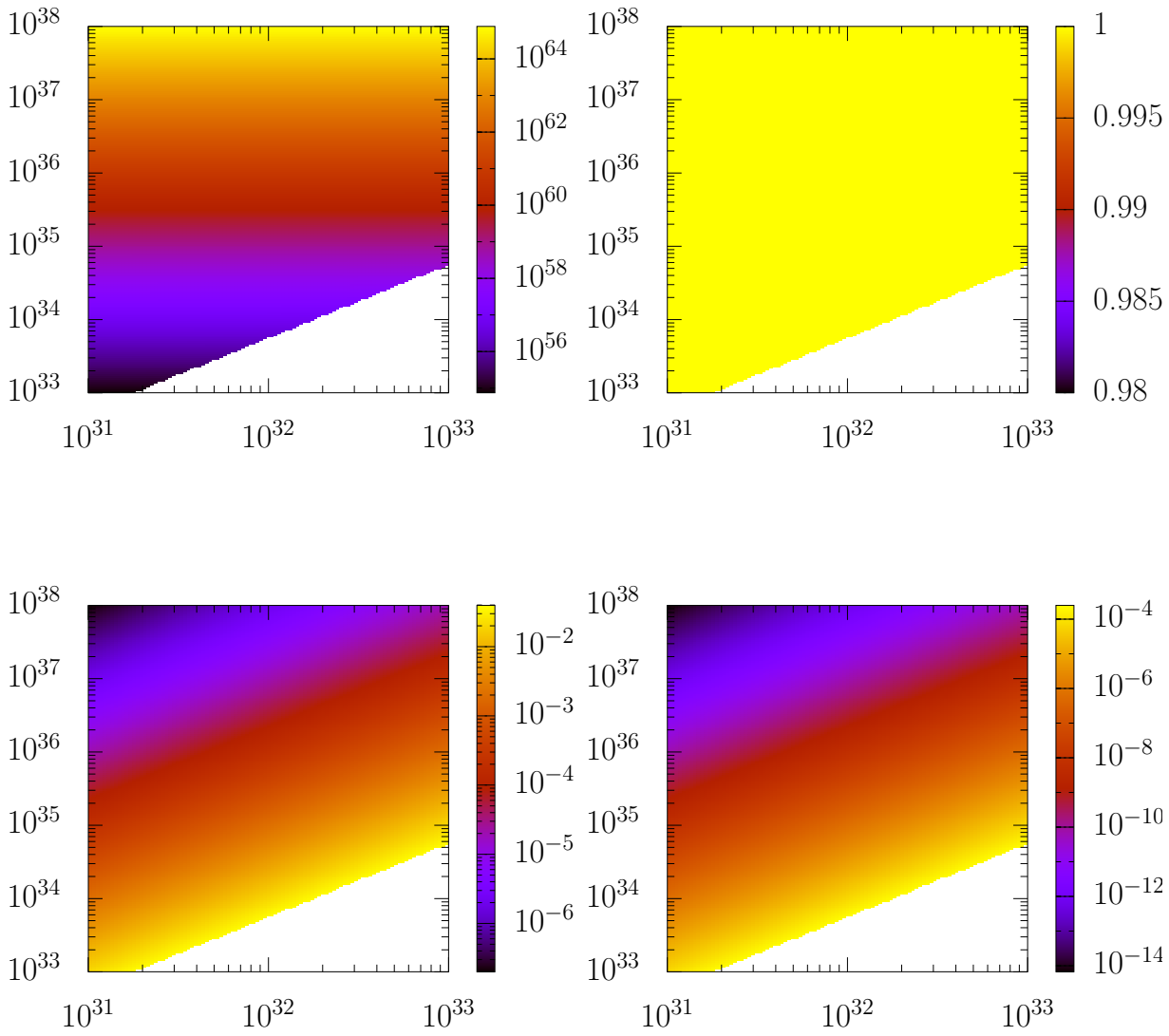


FIG. 5. Figure representing variation of inflaton potential and the three individual terms in its expression (e.g., Eqn.) with parameters at the onset of inflation. The horizontal and vertical axes on each sub panel represent γ_1 and γ_2 values respectively. The top left panel shows variation of potential (Eqn) at $t = t_i$. The color bar is in unit of GeV^4. The color distribution on the top left panel indicates that the potential at the beginning of inflation is determined by γ_2 with a good accuracy. The top right, bottom left and bottom right panels show ratio of first , second and third term of Eqn. with the potential plotted in the top left panel. Again these three sub panels show that the potential is dominantly determined by the first term alone while the last two terms are significantly smaller than the first term.

of t_r before which the reheating should start. One can see that this is a perfectly acceptable time limit. The value t_r is well above the time when minimum of the potential energy is attained (e.g., Eqn. 34). Therefore, the inflationary model of this work can serve as a template for scale factor for both inflation, its end and until the time of start of reheating providing the later a widely flexible time range to take place.

We note that, at time t_m when the scale factor reaches maximum after the inflation $H(t_m) = 0$. Hence, $\rho(\phi(t_m)) = 0$ and obviously $\dot{\phi}^2/2 = V(\phi)$ at $t = t_m$. We emphasize that although $\rho(\phi(t_m)) = 0$ the universe obviously is not empty, the scalar field is present in it. The energy density always satisfies the weak energy condition in our model.

One can also obtain the two time values at which $V(t)$ crosses the zero line. A little algebra shows that $V(t) = 0$ at $t_\pm - t_i = 1/\gamma_1 - 2/[(2 \pm \sqrt{3})\gamma_2]$. Clearly $t_+ > t_-$

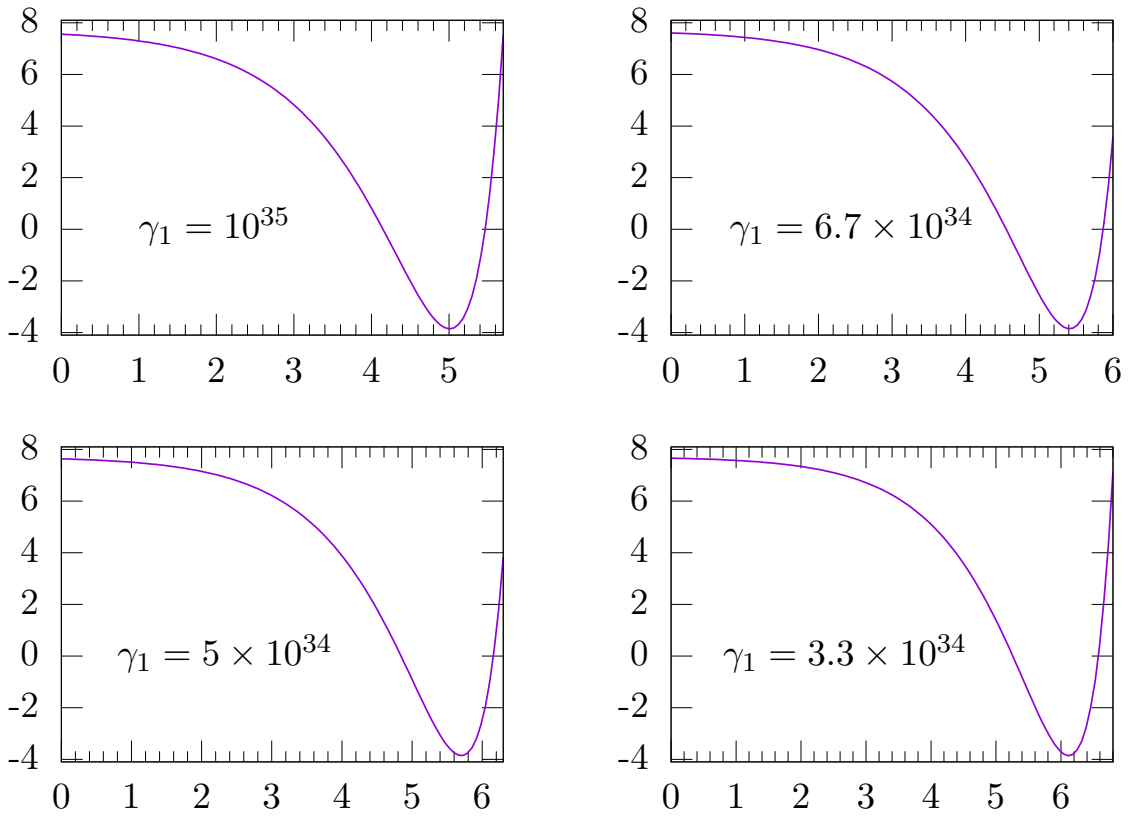


FIG. 6. In this figure we show variation of inflation potential $V(\phi)$ (in GeV^4 unit) with respect to dimensionless field variable $\sqrt{4\pi G}(\phi - \phi_i)$ for several choices of the ratio of γ_1 and γ_2 parameters. $\gamma_2 = 10^{37}s$ for all four sub panels. The γ_1 value varies between different sub panels and its value is mentioned in each sub panel in unit of second. For smaller values of γ_1 the minimum of the potential is shifted to the right and the slow roll happen for extended period indicated by the longer plateau region of the potential as predicted by analytical equations discussed in the text.

and $t_+ - t_i = 1/\gamma_1 - 0.423/\gamma_2$. Considering the earlier result $t_r - t_i = 1/\gamma_1 - 0.41/\gamma_2$ we conclude that during the post-inflation collapse when the scale factor returns to the value it had at the end of inflation the potential energy already turns positive.

F. Slow Roll Parameters

It is clear from the expression of $V(\phi)$ from Eqn. 16 that the ratio $(dV/d\phi)/V$ depends on the ratio γ_2/γ_1 apart from the dimensionless variable $x = \sqrt{4\pi G}(\phi - \phi_e)$. This implies slow roll parameter ϵ_V (and also η_V) depends on the same ratio and the variable x as we see in Eqn. 26. Interestingly, the variables ϵ_H and η_H as functions of variable x show similar behaviour (e.g., Eqn 25). In Fig. 7 we show the variation of $\epsilon_H(\phi)$ and $\epsilon_V(\phi)$ with $\sqrt{4\pi G}(\phi - \phi_e)$ for different values of γ_2/γ_1. From Eqn. 25 it is easy to conclude that slow roll is satisfied as long as $\frac{\gamma_2}{\gamma_1} \exp\left(-\sqrt{4\pi G}(\phi - \phi_i)\right) \gg 1$. Thus the slow roll conditions is satisfied for large ratio of γ_2/γ_1 and small x. Using Eqn. 26 for $\epsilon_V(\phi)$ we can show that $\epsilon_V \ll 1$ when the same condition is satisfied. However, we note that the $\epsilon_H = 1$ when $\gamma_2/\gamma_1 \exp(-x) = 2$ whereas $\epsilon_V = 4$ when $\gamma_2/\gamma_1 \exp(-x) = 2$. Therefore, ϵ_V crosses unity from below earlier than ϵ_H. One can show that ϵ_V crosses unity from below at $\gamma_2/\gamma_1 \exp(-x) = 3/2 - \sqrt{11/12}$.

For each ratio the two slow roll parameters ϵ_V and ϵ_H agree well with each other at small field values $\sqrt{4\pi G}(\phi - \phi_e) \lesssim 2$. This region may well be regarded as the slow roll epoch when two types of ϵ parameters agree. Beyond slow roll regime, the two parameters start producing deviation at higher field with ϵ_V increasing faster than ϵ_H. The faster increase for ϵ_V is due to the fact that ϵ_V is larger than ϵ_H in our inflation model. This can be easily proven using Eqns. 23 and 25. The horizontal coordinate of the intersection point of solid lines with the dashed horizontal line represents the end of inflation for each value of γ_2/γ_1. Considering Fig. 7 one can identify the slow roll regime and end of inflation of epoch for each sub panel of Fig. 6. Also, revisiting Fig. 2 we conclude that the present day cosmologically important modes exit horizon during inflation during some interval of time when slow roll approximation was very well valid.

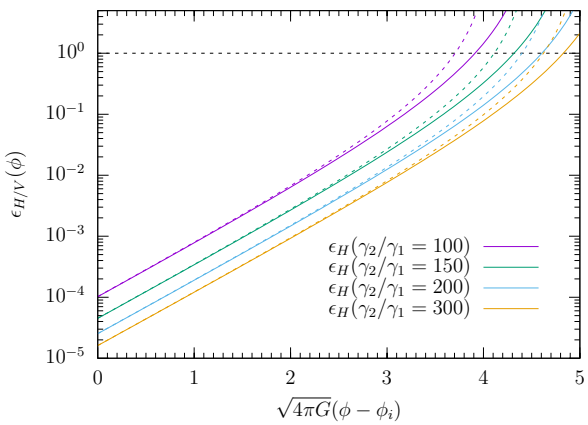


FIG. 7. Figure showing variation of slow roll parameters $\epsilon_H(\phi)$ and $\epsilon_V(\phi)$ with dimensionless variable $\sqrt{4\pi G}(\phi-\phi_i)$ for different choice of the ratio γ_2/γ_1. The dashed horizontal line indicates a slow roll value of unity and horizontal coordinate of the intersection of this line and ϵ_H represents the end of inflation. The slow roll regime corresponding to small value of slow roll parameters can be made out. As field increases the slow roll approximations become no longer and valid. ϵ_V increases faster than ϵ_H as expected.

VI. PREDICTION VS OBSERVATION

Within the approximation of first order slow roll parameters the spectral index (n_s) and the tensor to scalar ratio (r) follow

$$\begin{aligned} n_s &= 1-4\epsilon_H+2\eta_H \approx 1-6\epsilon_V+2\eta_V \\ r &= 16\epsilon_H \approx 16\epsilon_V \ , \end{aligned} \tag{36}$$

where we have used $\eta_V=\epsilon_H+\eta_H$ in the first line [50, 51].

An interesting result in our inflationary model is that the Eqn. 33 representing the variation of number of e-folds with $y=(\gamma_2/\gamma_1)\exp(-\sqrt{4\pi G}(\phi-\phi_i))$ can be inverted analytically to express y in terms of N. Using Lambert W function after some algebra one can show that $y=-W(-\exp(-\tilde{N}))$ where

$$\tilde{N}=N+2-\log(2) \ . \tag{37}$$

Since we are interested in N corresponding to the time of Horizon exit of cosmologically relevant modes $N\in(50,60)$ implying the argument of the W is very small in magnitude and is negative [52]. Using the asymptotic expression of $W(v)$ for $v\to 0_-$ we obtain,

$$\begin{aligned} y =& \tilde{N}+\log(\tilde{N})+\frac{\log(\tilde{N})}{\tilde{N}}-\frac{\log(\tilde{N})}{\tilde{N}^2}\left[\frac{\log(\tilde{N})}{2}-1\right] \\ &+\frac{\log(\tilde{N})}{6\tilde{N}^3}\left[6-9\log(\tilde{N})+2[\log(\tilde{N})]^2\right] \ . \end{aligned} \tag{38}$$

Hence the variable y is effectively expressed as function of $N=\tilde{N}-2+\log(2)$. Also using expressions of ϵ_V and η_V from Eqns. 26 and 27 one can show that each of these two slow roll variables are, in fact, dependent only on y. Specifically,

$$\begin{aligned} \epsilon_V &= \left[\frac{2-3y}{3y^2-6y+2}\right]^2 \\ \eta_V &= \left[\frac{4-3y}{3y^2-6y+2}\right] \ . \end{aligned} \tag{39}$$

Using Eqn. 39 in Eqn. 36 we obtain

$$\begin{aligned} n_s =& 1-\frac{2(9y^3-3y^2-6y+4)}{(3y^2-6y+2)^2} \\ r =& 16\left[\frac{2-3y}{3y^2-6y+2}\right]^2 \ . \end{aligned} \tag{40}$$

Using Eqn. 37 in Eqn. 38 and the resulting equations in Eqn. 40 one can obtain expressions of n_s and r as functions of a single variable N.

In Fig. 8 we show the n_s and r values predicted by our theory for different values of N between 60 and 50 in black line. Also shown in this figure are one and two sigma confidence regions on the n_s-r plane using Planck and ACT observations, each combined with Keck-BiCEP observations. From this figure we conclude that our predictions for the n_s and r are in well agreement with the latest cosmological observations.

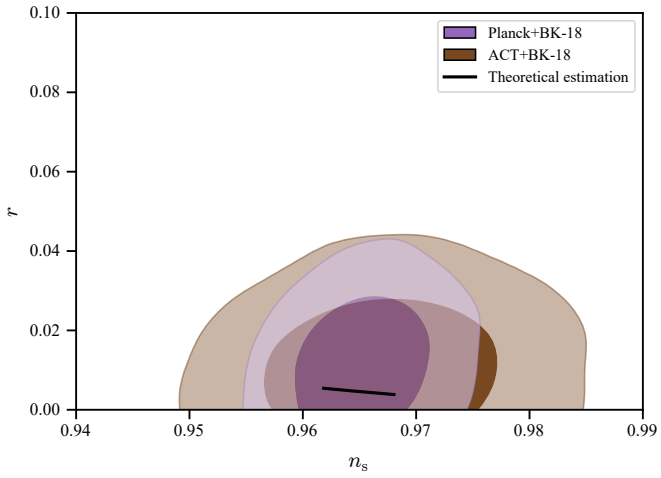


FIG. 8. Figure showing n_s vs r curve (in black) following the predictions of this work. One and two sigma confidence intervals of Planck and ACT CMB observations (each combined with BICEP-KECK) are also shown. Clearly our predictions are in good agreement with the observations [53].

As we have shown in Eqn. 7 γ_1 and γ_2 are two underlying parameters of our inflationary model. Yet, the final predictions of n_s and r values of the theory appear to be independent on these parameters. In fact, the variable y depends on γ_2/γ_1 and $\exp(-\sqrt{4\pi G}(\phi-\phi_i)$ both. Hence our predictions are indeed dependent on γ_2/γ_1. It is a remarkable property of the inflationary model of this

work that the variable y can be expressed as a closed form analytical expression involving N alone asymptotically. The asymptotic expression is relatively simple since Lambert's W function admits a negative argument with a very small magnitude for the cosmologically relevant number of e-folds for our inflationary model.

VII. DISCUSSION

It is well known that simple model of inflation with exactly exponential expansion of scale factor can not end inflation. This problem of persistent inflation occurs even if we make a little modification to quasi-exponential form $a(t) \sim \exp(\beta t^n)$, where β is a dimension full constant and $0 < n < 1$ [54]. A pertinent question is how can one device a scale factor during inflation which leads to a successful end of the inflationary epoch?

In this work, we develop a new method for building inflationary model. Instead of first formulating a potential and thereafter finding expressions for the slow-roll parameters using the potential, we follow a more general framework. We make an ansatz for the scale factor motivated by the expectation of a near exponential expansion during initial epochs of inflation and thereafter gradual slow down with time leading to end of inflation at certain instant where the scale factor stops accelerating. In this context we note that in some models of inflations where one starts with a potential energy of the scalar field graceful exit from inflation does not occur. The model scale factor of the current work, has the desired feature of graceful exit from inflation while simultaneously being favoured by the recent cosmological observations (PLANCK satellite mission, ground bases ACT observations and KECK-BICEP-2018). *The scale factor used in the work can be used as a template model to render the end of inflation in varoious cases of inflationary potentials in cases of minimal or non-minimal coupling with the gravity sector.*

Assuming a minimally coupled scalar field with gravity we use Friedman equations and continuity equations and present a complete analytical framework for inflationary dynamics including the scalar field and the gravity sector. We recover an exact analytical expression for the scalar field potential driving the inflation. This is an important advantage of our method. Another interesting point to note is the shape of the potential energy function has a flat slope for certain range of values of ϕ, followed by a steepening decay and an eventual minimum, beyond which the potential energy increases again. The flat slope helps achieving enough number of e-folds for inflation so as to solve the common cosmological problems [1, 55, 56]. The steepening slope results in end of inflation. The subsequent minimum in the potential is useful for reheating to take place. Our work, therefore, connects the features of the scale factor to features of inflationary potential.

Since our method is exact we do not need to make any assumption about the slow roll (i.e. neglecting the kinetic energy of the inflaton field with respect to the potential energy) to solve for $\phi(t)$ and other variables. The ansatz for the scale factor directly leads to a slow roll regime during some initial interval of the inflation. In this sense our results are most general. We note that, for calculations which use explicit assumption of slow roll ϵ_H, η_H and $N(t)$ become (approximately) equal to ϵ_V, η_V and $N(\phi)$ respectively. However such assumptions do not hold towards the later epoch of inflation where slow roll is expected to be violated for the onset of the end of the inflation.

Our work demonstrates that it is possible to device an exact yet simple analytical model of the scale factor that predicts the form of the inflaton potential in a manner that is is very well motivated from the requirements of both inflation theory and reheating mechanisms.

There are several inflation models [57–60] in which the predictions of n_s and r can be given in terms of number of e-folds, N. It is interesting to note that our predictions for these parameters are also given as analytical expressions of N alone. Such analytical expressions are advantageous since they encapsulate the any dependency of the relevant variables on the individual parameter values in terms of a single physically motivated variable (e.g., N). Therefore, one can estimate the predictions of the inflationary theory of this work independent of any explicit values of the parameters γ_1 and γ_2. In this context it may also be noted that our formalism also allows to express potential energy V (Eqn. 16) as $V(N)$, Hubble parameter H (Eqn. 17) as $H(N)$ (both upto some but different normalization constants). Further all the variables discussed in this work can be represented by functions of time or the scalar field as desired. Such alternative representations of any given dynamical variable over multiple domain is an useful feature of our model for gaining insights by investigating corresponding analytical expressions.

ACKNOWLEDGEMENT

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