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EXPANDED MIXED VIRTUAL ELEMENT METHODS FOR THE SOBOLEV EQUATION WITH A CONVECTION TERM

ANKIT KUMAR*, SANGITA YADAV*, AND SARVESH KUMAR†

Abstract. This article develops and analyzes an expanded mixed virtual element method for the Sobolev equation with a convection term. The proposed formulation relaxes certain restrictions inherent in the classical mixed approach and provides a flexible framework for the underlying model problem. The well-posedness of the resulting semi-discrete problem is established. For temporal discretization, a fully discrete scheme is constructed using the backward Euler method, ensuring stability and robustness. Optimal a priori error estimates are derived for both the semi-discrete and fully discrete schemes through the introduction of a novel intermediate projection operator, along with its corresponding approximation properties. Lastly, a set of numerical experiments is presented to support the theoretical findings.

Key words. Expanded mixed virtual element method, Sobolev equation, Convection term, A priori error estimates.

AMS subject classifications. 65M60, 65M15, 65M12.

1. Introduction. The Sobolev-type equations involve mixed derivatives with respect to both space and time in the highest-order term and are often known as pseudo-parabolic problems. In this article, we consider a second-order Sobolev equation with a convection term. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded polygonal domain with a Lipschitz boundary $\partial\Omega$. Then, for a given source term f, we consider the following Sobolev equation

$$\begin{aligned} u_t(\boldsymbol{x},t) + \mathcal{L}(u) &= f(\boldsymbol{x},t) \text{ in } \Omega \times (0,T], \\ u(\boldsymbol{x},t) &= 0 \qquad \text{on } \partial\Omega \times (0,T], \\ u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}) \qquad\quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\mathcal{L}(u) = -\nabla\cdot(\mu(\boldsymbol{x})\nabla u_t(\boldsymbol{x},t) + \epsilon(\boldsymbol{x})\nabla u(\boldsymbol{x},t)) + \boldsymbol{\beta}(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x},t) + \gamma(\boldsymbol{x})u(\boldsymbol{x},t)$, $T > 0$ represents the final time, u_0 be the given sufficiently smooth initial data, $\mu(\boldsymbol{x}), \epsilon(\boldsymbol{x})$ and $\gamma(\boldsymbol{x})$ be the strictly positive variable coefficients. Here, $\mu(\boldsymbol{x}), \epsilon(\boldsymbol{x}) \in L^\infty(\Omega)$ such that $0 < \underline{\mu} \le \mu(\boldsymbol{x}) \le \overline{\mu}$ and $0 < \underline{\epsilon} \le \epsilon(\boldsymbol{x}) \le \overline{\epsilon}$ for all $\boldsymbol{x} \in \Omega$, corresponds to the damping and diffusion term, respectively, while $\gamma(\boldsymbol{x}) \in L^\infty(\Omega)$ represents the reaction coefficient such that $0 \le \underline{\gamma} \le \gamma(\boldsymbol{x}) \le \overline{\gamma}$ for all $\boldsymbol{x} \in \Omega$. The vector valued function $\boldsymbol{\beta}(\boldsymbol{x}) : \Omega \to \mathbb{R}^2$ is the convection coefficient such that $\boldsymbol{\beta}(\boldsymbol{x}) \in [L^\infty(\Omega)]^2$ with $\overline{\boldsymbol{\beta}} = \sup_{\boldsymbol{x}\in\Omega} \boldsymbol{\beta}(\boldsymbol{x})$. These assumptions are made to ensure the well-posedness of the problem (1.1), see [23].

The ability of the Sobolev equation to model complex phenomena that involve both diffusion and inertial effects enables researchers to utilize it in various engineering disciplines. Such as modeling problems of fluid flow through fissured materials, transport of moisture through soil [4], heat conduction processes in which both thermodynamic temperature and conductive temperature coexist [33]. Its applications can also be found in semiconductor theory, where the behavior of charge carriers changes with time as well [22]. Over the past few decades, considerable attention has been devoted to the numerical approximation of solutions to Sobolev-type equations. Early contributions primarily focused on finite difference schemes [16, 18], where stability and convergence properties of the approximations were rigorously established. Subsequently, Ewing [17] developed Galerkin-based approaches to handle Sobolev equations involving nonlinear terms. With further advancements, the finite element method (FEM) emerged as a powerful and flexible framework for approximating such problems [25]. Various FEM-based techniques have since been proposed to address nonlinear Sobolev equations [7, 27], which in turn motivated the development of several non-standard finite element approaches. For instance, Shi and Zhang [29] investigated a mixed FEM formulation employing nonconforming spaces. Later, Shi et al. extended this work by performing superconvergence analysis for mixed FEM applied to equations with nonlinear diffusion and damping, utilizing specially designed postprocessing interpolation operators. Additionally, Gao et al. [19] proposed a higher-order weak Galerkin FEM ensuring stability for a class of Sobolev equations. More recently, Xie et al. [36] explored hybrid

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high-order methods, achieving enhanced convergence rates in both the L^2-norm and energy norm through rigorous analysis. Furthermore, several studies have addressed the challenges posed by convection terms in Sobolev equations, leading to various numerical strategies [20, 37].

In 1992, a new formulation, known as the expanded mixed FEM, was introduced by Wheeler [35]. Later, Chen [9] extended this idea to general elliptic problems by introducing two auxiliary variables corresponding to the gradients and fluxes. This approach was further developed for quasi-linear elliptic problems in [10]. The expanded mixed FEM is particularly suitable for problems involving small diffusion or low-permeability media. It has also been analyzed for parabolic integro-differential equations with non-smooth initial data in [21]. Subsequently, this framework was extended to approximate solutions of nonlinear and nonlocal parabolic problems using lowest-order Raviart–Thomas elements [28]. Moreover, the method can achieve optimal orders of convergence for certain nonlinear problems where the standard mixed FEM fails to do so [8]. More recently, in 2019, Li et al. introduced a four-field formulation by incorporating three auxiliary variables and established optimal error estimates for the corresponding expanded mixed FEM [24].

A recent advancement in numerical discretization is the Virtual Element Method (VEM), introduced in [1, 11] as an extension of the classical finite element method that accommodates general polygonal meshes. One of the key strengths of VEM lies in its flexibility with respect to mesh geometry, along with its ability to bypass the explicit construction of complex local basis functions. Instead, the method relies on suitable projection operators onto polynomial spaces, allowing the computation of local bilinear forms directly from the available degrees of freedom. Over the last decade, VEM has attracted considerable interest and has been successfully applied to a wide range of problems, including elliptic and parabolic equations [15, 6, 34], elasticity models [12], the Stokes problem [2], and the Navier–Stokes equations [3]. It has also been employed for more complex models such as strongly damped wave equations [31] and von Kármán equations [30]. In the context of Sobolev-type equations, several VEM-based discretization approaches have been proposed [38, 39]. Furthermore, Liu et al. [26] analyzed models incorporating nonlinear diffusion and damping effects. In our recent work [23], we investigated a VEM formulation corresponding to problem (1.1), where optimal error estimates were established using a newly developed intermediate projection operator. The extension of VEM to mixed formulations was first presented in [5], where a Poisson-type problem was used to demonstrate the fundamental concepts. This framework was subsequently generalized to cover second-order elliptic partial differential equations in [13], along with a detailed error analysis.

This paper presents the analysis of an expanded mixed VEM for the Sobolev equation with a convection term involving variable coefficients. The classical mixed formulation requires the invertibility of the diffusion and damping coefficients, namely $\epsilon(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$, which is relaxed in the framework of the expanded mixed method. In this approach, an additional auxiliary variable is introduced, leading to an expanded mixed formulation of the model problem. The resulting system is formulated in the function spaces $H(\mathrm{div}, \Omega)$, $[L^2(\Omega)]^2$, and $L^2(\Omega)$. We first establish the well-posedness of the semi-discrete problem posed on suitably constructed discrete spaces. For the temporal discretization, the backward Euler time-stepping scheme is employed to ensure stability and robustness. Furthermore, optimal a priori error estimates are derived for both the semi-discrete and fully discrete schemes by introducing a novel intermediate projection operator. Approximation properties of this newly introduced projection operator are also derived. Finally, numerical experiments are presented to validate the theoretical findings. The remainder of this article is organized as follows. In Section 2, we introduce an expanded mixed formulation corresponding to the problem (1.1). Section 3 presents the virtual element discretization of the expanded mixed formulation, where we also formulate the corresponding semi-discrete problem and establish its well-posedness. Section 4 is divided into two parts: in the first part, we present some preliminary results and derive the approximation properties of the intermediate projection operator; in the second part, we prove a priori error estimates of optimal order using this projection operator. In Section 5, several numerical experiments are provided to validate the theoretical results. Section 6 concludes the paper with remarks and discusses possible future extensions of the proposed work.

2. Basic notations and the mixed formulation. For any bounded polygonal domain $\mathcal{O}$, we stick to the standard definitions of the functional spaces. Let $s \geq 0$ be an integer, then the notations $\|\cdot\|_{s,\mathcal{O}}$ and $|\cdot|_{s,\mathcal{O}}$ will represent the norm and semi-norm on the Sobolev space $H^s(\mathcal{O})$, respectively. As usual, $\|\cdot\|_{\mathcal{O}}$ will denote the norm in $L^2(\mathcal{O})$ space. The space $\mathbb{P}_s(\mathcal{O})$ refers to the space of all polynomials of degree less

than or equal to s. Additionally, the Bochner space $L^p(H^s(\mathcal{O}))$, defined as:

$$L^p(H^s(\mathcal{O})) = \left\{v : [0,T] \to H^s(\mathcal{O}) : \|v\|_{L^p(H^s(\mathcal{O}))} < \infty\right\},$$

equipped with the norm $\|v\|_{L^p(H^s(\mathcal{O}))} = \left(\int_0^T \|v(\cdot,t)\|_{s,\mathcal{O}}^p dt\right)^{\frac{1}{p}}$, $1 \le p < \infty$ and for $p = \infty$, $\|v\|_{L^\infty(H^s(\mathcal{O}))} = \operatorname*{ess\,sup}_{0\le t\le T} \|v(\cdot,t)\|_{s,\mathcal{O}}$. We will also use the following Sobolev spaces:

$$H(\mathrm{div},\mathcal{O}) = \left\{\boldsymbol{\tau} \in (L^2(\mathcal{O}))^2 : \mathrm{div}\ \boldsymbol{\tau} \in L^2(\mathcal{O})\right\},$$
$$H(\mathrm{rot},\mathcal{O}) = \left\{\boldsymbol{\tau} \in (L^2(\mathcal{O}))^2 : \mathrm{rot}\ \boldsymbol{\tau} \in L^2(\mathcal{O})\right\},$$

where for any $\boldsymbol{\tau} = (\tau_1, \tau_2)$, we define $\mathrm{div}\ \boldsymbol{\tau} = \frac{\partial \tau_1}{\partial x} + \frac{\partial \tau_2}{\partial y}$ and $\mathrm{rot}\ \boldsymbol{\tau} = \frac{\partial \tau_2}{\partial x} - \frac{\partial \tau_1}{\partial y}$. The subscript $\mathcal{O}$ will be omitted from the notation of norms and semi-norms when $\mathcal{O} = \Omega$.

Now, let us introduce the auxiliary variables as $\boldsymbol{\lambda}(\boldsymbol{x},t) = \nabla u(\boldsymbol{x},t)$ and $\boldsymbol{p}(\boldsymbol{x},t) = \mu(\boldsymbol{x})\nabla u_t(\boldsymbol{x},t) + \epsilon(\boldsymbol{x})\nabla u(\boldsymbol{x},t)$, then (1.1) reduces to the following formulation: For all $t \in (0,T]$,

$$\boldsymbol{\lambda}(\boldsymbol{x},t) - \nabla u(\boldsymbol{x},t) = 0, \tag{2.1a}$$
$$\boldsymbol{p}(\boldsymbol{x},t) - \mu(\boldsymbol{x})\boldsymbol{\lambda}_t(\boldsymbol{x},t) - \epsilon(\boldsymbol{x})\boldsymbol{\lambda}(\boldsymbol{x},t) = 0, \tag{2.1b}$$
$$u_t(\boldsymbol{x},t) - \nabla\cdot\boldsymbol{p}(\boldsymbol{x},t) + \boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{\lambda}(\boldsymbol{x},t) + \gamma(\boldsymbol{x})u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \tag{2.1c}$$

with the initial conditions $u(\boldsymbol{x},0) = u_0(\boldsymbol{x})$ and $\boldsymbol{\lambda}(\boldsymbol{x},0) = \boldsymbol{\lambda}_0(\boldsymbol{x}) = \nabla u_0(\boldsymbol{x})$.

Denote the Sobolev spaces as $Q = H(\mathrm{div};\Omega)$, $V = [L^2(\Omega)]^2$ and $W = L^2(\Omega)$. Then the expanded mixed weak formulation corresponding to the problem (2.1) seeks the pair $(\boldsymbol{p},\boldsymbol{\lambda},u) \in Q \times V \times W$ for all $t \in (0,T]$ such that

$$a(\boldsymbol{\lambda},\boldsymbol{\zeta}) + (u, \nabla\cdot\boldsymbol{\zeta}) = 0 \qquad \forall\ \boldsymbol{\zeta} \in Q, \tag{2.2a}$$
$$a(\boldsymbol{\tau},\boldsymbol{p}) - m_1(\boldsymbol{\lambda}_t,\boldsymbol{\tau}) - m_2(\boldsymbol{\lambda},\boldsymbol{\tau}) = 0 \qquad \forall\ \boldsymbol{\tau} \in V, \tag{2.2b}$$
$$(u_t, w) - (\nabla\cdot\boldsymbol{p}, w) + (\boldsymbol{\beta}\cdot\boldsymbol{\lambda}, w) + (\gamma u, w) = (f, w) \quad \forall\ w \in W, \tag{2.2c}$$

where the bilinear forms $a(\bullet,\bullet) : V \times V \to \mathbb{R}$, $m_1(\bullet,\bullet) : V \times V \to \mathbb{R}$ and $m_2(\bullet,\bullet) : V \times V \to \mathbb{R}$ are defined as $a(\boldsymbol{\lambda},\boldsymbol{\zeta}) = (\boldsymbol{\lambda},\boldsymbol{\zeta})$, $m_1(\boldsymbol{\lambda},\boldsymbol{\tau}) = (\mu\boldsymbol{\lambda},\boldsymbol{\tau})$ and $m_2(\boldsymbol{\lambda},\boldsymbol{\tau}) = (\epsilon\boldsymbol{\lambda},\boldsymbol{\tau})$. Owing to the bounds of the damping and diffusion coefficients, it can be readily observe that

$$m_1(\boldsymbol{\lambda},\boldsymbol{\tau}) \le \overline{\mu}\|\boldsymbol{\lambda}\|\|\boldsymbol{\tau}\|, \quad m_2(\boldsymbol{\lambda},\boldsymbol{\tau}) \le \overline{\epsilon}\|\boldsymbol{\lambda}\|\|\boldsymbol{\tau}\| \quad \text{for } \boldsymbol{\lambda},\boldsymbol{\tau} \in V.$$

And for all $\boldsymbol{\tau} \in V$

$$m_1(\boldsymbol{\tau},\boldsymbol{\tau}) \ge \underline{\mu}\|\boldsymbol{\tau}\|^2, \quad m_2(\boldsymbol{\tau},\boldsymbol{\tau}) \ge \underline{\epsilon}\|\boldsymbol{\tau}\|^2,$$

Also, there exists a $\alpha > 0$ such that

$$\alpha\|w\| \le \sup_{\boldsymbol{p}\in Q\setminus\{0\}} \frac{(\nabla\cdot\boldsymbol{p}, w)}{\|\boldsymbol{p}\|_Q} \quad \text{for all } w \in W.$$

Since the mixed weak formulation (2.2) is obtained from the model problem (1.1), any solution $(\boldsymbol{p},\boldsymbol{\lambda},u)$ of (2.2) yields a function u that satisfies the problem (1.1). Conversely, if u is the unique solution of (1.1), then the auxiliary variables $\boldsymbol{p}$ and $\boldsymbol{\lambda}$ can be uniquely defined in terms of u. Therefore, the mixed formulation (2.2) is equivalent to the model problem (1.1). Consequently, the existence and uniqueness of the solution to (1.1) imply the existence and uniqueness of the solution to (2.2).

3. Virtual element approximation. We now proceed to derive the expanded mixed VEM formulation corresponding to the problem (2.2). Let $\mathcal{T}_h$ be a decomposition of the domain Ω into simply connected polygonal elements K with boundary ∂K. Let $\mathcal{E}_h$ denote the set of all edges of the mesh $\mathcal{T}_h$. The diameter of an element K is denoted by h_K, and the mesh size h is defined as $h = \max_{K\in\mathcal{T}_h} h_K$. Moreover, $\boldsymbol{n}$ denotes the unit outward normal vector to each edge e of an element K. Additionally, we assume that for every element K there exists a positive constant ∂K such that

- K is star-shaped with respect to the ball of radius $\delta^K h_K$,
- $h_e \geq \delta^K h_K$, where h_e denotes the length of edge $e \in \partial K$.

For an interger $r \geq 1$, for every element K we denote

$$\mathcal{G}_r(K) = \nabla \mathbb{P}_{r+1}(K),$$

and the L^2-orthogonal complement

$$\mathcal{G}_r^{\perp}(K) = (\mathcal{G}_r(K))^{\perp} \subset [\mathbb{P}_r(K)]^2.$$

Hence, the vector polynomial space admits, $[\mathbb{P}_r(K)]^2 = \mathcal{G}_r(K) \oplus \mathcal{G}_r^{\perp}(K)$. We now introduce the local virtual element space used for the approximation of the flux variable. For each element $K \in \mathcal{T}_h$, the local virtual space is defined as follows [13]:

$$Q_h^r(K) = \{\boldsymbol{\zeta} \in H(\mathrm{div}; K) \cap H(\mathrm{rot}; K) | \boldsymbol{\zeta} \cdot \boldsymbol{n}|_e \in \mathbb{P}_r(e) \ \forall \ e \in \partial K, \nabla \cdot \boldsymbol{\zeta} \in \mathbb{P}_r(K), \text{and rot } \boldsymbol{\zeta} \in \mathbb{P}_{r-1}(K)\},$$

where $\mathbb{P}_{-1}(K) = \{0\}$. This space consists of vector fields whose normal component on each edge is a polynomial of degree at most k, whose divergence is a polynomial of degree at most k inside the element, and whose rotation is a polynomial of degree at most $r-1$. The inclusion $Q_h^r(K) \subset H(\mathrm{div}; K)$ ensures the continuity of the normal component across element interfaces, which is a key requirement for mixed formulations. Next, in order to uniquely characterize the functions in $Q_h^r(K)$, we define an appropriate set of degrees of freedom following [5]. These degrees of freedom are chosen so that they are computable and guarantee the unisolvability of the space. For any $\boldsymbol{\zeta} \in Q_h^r(K)$, the degrees of freedom are given by

- $\displaystyle\int_e \boldsymbol{\zeta} \cdot \boldsymbol{n} q_r \, \mathrm{d}e$ for all edges e, for all $q_r \in \mathbb{P}_r(e)$,
- $\displaystyle\int_K \boldsymbol{\zeta} \cdot \boldsymbol{g}_{r-1} \, \mathrm{d}\boldsymbol{x}$ for all element K, for all $\boldsymbol{g}_{r-1} \in \mathcal{G}_{r-1}(K)$,
- $\displaystyle\int_K \boldsymbol{\zeta} \cdot \boldsymbol{g}_r^{\perp} \, \mathrm{d}\boldsymbol{x}$ for all element K, for all $\boldsymbol{g}_r^{\perp} \in \mathcal{G}_r^{\perp}(K)$.

After defining the local virtual spaces, we construct the corresponding global discrete spaces by assembling the local spaces over all mesh elements. The global space for the flux variable is defined as

$$Q_h^r = \{\boldsymbol{\zeta} \in H(\mathrm{div}; \Omega) : \boldsymbol{\zeta}|_K \in Q_h^r(K) \ \forall \ K \in \mathcal{T}_h\}.$$

Next, we define the discrete space for the auxiliary vector variable as the space of piecewise polynomial vector fields,

$$V_h^r = \left\{\boldsymbol{\tau} \in [L^2(\Omega)]^2 : \boldsymbol{\tau}|_K \in Q_h^r(K) \ \forall \ K \in \mathcal{T}_h\right\}.$$

Finally, the discrete space for the scalar variable is given by

$$W_h^r = \left\{w \in L^2(\Omega) : w|_K \in \mathbb{P}_r(K) \ \forall \ K \in \mathcal{T}_h\right\}.$$

Throughout the analysis, we will be making use of the L^2-projection operators $\boldsymbol{\Pi}_r^0 : V \to [\mathbb{P}_r(K)]^2$ and $\Pi_r^0 : W \to \mathbb{P}_r(K)$, which are defined as: for any $\boldsymbol{\tau} \in V$ and $w \in W$

$$\int_K (\boldsymbol{\tau} - \boldsymbol{\Pi}_r^0 \boldsymbol{\tau}) \boldsymbol{p_r} d\boldsymbol{x} = 0 \quad \forall \ \boldsymbol{p}_r \in [\mathbb{P}_r(K)]^2, \quad \forall \ K \in \mathcal{T}_h,$$
$$\int_K (w - \Pi_r^0 w) p_r d\boldsymbol{x} = 0 \quad \forall \ p_r \in \mathbb{P}_r(K), \quad \forall \ K \in \mathcal{T}_h.$$

These projection operators are critical to construct the computable bilinear forms. Next, we define the Fortin operator [13] $\boldsymbol{\mathcal{I}}_h^F : [H^1(\Omega)]^2 \to Q_h^r$ which is constructed by exploiting the degrees of freedom of the local virtual element space Q_h^r. For any $\boldsymbol{\zeta} \in [H^1(\Omega)]^2$, the operator $\boldsymbol{\mathcal{I}}_h^F \boldsymbol{\zeta} \in Q_h^r$ is defined by imposing that the following conditions hold on each element $K \in \mathcal{T}_h$:

- $\displaystyle\int_e \left(\boldsymbol{\zeta} - \boldsymbol{\mathcal{I}}_h^F \boldsymbol{\zeta}\right) \cdot \boldsymbol{n} q_r \, \mathrm{d}e$ for all edges e, for all $q_r \in \mathbb{P}_r(e)$,

- $\displaystyle\int_K \left(\boldsymbol{\zeta} - \boldsymbol{\mathcal{I}}_h^F \boldsymbol{\zeta}\right) \cdot \boldsymbol{g}_{r-1}\, \mathrm{d}\boldsymbol{x}$ for all element K, for all $\boldsymbol{g}_{r-1} \in \mathcal{G}_{r-1}(K)$,
- $\displaystyle\int_K \left(\boldsymbol{\zeta} - \boldsymbol{\mathcal{I}}_h^F \boldsymbol{\zeta}\right) \cdot \boldsymbol{g}_r^{\perp}\, \mathrm{d}\boldsymbol{x}$ for all element K, for all $\boldsymbol{g}_r^{\perp} \in \mathcal{G}_r^{\perp}(K)$.

The Fortin operator also satisfies the following property:

$$\nabla \cdot \boldsymbol{\mathcal{I}}_h^F \boldsymbol{\tau} = \Pi_r^0 \nabla \cdot \boldsymbol{\tau}. \tag{3.1}$$

Moreover, the following estimates hold true:

$$\|\boldsymbol{\tau} - \boldsymbol{\mathcal{I}}_h^F \boldsymbol{\tau}\| \leq C h^{r+1} |\boldsymbol{\tau}|_{r+1}, \quad \|\nabla \cdot (\boldsymbol{\tau} - \boldsymbol{\mathcal{I}}_h^F \boldsymbol{\tau})\| \leq C h^{r+1} |\nabla \cdot \boldsymbol{\tau}|_{r+1}. \tag{3.2}$$

To state the semi-discrete formulation, we first introduce discrete bilinear forms that approximate the corresponding continuous bilinear forms. For every $\boldsymbol{\lambda}_h, \boldsymbol{\tau}_h \in V_h^r|_K$, the local discrete bilinear forms on an element $K \in \mathcal{T}_h$ are defined by:

$$\begin{aligned} m_{1,h}^K(\boldsymbol{\lambda}_h, \boldsymbol{\tau}_h) &= (\mu \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}_h, \boldsymbol{\Pi}_r^0 \boldsymbol{\tau}_h)_K + \mu_K S^K((I - \boldsymbol{\Pi}_r^0)\boldsymbol{\lambda}_h, (I - \boldsymbol{\Pi}_r^0)\boldsymbol{\tau}_h), \\ m_{2,h}^K(\boldsymbol{\lambda}_h, \boldsymbol{\tau}_h) &= (\epsilon \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}_h, \boldsymbol{\Pi}_r^0 \boldsymbol{\tau}_h)_K + \epsilon_K S^K((I - \boldsymbol{\Pi}_r^0)\boldsymbol{\lambda}_h, (I - \boldsymbol{\Pi}_r^0)\boldsymbol{\tau}_h). \end{aligned}$$

Here, μ_K and ϵ_K are some positive constant approximations of μ and ϵ, respectively, on each element K of $\mathcal{T}_h$. And, $S^K(\bullet, \bullet)$ is the stabilization term such that for all $\boldsymbol{\tau}_h \in V_h^r$, there exist $\underline{\alpha}$ and $\overline{\alpha}$ independent of h_K satisfying

$$\underline{\alpha} \|\boldsymbol{\tau}_h\|_K^2 \leq S^K(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \leq \overline{\alpha} \|\boldsymbol{\tau}_h\|_K^2. \tag{3.3}$$

The stabilization term $S^K(\bullet, \bullet)$ is defined as:

$$S^K(\boldsymbol{\lambda}_h, \boldsymbol{\tau}_h) = |K| \sum_{i=1}^{N_V^K} \mathcal{X}_i(\boldsymbol{\lambda}_h) \mathcal{X}_i(\boldsymbol{\tau}_h), \quad \text{for all } \boldsymbol{\lambda}_h, \boldsymbol{\tau}_h \in V_h^r(K),$$

where N_V^K stands for the dimension of the local virtual space $V_h^r(K)$, $\mathcal{X}_i$ is the i-th degree of freedom of the $V_h^r(K)$. Note that the bound (3.3) will help to obtain the continuity and coercivity bounds of $m_{1,h}(\bullet, \bullet)$ and $m_{2,h}(\bullet, \bullet)$ stated in the subsequent lemmas, for proof we refer to [13].

LEMMA 3.1. *For the bilinear form $m_{1,h}(\bullet, \bullet)$ and $m_{2,h}(\bullet, \bullet)$, we have the following continuity bounds:*

$$m_{1,h}(\boldsymbol{\lambda}_h, \boldsymbol{\tau}_h) \leq \overline{M}_1 \|\boldsymbol{\lambda}_h\| \|\boldsymbol{\tau}_h\| \quad \textit{for all } \boldsymbol{q}_h, \boldsymbol{\tau}_h \in V_h^r, \quad m_{2,h}(\boldsymbol{\lambda}_h, \boldsymbol{\tau}_h) \leq \overline{M}_2 \|\boldsymbol{\lambda}_h\| \|\boldsymbol{\tau}_h\| \quad \textit{for all } \boldsymbol{\lambda}_h, \boldsymbol{\tau}_h \in V_h^r,$$

with $\overline{M}_1$ and $\overline{M}_2$ depending on μ and ϵ, respectively.

LEMMA 3.2. *For the bilinear form $m_{1,h}(\bullet, \bullet)$ and $m_{2,h}(\bullet, \bullet)$,we have the following coercivity bounds:*

$$m_{1,h}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \underline{M}_1 \|\boldsymbol{\tau}_h\|^2, \qquad m_{2,h}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \underline{M}_2 \|\boldsymbol{\tau}_h\|^2 \quad \textit{for all } \boldsymbol{\tau}_h \in V_h^r,$$

with $\underline{M}_1$ and $\underline{M}_2$ depending on μ and ϵ, respectively.

We recall that the pair of discrete spaces $Q_h^r \times W_h^r$ satisfies the discrete inf–sup condition. In particular, there exists a positive constant $\tilde{\alpha}$, independent of the mesh size h, such that (see [5]):

$$\tilde{\alpha} \|w_h\| \leq \sup_{\boldsymbol{p}_h \in Q_h^r \setminus \{0\}} \frac{(\nabla \cdot \boldsymbol{p}_h, w_h)}{\|\boldsymbol{p}_h\|_{Q_h^r}} \quad \text{for all } w_h \in W_h^r. \tag{3.4}$$

3.1. Semi-discrete problem. We are now in a position to formulate the semi-discrete virtual element approximation of the mixed problem (2.2). The spatial discretization is performed using the previously defined discrete spaces Q_h^r, V_h^r, and W_h^r, while the time variable remains continuous. The discrete formulation is obtained by replacing the continuous spaces and bilinear forms in (2.2) with their corresponding virtual element counterparts.

The semi-discrete problem reads as follows: for each $t \in (0,T]$, find $(\boldsymbol{p}_h, \boldsymbol{\lambda}_h, u_h) \in Q_h^r \times V_h^r \times W_h^r$ such that

$$a_h(\boldsymbol{\lambda}_h, \boldsymbol{\zeta}_h) + (u_h, \nabla \cdot \boldsymbol{\zeta}_h) = 0 \qquad \forall\ \boldsymbol{\zeta}_h \in Q_h^r, \tag{3.5a}$$

$$a_h(\boldsymbol{\tau}_h, \boldsymbol{p}_h) - m_{1,h}(\boldsymbol{\lambda}_{h,t}, \boldsymbol{\tau}_h) - m_{2,h}(\boldsymbol{\lambda}_h, \boldsymbol{\tau}_h) = 0 \qquad \forall\ \boldsymbol{\tau}_h \in V_h^r, \tag{3.5b}$$

$$(u_{h,t}, w_h) - (\nabla \cdot \boldsymbol{p}_h, w_h) + (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}_h, w_h) + (\gamma u_h, w_h) = (f, w_h) \quad \forall\ w_h \in W_h^r. \tag{3.5c}$$

The system is supplemented with the initial conditions $u_h(\boldsymbol{x}, 0) = \Pi_r^0 u_0$ and $\boldsymbol{\lambda}_h(\boldsymbol{x}, 0) = \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}_0$. Here, $a_h(\bullet, \bullet) : V_h^r \times V_h^r \to \mathbb{R}$ is defined as $a_h(\boldsymbol{\lambda}_h, \boldsymbol{\zeta}_h) = \sum_{K \in \mathcal{T}_h} a_h^K(\boldsymbol{\lambda}_h, \boldsymbol{\zeta}_h)$ with $a_h^K(\boldsymbol{\lambda}_h, \boldsymbol{\zeta}_h) = (\boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}_h, \boldsymbol{\Pi}_r^0 \boldsymbol{\zeta}_h)_K + S^K((I - \boldsymbol{\Pi}_r^0)\boldsymbol{\lambda}_h, (I - \boldsymbol{\Pi}_r^0)\boldsymbol{\tau}_h)$. It is clear from the definition of the L^2-projection operator that $\boldsymbol{\Pi}_r^0 \boldsymbol{q}_r = \boldsymbol{q}_r$ for all $\boldsymbol{q}_r \in \mathbb{P}_r(K)$. Then for any $\boldsymbol{\tau}_h \in V_h^r(K)$, the following polynomial consistency property is satisfied:

$$a_h^K(\boldsymbol{q}_r, \boldsymbol{\tau}_h) = a^K(\boldsymbol{q}_r, \boldsymbol{\tau}_h),\ \forall\ \boldsymbol{q}_r \in \mathbb{P}_r(K). \tag{3.6}$$

Using the continuity and coercivity bounds of the biliner forms $m_{1,h}(\bullet, \bullet)$ and $m_{2,h}(\bullet, \bullet)$ and the discrete inf-sup condition (3.4), we will establish the well-posedness of the semi-discrete problem. In this order, we will derive the stability bounds of the semi-discrete solution $(\boldsymbol{p}_h, \boldsymbol{\lambda}_h, u_h)$ in the subsequent theorem.

THEOREM 3.3 (Stability). *Let $(\boldsymbol{p}_h, \boldsymbol{\lambda}_h, u_h) \in Q_h^r \times V_h^r \times W_h^r$ be the solution to the problem (3.5). Then it holds that*

$$\|u_h(t)\|^2 + \|\boldsymbol{\lambda}_h(t)\|^2 + \|\boldsymbol{p}_h(t)\|^2 \leq C \left(\|u_h(0)\|^2 + \|\boldsymbol{\lambda}_h(0)\|^2 + \|f\|^2_{H^1(L^2(\Omega))} \right).$$

Proof. Take $\boldsymbol{\zeta}_h = \boldsymbol{p}_h$ in (3.5a), $\boldsymbol{\tau}_h = \boldsymbol{\lambda}_h$ in (3.5b) and $w_h = u_h$ in (3.5c), then combine them to get

$$(u_{h,t}, u_h) + m_{1,h}(\boldsymbol{\lambda}_{h,t}, \boldsymbol{\lambda}_h) + m_{2,h}(\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_h) + (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}_h, u_h) + (\gamma u_h, u_h) = (f, u_h).$$

From the bounds of the bilinear form $m_{1,h}(\bullet, \bullet)$ and $m_{2,h}(\bullet, \bullet)$ from Lemma 3.1 and Lemma 3.2, the above equation yields that

$$\frac{1}{2}\frac{d}{dt}\left(\|u_h\|^2 + \underline{M}_1 \|\boldsymbol{\lambda}_h\|^2 \right) + \underline{M}_2 \|\boldsymbol{\lambda}_h\|^2 + \underline{\gamma} \|u_h\|^2 \leq \|f\| \|u_h\| + \overline{\boldsymbol{\beta}} \|\boldsymbol{\lambda}_h\| \|u_h\|.$$

Using Young's inequality and on multiplying by 2 on both sides, we get

$$\frac{d}{dt}\left(\|u_h\|^2 + 2\underline{M}_1 \|\boldsymbol{\lambda}_h\|^2 \right) \leq \|f\|^2 + 2\|u_h\|^2 + \overline{\boldsymbol{\beta}}^2 \|\boldsymbol{\lambda}_h\|^2.$$

Integrating with respect to time from 0 to t on both sides, we deduce that

$$\|u_h(t)\|^2 + \|\boldsymbol{\lambda}_h(t)\|^2 \leq C \left(\|u_h(0)\|^2 + \|\boldsymbol{\lambda}_h(0)\|^2 + \int_0^t \left(\|f(s)\|^2 + \|u_h(s)\|^2 + \|\boldsymbol{\lambda}_h(s)\|^2 \right) \mathrm{d}s \right).$$

Using Grönwall's lemma, we achieve

$$\|u_h(t)\|^2 + \|\boldsymbol{\lambda}_h(t)\|^2 \leq C \left(\|u_h(0)\|^2 + \|\boldsymbol{\lambda}_h(0)\|^2 + \int_0^t \|f(s)\|^2 \, \mathrm{d}s \right). \tag{3.7}$$

On differentiating (3.5) with respect to time and following the same procedure to obtain (3.7), we will get

$$\|u_{h,t}(t)\|^2 + \|\boldsymbol{\lambda}_{h,t}(t)\|^2 \leq C \left(\|u_h(0)\|^2 + \|\boldsymbol{\lambda}_h(0)\|^2 + \|f\|^2_{H^1(L^2(\Omega))} \right). \tag{3.8}$$

Now, on taking $\boldsymbol{\tau}_h = \boldsymbol{p}_h$ in (3.5b), we infer that

$$\|\boldsymbol{p}_h\|^2 \leq \overline{M}_1 \|\boldsymbol{\lambda}_{h,t}\| \|\boldsymbol{p}_h\| + \overline{M}_2 \|\boldsymbol{\lambda}_h\| \|\boldsymbol{p}_h\|,$$

using Young's inequality

$$\|\boldsymbol{p}_h\|^2 \leq \left(\overline{M}_1^2 \|\boldsymbol{\lambda}_{h,t}\|^2 + \overline{M}_2^2 \|\boldsymbol{\lambda}_h\|^2 \right) + \frac{1}{2} \|\boldsymbol{p}_h\|^2.$$

From (3.7) and (3.8), we obtain

$$\|\boldsymbol{p}_h\|^2 \leq C\left(\|u_h(0)\|^2 + \|\boldsymbol{\lambda}_h(0)\|^2 + \|f\|^2_{H^1(L^2(\Omega))}\right). \tag{3.9}$$

Finally, we conclude the proof by combining the Equations (3.7) and (3.9). ∎

Existence and uniqueness: Let $\{\Phi_i\}_{i=1}^{N_Q}$, $\{\Psi_j\}_{j=1}^{N_V}$ and $\{\phi\}_{k=1}^{N_W}$ be the basis for the finite-dimensional spaces Q_h^r, V_h^r and W_h^r, respectively, where $N_Q = dim\,(Q_h^r), N_V = dim\,(V_h^r)$ and $N_W = dim\,(W_h^r)$. Then the discrete unknowns $\boldsymbol{p}_h, \boldsymbol{\lambda}_h, u_h$ can be expanded in terms of these bases as

$$\boldsymbol{p}_h = \sum_{i=1}^{N_Q} \tilde{\boldsymbol{p}}_i(t)\Phi_i, \quad \boldsymbol{\lambda}_h = \sum_{j=1}^{N_V} \tilde{\boldsymbol{\lambda}}_j(t)\Psi_j, \quad u_h = \sum_{k=1}^{N_W} \tilde{u}_k(t)\phi_k,$$

where $\{\tilde{\boldsymbol{p}}_i(t)\}_{i=1}^{N_Q}$, $\left\{\tilde{\boldsymbol{\lambda}}_j(t)\right\}_{j=1}^{N_V}$ and $\{\tilde{u}_k(t)\}_{k=1}^{N_W}$ are the corresponding time-dependent coefficients. Substituting these expansions into the semi-discrete formulation (3.5) and testing with the corresponding basis functions yields the following system of ordinary differential equations for $t \in (0, T]$:

$$A\tilde{\boldsymbol{\lambda}}(t) + S^T\tilde{u}(t) = \boldsymbol{0}, \tag{3.10a}$$

$$A^T\tilde{\boldsymbol{p}}(t) - M_1\frac{d\tilde{\boldsymbol{\lambda}}(t)}{dt} - M_2\tilde{\boldsymbol{\lambda}}(t) = \boldsymbol{0}, \tag{3.10b}$$

$$H\frac{d\tilde{u}(t)}{dt} - S\tilde{\boldsymbol{p}}(t) + B\tilde{\boldsymbol{\lambda}}(t) + C\tilde{u}(t) = L, \tag{3.10c}$$

with $\tilde{u}(0) = \tilde{u}_0$ and $\tilde{\boldsymbol{\lambda}}(0) = \tilde{\boldsymbol{\lambda}}_0$, where $\tilde{u}_0$ and $\tilde{\boldsymbol{\lambda}}_0$ denote the given initial data. The matrices are defined as follows: for $i = 1, 2, \ldots, N_Q$ and $j = 1, 2, \ldots, N_V$ such that

$$A_{ij} = a_h(\Psi_j, \Phi_i), \quad M_{1ij} = m_{1,h}(\Psi_j, \Phi_i), \quad M_{2ij} = m_{2,h}(\Psi_j, \Phi_i),$$

and

$$\begin{array}{lll} H_{ij} = (\phi_j, \phi_i), & C_{ij} = (\gamma\phi_j, \phi_i), & i, j = 1, 2, \ldots, N_W \\ S_{ij} = (\nabla\cdot\Phi_j, \phi_i), & i = 1, 2, \ldots, N_W, & j = 1, 2, \ldots, N_Q \\ B_{ij} = (\boldsymbol{\beta}\cdot\boldsymbol{\Pi}_r^0\Phi_j, \phi_i), & i = 1, 2, \ldots, N_W, & j = 1, 2, \ldots, N_V. \end{array}$$

Finally, the load vector L associated with the source term is given by $L = (L_1, L_2, \ldots, L_{N_W})^T$ where $L_j(t) = (f, \phi_j),\ j = 1, 2, \ldots, N_W$. Note that from (3.10a) and (3.10b), it hold that

$$\tilde{\boldsymbol{\lambda}}(t) = -(A^TA)^{-1}S^T\tilde{u}(t),$$

$$\tilde{\boldsymbol{p}}(t) = (AA^T)^{-1}M_1\frac{d\tilde{\boldsymbol{\lambda}}(t)}{dt} + (AA^T)^{-1}M_2\tilde{\boldsymbol{\lambda}}(t).$$

Replacing $\tilde{\boldsymbol{p}}(t)$ and $\tilde{\boldsymbol{\lambda}}(t)$ in (3.10c) leads to the following differential equation involving only $\tilde{u}(t)$:

$$\begin{aligned}\left(H + S(AA^T)^{-1}M_1(A^TA)^{-1}S^T\right)\frac{d\tilde{u}(t)}{dt} + S(AA^T)^{-1}M_2(A^TA)^{-1}S^T\tilde{u}(t) \\ -B(A^TA)^{-1}S^T\tilde{u}(t) + C\tilde{u}(t) = L.\end{aligned} \tag{3.11}$$

Since the matrices H and M_1 are symmetric positive definite and the discrete spaces Q_h^r and W_h^r satisfy the discrete inf–sup condition (3.4), then there exists a unique solution to the system (3.11) with the initial condition $\tilde{u}(0) = \tilde{u}_0$ from Picard's theorem. Once $\tilde{u}(t)$ is determined, the vectors $\tilde{\boldsymbol{\lambda}}(t)$ and $\tilde{\boldsymbol{p}}(t)$ can be uniquely recovered from (3.10a) and (3.10b). Consequently, the semi-discrete problem (3.5) admits a unique solution and is therefore well-posed.

4. Error estimates. In this section, we derive the error estimates for the solution of the semi-discrete problem (3.5). The section is divided into two parts. In the first part, we present several preliminary results that are required for the analysis. These results are then used in the second part to establish the optimal a priori error estimates for the semi-discrete scheme.

4.1. Preliminary Results. In this section, we present some auxiliary results that will be used in the derivation of the error estimates. In particular, we recall the consistency error estimates associated with the discrete bilinear forms $m_{1,h}(\bullet,\bullet)$ and $m_{2,h}(\bullet,\bullet)$. The following lemma provides the required bounds; for the proof we refer to [Lemma 1, [32]].

LEMMA 4.1. *For every $K \in \mathcal{T}_h$, let μ and ϵ be smooth enough functions and assume that $\boldsymbol{\tau}_h \in V_h^r$. Then it holds that*

$$m_{1,h}^K(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\tau}_h) - m_1^K(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\tau}_h) \leq Ch_K^{r+1}|\boldsymbol{\lambda}|_{r+1,K}\|\boldsymbol{\tau}_h\|_K,$$
$$m_{2,h}^K(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\tau}_h) - m_2^K(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\tau}_h) \leq Ch_K^{r+1}|\boldsymbol{\lambda}|_{r+1,K}\|\boldsymbol{\tau}_h\|_K.$$

Furthermore, we introduce a projection operator $\mathcal{P}^h : [L^2(\Omega)]^2 \to V_h^r$ defined through the bilinear form $a_h(\bullet,\bullet)$. For $\boldsymbol{\lambda} \in [L^2(\Omega)]^2$, the projection $\mathcal{P}^h\boldsymbol{\lambda} \in V_h^r$ is defined by the relation

$$a_h(\mathcal{P}^h\boldsymbol{\lambda},\boldsymbol{\tau}_h) = a(\boldsymbol{\lambda},\boldsymbol{\tau}_h) \qquad \forall\ \boldsymbol{\tau}_h \in V_h^r. \tag{4.1}$$

The following lemma provides the approximation property of this projection operator.

LEMMA 4.2. *Let $\mathcal{P}^h\boldsymbol{\lambda}$ be the projection defined by (4.1). Then the following estimate holds:*

$$\|\mathcal{P}^h\boldsymbol{\lambda} - \boldsymbol{\lambda}\| \leq Ch^{r+1}|\boldsymbol{\lambda}|_{r+1}.$$

Proof. Let $\boldsymbol{\chi} = \mathcal{P}^h\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0\boldsymbol{\lambda}$. Then we first observe that

$$\begin{aligned} a_h(\boldsymbol{\chi},\boldsymbol{\chi}) &= a_h(\mathcal{P}^h\boldsymbol{\lambda},\boldsymbol{\chi}) - a_h(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\chi}) \\ &= a(\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\chi}) + a(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\chi}) - a_h(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda},\boldsymbol{\chi}). \end{aligned}$$

By exploiting the polynomial consistency (3.6) together with the continuity bounds of the bilinear forms $a(\bullet,\bullet)$, we arrive at

$$a_h(\boldsymbol{\chi},\boldsymbol{\chi}) \leq C\|\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0\boldsymbol{\lambda}\|\|\boldsymbol{\chi}\|.$$

Now, using the coercivity of the discrete bilinear form $a_h(\bullet,\bullet)$, we deduce

$$\|\boldsymbol{\chi}\| \leq C\|\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0\boldsymbol{\lambda}\|.$$

Finally, from the approximation property of the L^2-projection operator $\boldsymbol{\Pi}_r^0$, we have

$$\|\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0\boldsymbol{\lambda}\| \leq Ch^{r+1}|\boldsymbol{\lambda}|_{r+1}.$$

Hence,

$$\|\boldsymbol{\chi}\| \leq Ch^{r+1}|\boldsymbol{\lambda}|_{r+1}.$$

The desired estimate now follows from the triangle inequality. ∎

Let us now introduce the mixed intermediate projection operator as the map $(\mathcal{R}^h\boldsymbol{p},\mathcal{R}^h\boldsymbol{\lambda},\mathcal{R}^h u) : Q \times V \times W \to Q_h^r \times V_h^r \times W_h^r$ which will play an important role in the derivation of the error estimates. The operator is defined through the following system:

$$a_h(\mathcal{R}^h\boldsymbol{\lambda},\boldsymbol{\zeta}_h) + (\mathcal{R}^h u,\nabla\cdot\boldsymbol{\zeta}_h) = a(\boldsymbol{\lambda},\boldsymbol{\zeta}_h) + (u,\nabla\cdot\boldsymbol{\zeta}_h) \qquad \forall\ \boldsymbol{\zeta}_h \in Q_h^r, \tag{4.2a}$$

$$a_h(\boldsymbol{\tau}_h,\mathcal{R}^h\boldsymbol{p}) - m_{1,h}(\mathcal{R}^h\boldsymbol{\lambda}_t,\boldsymbol{\tau}_h) - m_{2,h}(\mathcal{R}^h\boldsymbol{\lambda},\boldsymbol{\tau}_h) = a(\boldsymbol{\tau}_h,\boldsymbol{p}) - m_1(\boldsymbol{\lambda}_t,\boldsymbol{\tau}_h) - m_2(\boldsymbol{\lambda},\boldsymbol{\tau}_h) \quad \forall\ \boldsymbol{\tau}_h \in V_h^r, \tag{4.2b}$$

$$(\nabla\cdot\mathcal{R}^h\boldsymbol{p},w_h) = (\nabla\cdot\boldsymbol{p},w_h) \qquad \forall\ w_h \in W_h^r, \tag{4.2c}$$

along with $\mathcal{R}^h u(\boldsymbol{x},0) = \Pi_r^0 u_0$ and $\mathcal{R}^h\boldsymbol{\lambda}(\boldsymbol{x},0) = \boldsymbol{\Pi}_r^0\boldsymbol{\lambda}_0$. The well-posedness of the problem (4.2) follows from the coercivity of the bilinear form $m_{2,h}(\bullet,\bullet)$ together with the discrete inf–sup condition (3.4). The approximation properties of this projection operator are stated in the following theorem.

THEOREM 4.3. *Let the pairs $(\boldsymbol{p}, \boldsymbol{\lambda}, u)$ and $(\mathcal{R}^h\boldsymbol{p}, \mathcal{R}^h\boldsymbol{\lambda}, \mathcal{R}^h u)$ be the solutions to the system (2.2) and (4.2), respectively. Then there exists a positive constant C independent of the mesh size h such that*

$$\|\mathcal{R}^h\boldsymbol{p} - \boldsymbol{p}\| \leq Ch^{r+1}\left(|\boldsymbol{p}|_{r+1} + |\boldsymbol{\lambda}|_{r+1} + |\boldsymbol{\lambda}_t|_{r+1} + \|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))}\right), \tag{4.3}$$

$$\|\mathcal{R}^h\boldsymbol{\lambda} - \boldsymbol{\lambda}\| \leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |\boldsymbol{\lambda}|_{r+1}\right), \tag{4.4}$$

$$\|\mathcal{R}^h u - u\| \leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u|_{r+1}\right). \tag{4.5}$$

Proof. Let us denote $\boldsymbol{\theta}_p = \mathcal{R}^h\boldsymbol{p} - \boldsymbol{p}$, $\boldsymbol{\theta}_\lambda = \mathcal{R}^h\boldsymbol{\lambda} - \boldsymbol{\lambda}$ and $\rho = \mathcal{R}^h u - u$. We now decompose $\boldsymbol{\theta}_p = \boldsymbol{\eta}_p + \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p} - \boldsymbol{p}$, $\boldsymbol{\theta}_\lambda = \boldsymbol{\eta}_\lambda + \mathcal{P}^h\boldsymbol{\lambda} - \boldsymbol{\lambda}$ and $\rho = \varsigma + \Pi_r^0 u - u$, where $\boldsymbol{\eta}_p = \mathcal{R}^h\boldsymbol{p} - \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p}$, $\boldsymbol{\eta}_\lambda = \mathcal{R}^h\boldsymbol{\lambda} - \mathcal{P}^h\boldsymbol{\lambda}$ and $\varsigma = \mathcal{R}^h u - \Pi_r^0 u$. Then we can write the equation (4.2) as:

$$a_h(\boldsymbol{\eta}_\lambda, \boldsymbol{\zeta}_h) + (\varsigma, \nabla\cdot\boldsymbol{\zeta}_h) = 0 \qquad \forall\ \boldsymbol{\zeta}_h \in Q_h^r, \tag{4.6a}$$

$$a_h(\boldsymbol{\tau}_h, \boldsymbol{\eta}_p) - m_{1,h}(\boldsymbol{\eta}_{\lambda,t}, \boldsymbol{\tau}_h) - m_{2,h}(\boldsymbol{\eta}_\lambda, \boldsymbol{\tau}_h) = a(\boldsymbol{\tau}_h, \boldsymbol{p}) - a_h(\boldsymbol{\tau}_h, \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p}) + m_{1,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}_t, \boldsymbol{\tau}_h) - m_1(\boldsymbol{\lambda}_t, \boldsymbol{\tau}_h) + m_{2,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}, \boldsymbol{\tau}_h) - m_2(\boldsymbol{\lambda}, \boldsymbol{\tau}_h)\ \forall\ \boldsymbol{\tau}_h \in V_h^r, \tag{4.6b}$$

$$(\nabla\cdot\boldsymbol{\eta}_p, w_h) = 0 \qquad \forall\ w_h \in W_h^r. \tag{4.6c}$$

Replacing $\boldsymbol{\zeta}_h$ by $\boldsymbol{\eta}_p$ in (4.6a), $\boldsymbol{\tau}_h$ by $\boldsymbol{\eta}_\lambda$ in (4.6b) and w_h by ς in (4.6c) and combining them together will results into

$$m_{1,h}(\boldsymbol{\eta}_{\lambda,t}, \boldsymbol{\eta}_\lambda) + m_{2,h}(\boldsymbol{\eta}_\lambda, \boldsymbol{\eta}_\lambda) = a_h(\boldsymbol{\eta}_\lambda, \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p}) - a(\boldsymbol{\eta}_\lambda, \boldsymbol{p}) + m_1(\boldsymbol{\lambda}_t, \boldsymbol{\eta}_\lambda) - m_{1,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}_t, \boldsymbol{\eta}_\lambda) + m_2(\boldsymbol{\lambda}, \boldsymbol{\eta}_\lambda) - m_{2,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}, \boldsymbol{\eta}_\lambda). \tag{4.7}$$

Now, one by one, we bound the terms on the right-hand side of the Equation (4.7). We start with

$$a_h(\boldsymbol{\eta}_\lambda, \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p}) - a(\boldsymbol{\eta}_\lambda, \boldsymbol{p}) = a_h(\boldsymbol{\eta}_\lambda, \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p} - \boldsymbol{\Pi}_r^0\boldsymbol{p}) - a(\boldsymbol{\eta}_\lambda, \boldsymbol{p} - \boldsymbol{\Pi}_r^0\boldsymbol{p}).$$

Using continuity bounds for the bilinear forms we can bound the right-hand side of the above equation as

$$a_h(\boldsymbol{\eta}_\lambda, \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p}) - a(\boldsymbol{\eta}_\lambda, \boldsymbol{p}) \leq C\left(\|\boldsymbol{\mathcal{I}}_h^F\boldsymbol{p} - \boldsymbol{\Pi}_r^0\boldsymbol{p}\|\|\boldsymbol{\eta}_\lambda\| + \|\boldsymbol{p} - \boldsymbol{\Pi}_r^0\boldsymbol{p}\|\|\boldsymbol{\eta}_\lambda\|\right),$$

it yields that

$$a_h(\boldsymbol{\eta}_\lambda, \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p}) - a(\boldsymbol{\eta}_\lambda, \boldsymbol{p}) \leq Ch^{r+1}|\boldsymbol{p}|_{r+1}\|\boldsymbol{\eta}_\lambda\|. \tag{4.8}$$

The second and third terms can be directly bounded using the Lemma 4.1 as follows:

$$m_1(\boldsymbol{\lambda}_t, \boldsymbol{\eta}_\lambda) - m_{1,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}_t, \boldsymbol{\eta}_\lambda) \leq Ch^{r+1}|\boldsymbol{\lambda}_t|_{r+1}\|\boldsymbol{\eta}_\lambda\|. \tag{4.9}$$

$$m_2(\boldsymbol{\lambda}, \boldsymbol{\eta}_\lambda) - m_{2,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}, \boldsymbol{\eta}_\lambda) \leq Ch^{r+1}|\boldsymbol{\lambda}|_{r+1}\|\boldsymbol{\eta}_\lambda\|. \tag{4.10}$$

Putting all the values from Equations (4.8)-(4.10) into (4.7) and using coercivity of $m_{1,h}(\bullet,\bullet)$ and $m_{2,h}(\bullet,\bullet)$, we obtain the following bound:

$$\frac{d}{dt}\|\boldsymbol{\eta}_\lambda\|^2 + \|\boldsymbol{\eta}_\lambda\|^2 \leq Ch^{r+1}\left(|\boldsymbol{p}|_{r+1} + |\boldsymbol{\lambda}|_{r+1} + |\boldsymbol{\lambda}_t|_{r+1}\right)\|\boldsymbol{\eta}_\lambda\|.$$

On integrating from '0' to 't' and employing Grönwall's lemma, we get

$$\|\boldsymbol{\eta}_\lambda(t)\|^2 \leq Ch^{2(r+1)}\left(\|\boldsymbol{p}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))}\right). \tag{4.11}$$

Choose $\boldsymbol{\tau}_h = \boldsymbol{\eta}_p$ in (4.6b), then it can be easily written that

$$\|\boldsymbol{\eta}_p\|^2 \leq a(\boldsymbol{\eta}_p, \boldsymbol{p}) - a_h(\boldsymbol{\eta}_p, \boldsymbol{\mathcal{I}}_h^F\boldsymbol{p}) + m_{1,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}_t, \boldsymbol{\eta}_p) - m_1(\boldsymbol{\lambda}_t, \boldsymbol{\eta}_p) + m_{2,h}(\boldsymbol{\Pi}_r^0\boldsymbol{\lambda}, \boldsymbol{\eta}_p) - m_2(\boldsymbol{\lambda}, \boldsymbol{\eta}_p) + m_{2,h}(\boldsymbol{\eta}_\lambda, \boldsymbol{\eta}_p).$$

Utilizing the estimate in (4.8) and the Lemma 4.1 and the continuity bounds of the bilinear form $m_{2,h}(\bullet,\bullet)$, we infer that

$$\|\boldsymbol{\eta}_p\|^2 \leq Ch^{r+1}\left(|\boldsymbol{p}|_{r+1} + |\boldsymbol{\lambda}_t|_{r+1} + |\boldsymbol{\lambda}|_{r+1} + \|\boldsymbol{\eta}_\lambda\|\right)\|\boldsymbol{\eta}_p\|,$$

from the bound in (4.11), it implies that

$$\|\boldsymbol{\eta}_p\| \leq Ch^{r+1}\left(|\boldsymbol{p}|_{r+1} + |\boldsymbol{\lambda}|_{r+1} + |\boldsymbol{\lambda}_t|_{r+1} + \|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))}\right). \tag{4.12}$$

Now, the discrete inf-sup condition allows us to write

$$\|\varsigma\| \leq \sup_{\boldsymbol{\eta}_p \in Q_h^r \setminus \{0\}} \frac{(\varsigma, \nabla \cdot \boldsymbol{\eta}_p)}{\|\boldsymbol{\eta}_p\|_{Q_h^r}}.$$

From (4.6a) and the continuity bounds of the bilinear form $a_h(\bullet, \bullet)$, we get the following bound for ς:

$$\|\varsigma\| \leq \|\boldsymbol{\eta}_\lambda\|,$$

i.e.,

$$\|\varsigma\| \leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))}\right). \tag{4.13}$$

In order to determine the estimates for $\boldsymbol{\theta}_p, \boldsymbol{\theta}_\lambda$ and ρ, we use the triangle inequality along with the bounds in the equations (4.11), (4.12) and (4.13) in the following manner:

$$\|\boldsymbol{\theta}_p\| \leq \|\boldsymbol{\eta}_p\| + \|\boldsymbol{\mathcal{I}}_h^F \boldsymbol{p} - \boldsymbol{p}\|,$$

using (3.2), we obtain

$$\|\boldsymbol{\theta}_p\| \leq Ch^{r+1}\left(|\boldsymbol{p}|_{r+1} + |\boldsymbol{\lambda}|_{r+1} + |\boldsymbol{\lambda}_t|_{r+1} + \|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))}\right).$$

Consequently, we get the other two estimates as:

$$\|\boldsymbol{\theta}_\lambda\| \leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |\boldsymbol{\lambda}|_{r+1}\right).$$

And,

$$\|\rho\| \leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u|_{r+1}\right).$$

It completes the proof. ∎

4.2. A priori Error Estimates. In this subsection, we derive the a priori error estimates for the semi-discrete virtual element approximation. The following theorem provides the convergence rates of the discrete solutions $(\boldsymbol{p}_h, \boldsymbol{\lambda}_h, u_h)$ towards the exact solution $(\boldsymbol{p}, \boldsymbol{\lambda}, u)$ of the expanded mixed formulation.

THEOREM 4.4. *Let $(\boldsymbol{p}, \boldsymbol{\lambda}, u)$ and $(\boldsymbol{p}_h, \boldsymbol{\lambda}_h, u_h)$ denote the solutions to the problems (2.2) and (3.5), respectively. Then there exists a positive constant independent of the mesh size h such that*

$$\begin{aligned}
\|u_h - u\| &\leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|u\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))}\right.\\
&\quad \left. + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + \|u_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + |u|_{r+1}\right),\\
\|\boldsymbol{\lambda}_h - \boldsymbol{\lambda}\| &\leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|u\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))}\right.\\
&\quad \left. + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + \|u_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + |\boldsymbol{\lambda}|_{r+1}\right),
\end{aligned}$$

and

$$\begin{aligned}
\|p_h - p\| &\leq Ch^{r+1}\left(\|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|u\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))}\right.\\
&\quad + \|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|u_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_{tt}\|_{L^2(H^{r+1}(\Omega))}\\
&\quad \left. + \|u_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{ttt}\|_{L^2(H^{r+1}(\Omega))} + |\boldsymbol{p}|_{r+1} + |\boldsymbol{\lambda}|_{r+1} + |\boldsymbol{\lambda}_t|_{r+1}\right).
\end{aligned}$$

Proof. Let us split the error quantities as: $\boldsymbol{p}_h - \boldsymbol{p} = \boldsymbol{\vartheta}_p + \boldsymbol{\theta}_p$, $\boldsymbol{\lambda}_h - \boldsymbol{\lambda} = \boldsymbol{\vartheta}_\lambda + \boldsymbol{\theta}_\lambda$ and $u_h - u = \varrho + \rho$, where $\boldsymbol{\vartheta}_p = \boldsymbol{p}_h - \mathcal{R}^h \boldsymbol{p}$, $\boldsymbol{\vartheta}_\lambda = \boldsymbol{\lambda}_h - \mathcal{R}^h \boldsymbol{\lambda}$ and $\varrho = u_h - \mathcal{R}^h u$. Then using (3.5), we get the following error equation:

$$a_h(\boldsymbol{\vartheta}_\lambda, \boldsymbol{\zeta}_h) + (\varrho, \nabla \cdot \boldsymbol{\zeta}_h) = 0 \qquad \forall\ \boldsymbol{\zeta}_h \in Q_h^r, \tag{4.14a}$$

$$a_h(\boldsymbol{\tau}_h, \boldsymbol{\vartheta}_p) - m_{1,h}(\boldsymbol{\vartheta}_{\lambda,t}, \boldsymbol{\tau}_h) - m_{2,h}(\boldsymbol{\vartheta}_\lambda, \boldsymbol{\tau}_h) = 0 \qquad \forall\ \boldsymbol{\tau}_h \in V_h^r, \tag{4.14b}$$

$$\begin{aligned}
(\varrho_t, w_h) - (\nabla \cdot \boldsymbol{\vartheta}_p, w_h) + (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \boldsymbol{\vartheta}_\lambda, w_h) + (\gamma \varrho, w_h) &= (u_t, w_h) - (\mathcal{R}^h u_t, w_h)\\
&\quad + (\boldsymbol{\beta} \cdot \boldsymbol{\lambda}, w_h) - (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \mathcal{R}^h \boldsymbol{\lambda}, w_h) + (\gamma u, w_h) - (\gamma \mathcal{R}^h u, w_h) \quad \forall\ w_h \in W_h^r.
\end{aligned} \tag{4.14c}$$

Taking $\boldsymbol{\zeta}_h = \boldsymbol{\vartheta}_p$ in (4.14a), $\boldsymbol{\tau}_h = \boldsymbol{\vartheta}_\lambda$ in (4.14b) and $w_h = \varrho$ in (4.14c) and combining them all will result in the following equation:

$$
\begin{aligned}
(\varrho_t, \varrho) + m_{1.h}(\boldsymbol{\vartheta}_{\lambda,t}, \boldsymbol{\vartheta}_\lambda) + m_{2.h}(\boldsymbol{\vartheta}_\lambda, \boldsymbol{\vartheta}_\lambda) + (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \boldsymbol{\vartheta}_\lambda, \varrho) + (\gamma\varrho, \varrho) = (u_t, \varrho) - (\mathcal{R}^h u_t, \varrho) \\
+(\boldsymbol{\beta} \cdot \boldsymbol{\lambda}, \varrho) - (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \mathcal{R}^h \boldsymbol{\lambda}, \varrho) + (\gamma u, \varrho) - (\gamma \mathcal{R}^h u, \varrho).
\end{aligned} \tag{4.15}
$$

We now aim to bound the terms on the right-hand side of the Equation (4.15). Let us determine the estimates by starting with following term:

$$
\begin{aligned}
(u_t, \varrho) - (\mathcal{R}^h u_t, \varrho) &= (u_t - \mathcal{R}^h u_t, \varrho) \\
&\leq \|u_t - \mathcal{R}^h u_t\| \|\varrho\|,
\end{aligned}
$$

it results into

$$
(u_t, \varrho) - (\mathcal{R}^h u_t, \varrho) \leq C h^{r+1} \left(\|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + |u_t|_{r+1} \right) \|\varrho\|. \tag{4.16}
$$

Now, by exploiting properties of the L^2-projection operator, we can write the third term as

$$
\begin{aligned}
(\boldsymbol{\beta} \cdot \boldsymbol{\lambda}, \varrho) - (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \mathcal{R}^h \boldsymbol{\lambda}, \varrho) &= (\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}, \boldsymbol{\beta}\varrho) + (\boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}, \boldsymbol{\beta}\varrho) - (\boldsymbol{\Pi}_r^0 \mathcal{R}^h \boldsymbol{\lambda}, \boldsymbol{\Pi}_r^0(\boldsymbol{\beta}\varrho)) \\
&= (\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}, \boldsymbol{\beta}\varrho) + (\boldsymbol{\lambda} - \mathcal{R}^h \boldsymbol{\lambda}, \boldsymbol{\Pi}_r^0(\boldsymbol{\beta}\varrho)) \\
&\leq C \left(\|\boldsymbol{\lambda} - \boldsymbol{\Pi}_r^0 \boldsymbol{\lambda}\| + \|\boldsymbol{\lambda} - \mathcal{R}^h \boldsymbol{\lambda}\| \right) \|\varrho\|,
\end{aligned}
$$

it yields that

$$
\begin{aligned}
(\boldsymbol{\beta} \cdot \boldsymbol{\lambda}, \varrho) - (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}_r^0 \mathcal{R}^h \boldsymbol{\lambda}, \varrho) \leq C h^{r+1} \left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} \right. \\
\left. + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |\boldsymbol{\lambda}|_{r+1} \right) \|\varrho\|.
\end{aligned} \tag{4.17}
$$

Finally, the last term in the Equation (4.15) can be estimated as

$$
\begin{aligned}
(\gamma u, \varrho) - (\gamma \mathcal{R}^h u, \varrho) &= (\gamma(u - \mathcal{R}^h u), \varrho) \\
&\leq \overline{\gamma} \|u - \mathcal{R}^h u\| \|\varrho\|.
\end{aligned}
$$

From Theorem 4.3, we get

$$
(\gamma u, \varrho) - (\gamma \mathcal{R}^h u, \varrho) \leq C h^{r+1} \left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u|_{r+1} \right) \|\varrho\|. \tag{4.18}
$$

On plugging the bounds obtained in the equations (4.16)-(4.18) into the Equation (4.15), we achieve

$$
\begin{aligned}
(\varrho_t, \varrho) + m_{1.h}(\boldsymbol{\vartheta}_{\lambda,t}, \boldsymbol{\vartheta}_\lambda) + m_{2.h}(\boldsymbol{\vartheta}_\lambda, \boldsymbol{\vartheta}_\lambda) + (\gamma\varrho, \varrho) \leq C \left(h^{r+1} \left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} \right. \right. \\
+ \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} \\
\left. \left. + |u|_{r+1} + |u_t|_{r+1} + |\boldsymbol{\lambda}|_{r+1} \right) \|\varrho\| + \|\boldsymbol{\vartheta}_\lambda\| \|\varrho\| \right).
\end{aligned}
$$

By using coercivity of the bilinear forms the left-hand side of the above equation will reduce to

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left(\|\varrho\|^2 + \underline{M}_1 \|\boldsymbol{\vartheta}_\lambda\|^2 \right) + \underline{M}_2 \|\boldsymbol{\vartheta}_\lambda\|^2 + \underline{\gamma} \|\varrho\|^2 \leq C \left(h^{r+1} \left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} \right. \right. \\
+ \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} \\
\left. \left. + |u|_{r+1} + |u_t|_{r+1} + |\boldsymbol{\lambda}|_{r+1} \right) \|\varrho\| + \|\boldsymbol{\vartheta}_\lambda\| \|\varrho\| \right).
\end{aligned}
$$

Multiplying by 2 on both sides and application of Young's inequality helps us to write the following equation:

$$
\begin{aligned}
\frac{d}{dt} \left(\|\varrho\|^2 + \underline{M}_1 \|\boldsymbol{\vartheta}_\lambda\|^2 \right) + 2\underline{M}_2 \|\boldsymbol{\vartheta}_\lambda\|^2 + 2\underline{\gamma} \|\varrho\|^2 \leq C \left(h^{2(r+1)} \left(\|\boldsymbol{p}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^2(H^{r+1}(\Omega))} \right. \right. \\
+ \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} \\
\left. \left. + |u|^2_{r+1} + |u_t|^2_{r+1} + |\boldsymbol{\lambda}|^2_{r+1} \right) + \|\boldsymbol{\vartheta}_\lambda\|^2 + \|\varrho\|^2 \right).
\end{aligned}
$$

Integrating on both sides with respect to time from '0' to 't' and noting that $\varrho(0) = 0$ and $\boldsymbol{\vartheta}_\lambda(0) = 0$, we deduce that

$$\begin{aligned}
&\|\varrho(t)\|^2 + \|\boldsymbol{\vartheta}_\lambda(t)\|^2 \\
&\quad \leq C \int_0^t \Big(h^{2(r+1)} \Big(\|\boldsymbol{p}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|^2_{L^2(H^{r+1}(\Omega))} \\
&\qquad + \|\boldsymbol{\lambda}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} + |u|^2_{r+1} + |u_t|^2_{r+1} + |\boldsymbol{\lambda}|^2_{r+1} \Big) + \|\boldsymbol{\vartheta}_\lambda\|^2 + \|\varrho\|^2 \Big) \, \mathrm{d}s.
\end{aligned}$$

Applying Grönwall's lemma, we get

$$\begin{aligned}
\|\varrho(t)\|^2 + \|\boldsymbol{\vartheta}_\lambda(t)\|^2 \leq C h^{2(r+1)} &\Big(\|\boldsymbol{p}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|^2_{L^2(H^{r+1}(\Omega))} \\
&+ \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} + \|u\|^2_{L^2(H^{r+1}(\Omega))} + \|u_t\|^2_{L^2(H^{r+1}(\Omega))} \Big).
\end{aligned} \tag{4.19}$$

On differentiating (4.14) with respect to time and following the same procedure as in (4.15), we obtain

$$\begin{aligned}
(\varrho_{tt}, \varrho_t) + m_{1.h}(\boldsymbol{\vartheta}_{\lambda,tt}, \boldsymbol{\vartheta}_{\lambda,t}) + m_{2.h}(\boldsymbol{\vartheta}_{\lambda,t}, \boldsymbol{\vartheta}_{\lambda,t}) + (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}^0_r \boldsymbol{\vartheta}_{\lambda,t}, \varrho_t) + (\gamma \varrho_t, \varrho_t) \\
= m_{1,h}(\mathcal{R}^h \boldsymbol{\lambda}_{tt}, \boldsymbol{\vartheta}_{\lambda,t}) - m_1(\boldsymbol{\lambda}_{tt}, \boldsymbol{\vartheta}_{\lambda,t}) + (u_{tt}, \varrho_t) - (\mathcal{R}^h u_{tt}, \varrho_t) \\
+ (\boldsymbol{\beta} \cdot \boldsymbol{\lambda}_t, \varrho_t) - (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}^0_r \mathcal{R}^h \boldsymbol{\lambda}_t, \varrho_t) + (\gamma u_t, \varrho_t) - (\gamma \mathcal{R}^h u_t, \varrho_t).
\end{aligned} \tag{4.20}$$

The right-hand side can be bounded in the similar way as in equations (4.16)-(4.18) and we get the following bounds:

$$\begin{aligned}
(u_{tt}, \varrho_t) - (\mathcal{R}^h u_{tt}, \varrho_t) \leq C h^{r+1} &\big(\|\boldsymbol{p}_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} \\
&+ \|\boldsymbol{\lambda}_{ttt}\|_{L^2(H^{r+1}(\Omega))} + |u_{tt}|_{r+1} \big) \|\varrho_t\|,
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
(\boldsymbol{\beta} \cdot \boldsymbol{\lambda}_t, \varrho_t) - (\boldsymbol{\beta} \cdot \boldsymbol{\Pi}^0_r \mathcal{R}^h \boldsymbol{\lambda}_t, \varrho_t) \leq C h^{r+1} &\big(\|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} \\
&+ \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + |\boldsymbol{\lambda}_t|_{r+1} \big) \|\varrho_t\|,
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
(\gamma u_t, \varrho_t) - (\gamma \mathcal{R}^h u_t, \varrho_t) \leq C h^{r+1} &\big(\|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} \\
&+ \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + |u_t|_{r+1} \big) \|\varrho_t\|.
\end{aligned} \tag{4.23}$$

On putting the value of these bounds in (4.20) and using the coercivity of the bilinear forms, we end up with the following equation:

$$\begin{aligned}
\frac{d}{dt} \left(\|\varrho_t\|^2 + \underline{M}_1 \|\boldsymbol{\vartheta}_{\lambda,t}\|^2 \right) + 2\underline{M}_2 \|\boldsymbol{\vartheta}_{\lambda,t}\|^2 + 2\underline{\gamma} \|\varrho_t\|^2 \leq C \big(&h^{r+1} \big(\|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} \\
&+ \|\boldsymbol{p}_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{ttt}\|_{L^2(H^{r+1}(\Omega))} \\
&+ |u_t|_{r+1} + |u_{tt}|_{r+1} + |\boldsymbol{\lambda}_t|_{r+1} \big) \|\varrho_t\| + \|\boldsymbol{\vartheta}_{\lambda,t}\| \|\varrho_t\| \big).
\end{aligned}$$

Using Young's inequality, we get

$$\begin{aligned}
\frac{d}{dt} \left(\|\varrho_t\|^2 + \underline{M}_1 \|\boldsymbol{\vartheta}_{\lambda,t}\|^2 \right) + 2\underline{M}_2 \|\boldsymbol{\vartheta}_{\lambda,t}\|^2 + 2\underline{\gamma} \|\varrho_t\|^2 \leq C \Big(&h^{2(r+1)} \Big(\|\boldsymbol{p}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} \\
&+ \|\boldsymbol{p}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{ttt}\|^2_{L^2(H^{r+1}(\Omega))} \\
&+ |u_t|^2_{r+1} + |u_{tt}|^2_{r+1} + |\boldsymbol{\lambda}_t|^2_{r+1} \big) + \|\boldsymbol{\vartheta}_{\lambda,t}\|^2 + \|\varrho_t\|^2 \Big).
\end{aligned}$$

Integrating with respect to time from '0' to 't' and using Grönwall's lemma, we assert that

$$\begin{aligned}
\|\varrho_t(t)\|^2 + \|\boldsymbol{\vartheta}_{\lambda,t}(t)\|^2 \leq C h^{2(r+1)} &\Big(\|\boldsymbol{p}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|u_t\|^2_{L^2(H^{r+1}(\Omega))} \\
&+ \|\boldsymbol{\lambda}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{p}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} + \|u_{tt}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{ttt}\|^2_{L^2(H^{r+1}(\Omega))} \Big).
\end{aligned} \tag{4.24}$$

Now, take $\boldsymbol{\tau}_h = \boldsymbol{\vartheta}_p$ in (4.14b) and we can write it as

$$a_h(\boldsymbol{\vartheta}_p, \boldsymbol{\vartheta}_p) = m_{1,h}(\boldsymbol{\vartheta}_{\lambda,t}, \boldsymbol{\vartheta}_p) + m_{2,h}(\boldsymbol{\vartheta}_\lambda, \boldsymbol{\vartheta}_p). \tag{4.25}$$

Using the properties of bilinear forms, the Equation (4.25) implies that

$$\|\boldsymbol{\vartheta}_p\|^2 \leq C\left(\|\boldsymbol{\vartheta}_{\lambda,t}\| + \|\boldsymbol{\vartheta}_\lambda\|\right)\|\boldsymbol{\vartheta}_p\|.$$

From the estimates in (4.19) and (4.24), we conclude that

$$\begin{aligned}\|\boldsymbol{\vartheta}_p\| \leq Ch^{r+1}\big(&\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|u\|_{L^2(H^{r+1}(\Omega))}\\ &+\|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + \|u_t\|_{L^2(H^{r+1}(\Omega))}\\ &+\|\boldsymbol{p}_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|u_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{ttt}\|_{L^2(H^{r+1}(\Omega))}\big).\end{aligned} \tag{4.26}$$

Equations (4.19), (4.24) and (4.26) together with the triangle inequality complete the proof. ∎

5. Fully Discrete Scheme. The purpose of this section is to employ the Euler backward method to discretize in time and construct the fully discrete scheme. We will also derive a priori error estimates of the solution to the resulting fully discrete scheme.

Let the time interval $[0,T]$ be partitioned into N uniform subintervals, where N is a positive integer. The discrete time levels are defined by $t_n = n\Delta t$ for $n = 0,1,\ldots,N$, where Δt denotes the time step size given by $\Delta t = t_{n+1} - t_n$. For any function w, we introduce the following notations:

$$w^n = w(t_n), \qquad \partial_t w^n = \frac{w^{n+1} - w^n}{\Delta t}.$$

Furthermore, for any $n = 0,1\ldots,N$, we denote $P^n \approx \boldsymbol{p}_h(\cdot,t_n)$, $S^n \approx \boldsymbol{\lambda}_h(\cdot,t_n)$ and $U^n \approx u_h(\cdot,t_n)$. Then the fully discrete scheme is defined as: find $(P^n,S^n,U^n) \in Q_h^r \times V_h^r \times W_h^r$ such that

$$a_h(S^{n+1},\boldsymbol{\zeta}_h) + (U^{n+1},\nabla\cdot\boldsymbol{\zeta}_h) = 0 \qquad \forall\ \boldsymbol{\zeta}_h \in Q_h^r, \tag{5.1a}$$

$$a_h(\boldsymbol{\tau}_h,P^{n+1}) - m_{1,h}(\partial_t S^n,\boldsymbol{\tau}_h) - m_{2,h}(S^{n+1},\boldsymbol{\tau}_h) = 0 \qquad \forall\ \boldsymbol{\tau}_h \in V_h^r, \tag{5.1b}$$

$$(\partial_t U^n, w_h) - (\nabla\cdot P^{n+1}, w_h) + (\boldsymbol{\beta}\cdot\boldsymbol{\Pi}_r^0 S^{n+1}, w_h) + (\gamma U^{n+1}, w_h) = (f^{n+1}, w_h) \quad \forall\ w_h \in W_h^r. \tag{5.1c}$$

The above system is supplemented with the initial conditions $U^0 = \boldsymbol{\Pi}_r^0 u_0$, $S^0 = \boldsymbol{\Pi}_r^0\boldsymbol{\lambda}_0$. The convergence results of the fully discrete solution are given in the subsequent theorem.

THEOREM 5.1. *Let $(\boldsymbol{p}(t_n),\boldsymbol{\lambda}(t_n),u(t_n))$ denote the solution to the problem (2.2) whereas (P^n,S^n,U^n) denote the solution to the fully discrete problem (5.1). Then there exists a constant independent of the mesh size h and time step Δt such that*

$$\begin{aligned}\max_{1\leq n\leq N}\left(\|U^n - u^n\| + \|S^n - \boldsymbol{\lambda}^n\|\right) \leq Ch^{r+1}\big(&\|\boldsymbol{p}_t\|_{L^1(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^1(H^{r+1}(\Omega))} + \|u_t\|_{L^1(H^{r+1}(\Omega))}\\ &+\|\boldsymbol{p}\|_{L^\infty(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^\infty(H^{r+1}(\Omega))} + \|u\|_{L^\infty(H^{r+1}(\Omega))}\\ &+\|\boldsymbol{\lambda}_t\|_{L^\infty(H^{r+1}(\Omega))}\big) + C\Delta t\left(\|\boldsymbol{\lambda}_{tt}\|_{L^1(L^2(\Omega))} + \|u_{tt}\|_{L^1(L^2(\Omega))}\right).\end{aligned}$$

and

$$\begin{aligned}\max_{1\leq n\leq N}\|P^n - \boldsymbol{p}^n\| \leq Ch^{r+1}\big(&\|\boldsymbol{p}\|_{L^\infty(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^\infty(H^{r+1}(\Omega))} + \|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))}\\ &+\|\boldsymbol{\lambda}_t\|_{L^\infty(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))} + \|u\|_{L^\infty(H^{r+1}(\Omega))}\\ &+\|u_t\|_{L^\infty(H^{r+1}(\Omega))} + \|u_t\|_{L^1(H^{r+1}(\Omega))} + \|u_{tt}\|_{L^1(H^{r+1}(\Omega))}\big)\\ &+C\Delta t\left(\|u_{tt}\|_{L^1(L^2(\Omega))} + \|u_{ttt}\|_{L^1(L^2(\Omega))}\right).\end{aligned}$$

Proof. For any $j = 0,1,\ldots,N-1$, we decompose the error quantities as: $P^j - \boldsymbol{p}^j = \boldsymbol{\vartheta}_p^j + \boldsymbol{\theta}_p^j$, $S^j - \boldsymbol{\lambda}^j = \boldsymbol{\vartheta}_\lambda^j + \boldsymbol{\theta}_\lambda^j$ and $U^j - u^j = \varrho^j + \rho^j$, where $\boldsymbol{\vartheta}_p^j = P^j - \mathcal{R}^h\boldsymbol{p}^j$, $\boldsymbol{\vartheta}_\lambda^j = S^j - \mathcal{R}^h\boldsymbol{\lambda}^j$ and $\varrho^j = U^j - \mathcal{R}^h u^j$. Then from (5.1), we obtain the following error equation using (4.2):

$$a_h(\boldsymbol{\vartheta}_\lambda^{j+1},\boldsymbol{\zeta}_h) + (\varrho^{j+1},\nabla\cdot\boldsymbol{\zeta}_h) = 0 \qquad \forall\ \boldsymbol{\zeta}_h \in Q_h^r, \tag{5.2a}$$

$$a_h(\boldsymbol{\tau}_h,\boldsymbol{\vartheta}_p^{j+1}) - m_{1,h}(\partial_t\boldsymbol{\vartheta}_\lambda^j,\boldsymbol{\tau}_h) - m_{2,h}(\boldsymbol{\vartheta}_\lambda^{j+1},\boldsymbol{\tau}_h) = 0 \qquad \forall\ \boldsymbol{\tau}_h \in V_h^r, \tag{5.2b}$$

$$\begin{aligned}(\partial_t\varrho^j, w_h) - (\nabla\cdot\boldsymbol{\vartheta}_p^{j+1}, w_h) + (\boldsymbol{\beta}\cdot\boldsymbol{\Pi}_r^0\boldsymbol{\vartheta}_\lambda^{j+1}, w_h) + (\gamma\varrho^{j+1}, w_h) &= (u_t^{j+1} - \mathcal{R}^h\partial_t u^j, w_h)\\ &+(\boldsymbol{\beta}\cdot(\boldsymbol{\lambda}^{j+1} - \boldsymbol{\Pi}_r^0\mathcal{R}^h\boldsymbol{\lambda}^{j+1}), w_h)+(\gamma(u^{j+1} - \mathcal{R}^h u^{j+1}), w_h)\ \forall\ w_h \in W_h^r.\end{aligned} \tag{5.2c}$$

Put $\boldsymbol{\zeta}_h = \boldsymbol{\vartheta}_p^{j+1}$ in (5.2a), $\boldsymbol{\tau}_h = \boldsymbol{\vartheta}_\lambda^{j+1}$ in (5.2b) and $w_h = \varrho^{j+1}$ in (5.2c) and combining them all to obtain

$$\begin{aligned}(\partial_t \varrho^j, \varrho^{j+1}) + m_{1.h}(\partial_t \boldsymbol{\vartheta}_\lambda^j, \boldsymbol{\vartheta}_\lambda^{j+1}) + m_{2.h}(\boldsymbol{\vartheta}_\lambda^{j+1}, \boldsymbol{\vartheta}_\lambda^{j+1}) + (\boldsymbol{\beta}\cdot\boldsymbol{\Pi}_r^0\boldsymbol{\vartheta}_\lambda^{j+1}, \varrho^{j+1}) + (\gamma\varrho^{j+1}, \varrho^{j+1}) \\ = (u_t^{j+1} - \mathcal{R}^h\partial_t u^j, \varrho^{j+1}) + (\boldsymbol{\beta}\cdot(\boldsymbol{\lambda}^{j+1} - \boldsymbol{\Pi}_r^0\mathcal{R}^h\boldsymbol{\lambda}^{j+1}), \varrho^{j+1}) + (\gamma(u^{j+1} - \mathcal{R}^h u^{j+1}), \varrho^{j+1}).\end{aligned} \tag{5.3}$$

In order to bound the right-hand side of the equation (5.3), we proceed in the following manner:

$$\begin{aligned}(u_t^{j+1} - \mathcal{R}^h\partial_t u^j, \varrho^{j+1}) &= (u_t^{j+1} - \partial_t u^j, \varrho^{j+1}) + (\partial_t u^j - \mathcal{R}^h\partial_t u^j, \varrho^{j+1}) \\ &\leq \left(\|u_t^{j+1} - \partial_t u^j\| + \|\partial_t u^j - \mathcal{R}^h\partial_t u^j\|\right)\|\varrho^{j+1}\|,\end{aligned}$$

using the definition of $\partial_t u^j$, we obtain

$$(u_t^{j+1} - \mathcal{R}^h\partial_t u^j, \varrho^{j+1}) \leq \frac{1}{\Delta t}\left(\|\Delta t u_t^{j+1} - (u^{j+1} - u^j)\| + \|u^{j+1} - u^j - \mathcal{R}^h(u^{j+1} - u^j)\|\right)\|\varrho^{j+1}\|.$$

From the estimates in Lemma 4.3, we deduce that

$$\begin{aligned}(u_t^{j+1} - \mathcal{R}^h\partial_t u^j, \varrho^{j+1}) \leq C\frac{1}{\Delta t}\Big(\|\Delta t u_t^{j+1} - (u^{j+1} - u^j)\| + h^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))}\right. \\ \left. + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1} - u^j|_{r+1}\right)\Big)\|\varrho^{j+1}\|.\end{aligned} \tag{5.4}$$

The other terms will satisfy the following bounds:

$$\begin{aligned}(\gamma(u^{j+1} - \mathcal{R}^h u^{j+1}), \varrho^{j+1}) \leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))}\right. \\ \left. + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1}|_{r+1}\right)\|\varrho^{j+1}\|,\end{aligned} \tag{5.5}$$

and,

$$\begin{aligned}(\boldsymbol{\beta}\cdot(\boldsymbol{\lambda}^{j+1} - \boldsymbol{\Pi}_r^0\mathcal{R}^h\boldsymbol{\lambda}^{j+1}), \varrho^{j+1}) \leq Ch^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))}\right. \\ \left. + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |\boldsymbol{\lambda}^{j+1}|_{r+1}\right)\|\varrho^{j+1}\|.\end{aligned} \tag{5.6}$$

Plugging the values from equations (5.4)-(5.6) to the equation (5.3) and using coercivity of the bilinear forms, we obtain

$$\begin{aligned}&(\partial_t \varrho^j, \varrho^{j+1}) + m_{1.h}(\partial_t\boldsymbol{\vartheta}_\lambda^j, \boldsymbol{\vartheta}_\lambda^{j+1}) + \underline{M}_2\|\boldsymbol{\vartheta}_\lambda^{j+1}\|^2 + \underline{\gamma}\|\varrho^{j+1}\|^2 \\ &\leq C\Bigg(h^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1}|_{r+1} + |\boldsymbol{\lambda}^{j+1}|_{r+1}\right) \\ &+ \frac{1}{\Delta t}\Big(h^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1} - u^j|_{r+1}\right) \\ &+ \|\Delta t u_t^{j+1} - (u^{j+1} - u^j)\|\Big) + \|\boldsymbol{\vartheta}_\lambda^{j+1}\|\Bigg)\|\varrho^{j+1}\|.\end{aligned}$$

It is easy to observe that

$$\begin{aligned}&\frac{1}{2\Delta t}\left((\varrho^{j+1}, \varrho^{j+1}) - (\varrho^j, \varrho^j)\right) + \frac{1}{2\Delta t}\left(m_{1.h}(\boldsymbol{\vartheta}_\lambda^{j+1}, \boldsymbol{\vartheta}_\lambda^{j+1}) - m_{1.h}(\boldsymbol{\vartheta}_\lambda^j, \boldsymbol{\vartheta}_\lambda^j)\right) \\ &\leq C\Bigg(h^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1}|_{r+1} + |\boldsymbol{\lambda}^{j+1}|_{r+1}\right) \\ &+ \frac{1}{\Delta t}\Big(h^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1} - u^j|_{r+1}\right) \\ &+ \|\Delta t u_t^{j+1} - (u^{j+1} - u^j)\|\Big) + \|\boldsymbol{\vartheta}_\lambda^{j+1}\|\Bigg)\|\varrho^{j+1}\|.\end{aligned}$$

On multiplying by $2\Delta t$ on both sides, we can write

$$\begin{aligned}&(\varrho^{j+1}, \varrho^{j+1}) - (\varrho^j, \varrho^j) + m_{1.h}(\boldsymbol{\vartheta}_\lambda^{j+1}, \boldsymbol{\vartheta}_\lambda^{j+1}) - m_{1.h}(\boldsymbol{\vartheta}_\lambda^j, \boldsymbol{\vartheta}_\lambda^j) \\ &\leq C\Bigg(\Delta t h^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1}|_{r+1} + |\boldsymbol{\lambda}^{j+1}|_{r+1}\right) \\ &+ h^{r+1}\left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1} - u^j|_{r+1}\right) \\ &+ \|\Delta t u_t^{j+1} - (u^{j+1} - u^j)\| + \Delta t\|\boldsymbol{\vartheta}_\lambda^{j+1}\|\Bigg)\|\varrho^{j+1}\|.\end{aligned}$$

Taking the sum from 0 to $n-1$ on both sides and noting that $\varrho^0 = \boldsymbol{\vartheta}_\lambda^0 = 0$, we obtain

$$
\begin{aligned}
&\|\varrho^n\|^2 + \|\boldsymbol{\vartheta}_\lambda^n\|^2 \\
&\quad \leq C \sum_{j=0}^{n-1} \Bigg(\Delta t h^{r+1} \left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1}|_{r+1} \right. \\
&\quad \left. + |\boldsymbol{\lambda}^{j+1}|_{r+1} \right) + h^{r+1} \left(\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))} + |u^{j+1} - u^j|_{r+1} \right) \\
&\quad + \|\Delta t u_t^{j+1} - (u^{j+1} - u^j)\| + \Delta t \|\boldsymbol{\vartheta}_\lambda^{j+1}\| \Bigg) \|\varrho^{j+1}\|.
\end{aligned} \tag{5.7}
$$

From Taylor's series expansion, we get the following bounds for the terms on the right-hand side of the Equation (5.7):

$$
\sum_{j=0}^{n-1} \|\Delta t u_t^{j+1} - (u^{j+1} - u^j)\| \leq \Delta t \|u_{tt}\|_{L^1(L^2(\Omega))}, \tag{5.8}
$$

$$
\sum_{j=0}^{n-1} h^{r+1} |u^{j+1} - u^j|_{r+1} \leq h^{r+1} \|u_t\|_{L^1(H^{r+1}(\Omega))}, \tag{5.9}
$$

$$
\sum_{j=0}^{n-1} h^{r+1} \Delta t |u^{j+1}|_{r+1} \leq \Delta t n h^{r+1} \sup_{0 \leq j \leq n-1} |u^{j+1}|_{r+1} \leq t_n h^{r+1} \|u\|_{L^\infty(H^{r+1}(\Omega))}, \tag{5.10}
$$

$$
\sum_{j=0}^{n-1} h^{r+1} \Delta t |\boldsymbol{\lambda}^{j+1}|_{r+1} \leq \Delta t n h^{r+1} \sup_{0 \leq j \leq n-1} |\boldsymbol{\lambda}^{j+1}|_{r+1} \leq t_n h^{r+1} \|\boldsymbol{\lambda}\|_{L^\infty(H^{r+1}(\Omega))}, \tag{5.11}
$$

Using the values of bounds given in equations (5.8)-(5.11) in the Equation (5.7) along with the Young's inequality, we infer that

$$
\begin{aligned}
\frac{1}{2}\|\varrho^n\|^2 + (1 - C(\Delta t)^2)\|\boldsymbol{\vartheta}_\lambda^n\|^2 \leq C \Bigg(& h^{2(r+1)} \left(\|\boldsymbol{p}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} \right. \\
& \left. + \|u\|^2_{L^\infty(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^\infty(H^{r+1}(\Omega))} + \|u_t\|^2_{L^1(H^{r+1}(\Omega))} \right) \\
& + (\Delta t)^2 \|u_{tt}\|^2_{L^1(L^2(\Omega))} \Bigg) + \sum_{j=0}^{n-2} \left(\frac{1}{2}\|\varrho^{j+1}\|^2 + C(\Delta t)^2 \|\boldsymbol{\vartheta}_\lambda^{j+1}\|^2 \right).
\end{aligned}
$$

We choose Δt_0 such that $0 < \Delta t \leq \Delta t_0, \left(1 - C(\Delta t)^2\right) > 0$. Then, application of discrete Grönwall's lemma helps us to conclude that

$$
\begin{aligned}
\max_{1 \leq n \leq N} \left(\|\varrho^n\|^2 + \|\boldsymbol{\vartheta}_\lambda^n\|^2 \right) \leq C \Bigg(& h^{2(r+1)} \left(\|\boldsymbol{p}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} \right. \\
& \left. + \|u\|^2_{L^\infty(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}\|^2_{L^\infty(H^{r+1}(\Omega))} + \|u_t\|^2_{L^1(H^{r+1}(\Omega))} \right) + (\Delta t)^2 \|u_{tt}\|^2_{L^1(L^2(\Omega))} \Bigg).
\end{aligned} \tag{5.12}
$$

Consequently, we can obtain the following estimate as well:

$$
\begin{aligned}
\max_{1 \leq n \leq N} \left(\|\partial_t \varrho^n\|^2 + \|\partial_t \boldsymbol{\vartheta}_\lambda^n\|^2 \right) \leq C \Bigg(& h^{2(r+1)} \left(\|\boldsymbol{p}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^2(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_{tt}\|^2_{L^2(H^{r+1}(\Omega))} \right. \\
& \left. + \|u_t\|^2_{L^\infty(H^{r+1}(\Omega))} + \|\boldsymbol{\lambda}_t\|^2_{L^\infty(H^{r+1}(\Omega))} + \|u_{tt}\|^2_{L^1(H^{r+1}(\Omega))} \right) + (\Delta t)^2 \|u_{ttt}\|^2_{L^1(L^2(\Omega))} \Bigg).
\end{aligned} \tag{5.13}
$$

Now, take $\tau_h = \boldsymbol{\vartheta}_p^{j+1}$ in (5.2b) and we get

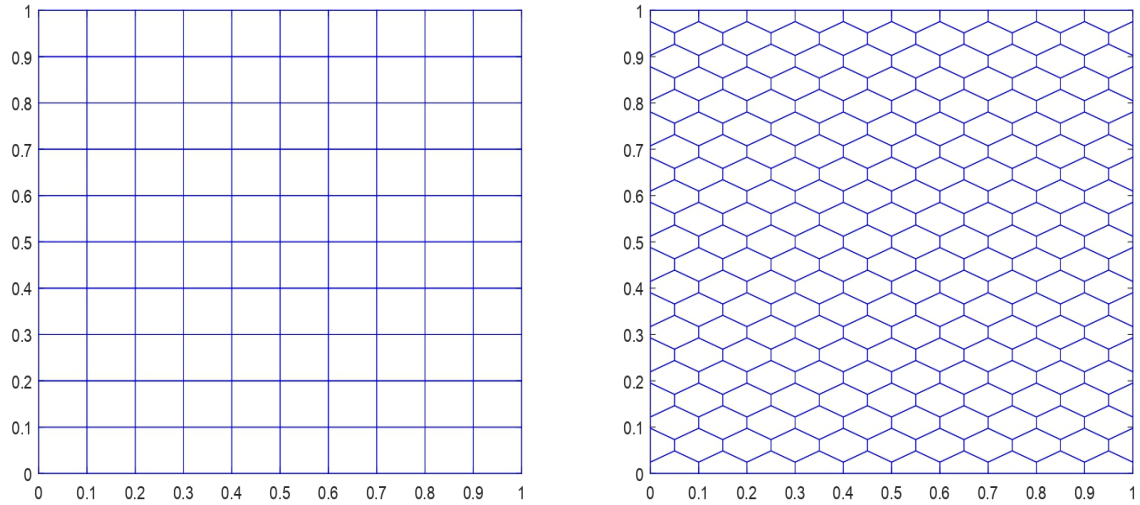
$$
a_h(\boldsymbol{\vartheta}_p^{j+1}, \boldsymbol{\vartheta}_p^{j+1}) = m_{1,h}(\partial_t \boldsymbol{\vartheta}_\lambda^j, \boldsymbol{\vartheta}_p^{j+1}) + m_{2,h}(\boldsymbol{\vartheta}_\lambda^{j+1}, \boldsymbol{\vartheta}_p^{j+1}).
$$

Taking the sum from 0 to $n-1$ on both sides and using Equation (5.12) and (5.13), we deduce that

$$\begin{aligned}\sum_{j=0}^{n-1}\|\boldsymbol{\vartheta}_p^{j+1}\| \leq C\,\big(h^{r+1}\,\big(&\|\boldsymbol{p}\|_{L^2(H^{r+1}(\Omega))}+\|\boldsymbol{\lambda}\|_{L^2(H^{r+1}(\Omega))}+\|\boldsymbol{p}_t\|_{L^2(H^{r+1}(\Omega))}\\ &+\|\boldsymbol{\lambda}_t\|_{L^2(H^{r+1}(\Omega))}+\|\boldsymbol{\lambda}_{tt}\|_{L^2(H^{r+1}(\Omega))}+\|u\|_{L^\infty(H^{r+1}(\Omega))}+\|\boldsymbol{\lambda}\|_{L^\infty(H^{r+1}(\Omega))}\\ &+\|u_t\|_{L^\infty(H^{r+1}(\Omega))}+\|\boldsymbol{\lambda}_t\|_{L^\infty(H^{r+1}(\Omega))}+\|u_t\|_{L^1(H^{r+1}(\Omega))}+\|u_{tt}\|_{L^1(H^{r+1}(\Omega))}\big)\\ &+\Delta t\,\big(\|u_{tt}\|_{L^1(L^2(\Omega))}+\|u_{ttt}\|_{L^1(L^2(\Omega))}\big)\big)\,.\end{aligned} \tag{5.14}$$

The equations (5.12), (5.13) and (5.14) together with the triangle inequality completes the proof. ∎

6. Numerical Experiments. In this section, we validate the theoretical results by presenting some numerical experiments. All the experiments have been performed on MATLAB software implemented by following the idea outlined in [14]. The spatial domain Ω is discretized using the square and hexagonal meshses as demonstrated in Figure 6.1. We have fixed the spatial domain $\Omega = [0,1]\times[0,1]$ and the temporal domain $[0,1]$. The error quantities have been computed at the final time $T = 1$.



(a) Square mesh (b) Hexagonal mesh

FIG. 6.1. *Demonstration of meshes used to discretize the domain.*

Example 1: We consider the problem (1.1) in $\Omega\times(0,T]$, have an analytical solution as follows:

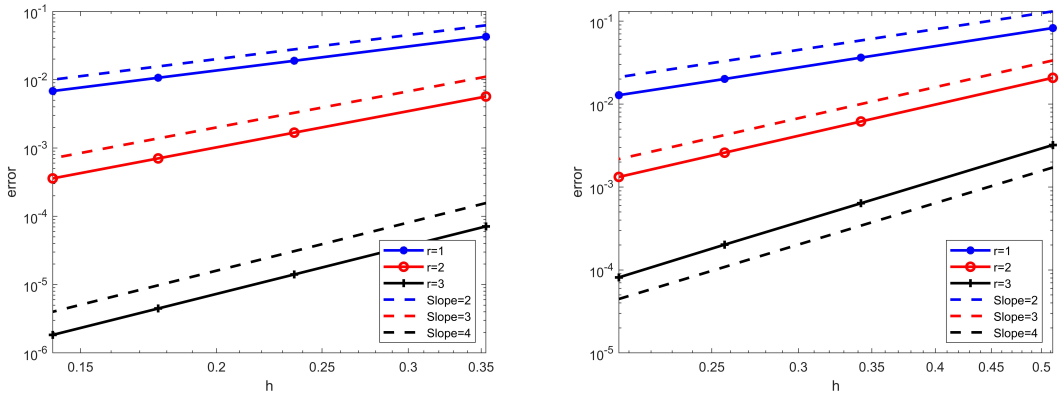
$$u(x,y,t) = t\sin(\pi x)\sin(\pi y),$$

with coefficients $\mu(\boldsymbol{x}) = x+y+1$, $\epsilon(\boldsymbol{x}) = x^2+1$, $\boldsymbol{\beta} = \begin{bmatrix}x\\ y\end{bmatrix}$ and $\gamma(\boldsymbol{x}) = x+y$. The source function $f(\boldsymbol{x},t)$ is computed accordingly. The experiment have been performed on both square and hexagonal meshes and the obtained results are displayed in Figure 6.2, 6.3 and 6.4 for the error in $u_h, \boldsymbol{\lambda}_h$ and $\boldsymbol{p}_h$, respectively, for $r = 1, 2$ and 3. The plots in all three figures confirm that in each case, we are getting the expected rate of convergence.

Example 2: Consider the problem (1.1) in $\Omega\times(0,T]$, with the exact solution as follows:

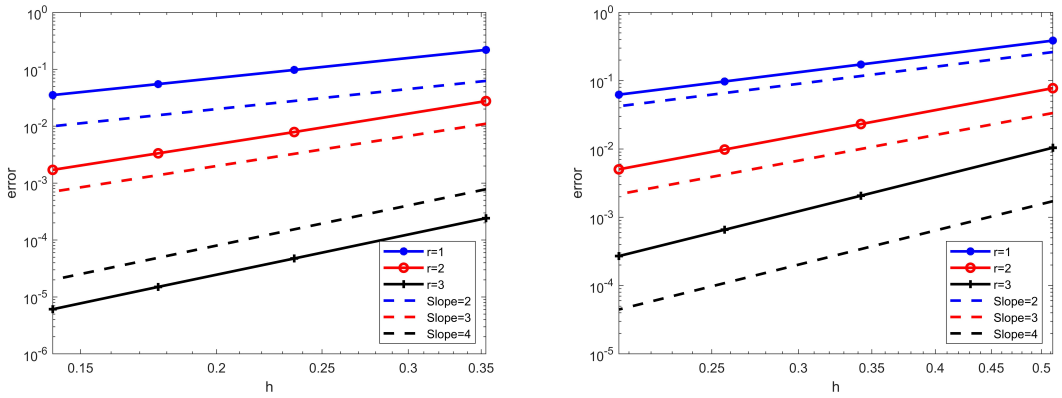
$$u(x,y,t) = e^t x^3(1-x)y^2(1-y),$$

with coefficients $\mu(\boldsymbol{x}) = 0.01$, $\epsilon(\boldsymbol{x}) = 0.01$, $\boldsymbol{\beta} = \begin{bmatrix}1\\ 1\end{bmatrix}$ and $\gamma(\boldsymbol{x}) = 1$. The source function $f(\boldsymbol{x},t)$ is computed accordingly. While performing the experiments, the spatial domain is discretized using the meshes as



(a) (b)

FIG. 6.2. *Plots showing error in u_h for Example 1 on square (left) and hexagonal (right) meshes.*



(a) (b)

FIG. 6.3. *Plots showing error in $\boldsymbol{\lambda}_h$ for Example 1 on square (left) and hexagonal (right) meshes.*

demonstrated in Figure 6.1 and the temporal domain is discretized by taking a step size of 0.001. It can be clearly observed in Figure 6.5, 6.6 and 6.7 for the error in $u_h, \boldsymbol{\lambda}_h$, and $\boldsymbol{p}_h$, respectively, for $r = 1, 2$ and 3 that we have achieved the optimal order of convergence as claimed in the theoretical results.

7. Conclusion. In this work, we have developed and analyzed a new virtual element formulation for the Sobolev equation with a convection term, namely the expanded mixed VEM. The main difficulty arising from the presence of distinct diffusion and damping coefficients has been effectively addressed through the introduction of an additional auxiliary variable. The well-posedness of the proposed formulation has been established, and a backward Euler scheme has been employed for the temporal discretization, ensuring stability. By introducing a suitable intermediate projection operator and carefully designing the discrete bilinear forms, we have derived optimal a priori error estimates for both the semi-discrete and fully discrete schemes. Numerical experiments have been presented to validate the theoretical results and demonstrate the effectiveness of the proposed method.

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Author Contributions All authors contributed equally to this work.

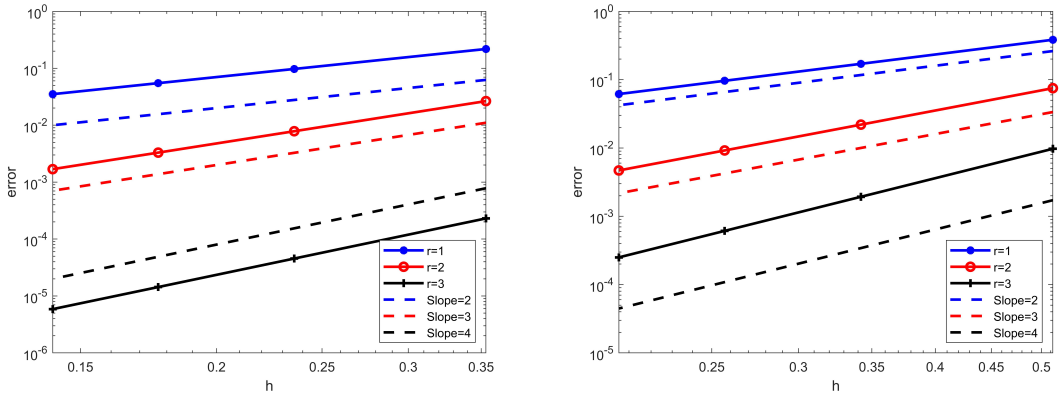


FIG. 6.4. *Plots showing error in* $\boldsymbol{p}_h$ *for Example 1 on square (left) and hexagonal (right) meshes.*

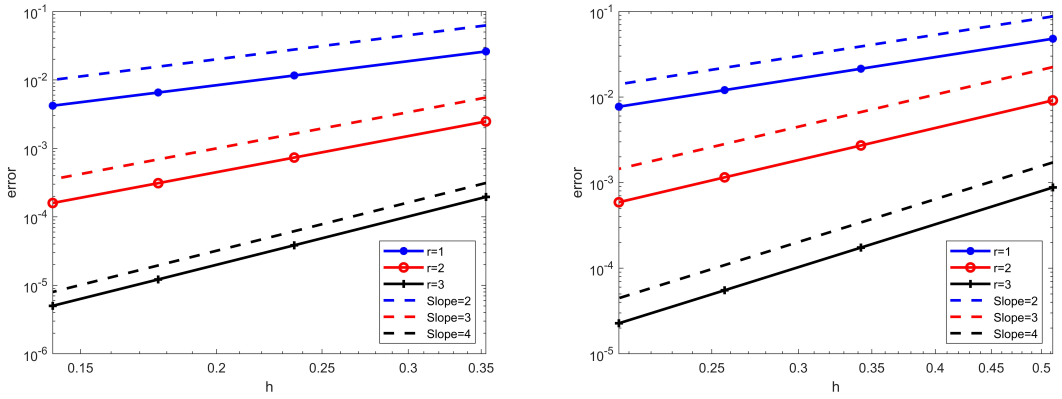


FIG. 6.5. *Plots showing error in* u_h *for Example 2 on square (left) and hexagonal (right) meshes.*

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Data Availability No datasets were generated or analyzed during the current study.

Declarations

Competing Interest The authors declare no competing interests.

Ethics and Consent to Participate Not applicable.

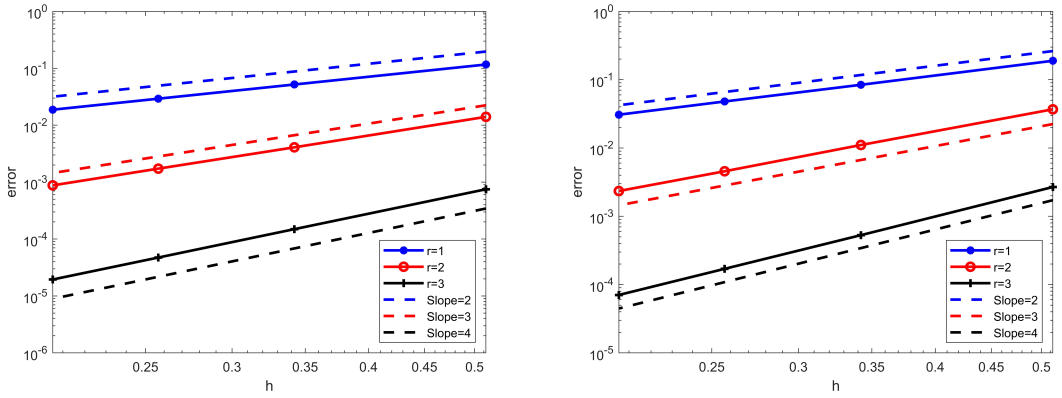
REFERENCES

[1] B. AHMAD, A. ALSAEDI, F. BREZZI, L. D. MARINI, AND A. RUSSO, *Equivalent projectors for virtual element methods*, Computers & Mathematics with Applications, 66 (2013), pp. 376–391.

[2] P. F. ANTONIETTI, L. B. DA VEIGA, D. MORA, AND M. VERANI, *A stream virtual element formulation of the Stokes problem on polygonal meshes*, SIAM Journal on Numerical Analysis, 52 (2014), pp. 386–404.

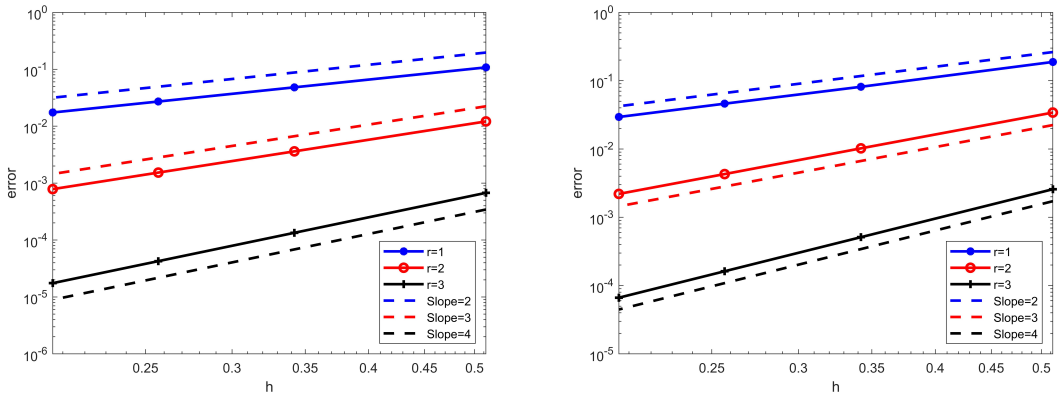
[3] P. F. ANTONIETTI, G. VACCA, AND M. VERANI, *Virtual element method for the Navier–Stokes equation coupled with the heat equation*, IMA Journal of Numerical Analysis, 43 (2023), pp. 3396–3429.

[4] G. I. BARENBLATT, I. P. ZHELTOV, AND I. KOCHINA, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata]*, Journal of applied mathematics and mechanics, 24 (1960), pp. 1286–1303.



(a) (b)

FIG. 6.6. *Plots showing error in* $\boldsymbol{\lambda}_h$ *for Example 2 on square (left) and hexagonal (right) meshes.*



(a) (b)

FIG. 6.7. *Plots showing error in* $\boldsymbol{p}_h$ *for Example 2 on square (left) and hexagonal (right) meshes.*

[5] F. BREZZI, R. S. FALK, AND L. D. MARINI, *Basic principles of mixed virtual element methods*, ESAIM: Mathematical Modelling and Numerical Analysis, 48 (2014), pp. 1227–1240.

[6] A. CANGIANI, G. MANZINI, AND O. J. SUTTON, *Conforming and nonconforming virtual element methods for elliptic problems*, IMA Journal of Numerical Analysis, 37 (2017), pp. 1317–1354.

[7] C. CHEN, K. LI, Y. CHEN, AND Y. HUANG, *Two-grid finite element methods combined with Crank-Nicolson scheme for nonlinear Sobolev equations*, Advances in Computational Mathematics, 45 (2019), pp. 611–630.

[8] Z. CHEN, *BDM mixed methods for a nonlinear elliptic problem*, Journal of computational and applied mathematics, 53 (1994), pp. 207–223.

[9] ———, *Expanded mixed finite element methods for linear second-order elliptic problems, I*, ESAIM: Mathematical Modelling and Numerical Analysis, 32 (1998), pp. 479–499.

[10] ———, *Expanded mixed finite element methods for quasilinear second order elliptic problems, II*, ESAIM: Mathematical Modelling and Numerical Analysis, 32 (1998), pp. 501–520.

[11] L. B. DA VEIGA, F. BREZZI, A. CANGIANI, G. MANZINI, L. D. MARINI, AND A. RUSSO, *Basic principles of virtual element methods*, Mathematical Models and Methods in Applied Sciences, 23 (2013), pp. 199–214.

[12] L. B. DA VEIGA, F. BREZZI, AND L. D. MARINI, *Virtual elements for linear elasticity problems*, SIAM Journal on Numerical Analysis, 51 (2013), pp. 794–812.

[13] L. B. DA VEIGA, F. BREZZI, L. D. MARINI, AND A. RUSSO, *Mixed virtual element methods for general second order elliptic problems on polygonal meshes*, ESAIM: Mathematical Modelling and Numerical Analysis, 50 (2016), pp. 727–747.

[14] L. B. DA VEIGA, F. BREZZI, L. D. MARINI, AND A. RUSSO, *Virtual element implementation for general elliptic equations*, in Building bridges: connections and challenges in modern approaches to numerical partial differential equations, Springer, 2016, pp. 39–71.

[15] ———, *Virtual element method for general second-order elliptic problems on polygonal meshes*, Mathematical Models and Methods in Applied Sciences, 26 (2016), pp. 729–750.

[16] R. E. EWING, *The approximation of certain parabolic equations backward in time by Sobolev equations*, SIAM Journal on Mathematical Analysis, 6 (1975), pp. 283–294.

[17] ———, *Time-stepping Galerkin methods for nonlinear Sobolev partial differential equations*, SIAM Journal on Numerical Analysis, 15 (1978), pp. 1125–1150.
[18] W. H. Ford and T. Ting, *Stability and convergence of difference approximations to pseudo-parabolic partial differential equations*, Mathematics of Computation, 27 (1973), pp. 737–743.
[19] F. Gao, J. Cui, and G. Zhao, *Weak Galerkin finite element methods for Sobolev equation*, Journal of Computational and Applied Mathematics, 317 (2017), pp. 188–202.
[20] F. Gao and H. Rui, *A split least-squares characteristic mixed finite element method for Sobolev equations with convection term*, Mathematics and Computers in Simulation, 80 (2009), pp. 341–351.
[21] D. Goswami, A. K. Pani, and S. Yadav, *Optimal error estimates of two mixed finite element methods for parabolic integro-differential equations with nonsmooth initial data*, Journal of Scientific Computing, 56 (2013), pp. 131–164.
[22] R. Huilgol, *A second order fluid of the differential type*, International Journal of Non-Linear Mechanics, 3 (1968), pp. 471–482.
[23] A. Kumar, S. Kumar, and S. Yadav, *Convergence analysis of virtual element methods for the sobolev equation with convection*, Journal of Applied Mathematics and Computing, 71 (2025), pp. 8163–8193.
[24] N. Li, P. Lin, and F. Gao, *An expanded mixed finite element method for two-dimensional Sobolev equations*, Journal of Computational and Applied Mathematics, 348 (2019), pp. 342–355.
[25] T. Liu, Y.-p. Lin, M. Rao, and J. Cannon, *Finite element methods for sobolev equations*, Journal of Computational Mathematics, (2002), pp. 627–642.
[26] W. Liu, Y. Chen, Q. Gu, and Y. Huang, *Virtual element method for nonlinear Sobolev equation on polygonal meshes*, Numerical Algorithms, 94 (2023), pp. 1731–1761.
[27] A. K. Pany, S. Bajpai, and S. Mishra, *Finite element galerkin method for 2d sobolev equations with burgers' type nonlinearity*, Applied Mathematics and Computation, 387 (2020), p. 125113.
[28] N. Sharma, A. K. Pani, and K. K. Sharma, *Expanded mixed fem with lowest order RT elements for nonlinear and nonlocal parabolic problems*, Advances in Computational Mathematics, 44 (2018), pp. 1537–1571.
[29] D. Shi and Y. Zhang, *High accuracy analysis of a new nonconforming mixed finite element scheme for sobolev equations*, Applied Mathematics and Computation, 218 (2011), pp. 3176–3186.
[30] D. Shylaja and S. Kumar, *Morley type virtual element method for von Kármán equations*, Advances in Computational Mathematics, 50 (2024), p. 94.
[31] M. Suthar, A. Kumar, and S. Yadav, *Mixed virtual element method for a strongly damped wave equation*, Journal of Scientific Computing, 104 (2025), pp. 1–24.
[32] M. Suthar and S. Yadav, *Mixed virtual element method for linear parabolic integro-differential equations.*, International Journal of Numerical Analysis & Modeling, 21 (2024).
[33] T. W. Ting, *A cooling process according to two-temperature theory of heat conduction*, Journal of Mathematical Analysis and Applications, 45 (1974), pp. 23–31.
[34] G. Vacca and L. B. da Veiga, *Virtual element methods for parabolic problems on polygonal meshes*, Numerical Methods for Partial Differential Equations, 31 (2015), pp. 2110–2134.
[35] M. F. Wheeler, K. Roberson, and A. Chilakapati, *Three-dimensional bioremediation modeling in heterogeneous porous media*, tech. rep., Pacific Northwest Lab., Richland, WA (United States), 1992.
[36] C.-M. Xie, M.-F. Feng, and Y. Luo, *A hybrid high-order method for the Sobolev equation*, Applied Numerical Mathematics, 178 (2022), pp. 84–97.
[37] C.-M. Xie, M.-F. Feng, Y. Luo, and L. Zhang, *A hybrid high-order method for Sobolev equation with convection-dominated term*, Computers & Mathematics with Applications, 118 (2022), pp. 85–94.
[38] Y. Xu, Z. Zhou, and J. Zhao, *Conforming virtual element methods for Sobolev equations*, Journal of Scientific Computing, 93 (2022), p. 32.
[39] B. Zhang, J. Zhao, and S. Chen, *Virtual element method for the Sobolev equations*, Mathematical Methods in the Applied Sciences, 46 (2023), pp. 1266–1281.