Supplmentary Material

Article title. Comparison of the blocked sampling approaches for the spatial dynamic panel data model

Journal name. Journal of Geographical Systems

This file comprises two parts: the first is the sampling procedure for the parameters, excluding the space-time correlation in the spatial dynamic panel data (SDPD) model; the second is the evaluation of marginal effects.

1 Sampling procedure for the parameters except space-time correlation $\boldsymbol{\psi}$

The SDPD model is defined as follows:

$$\mathbf{y}_t = \rho \mathbf{W}\mathbf{y}_t + \phi \mathbf{y}_{t-1} + \theta \mathbf{W}\mathbf{y}_{t-1} + \alpha \boldsymbol{\iota}_n + \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\eta}_t, \tag{1}$$

$$\boldsymbol{\eta}_t = \boldsymbol{\mu} + \sigma \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, \boldsymbol{\Lambda}), \quad \boldsymbol{\Lambda} = diag(\lambda_1, \ldots, \lambda_n), \tag{2}$$

First, we assume the following prior distributions,

$$\boldsymbol{\beta} \sim \mathcal{N}(\boldsymbol{\beta}_0, \boldsymbol{\Sigma}_{\boldsymbol{\beta}_0}), \quad \alpha \sim \mathcal{N}(\alpha_0, \sigma^2_{\alpha_0}), \quad \tau^2 \sim \mathcal{IG}(\delta_{\tau_0}/2, s_{\tau_0}/2),$$
$$\sigma^2 \sim \mathcal{IG}(\delta_{\sigma_0}/2, s_{\sigma_0}/2), \quad \nu \sim \mathcal{G}(\delta_{\nu_0}, s_{\nu_0}) I[\nu > 2]$$

where $\mathcal{G}$ denotes a gamma distribution, and we assume the truncated gamma distribution to satisfy a finite variance.

Let $\mathbf{y}_{\boldsymbol{\beta}} = \{\boldsymbol{\iota}_T \otimes (\mathbf{I}_n - \rho \mathbf{W})\} y - \alpha \boldsymbol{\iota}_{nT}$. The full conditional distribution of $\boldsymbol{\beta}$ and σ^2 are given by

$$\boldsymbol{\beta} | \boldsymbol{\vartheta}_{-\boldsymbol{\beta}}, \mathbf{y}, \mathbf{y}_0, \mathbf{X}, \mathbf{W} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}_{\boldsymbol{\beta}}, \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}}), \quad \sigma^2 | \boldsymbol{\vartheta}_{-\sigma^2}, \mathbf{y}, \mathbf{y}_0, \mathbf{X}, \mathbf{W} \sim \mathcal{IG}(\hat{\delta}_\sigma/2, \ \hat{s}_\sigma/2),$$

where $\hat{\boldsymbol{\mu}}_{\boldsymbol{\beta}} = \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y}_{\boldsymbol{\beta}} + \boldsymbol{\Sigma}^{-1}_{\boldsymbol{\beta}_0}\boldsymbol{\beta}_0)$, $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}} = (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X} + \boldsymbol{\Sigma}^{-1}_{\boldsymbol{\beta}_0})^{-1}$, $\hat{\delta}_\sigma = \mathbf{e}'(\mathbf{I}_T \otimes \boldsymbol{\Lambda}^{-1})\mathbf{e} + \hat{\delta}_{\sigma_0}$ and $\hat{s}_\sigma = nT + s_{\sigma_0}$.

Next, we focus on sampling procedures for μ and τ^2. Define μ_i and $\mathbf{x}_i$ for $i = 1, \ldots, n$ as i the component of the vector $\boldsymbol{\mu}$ and $\mathbf{X}_i$, respectively. Let $\mathbf{y}_i = (y_{i1}, \ldots, y_{iT})'$ and $\mathbf{X}_i = (\mathbf{x}'_{i1}, \ldots, \mathbf{x}'_{iT})'$. The model (1) and (2) becomes

$$\bar{\mathbf{y}}_i = \mu_i \iota_T + \sigma \boldsymbol{\epsilon}_i, \quad \boldsymbol{\epsilon}_i \sim \mathcal{N}(0, \lambda_i \mathbf{I}_T)$$

where $\bar{\mathbf{y}}_i = (\bar{y}_{i1}, \ldots, \bar{y}_{iT})'$, $\bar{y}_{it} = y_{it} - \rho \sum_{j=1}^n w_{ij} y_{jt} - \phi y_{i,t-1} - \theta \sum_{j=1}^n w_{ij} y_{j,t-1} - \alpha - \mathbf{x}_{it}\beta$, and $\boldsymbol{\epsilon}_i = (\epsilon_{i1}, \ldots, \epsilon_{iT})'$. The full conditional distribution of μ_i and τ^2 are written as

$$\mu_i | \boldsymbol{\vartheta}_{-\mu_i}, \mathbf{y}, \mathbf{y}_0, \mathbf{X}, \mathbf{W} \sim \mathcal{N}(\hat{\mu}_i, \hat{\sigma}^2_{\mu_i}), \quad \tau^2 | \boldsymbol{\vartheta}_{-\tau^2} \sim \mathcal{IG}(\hat{\delta}_\tau/2, \hat{s}_\tau/2),$$

where $\hat{\mu}_i = \hat{\sigma}^2_{\mu_i}(\sigma^{-2}\lambda_i^{-1}\bar{\mathbf{y}}_i'\iota_T)$ and $\hat{\sigma}^2_{\mu_i} = (\sigma^{-2}\lambda_i^{-1}T + \tau^{-2})^{-1}$, $\hat{\delta}_\tau = \boldsymbol{\mu}'\boldsymbol{\mu} + \delta_{\tau_0}$ and $\hat{s}_\tau = n + s_{\tau_0}$.

For the constant term α, let $\mathbf{y}_\alpha = \{\boldsymbol{\iota}_T \otimes (\mathbf{I}_n - \rho\mathbf{W})\}\mathbf{y} - \{\boldsymbol{\iota}_T \otimes (\phi\mathbf{I}_n + \theta\mathbf{W})\}\mathbf{y}_- - \mathbf{X}\boldsymbol{\beta}$, and the full conditional distribution of α is given as

$$\alpha | \boldsymbol{\vartheta}_{-\alpha}, \mathbf{y}_0, \mathbf{y}, \mathbf{W}, \mathbf{X} \sim \mathcal{N}(\hat{\alpha}, \hat{\sigma}^2_\alpha),$$

where $\hat{\alpha} = \hat{\sigma}^2_\alpha(\boldsymbol{\iota}_{nT}'\boldsymbol{\Omega}^{-1}\mathbf{y}_\alpha + \sigma^{-2}_{\alpha_0}\alpha_0)$ and $\hat{\sigma}^2_\alpha = (\boldsymbol{\iota}_{nT}'\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}_{nT} + \sigma^{-2}_{\alpha_0})^{-1}$.

For the heteroskedasticity parameters, we use the Gibbs sampler and acceptance-rejection-MH (AR-MH) to generate $\boldsymbol{\lambda}$ and ν, respectively. Define $\mathbf{e}_i = (e_{i1}, \ldots, e_{iT})'$ and $e_{it} = y_{it} - \rho \sum_{j=1}^n w_{ij} y_{jt} - \phi y_{i,t-1} - \theta \sum_{j=1}^n w_{ij} y_{j,t-1} - \alpha - \mu_i - \mathbf{x}_{it}\beta$. Conditional on ν, the $\hat{\lambda}_i = (\mathbf{e}_i'\mathbf{e}_i + \nu - 2)/\lambda_i$ follows an independent $\chi^2(\nu + T)$ distribution. Hence, λ_i^* is sampled from $\chi^2(\nu + T)$ distribution. We calculate λ_i using λ_i^* and $\hat{\lambda}_i$. Finally, the log of the full conditional distribution of ν is given by

$$\ln p(\nu | \boldsymbol{\vartheta}_{-\nu}, \mathbf{y}_0, \mathbf{y}, \mathbf{W}, \mathbf{X}) \propto \frac{T\nu}{2} \ln\left(\frac{\nu - 2}{2}\right) - T \ln \Gamma\left(\frac{\nu}{2}\right) - \zeta\nu + (\delta_{\nu_0}) \ln(\nu),$$

where

$$\zeta = \frac{1}{2}\sum_{i=1}^n \left\{\ln(\lambda_i) + \frac{1}{\lambda_i}\right\} + s_{\nu_0}.$$

It is difficult to generate the parameter from the distribution. Thus, we employ the AR-MH algorithm extended by Watanabe (2001). This algorithm generates the parameter from a proposal density by applying the second-order Taylor expansion to the log of $p(\nu | \boldsymbol{\vartheta}_{-\nu}, \mathbf{y}_0, \mathbf{y}, \mathbf{W}, \mathbf{X})$ and evaluating the sampled parameter with AR and MH steps. See the details of the AR-MH algorithm for the degree of freedom in Watanabe (2001).

2 Summary measures of direct and indirect effects

The estimated parameters can be interpreted as the partial derivatives of the dependent variables with respect to the explanatory variables. Spatial models extend this framework by incorporating information from neighboring regions, enabling separate measurement of direct and indirect effects on the partial derivative. Let $\mathbf{X}_t = (\mathbf{x}_{t1}, \ldots, \mathbf{x}_{tk})$ and $\mathbf{x}_{tl} = (x_{1tl}, \ldots, x_{ntl})'$ $(l = 1, \ldots, k)$. The direct effect means the marginal effect of the l-th explanatory variable x_{il} on y_i, as expressed by $\frac{\partial y_i}{\partial x_{il}}$. On the other hand, the indirect effect is expressed as $\frac{\partial y_i}{\partial x_{jl}}$ $(i \neq j)$ and refers to the marginal effect of the l-th explanatory variable x_{jl} of another region j on y_i. Furthermore, because the SDPD model has a spatio-temporal structure, marginal effects can be estimated in the short and long term.

To evaluate the effects of these, consider the SDPD model of the equation (1), which we have rewritten as

$$\mathbf{y}_t = S_\rho(\mathbf{W})(\phi \mathbf{I}_n + \theta \mathbf{W})\mathbf{y}_{t-1} + S_\rho(\mathbf{W})\mathbf{X}_t \boldsymbol{\beta} + \mathbf{C}$$

where $S_\rho(\mathbf{W}) = (\mathbf{I}_n - \rho \mathbf{W})^{-1}$ and $\mathbf{C}$ is a rest term containing the intercept and error terms. Thus, the short-term marginal effects of the l th explanatory variable at time t are defined as:

$$S_l(\mathbf{W}) = \left[\frac{\partial y_{it}}{\partial x_{jtl}} \right] = \left[S_\rho(\mathbf{W})_{ij} \beta_l \right],$$

Moreover, the long-term marginal effects of the l th explanatory variable are given by

$$S_l^*(\mathbf{W}) = \left[\frac{\partial y_i}{\partial x_{jl}} \right] = \left[S_\psi(\mathbf{W})_{ij} \beta_l \right],$$

where $S_\psi(\mathbf{W}) = [(1-\phi)\mathbf{I}_n - (\rho+\theta)\mathbf{W}]^{-1}$. The diagonal elements of the matrix $S_l(\mathbf{W})_{ii}$ ($S_l^*(\mathbf{W})_{ii}$) contain the direct effects, and the off-diagonal elements denote the indirect effects.

As an overall assessment of marginal effects, LeSage and Pace (2009) suggests summary measures, such as the average total impact, average direct impact, and average indirect impact, calculated by

$$\begin{aligned}
\bar{M}_{total}(l) &= n^{-1} \iota_n' S_l(\mathbf{W}) \iota_n, \\
\bar{M}_{direct}(l) &= n^{-1} trace(S_l(\mathbf{W})), \\
\bar{M}_{indirect}(l) &= M_{total}(l) - M_{direct}(l).
\end{aligned}$$

Long-term marginal effects can be calculated by replacing $S_l^*(\mathbf{W})$.

References

LeSage J, Pace RK (2009) Introduction to spatial econometrics. Chapman&Hall/CRC

Watanabe T (2001) On sampling the degree-of-freedom of student's-t disturbances. Statistics and Probability Letters 52(2):177–181