

# Hybrid Integral Transform Solution of a Nonlinear Core-Flooding Model with Capillary Pressure and End Effects

**João Quaresma**

Federal University of Rio de Janeiro

**Renato Cotta**

Federal University of Rio de Janeiro

**Gianfranco Stieven**

Federal University of Rio de Janeiro

**Felipe Eler**

Federal University of Rio de Janeiro

**Paulo Couto**

`pcouto@petroleo.ufrj.br`

Federal University of Rio de Janeiro

---

## Research Article

### Keywords:

**Posted Date:** May 13th, 2026

**DOI:** <https://doi.org/10.21203/rs.3.rs-9555718/v1>

**License:**   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

**Additional Declarations:** No competing interests reported.

---

**Version of Record:** A version of this preprint was published at Transport in Porous Media on September 3rd, 2026. See the published version at <https://doi.org/10.1007/s11242-026-02351-z>.

# Hybrid Integral Transform Solution of a Nonlinear Core-Flooding Model with Capillary Pressure and End Effects

João N. N. Quaresma<sup>1,2</sup>, Renato M. Cotta<sup>1,3</sup>, Gianfranco M. Stieven<sup>1</sup>,  
Felipe M. Eler<sup>4</sup>, Paulo Couto<sup>4\*</sup>

<sup>1</sup>Mechanical Engineering Dept., PEM/COPPE, Universidade Federal do Rio de Janeiro, UFRJ, Rio de Janeiro, RJ, Brazil.

<sup>2</sup>School of Chemical Engineering, Universidade Federal do Pará, UFPA, Belém, PA, Brazil.

<sup>3</sup>General Directorate of Nuclear and Technological Development, DGDNTM, Brazilian Navy, IPqM-CTMRJ, Rio de Janeiro, RJ, Brazil.

<sup>4</sup>Civil Engineering Department, PEC/COPPE, Universidade Federal do Rio de Janeiro (UFRJ), Rio de Janeiro, RJ, Brazil.

\*Corresponding author(s). E-mail(s): [pcouto@poli.ufrj.br](mailto:pcouto@poli.ufrj.br); Contributing authors: [quaresma@ufpa.br](mailto:quaresma@ufpa.br); [cotta@mecanica.coppe.ufrj.br](mailto:cotta@mecanica.coppe.ufrj.br); [stieven@petroleo.ufrj.br](mailto:stieven@petroleo.ufrj.br);

## Abstract

Oil-water displacement in coreflooding experiments is governed by a nonlinear convection-diffusion equation derived from the extended Buckley-Leverett formulation that accounts for capillary pressure effects. When realistic constitutive relations are combined with nonlinear capillary end-effect boundary conditions, such as those proposed by Chang and Yortsos, the problem becomes significantly more challenging due to the strong coupling between saturation and capillary pressure gradients at the inlet and outlet boundaries. In this work, a hybrid numerical-analytical solution based on the Generalized Integral Transform Technique (GITT) is developed for the one-dimensional immiscible displacement problem, incorporating the nonlinear Chang-Yortsos boundary conditions into the mathematical formulation. The governing partial differential equation is integral transformed with just a simple filter to make the boundary conditions homogeneous and reduced to a coupled system of nonlinear ordinary differential equations in modal space via eigenfunction expansion. Nonlinear diffusion and boundary flux contributions are preserved explicitly through the application of Green's second identity, thereby maintaining the physical structure of capillary end effects. Validation against a literature benchmark core-flood case demonstrates excellent agreement and rapid spectral convergence. The resulting reduced-order framework provides an accurate and computationally efficient tool for forward simulation of unsteady-state coreflooding experiments.

## Nomenclature

|                                   |                                                                     |
|-----------------------------------|---------------------------------------------------------------------|
| $A$                               | Core cross-sectional area                                           |
| $A_i$                             | Integral coefficient related to transformed terms                   |
| $B_{ij}$                          | Integral coefficient related to transformed terms                   |
| $D$                               | Core diameter                                                       |
| $D_c(S_w)$                        | Nonlinear capillary diffusion coefficient                           |
| $f_w(S_w)$                        | Fractional flow function                                            |
| $\bar{G}_{ni}(t), n = 1, 2$       | Integral coefficient related to transformed terms                   |
| $i, j$                            | Index                                                               |
| $K$                               | Absolute permeability                                               |
| $k_{rw}(S_w)$                     | Relative permeability of water                                      |
| $k_{rw}^{\max}$                   | Maximum water relative permeability                                 |
| $k_{ro}(S_w)$                     | Relative permeability of oil                                        |
| $k_{ro}^{\max}$                   | Maximum oil relative permeability                                   |
| $L$                               | Core length                                                         |
| $\mu_{ni}, n = 1, 2$              | Eigenvalues                                                         |
| $\mu_w$                           | Water viscosity                                                     |
| $\mu_o$                           | Oil viscosity                                                       |
| $n_w$                             | Exponent in water relative permeability in the Brooks & Corey model |
| $n_o$                             | Exponent in oil relative permeability in the Brooks & Corey model   |
| $N$                               | Truncation order in eigenfunction expansions                        |
| $N_k$                             | Number of intervals in semi-analytical integration                  |
| $N_{ni}, n = 1, 2$                | Normalization integrals                                             |
| $P_c(S_w)$                        | Capillary pressure                                                  |
| $q$                               | Injected water flow rate                                            |
| $S_{wi}$                          | Irreducible water saturation                                        |
| $S_{or}$                          | Residual oil saturation                                             |
| $S_w(X, t)$                       | Water saturation                                                    |
| $S_{w,av}(t)$                     | Average water saturation                                            |
| $t$                               | Time                                                                |
| $t_{bt}$                          | Breakthrough time                                                   |
| $u_t$                             | Injected flow velocity                                              |
| $X$                               | Dimensionless spatial coordinate                                    |
| $x$                               | Dimensional spatial coordinate                                      |
| $\phi$                            | Porosity                                                            |
| $\psi_{n,i}(X), n = 1, 2$         | Eigenfunctions                                                      |
| $\tilde{\psi}_{n,i}(X), n = 1, 2$ | Normalized eigenfunctions                                           |

## 1. Introduction

The mathematical modelling of immiscible two-phase flow in porous media underlies fundamental processes in petroleum engineering, hydrogeology, and subsurface energy applications. Of particular interest is the displacement of oil by water, both at the field scale and in laboratory coreflooding experiments. One-dimensional coreflood tests provide controlled environments in which viscous, capillary, and sometimes gravitational forces interact, producing behaviours that reflect essential physics relevant to enhanced oil recovery and reservoir characterization. Despite the reduced dimensionality of these problems, the governing equations remain strongly nonlinear due to the saturation dependence of relative permeability and capillary pressure functions, posing challenges for both analytical and numerical solution strategies.

The early analytical foundation for two-phase immiscible displacement was established in the classical work of Buckley and Leverett [1], who considered one-dimensional waterflooding in the absence of capillary pressure and derived a solution based on fractional-flow theory that predicts shock and rarefaction structures in saturation profiles. While the Buckley-Leverett model serves as a cornerstone in reservoir engineering, its neglect of capillarity limits its applicability to high-capillary-number displacements. Subsequent research extended this framework to incorporate capillary pressure effects. Incorporating capillarity transforms the advective Buckley-Leverett equation into a nonlinear convection–diffusion equation, wherein the diffusion term is governed by the derivative of the capillary pressure function and the relative permeabilities [2,3], and diffusion vanishes at residual saturations. Early analytical treatments of this extended problem relied on simplifying assumptions or piecewise linearization of constitutive relations, leading to what are often termed “*capillary-modified Buckley-Leverett solutions*.” Welge’s early fractional flow formulation [4] established the groundwork for analytical displacement theory, whereas perturbation analyses by Killough [5] and semi-analytical treatments by Morrow and McCool [6] explored the effects of capillary pressure on saturation profiles. Further, de Swaan’s similarity solutions [7] provided insight into capillary diffusion behavior. Extensions to heterogeneous and composite systems were addressed analytically by Wu et al. [8], who derived generalized fractional-flow constructions for layered porous media. More recently, Pasquier et al. [9] revisited the theoretical structure of the Buckley-Leverett framework by incorporating explicit phase-coupling terms, clarifying the limits of classical Darcy-based assumptions, while Kaasschieter [10] analyzed entropy-consistent solutions of gravity-modified Buckley-Leverett equations with capillary regularization, establishing conditions for uniqueness and well-posedness. Collectively, these studies demonstrate that the Buckley-Leverett equation admits rigorous analytical and semi-analytical generalizations that incorporate capillarity, gravity, heterogeneity, and coupling effects, providing a structured theoretical foundation for the considered formulation.

Despite these advances, exact analytical solutions for fully nonlinear extended Buckley-Leverett problems with realistic constitutive curves remain unavailable. This has driven

extensive research into numerical methods capable of handling general constitutive relations and complex boundary conditions. Classical numerical strategies include finite-difference formulations that discretize both the saturation and pressure equations, with upwinding and implicit time-stepping to enhance stability. The influential work of Aziz and Settari [11] provided an early systematic treatment of fully implicit finite difference reservoir simulators for multiphase, multicomponent flows, handling strong nonlinearities via Newton–Raphson iterations. Finite volume methods have also been widely adopted, offering local mass conservation and flexibility in handling irregular grids; contributions by Huyakorn and Pinder [12] and later by Eymard et al. [13] established robust finite volume frameworks for immiscible and compressible flows in heterogeneous porous media. More recently, discontinuous Galerkin and mixed finite element methods have gained prominence for their ability to capture sharp saturation fronts and variable-flux continuity, with notable works by Cesmeci and Wheeler [14] and Sun and Wheeler [15] applying these methods to two-phase systems with capillarity.

Across numerical strategies, a persistent difficulty is the treatment of the nonlinear diffusion term and nonlinear boundary conditions that arise in realistic core and reservoir problems. Capillary pressure gradients near the inlet and outlet faces of a core — often referred to as capillary end effects — have been shown by Chang and Yortsos [16] to significantly influence displacement behaviour, particularly when the core length is short relative to capillary length scales. Their model rigorously incorporates capillary end effects and enforces thermodynamic consistency between saturation, phase fluxes, and capillary pressure gradients at both inlet and outlet boundaries. The formulation couples saturation and pressure at the boundaries nonlinearly, which, while physically necessary, complicates numerical stability and convergence. Perhaps for this reason, this elegant and compact boundary condition representation has not been widely adopted in the numerical methods literature for this class of problems, which prefers more straightforward representations for discrete approaches to end effects, based on fictitious blocks both upstream and downstream.

An alternative analytical pathway that combines the accuracy of analytical methods with the flexibility of numerical techniques is the Generalized Integral Transform Technique (GITT). Classical transforms have long been used in linear diffusion and advective problems, but their direct application to nonlinear, saturation-dependent transport equations is generally not feasible without prior linearization or transformation. GITT was originally developed for heat and fluid flow problems and provides an alternative hybrid analytical-numerical framework for handling partial differential equations [17-26]. By expanding the solution in terms of eigenfunctions of an auxiliary Sturm-Liouville problem, the governing partial differential equations are converted into a coupled system of nonlinear ordinary differential equations. Applications to nonlinear diffusion and convection–diffusion problems have demonstrated remarkable convergence characteristics and reduced computational cost. The application of GITT to nonlinear PDEs can be found in a number of classical porous media problems in soil

physics and petroleum reservoir engineering [27-33], but its use in two-phase porous media flow has been more limited.

To address these gaps, the present work develops a hybrid numerical-analytical framework for one-dimensional oil-water displacement that incorporates realistic capillary pressure functions and the nonlinear boundary conditions of Chang and Yortsos [16]. These effects are known to influence laboratory displacement experiments and transient saturation profiles. A simple filtering solution is adopted to homogenize the boundary conditions, and the nonlinear boundary flux contributions appear explicitly in the modal equations after an integral transformation and the use of Green's second identity. This preserves the physical meaning of capillary end effects while maintaining mathematical consistency. The resulting methodology not only advances the analytical understanding of extended Buckley-Leverett formulations with capillary effects, but also provides a high-fidelity benchmark for numerical simulators and a tool for interpreting laboratory core-flooding data. In several nonlinear transport applications, implicit nonlinear filtering was used to accelerate convergence, and auxiliary transformations, such as the Kirchhoff transformation, were applied to simplify the diffusion operator prior to the integral transformation [34-35]. However, the present investigation demonstrates that, for the one-dimensional oil-water displacement problem, including capillary pressure and its end effects via appropriate nonlinear boundary conditions, such methodological variants are unnecessary to achieve satisfactory convergence via GITT. Direct application of the integral transform to the original nonlinear governing equation yields sufficiently rapid convergence for moderate truncation orders. This implementation was thoroughly tested against the benchmark proposed by Lenormand et al. [36].

## 2. Problem Formulation

Consider the two-phase oil-water flow problem in a heterogeneous porous medium as shown in Eq. (1a):

$$\phi \frac{\partial S_w(X,t)}{\partial t} = - \frac{1}{L} \frac{\partial F_w(X,t)}{\partial X} \quad (1a)$$

where,  $X$  represents the dimensionless spatial coordinate, such that the dimensional spatial coordinate  $x$  is defined as  $x = XL$ , with  $L$  being the length of the core,  $t$  the time,  $S_w$  the water saturation such that  $S_w \equiv S_w(X,t)$ ,  $\phi$  the core porosity,  $u_i$  the injected flow velocity, and  $f_w(S_w)$  the water fraction. The total water flux is written as:

$$F_w(X,t) = u_i f_w(S_w) + D_c(S_w) \frac{\partial S_w(X,t)}{\partial X} \quad (1b)$$

Equation (1a) is subject to the initial condition:

$$S_w(X, 0) = S_{wi} \quad (2a)$$

Following Chang and Yortsos [16], the physically consistent nonlinear boundary conditions are expressed in terms of the total water flux. At the inlet  $X = 0$ ,

$$F_w(0, t) = F_{in}(t) \quad (2b)$$

and at the outlet  $X = 1$ ,

$$F_w(1, t) = F_{out}(t) \quad (2c)$$

The outlet condition accounts for capillary end effects through the pressure balance across the core-boundary interface. Both conditions are nonlinear and couple advective and diffusive contributions. The water fraction,  $f_w(S_w)$  is given as:

$$f_w(S_w) = \frac{\frac{k_{rw}(S_w)}{\mu_w}}{\frac{k_{rw}(S_w)}{\mu_w} + \frac{k_{ro}(S_w)}{\mu_o}} \quad (3)$$

where  $k_{rw}(S_w)$  and  $k_{ro}(S_w)$  are the relative permeabilities of water and oil, respectively, and  $\mu_w$  and  $\mu_o$  are the dynamic viscosities of water and oil, respectively. The term  $D_c(S_w)$ , which is expressed as:

$$D_c(S_w) = \frac{K}{L} \lambda_o(S_w) f_w(S_w) \frac{dP_c(S_w)}{dS_w} \quad (4)$$

is the nonlinear diffusion coefficient due to capillary pressure, with  $K$  being the absolute permeability,  $\lambda_o(S_w)$  the oil mobility, and  $P_c(S_w)$  the capillary pressure, and  $S_{wi}$  and  $S_{or}$  the residual water and oil saturation, respectively.

The relative permeabilities are defined according to the Brooks & Corey correlation [2]:

$$k_{rw}(S_w) = k_{rw}^{\max} \left( \frac{S_w - S_{wi}}{1 - S_{wi} - S_{or}} \right)^{n_w}, \quad k_{ro}(S_w) = k_{ro}^{\max} \left( \frac{1 - S_w - S_{or}}{1 - S_{wi} - S_{or}} \right)^{n_o} \quad (5)$$

where  $S_{wi}$  and  $S_{or}$  represent the irreducible water and oil saturations, respectively, and  $n_w$  and  $n_o$  are real-valued exponents fitted to experimental observations. The maximum relative

permeabilities of water and oil are denoted by  $k_{rw}^{\max}$  and  $k_{ro}^{\max}$ , respectively.

Here, the alternative of fitting a prescribed functional form to the capillary pressure data points was explored, which leads to smooth behavior of the nonlinear diffusion coefficient, following the expression first proposed by Skjaeveland et al. [37] and used by Berg et al. [38, 39], as:

$$P_c(S_w) = \frac{c_w}{\left(\frac{S_w - S_{wi}}{1 - S_{wi}}\right)^{a_w}} + \frac{c_n}{\left(\frac{1 - S_{or} - S_w}{1 - S_{or}}\right)^{a_n}} \quad (6)$$

Although the mathematical formulation given by Eqs. (1)-(2) is consistent and physically sound, deriving an explicit functional form for the outlet rate,  $F_{out}(t)$ , is not straightforward. This difficulty stems from the nonlinear coupling between capillary pressure, relative permeabilities, and saturation gradients at the outlet boundary, particularly after breakthrough. To address this issue, the problem is reformulated into two temporal regimes separated by the breakthrough time,  $t_{bt}$ . For  $0 \leq t \leq t_{bt}$ , the governing equations are:

$$\phi \frac{\partial S_{w1}(X, t)}{\partial t} = -\frac{1}{L} \frac{\partial F_{w1}(X, t)}{\partial X}, \quad 0 < X < 1, \quad 0 < t < t_{bt} \quad (7a)$$

$$S_{w1}(X, 0) = S_{wi}, \quad 0 \leq X \leq 1 \quad (7b)$$

$$F_{w1}(0, t) = u_t, \quad F_{w1}(1, t) = 0, \quad 0 < t < t_{bt} \quad (7c, d)$$

$$F_{w1}(X, t) = u_t f_w(S_{w1}) + D_c(S_{w1}) \frac{\partial S_{w1}(X, t)}{\partial X} \quad (7e)$$

In this regime, no water is produced at the outlet, which is physically consistent prior to breakthrough. For the second period,  $t > t_{bt}$ , the formulation is updated to account for the arrival of the water front. Therefore, the governing equations are:

$$\phi \frac{\partial S_{w2}(X, t)}{\partial t} = -\frac{1}{L} \frac{\partial F_{w2}(X, t)}{\partial X}, \quad 0 < X < 1, \quad t > t_{bt} \quad (8a)$$

$$S_{w2}(X, t_{bt}) = S_{w1}(X, t_{bt}), \quad 0 \leq X \leq 1 \quad (8b)$$

$$F_{w2}(0, t) = u_t, \quad S_{w2}(1, t) = S_{wL}, \quad t > t_{bt} \quad (8c, d)$$

$$F_{w2}(X, t) = u_t f_w(S_{w2}) + D_c(S_{w2}) \frac{\partial S_{w2}(X, t)}{\partial X} \quad (8e)$$

Here,  $S_{wL}$  is defined such that  $P_c(S_{wL}) = 0$ . In the formulations given by Eqs. (7) and (8),  $S_{w1}(X, t)$  and  $S_{w2}(X, t)$  denote the water saturation fields before and after breakthrough,

respectively. This two-stage formulation provides a numerically tractable and physically consistent alternative to directly prescribing  $F_{out}(t)$ , while naturally capturing the transition in the outlet behavior associated with breakthrough.

### 3. Solution Methodology

Before applying the solution methodology for Eqs. (7) and (8), some algebraic manipulations can be implemented to improve computational performance. For the first period,  $0 \leq t \leq t_{bt}$ , the average water saturation can be obtained from the integration of Eq. (7a) in the domain  $[0, 1]$  in the  $X$ -direction, and utilizing the boundary conditions (7c,d), to obtain:

$$\begin{aligned} \frac{dS_{w1,av}(t)}{dt} &= \frac{1}{L\phi} [F_{w1}(0,t) - F_{w1}(1,t)] = \frac{u_t}{L\phi}, \quad 0 < t < t_{bt} \\ S_{w1,av}(0) &= S_{wi} \end{aligned} \quad (9)$$

The initial condition for  $S_{w1,av}(t)$ , is given from the initial condition (7b) and from the following definition:

$$S_{w1,av}(t) = \int_0^1 S_{w1}(X,t) dX \quad (10)$$

Then, integrating Eqs. (9), one obtains:

$$S_{w1,av}(t) = S_{wi} + \frac{u_t t}{L\phi} \quad (11)$$

Thus, we split the potential  $S_{w1}(X,t)$  as

$$S_{w1}(X,t) = S_{w1,av}(t) + S_{w1,F}(X,t) \quad (12)$$

Introducing Eq. (12) into Eqs. (7), results in:

$$\phi \frac{\partial S_{w1,F}(X,t)}{\partial t} = -\frac{u_t}{L} - \frac{1}{L} \frac{\partial F_{w1,F}(X,t)}{\partial X}, \quad 0 < X < 1, \quad 0 < t < t_{bt} \quad (13a)$$

$$S_{w1,F}(X,0) = 0, \quad 0 \leq X \leq 1 \quad (13b)$$

$$F_{w1,F}(0,t) = u_t, \quad F_{w1,F}(1,t) = 0, \quad 0 < t < t_{bt} \quad (13c,d)$$

$$F_{w1,F}(X,t) = u_t f_w(S_{w1}) + D_c(S_{w1}) \frac{\partial S_{w1,F}(X,t)}{\partial X} \quad (13e)$$

For the second period,  $t > t_{bt}$ , the following filter is proposed:

$$S_{w2}(X,t) = S_{wL} + S_{w2,F}(X,t) \quad (14)$$

Now we introduce Eq. (14) into Eqs. (8), to yield

$$\phi \frac{\partial S_{w2,F}(X,t)}{\partial t} = -\frac{1}{L} \frac{\partial F_{w2,F}(X,t)}{\partial X}, \quad 0 < X < 1, \quad t > t_{bt} \quad (15a)$$

$$S_{w2,F}(X, t_{bt}) = S_{w1}(X, t_{bt}) - S_{wL}, \quad 0 \leq X \leq 1 \quad (15b)$$

$$F_{w2,L}(0,t) = u_t, \quad S_{w2,F}(1,t) = 0, \quad t > t_{bt} \quad (15c,d)$$

$$F_{w2,F}(X,t) = u_t f_w(S_{w2}) + D_c(S_{w2}) \frac{\partial S_{w2,F}(X,t)}{\partial X} \quad (15e)$$

In the formulations given by Eqs. (13) and (15),  $S_{w1,F}(X,t)$ , and  $S_{w2,F}(X,t)$  are the new filtered potentials to be computed through the GITT approach [17-26]. We now proceed with the steps for applying this integral transform methodology to solve the filtered potentials over the two time periods as follows.

### 3.1. For the first period, $0 \leq t \leq t_{bt}$ :

To construct the integral transform basis without imposing artificial restrictions on the physical boundary behavior, an auxiliary Sturm-Liouville problem with homogeneous boundary conditions of the second kind is adopted:

$$\frac{d^2 \psi_{1i}(X)}{dX^2} + \mu_{1i}^2 \psi_{1i}(X) = 0, \quad 0 < X < 1 \quad (16a)$$

$$\frac{d\psi_{1i}(0)}{dX} = 0, \quad \frac{d\psi_{1i}(1)}{dX} = 0 \quad (16b,c)$$

The eigenquantities and orthogonality property related to this eigenvalue problem are analytically obtained as

$$\psi_{1i}(X) = \cos(\mu_{1i}X), \quad \mu_{1i} = i\pi, \quad i=1,2,\dots \quad (16d,e)$$

$$\int_0^1 \psi_{1i}(X) \psi_{1j}(X) dX = \begin{cases} 0, & i \neq j \\ N_{1i}, & i = j \end{cases}, \quad N_{1i} = \int_0^1 \psi_{1i}^2(X) dX = \frac{1}{2}, \quad \tilde{\psi}_{1i}(X) = \frac{\psi_{1i}(X)}{\sqrt{N_{1i}}} \quad (16f-j)$$

The eigenvalue problem defined by Eqs. (16) allows for the proposition of the following integral-transform pair:

$$\bar{S}_{w1i}(t) = \int_0^1 \tilde{\psi}_{1i}(X) S_{w1,F}(X, t) dX, \quad \text{transform} \quad (17a)$$

$$S_{w1,F}(X, t) = \sum_{i=1}^{\infty} \tilde{\psi}_{1i}(X) \bar{S}_{w1i}(t), \quad \text{inverse} \quad (17b)$$

The integral transformation of Eqs. (13) is now performed. For this purpose, Eqs. (13a,b) are multiplied by the normalized eigenfunctions,  $\tilde{\psi}_{1i}(X)$ , and integrated over the domain  $[0,1]$  in the  $X$ -direction. After some algebraic manipulations, making use of the inverse formula, Eq. (17b), and of the boundary conditions, Eqs. (13c,d), we obtain the following coupled system of nonlinear ordinary differential equations for the computation of the transformed potentials  $\bar{S}_{w1i}(t)$ :

$$\begin{aligned} \frac{d\bar{S}_{w1i}(t)}{dt} &= \bar{G}_{1i}(t) \\ \bar{S}_{w1i}(0) &= 0 \end{aligned} \quad (18a,b)$$

where,

$$\bar{G}_{1i}(t) = \frac{\sqrt{2}u_t}{L\phi} + \bar{g}_{1i}(t), \quad \bar{g}_{1i}(t) = \frac{1}{L\phi} \int_0^1 \frac{d\tilde{\psi}_{1i}(X)}{dX} F_{w1,F}(X, t) dX \quad (18c,d)$$

The coupled system of nonlinear ordinary differential equations given by Eqs. (18) should be numerically solved with appropriate subroutines such as DIVPAG from the IMSL Library [40], which encompasses algorithms for solving both stiff and non-stiff initial value problems. It should be recalled that as the solution marches along time, the initial conditions and the coefficients in the transformed system are redefined to account for the new injected water flow rate at the corresponding injection times, and the initial value problem solver is then reinitiated at this new starting time.

A characteristic in the system given by Eqs. (18) is that the integral coefficient  $\bar{g}_{1i}(t)$  is

nonlinear and time-dependent, and for this reason, should be integrated along the ODE system solution itself. Traditional quadrature subroutines can be used to numerically integrate such a coefficient, but the computational cost could be unnecessarily high in some cases. Since closed-form evaluation of these integral coefficients is, in general, not feasible due to nonlinearities and/or complex functional dependencies, a semi-analytical integration scheme [41,42] is employed and is summarized here. In this approach, the eigenfunctions are treated analytically, while the remaining spatial dependence is handled numerically. First, the spatial domain  $X \in [0,1]$  is partitioned into  $N_k$  subintervals, such that  $0 = X_0 < X_1 < \dots < X_{N_k} = 1$ . Within each subinterval  $[X_{k-1}, X_k]$ , the function  $F_{w1,F}(X, t)$  is locally approximated by a linear profile (or higher order approximation):

$$F_{w1,F}^k(X, t) \cong a_k(t)X + b_k(t), \quad X \in [X_{k-1}, X_k] \quad (19)$$

where the coefficients  $a_k(t)$  and  $b_k(t)$  are determined from the nodal values of  $F_{w1,F}(X, t)$  as,

$$a_k(t) = \frac{F_{w1,F}(X_k, t) - F_{w1,F}(X_{k-1}, t)}{X_k - X_{k-1}}, \quad b_k(t) = \frac{X_k F_{w1,F}(X_{k-1}, t) - X_{k-1} F_{w1,F}(X_k, t)}{X_k - X_{k-1}} \quad (20a,b)$$

Substituting this approximation into the definition of  $\bar{g}_{li}(t)$ , one obtains the following decomposition:

$$\bar{g}_{li}(t) \cong \frac{1}{L\phi} \sum_{k=1}^{N_k} \int_{X_{k-1}}^{X_k} [a_k(t)X + b_k(t)] \frac{d\tilde{\psi}_{li}(X)}{dX} dX \quad (21)$$

Exploiting linearity, each subinterval contribution can be written as

$$\bar{g}_{li}(t) \cong \frac{1}{L\phi} \sum_{k=1}^{N_k} [a_k(t)A_{ik} + b_k(t)B_{ik}] \quad (22)$$

where the coefficient integrals in Eq. (22) are

$$A_{ik} = \int_{X_{k-1}}^{X_k} X \frac{d\tilde{\psi}_{li}(X)}{dX} dX, \quad B_{ik} = \int_{X_{k-1}}^{X_k} \frac{d\tilde{\psi}_{li}(X)}{dX} dX \quad (23a,b)$$

These coefficients in Eqs. (23) are computed analytically since we have the closed-form expression for  $\tilde{\psi}_{li}(X)$  and consequently its first derivative. These quantities depend only on

the eigenfunctions and the selected domain points and can therefore be precomputed and stored for all required indices  $i$  and subintervals  $N_k$ . This semi-analytical procedure significantly reduces numerical quadrature errors and improves convergence characteristics, particularly for highly oscillatory eigenfunctions at large orders  $i$ . Moreover, it preserves the spectral accuracy of the GITT framework while maintaining computational efficiency.

### 3.2. For the second period, $t > t_{bt}$ :

Similarly, to obtain the transformed system for the first period, the steps of the GITT approach are stated as follows. First, the chosen eigenvalue problem and its analytical solution, and respective eigenquantities are:

$$\frac{d^2\psi_{2i}(X)}{dX^2} + \mu_{2i}^2\psi_{2i}(X) = 0, \quad 0 < X < 1 \quad (24a)$$

$$\frac{d\psi_{2i}(0)}{dX} = 0, \quad \psi_{2i}(1) = 0 \quad (24b,c)$$

$$\psi_{2i}(X) = \cos(\mu_{2i}X), \quad \mu_{2i} = \frac{(2i-1)\pi}{2}, \quad i=1,2,\dots \quad (24d,e)$$

$$\int_0^1 \psi_{2i}(X)\psi_{2j}(X)dX = \begin{cases} 0, & i \neq j \\ N_{2i}, & i = j \end{cases}, \quad N_{2i} = \int_0^1 \psi_{2i}^2(X)dX = \frac{1}{2}, \quad \tilde{\psi}_{2i}(X) = \frac{\psi_{2i}(X)}{\sqrt{N_{2i}}} \quad (24f-h)$$

In the second time period, since the boundary condition at the outlet end is prescribed water saturation, the eigenvalue problem's boundary condition is of the first kind. The eigenvalue problem defined by Eqs. (24) allows for the proposition of the following integral-transform pair:

$$\bar{S}_{w2i}(t) = \int_0^1 \tilde{\psi}_{2i}(X) S_{w2,F}(X,t) dX, \quad \text{transform} \quad (25a)$$

$$S_{w2,F}(X,t) = \sum_{i=1}^{\infty} \tilde{\psi}_{2i}(X) \bar{S}_{w2i}(t), \quad \text{inverse} \quad (25b)$$

To perform the integral transformation of Eqs. (15a,b), we multiply them by the normalized eigenfunctions,  $\tilde{\psi}_{2i}(X)$ , and integrate over the domain  $[0, 1]$  in the  $X$ -direction. Therefore, we make use of the inverse formula, Eq. (25b), and of the boundary conditions, Eqs. (15c,d). Then, after some algebraic manipulations, we obtain the following coupled system of nonlinear ODEs for the computation of the transformed potentials  $\bar{S}_{w2i}(t)$ :

$$\frac{d\bar{S}_{w2i}(t)}{dt} = \bar{G}_{2i}(t) \quad (26a,b)$$

$$\bar{S}_{w2i}(t_{bt}) = [S_{w1,av}(t_{bt}) - S_{wL}]A_i + \sum_{j=1}^{\infty} B_{ij}\bar{S}_{w1j}(t_{bt})$$

where,

$$\bar{G}_{2i}(t) = \frac{\sqrt{2}u_t}{L\phi} + \bar{g}_{2i}(t), \quad \bar{g}_{2i}(t) = \frac{1}{L\phi} \int_0^1 \frac{d\tilde{\psi}_{2i}(X)}{dX} F_{w2,F}(X,t) dX \quad (26c,d)$$

$$A_i = \int_0^1 \tilde{\psi}_{2i}(X) dX, \quad B_{ij} = \int_0^1 \tilde{\psi}_{2i}(X) \tilde{\psi}_{1j}(X) dX \quad (26e,f)$$

The integral coefficient  $\bar{g}_{2i}(t)$  is computed in a similar manner to  $\bar{g}_{1i}(t)$  using the semi-analytical scheme previously described.

#### 4. Computation of Average Water Saturation, Pressure Drops, Production Rate, and Oil Volume

The evaluation of the hydrodynamic and production quantities of practical interest, namely, the average water saturation, the water and oil pressure drops ( $\Delta P_w$  and  $\Delta P_o$ ), the production rate, and the oil volume, follows directly from the saturation field  $S_w(X, t)$  and the associated constitutive relations.

##### 4.1. Average Water Saturation

The average water saturation is calculated from its usual definition as shown in Eq. (10). Here, we compute this quantity for the second period, since for the first one it was determined a priori as previously discussed and shown in Eq. (11). Therefore, for  $t > t_{bt}$ , after introducing Eq. (14), and the inverse formula (25b) into the definition for  $S_{w2,av}$ , one obtains:

$$S_{w2,av}(t) = \int_0^1 S_{w2}(X,t) dX = S_{wL} + \sum_{i=1}^{\infty} A_i \bar{S}_{w2i}(t) \quad (27)$$

## 4.2. Phase Pressure Drops

The pressure drop of each phase along the porous core is obtained from Darcy's law. For a one-dimensional system of length  $L$ , the pressure gradients are expressed as:

$$u_w = -\frac{K\lambda_w}{L} \frac{\partial P_w}{\partial X}, \quad u_o = -\frac{K\lambda_o}{L} \frac{\partial P_o}{\partial X}, \quad P_c = P_o - P_w \quad (28a-c)$$

After algebraic manipulations and integrating the resulting equations, one obtains the following expressions for the computation of  $\Delta P_w$  and  $\Delta P_o$ , respectively:

$$\Delta P_o = f_w[S_w(0,t)]P_c[S_w(0,t)] - f_w[S_w(1,t)]P_c[S_w(1,t)] + \int_0^1 \left[ \frac{Lu_t}{K\lambda_t} + P_c(S_w) \frac{df_w(S_w)}{dS_w} \frac{\partial S_w}{\partial X} \right] dX \quad (29a)$$

$$\Delta P_w = \Delta P_o - \{P_c[S_w(0,t)] - P_c[S_w(1,t)]\} \quad (29b)$$

## 4.3. Production Rate

The oil production rate is computed using the material balance formulation based on the average saturation as:

$$\text{Rate}(t) = PV \frac{dS_{w,av}(t)}{dt} \quad (30)$$

where  $PV = \phi AL$  is the porous volume. Numerical differentiation of  $S_{w,av}(t)$  in the second time period should be performed using appropriate smoothing or filtering techniques to avoid oscillations, such as the function DCSDER from the IMSL Library [40], which evaluates the derivative of a spline piecewise polynomial approximation at a given point.

## 4.4. Oil Volume

The oil production is then obtained from:

$$V_o^{prod}(t) = PV[S_{w,av}(t) - S_{wi}] \quad (31)$$

## 5. Test Case

To demonstrate the solution methodology presented in this work, a test case was selected based on the study by Lenormand et al. [36], specifically Case 3 of the Ref. [36], in which  $P_c(S_w) \neq 0$ . The capillary pressure is provided in the form of a data table (blue dots in Fig.1),

which was fitted with the proposed Eq. (6) above (black curve), as shown in Fig. 1, for both the raw data and the fitted curve within the physically admissible saturation interval  $S_{wi} \leq S_w \leq 1 - S_{or}$ . The following numerical values for the correlation coefficients have been adopted here:  $c_w = 8.55717 \times 10^6$ ,  $a_w = 0.00560205$ ,  $c_n = -8.55329 \times 10^6$ , and  $a_n = 0.00596085$ . Table 1 presents the input data used for the test case, including in the last two rows the different times and injected water flow rates, which vary along the core flooding test.

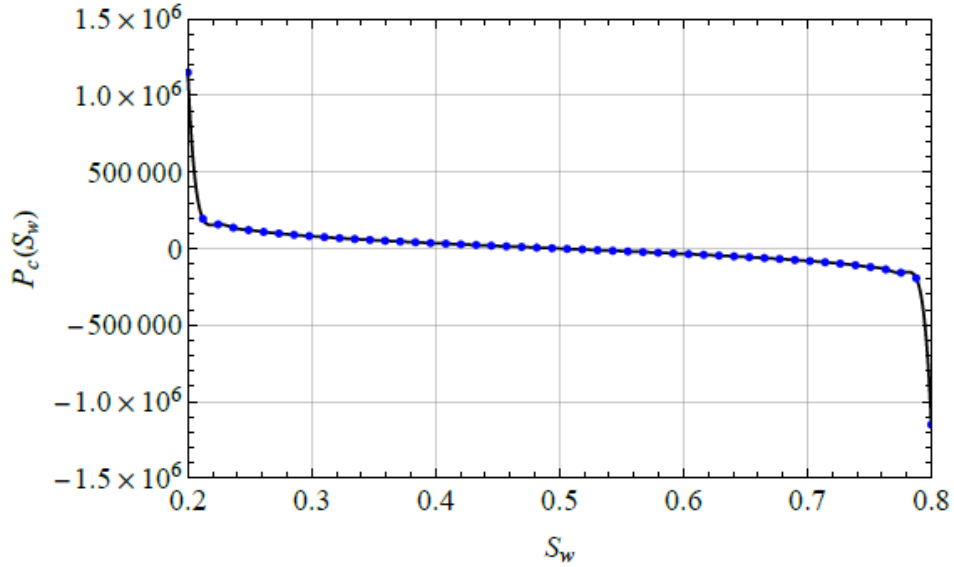


**Fig. 1.** Original  $P_c(S_w)$  data in Lenormand et al. [36] (blue dots) and the curve fit used here (black curve).

**Table 1.** Physical properties and other input data [36].

| Property                                | Symbol          | Value                 | Unit                             |
|-----------------------------------------|-----------------|-----------------------|----------------------------------|
| Maximum water relative permeability     | $k_{rw}^{\max}$ | 0.5                   | —                                |
| Maximum oil relative permeability       | $k_{ro}^{\max}$ | 0.5                   | —                                |
| Exponent in $k_{rw}$ model              | $n_w$           | 3                     | —                                |
| Exponent in $k_{ro}$ model              | $n_o$           | 3                     | —                                |
| Irreducible oil saturation              | $S_{or}$        | 0.2                   | —                                |
| Irreducible water saturation            | $S_{wi}$        | 0.2                   | —                                |
| Porosity                                | $\phi$          | 0.25                  | —                                |
| Water viscosity                         | $\mu_w$         | $1 \times 10^{-3}$    | $\text{kg.m}^{-1}.\text{s}^{-1}$ |
| Oil viscosity                           | $\mu_o$         | $5 \times 10^{-3}$    | $\text{kg.m}^{-1}.\text{s}^{-1}$ |
| Core diameter                           | $D$             | 0.04                  | m                                |
| Absolute permeability                   | $K$             | 100                   | mD                               |
| Core length                             | $L$             | 0.08                  | m                                |
| Time of injected flow rate modification | $t_{inj}$       | {10, 13, 16, 21, 26}  | h                                |
| Injected water flow rates               | $q$             | {1, 10, 70, 200, 300} | $\text{cm}^3.\text{h}^{-1}$      |

## 6. Results and Discussion

A computational code was developed in FORTRAN 95/2003 to solve the physical problem under investigation. The governing partial differential equations were transformed into coupled systems of ordinary differential equations using the Generalized Integral Transform Technique (GITT). The resulting truncated ODE systems [Eqs. (18) and (26)] were solved using the IMSL DIVPAG subroutine [40], with a prescribed user tolerance of  $10^{-8}$  to ensure high numerical accuracy in the time integration. As previously discussed, we use a semi-analytical formulation to evaluate the integral coefficients arising from the transformation. In this context,  $N_k$  denotes the number of subintervals employed in this semi-analytical integration scheme.

The following discussion focuses on the convergence behavior of the GITT solution and its verification against an in-house numerical code based on the Method of Lines combined with the Finite Volume Method (MOL/FVM) and the SCORES code by Lenormand et al. [36].

At early times (Table 2), the solution exhibits a strong dependence on the truncation order of the eigenfunction expansion ( $N$ ), as expected for this class of methods. For  $N = 10$ , the results deviate significantly from the numerical solution (MOL/FVM), particularly near the inlet ( $X = 0.1$ ), indicating an insufficient modal representation. As  $N$  increases from 10 to 20, a substantial improvement is observed. From  $N = 30$  onward, the solution converges, and differences become negligible. For  $N \geq 40$ , the results match the MOL/FVM solution to within 4 significant digits, demonstrating rapid spectral convergence. This behavior is consistent with the GITT methodology, in which early-time solutions tend to have sharper gradients (front propagation), requiring higher truncation orders to accurately capture the spatial variations.

Additionally, the comparison with the results obtained using the code from Berg et al. [38], available at <https://github.com/sede-open/Core2Relperm>, shows very good agreement. The values reported from Berg et al. [38], obtained through interpolation at  $X = 0.1, 0.3, 0.5, 0.7$ , and  $0.9$ , differ slightly from both the present GITT solution and the MOL/FVM results, particularly near the outlet ( $X = 0.9$ ). These differences remain within approximately  $10^{-3}$  to  $10^{-2}$ , which is consistent with differences in numerical discretization, interpolation procedures, and nonlinear solver implementations.

At intermediate times (Table 3), the convergence is even more remarkable. The transition from  $N = 10$  to  $N = 20$  again produces a marked correction, but from  $N = 20$  onward, the solution is essentially converged. For  $N \geq 20$ , the results remain unchanged up to the displayed precision, indicating that three to four significant digits are already achieved at moderate truncation orders. A slight discrepancy with MOL/FVM (on the order of  $10^{-3}$ )

persists, as expected due to differences in numerical formulations (spectral vs. discretization-based), possible numerical diffusion in FVM, and distinct treatment of nonlinearities.

The comparison with Berg et al. [38] also shows excellent consistency. The interpolated values from the Core2Relperm code are slightly lower than the present GITT results at most spatial locations, but the differences remain small and systematic, indicating that all three approaches capture the same physical behavior. These results further validate the accuracy and robustness of the present GITT implementation.

At late times (Table 4), the solution is already relatively smooth due to diffusive and capillary effects that reduce spatial gradients. Therefore,  $N = 20$  already provides a reasonable approximation, and convergence is achieved for  $N \geq 30$ . The oscillatory adjustment between  $N = 20$  and  $N = 50$  is minimal, indicating convergence to within 3 to 4 significant digits. The agreement with MOL/FVM is excellent, confirming that the implemented GITT code also captures the long-time behavior with high fidelity.

The comparison with the Berg et al. [38] results again shows very good agreement. Minor deviations appear mainly at the downstream locations ( $X = 0.7$  and  $0.9$ ), but these remain small and consistent with expected numerical differences. The overall agreement among GITT, MOL/FVM, and Core2Relperm results provides strong verification of the present formulation.

**Table 2.** Convergence of  $S_w(X, t)$  for  $t = 7800$  s and  $N$  up to 50 ( $N_k = 200$ ).

| $N$                     | $X = 0.1$ | $X = 0.3$ | $X = 0.5$ | $X = 0.7$ | $X = 0.9$ |
|-------------------------|-----------|-----------|-----------|-----------|-----------|
| <b>10</b>               | 0.2491    | 0.2013    | 0.2012    | 0.2012    | 0.2012    |
| <b>20</b>               | 0.3300    | 0.3147    | 0.2960    | 0.2707    | 0.2184    |
| <b>30</b>               | 0.3299    | 0.3147    | 0.2960    | 0.2708    | 0.2186    |
| <b>40</b>               | 0.3299    | 0.3147    | 0.2960    | 0.2708    | 0.2195    |
| <b>50</b>               | 0.3299    | 0.3146    | 0.2960    | 0.2708    | 0.2205    |
| <b>MOL/FVM</b>          | 0.3299    | 0.3146    | 0.2960    | 0.2708    | 0.2206    |
| <b>Berg et al. [38]</b> | 0.3281    | 0.3128    | 0.2940    | 0.2689    | 0.2223    |

**Table 3.** Convergence of  $S_w(X, t)$  for  $t = 40900$  s and  $N$  up to 50 ( $N_k = 200$ ).

| $N$                     | $X = 0.1$ | $X = 0.3$ | $X = 0.5$ | $X = 0.7$ | $X = 0.9$ |
|-------------------------|-----------|-----------|-----------|-----------|-----------|
| <b>10</b>               | 0.4991    | 0.4982    | 0.4981    | 0.4985    | 0.4995    |
| <b>20</b>               | 0.5552    | 0.5450    | 0.5341    | 0.5218    | 0.5079    |
| <b>30</b>               | 0.5552    | 0.5450    | 0.5341    | 0.5218    | 0.5079    |
| <b>40</b>               | 0.5552    | 0.5450    | 0.5341    | 0.5218    | 0.5079    |
| <b>50</b>               | 0.5552    | 0.5450    | 0.5341    | 0.5218    | 0.5079    |
| <b>MOL/FVM</b>          | 0.5566    | 0.5466    | 0.5358    | 0.5239    | 0.5103    |
| <b>Berg et al. [38]</b> | 0.5548    | 0.5447    | 0.5337    | 0.5216    | 0.5077    |

**Table 4.** Convergence of  $S_w(X, t)$  for  $t = 61900$  s and  $N$  up to 50 ( $N_k = 200$ ).

| $N$                     | $X = 0.1$ | $X = 0.3$ | $X = 0.5$ | $X = 0.7$ | $X = 0.9$ |
|-------------------------|-----------|-----------|-----------|-----------|-----------|
| <b>10</b>               | 0.5048    | 0.5034    | 0.5023    | 0.5013    | 0.5004    |
| <b>20</b>               | 0.7263    | 0.7111    | 0.6961    | 0.6696    | 0.5982    |
| <b>30</b>               | 0.7270    | 0.7117    | 0.6966    | 0.6701    | 0.5991    |
| <b>40</b>               | 0.7267    | 0.7115    | 0.6964    | 0.6699    | 0.5987    |
| <b>50</b>               | 0.7269    | 0.7116    | 0.6965    | 0.6700    | 0.5988    |
| <b>MOL/FVM</b>          | 0.7267    | 0.7114    | 0.6963    | 0.6697    | 0.5991    |
| <b>Berg et al. [38]</b> | 0.7264    | 0.7110    | 0.6958    | 0.6692    | 0.5974    |

Tables 5-7 analyze the sensitivity of the solution to the number of subintervals  $N_k$  used in evaluating the integral coefficients. For all times analyzed ( $t = 7800$ ,  $40900$ , and  $61900$  s), the results are practically insensitive to  $N_k$  once  $N_k \geq 100$ . Differences between  $N_k = 100$ ,  $150$ , and  $200$  occur only in the fourth or fifth decimal place. This analysis indicates that the semi-analytical integration is well-resolved even with relatively coarse quadrature. Therefore, this relevant computational result suggests that within the range of governing parameters analyzed, the computational efficiency can be improved by selecting moderate values of  $N_k$  without compromising accuracy.

The comparison with the results obtained from the code in Berg et al. [38] further confirms this conclusion. For all analyzed times, the interpolated values from the Core2Relperm code remain close to the present GITT results, independently of  $N_k$ . This behavior indicates that the numerical accuracy of the present method is not limited by the semi-analytical integration procedure once moderate  $N_k$  values are adopted. At  $t = 7800$  s (Table 5), small differences between the present GITT results and those from Berg et al. [38] are observed, particularly near the outlet. However, these discrepancies are comparable to those observed between GITT and MOL/FVM, indicating that all approaches provide consistent solutions. At  $t = 40900$  s (Table 6), the agreement improves further, and all approaches show nearly identical spatial trends. The independence of the solution with respect to  $N_k$  is clearly confirmed. At  $t = 61900$  s (Table 7), the solution remains insensitive to  $N_k$ , and the comparison with Berg et al. [38] again demonstrates excellent agreement. The small observable differences are consistent with expected numerical deviations among different solution strategies.

Overall, the comparisons among GITT, MOL/FVM, and the Core2Relperm code reconfirm the reliability of the present implementation and that moderate values of  $N_k$  provide an optimal compromise between accuracy and computational cost.

**Table 5.** Convergence of  $S_w(X, t)$  for  $t = 7800$  s and  $N = 50$  and different intervals  $N_k$  in the semi-analytical integration of the source term.

| $N_k$                   | $X = 0.1$ | $X = 0.3$ | $X = 0.5$ | $X = 0.7$ | $X = 0.9$ |
|-------------------------|-----------|-----------|-----------|-----------|-----------|
| <b>100</b>              | 0.3299    | 0.3146    | 0.2960    | 0.2708    | 0.2203    |
| <b>150</b>              | 0.3299    | 0.3146    | 0.2960    | 0.2708    | 0.2205    |
| <b>200</b>              | 0.3299    | 0.3146    | 0.2960    | 0.2708    | 0.2205    |
| <b>MOL/FVM</b>          | 0.3299    | 0.3146    | 0.2960    | 0.2708    | 0.2206    |
| <b>Berg et al. [38]</b> | 0.3281    | 0.3128    | 0.2940    | 0.2689    | 0.2223    |

**Table 6.** Convergence of  $S_w(X, t)$  for  $t = 40900$  s and  $N = 50$  and different intervals  $N_k$  in the semi-analytical integration of the source term.

| $N_k$                   | $X = 0.1$ | $X = 0.3$ | $X = 0.5$ | $X = 0.7$ | $X = 0.9$ |
|-------------------------|-----------|-----------|-----------|-----------|-----------|
| <b>100</b>              | 0.5552    | 0.5450    | 0.5341    | 0.5218    | 0.5079    |
| <b>150</b>              | 0.5552    | 0.5450    | 0.5341    | 0.5218    | 0.5079    |
| <b>200</b>              | 0.5552    | 0.5450    | 0.5341    | 0.5218    | 0.5079    |
| <b>MOL/FVM</b>          | 0.5566    | 0.5466    | 0.5358    | 0.5239    | 0.5103    |
| <b>Berg et al. [38]</b> | 0.5548    | 0.5447    | 0.5337    | 0.5216    | 0.5077    |

**Table 7.** Convergence of  $S_w(X, t)$  for  $t = 61900$  s and  $N = 50$  and different intervals  $N_k$  in the semi-analytical integration of the source term.

| $N_k$                   | $X = 0.1$ | $X = 0.3$ | $X = 0.5$ | $X = 0.7$ | $X = 0.9$ |
|-------------------------|-----------|-----------|-----------|-----------|-----------|
| <b>100</b>              | 0.7269    | 0.7116    | 0.6965    | 0.6700    | 0.5989    |
| <b>150</b>              | 0.7269    | 0.7116    | 0.6965    | 0.6700    | 0.5988    |
| <b>200</b>              | 0.7269    | 0.7116    | 0.6965    | 0.6700    | 0.5988    |
| <b>MOL/FVM</b>          | 0.7267    | 0.7114    | 0.6963    | 0.6697    | 0.5991    |
| <b>Berg et al. [38]</b> | 0.7264    | 0.7110    | 0.6958    | 0.6692    | 0.5974    |

Figure 2 graphically compares the GITT and MOL/FVM solutions and demonstrates excellent agreement across both spatial and temporal domains. This comparison serves as an independent verification of the implemented code. From a physical standpoint, the profiles reflect the propagation of the saturation front, with gradual smoothing due to capillary effects. The GITT solution accurately captures both the steep gradients near the advancing front and the smooth regions behind the front.

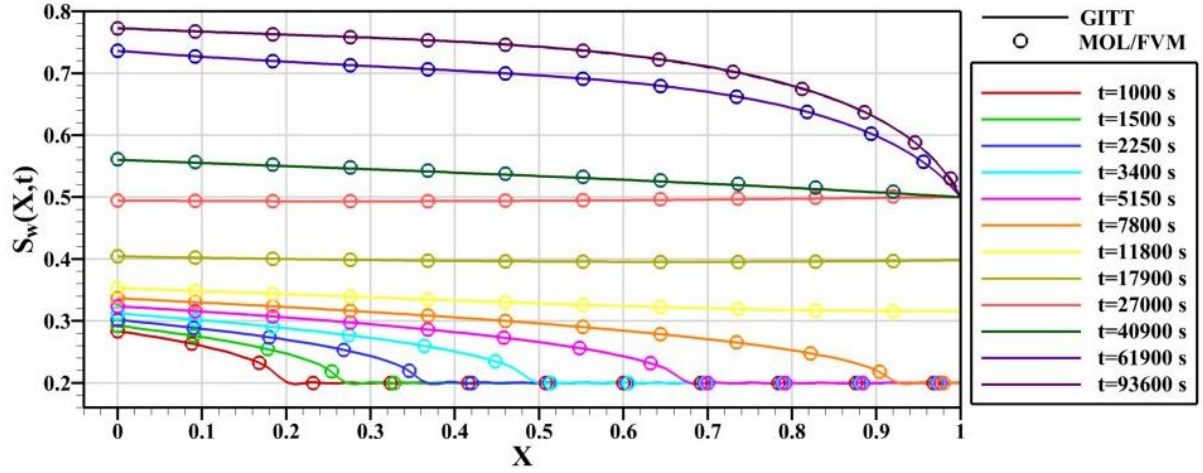


**Fig. 2.** Verification of the GITT results for spatial profiles  $S_w(X, t)$  with those obtained with the Method of Lines combined with the Finite Volume Method (MOL/FVM) for different times.

In Fig. 3, we show that the agreement among GITT, MOL/FVM, and the SCORES code results for the average water saturation is quite good to the graph scale, confirming that the global mass balance is correctly obeyed. Such a comparison also verifies the present GITT code beyond academic benchmarks, since the SCORES code is widely used in reservoir simulation. Physically,  $S_{w,av}(t)$  reflects the cumulative displacement efficiency. Therefore, the initial plateau corresponds to pre-breakthrough conditions with a slow increase. The smooth transition region indicates front arrival and oil displacement dynamics, showing a rapid increase that evidences the strong influence of capillary diffusion and the absence of shock-like discontinuities. The asymptotic behavior (post-breakthrough regime) corresponds to the approach to residual oil saturation. The ability of GITT to reproduce SCORES results demonstrates that it is not only mathematically consistent but also physically predictive.

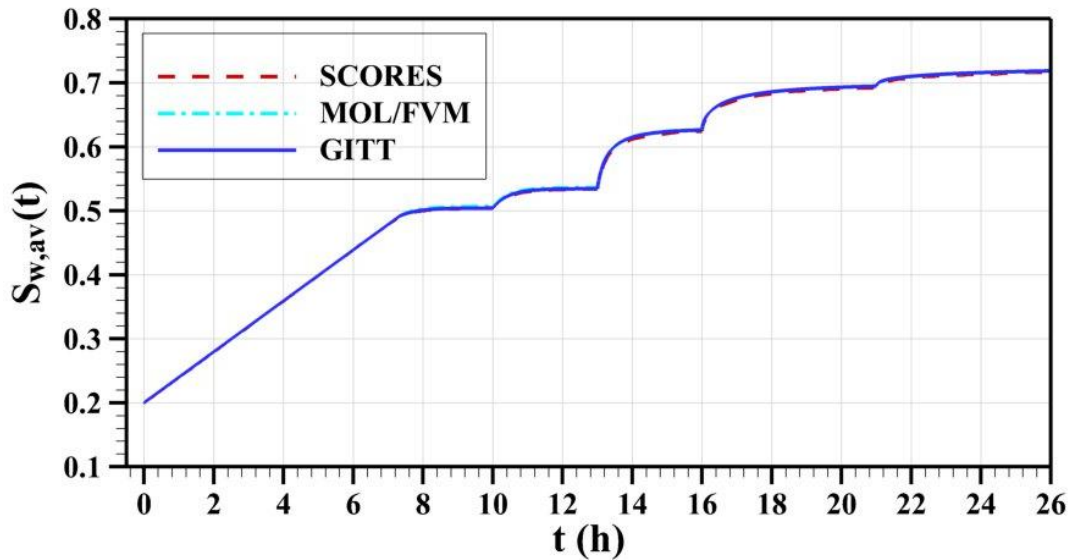


**Fig. 3.** Verification of the GITT results for the  $S_{w,av}(t)$  with those obtained with MOL/FVM and the SCORES [36] code.

Figure 4 shows the evolution of pressure drop along the core and reflects changes in the mobility ratio as water displaces oil. The pressure-drop distribution obtained with the GITT approach exhibits smooth spatial profiles without spurious oscillations, indicating numerical stability and proper coupling between the saturation and pressure fields. The results are consistent with those obtained via the MOL/FVM scheme, confirming that the truncation order and time integration tolerance are sufficient to accurately resolve the pressure field. Since the pressure drop evolution reflects changes in total mobility as the displacement progresses, we observe higher resistance at early times due to oil dominance, followed by a gradual redistribution of pressure gradients as water saturation increases. This behavior is consistent with the expected coupling between saturation-dependent permeability and flow resistance in porous media.

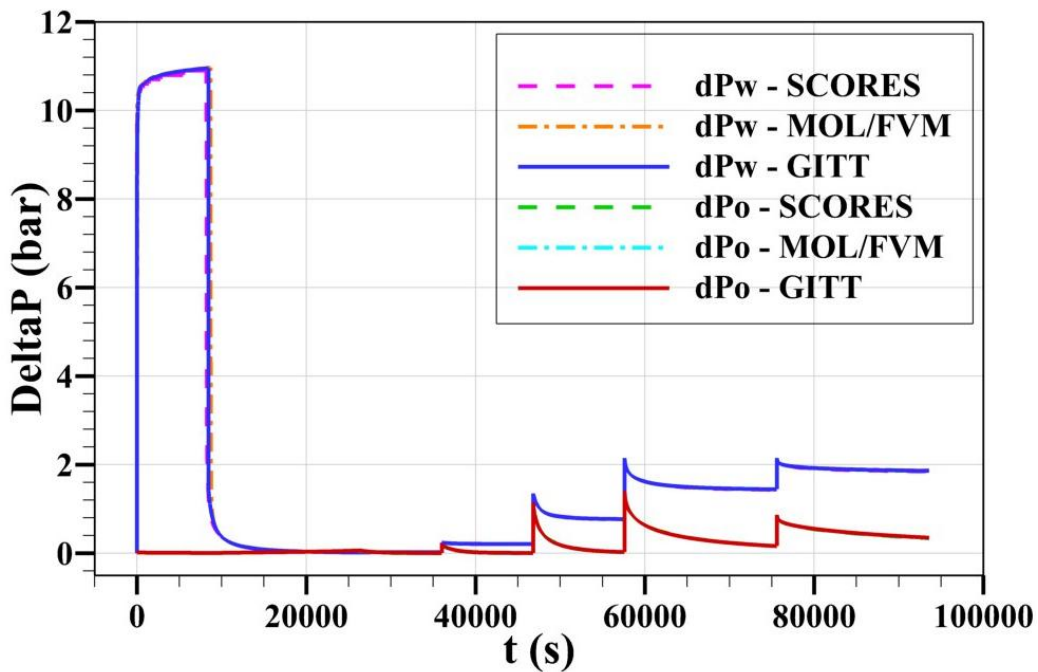


**Fig. 4.** Comparison of pressure drop distribution along the core length against those obtained with MOL/FVM and the SCORES [36] code.

The production rate curve computed via GITT is smooth and free of numerical noise, demonstrating that the temporal derivative of the average saturation is accurately captured (Fig. 5). The agreement with MOL/FVM and SCORES results confirms the robustness of the method in predicting transient production dynamics. The production rate exhibits a pronounced increase near breakthrough, corresponding to the arrival of the displacement front at the outlet, followed by a gradual decline as water production increases and oil mobility decreases. The smooth transition of the curve indicates the influence of capillary effects, which mitigate sharp discontinuities and promote a more diffused front behavior.

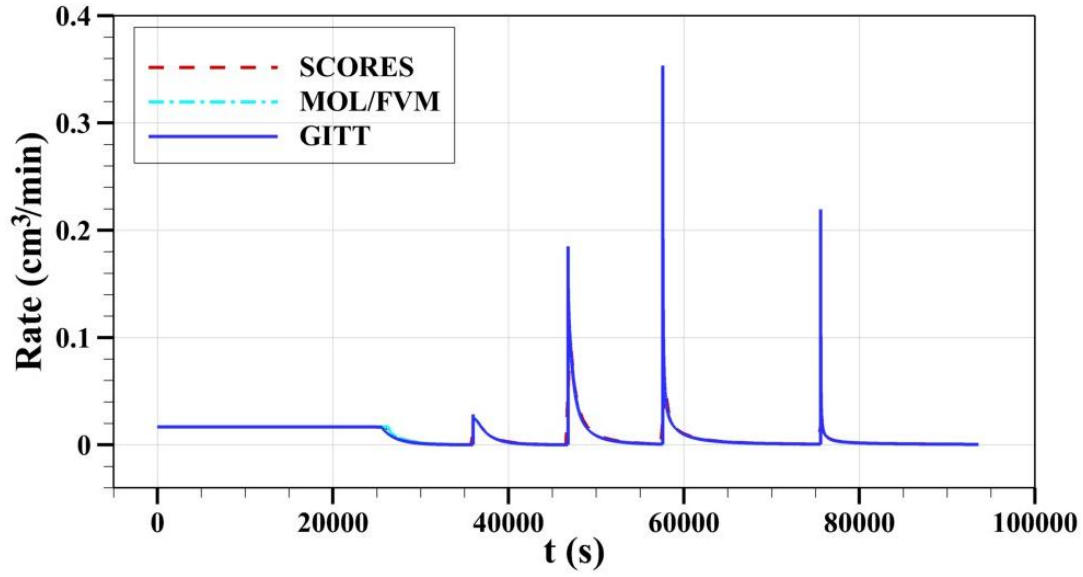


**Fig. 5.** Production rate distribution along the core length as a function of time: Comparison against results obtained with MOL/FVM and the SCORES [36] code.

In Fig. 6, the cumulative oil production predicted by GITT shows excellent agreement with the MOL/FVM and SCORES solutions, indicating accurate time integration and absence of cumulative numerical errors. The stability of the curve further confirms that the method reliably captures long-time behavior without loss of precision. The cumulative production curve initially exhibits a near-linear increase, corresponding to efficient oil displacement prior to breakthrough, followed by a reduction in slope as water production becomes dominant and oil recovery efficiency decreases. This behavior reflects the transition from oil-dominated to water-dominated flow, characteristic of immiscible displacement processes in porous media.

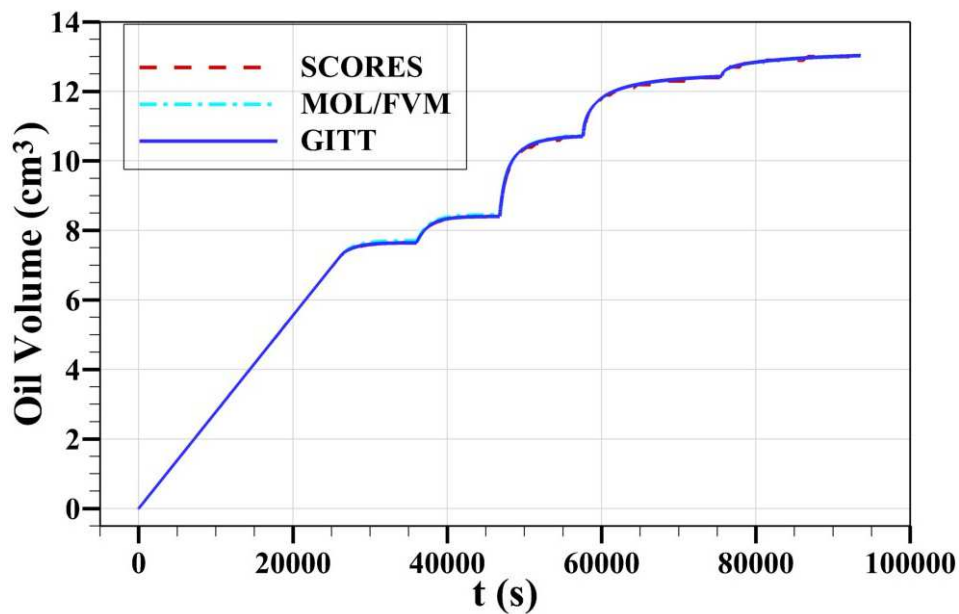


**Fig. 6.** Oil volume production along the core length as a function of time: Comparison against results obtained with MOL/FVM and the SCORES [36] code.

The final remarks on results and discussion show that, from a numerical perspective, the present GITT implementation is highly robust and efficient, achieving rapid spectral convergence with respect to the truncation order  $N$  while exhibiting minimal sensitivity to the quadrature parameter  $N_k$ . The hybrid analytical-numerical structure of the method allows for the accurate solution of the governing equations at relatively low computational cost, as only a moderate number of modes is required to achieve convergence to several significant digits. The use of the DIVPAG solver with a strict tolerance of  $10^{-8}$  ensures that temporal integration errors remain negligible, allowing the truncation error to dominate but in a controlled manner. Furthermore, the excellent agreement with both the MOL/FVM scheme and the SCORES [36] code confirms the reliability, stability, and consistency of the implementation, as well as the absence of spurious numerical diffusion or oscillations.

From a physical perspective, the GITT-based solution accurately captures the fundamental mechanisms governing immiscible displacement in porous media, including the nonlinear coupling between saturation and pressure fields, the smoothing of the displacement front by capillary pressure, and the dynamic evolution of production-related variables. The results consistently reproduce key flow features, including front propagation, mobility-driven redistribution of pressure gradients, and the transition from oil-dominated to water-dominated production regimes. In addition, the method provides a physically consistent description of both local and global quantities, such as spatial saturation profiles, pressure drop, production rate, and cumulative oil recovery, demonstrating its reliability and predictive capability for analyzing complex transport phenomena in porous media.

## 7. Conclusions

The present work develops a hybrid analytical–numerical solution for one-dimensional immiscible oil–water displacement that integrates realistic relative permeability and capillary pressure relations with the nonlinear capillary end-effect boundary conditions proposed by Chang and Yortsos [16]. The governing nonlinear convection-diffusion equation is treated directly using the Generalized Integral Transform Technique, without a prior Kirchhoff transformation or implicit filters for homogenizing nonlinear boundary conditions. By expanding the saturation field in terms of eigenfunctions of an auxiliary Sturm-Liouville problem, the original partial differential equation is reduced to a coupled system of nonlinear ordinary differential equations in modal space, in which nonlinear diffusion and boundary flux contributions are retained explicitly. The use of fictitious regions downstream and upstream to

accommodate end effects is therefore totally avoided, and the formal mathematical treatment of the boundary conditions is warranted. Thus, this approach preserves the physical structure of capillary end effects while providing controlled spectral convergence and high numerical accuracy. The methodology therefore bridges classical analytical displacement theory and modern computational simulation, offering both physical insight and a high-fidelity benchmark tool for laboratory core-flood interpretation and simulator validation. For the benchmark case considered, accurate saturation profiles were obtained with truncation orders below  $N=40$ , demonstrating rapid spectral convergence and significant computational savings relative to fully discretized schemes.

Beyond its role as a forward simulator, the present formulation is particularly relevant to inverse-problem analysis, where the objective is to determine the relative permeability and capillary pressure functions from unsteady-state core-flood measurements. Recent studies have emphasized the non-uniqueness and uncertainty associated with relative permeability and capillary pressure measurements [38,39,43], reinforcing the importance of high-fidelity forward models for experimental interpretation. In this context, the semi-analytical structure provided by the Generalized Integral Transform Technique offers a significant methodological advantage, as demonstrated here: accurate solutions can be obtained with relatively low truncation orders while maintaining strict control of convergence and stability. Such low-order truncations naturally lead to reduced-order models, which are particularly attractive for inverse analysis because they drastically reduce computational cost compared with fully discrete simulations. Therefore, the present GITT approach may provide a computationally efficient surrogate that accelerates the inverse determination of relative permeability and capillary pressure parameters while retaining the essential nonlinear physics of capillary-driven oil-water displacement.

## **Acknowledgments**

RMC would like to acknowledge partial financial support from CNPq, FAPERJ, and CAPES, all of which are research funding agencies in Brazil. PC would like to acknowledge CNPq through process no. 311143/2022-9. This research is associated with the ongoing R&D project registered as ANP no 23247-0, “Evaluation of Methodologies for Interpretation of Relative Permeability Curves in Heterogeneous Porous Media” (UFRJ/Petrobras/ANP), sponsored by Petrobras S/A under the ANP R&D levy as “Investment Commitment to Research and

Development”. The authors are grateful to Prof. Yannis Yortsos for kindly providing additional information on the nonlinear boundary conditions proposed in [16].

## References

1. Buckley, S. E., and Leverett, M. C., “Mechanism of Fluid Displacement in Sands,” Trans. AIME, Vol. 146, pp. 107–116, 1942.
2. Brooks, R. H., and Corey, A. T., Hydraulic Properties of Porous Media, Hydrology Paper no.3, p.37, Colorado State University, Fort Collins, 1964.
3. Bear, J., Dynamics of Fluids in Porous Media. Elsevier, New York, 1972.
4. Welge, H. L. (1952). A simplified method for computing oil recovery by gas or water drive. Transactions of the AIME, 195, 91–98.
5. Killough, J. E. (1976). Capillary pressure effects on one-dimensional two-phase flow. SPE Journal, 16(6), 353–362.
6. Morrow, N. R., and McCool, J. I. (1979). The influence of capillary pressure on laboratory displacement in consolidated porous media. SPE Journal, 19(3), 209–220.
7. de Swaan O. (1980). The effect of capillary pressure on immiscible displacement in porous media. Transport in Porous Media, 5(2), 85–105.
8. Wu, Y.-S., Chen, Z. X., and Pruess, K. (1993). Buckley-Leverett flow in composite porous media. SPE Advanced Technology Series, 1(1), 91–102. SPE-22329-PA.
9. Pasquier, S., Quintard, M., and Davit, Y. (2017). Modeling two-phase flow of immiscible fluids in porous media: Buckley-Leverett theory with explicit coupling terms. Physical Review Fluids, 2, 104101.
10. Kaasschieter, E. F. (1999). Solving the Buckley-Leverett equation with gravity in a heterogeneous porous medium. Computational Geosciences, 3, 23–48.
11. Aziz, K., and Settari, A., Petroleum Reservoir Simulation, Applied Science Publishers, 1979.
12. Huyakorn, P. S., and Pinder, G. F., Computational Methods in Subsurface Flow, Academic Press, 1983.
13. Eymard, R., Gallouët, T., and Herbin, R., “Finite Volume Methods,” Handbook of Numerical Analysis, Vol. VII, 2000.
14. Cesmeci, S., and Wheeler, M. F., “Mixed Finite Element Methods for Two-Phase Flow in Porous Media,” Computer Methods in Applied Mechanics and Engineering, 1992.
15. Sun, S., and Wheeler, M. F., “Discontinuous Galerkin Methods for Immiscible Two-Phase Flow in Porous Media,” SIAM Journal on Numerical Analysis, 2006.
16. Chang, J., and Yortsos, Y. C., “Effect of Capillary End Effects on Immiscible Displacement in Porous Media,” SPE Reservoir Engineering, Vol. 7, 1992, pp. 285–292.
17. Cotta, R.M., Hybrid numerical/analytical approach to nonlinear diffusion problems. Numerical Heat Transfer 17(2), 217–226, 1990.

18. Serfaty, R., and Cotta, R.M.: Hybrid analysis of transient non-linear convection-diffusion problems. *International Journal of Numerical Methods for Heat & Fluid Flow* 2(1), 55–62 (1992)
19. Cotta, R.M., *Integral Transforms in Computational Heat and Fluid Flow*. CRC Press, Boca Raton, FL, 1993.
20. Cotta, R.M., and Mikhailov, M.: Integral transform method. *Applied Mathematical Modelling* 17(3), 156–161 (1993)
21. Cotta, R.M., and Mikhailov, M.D.: *Heat Conduction: Lumped Analysis, Integral Transforms, Symbolic Computation*. John Wiley & Sons, New York (1997)
22. Cotta, R.M., *The Integral Transform Method in Thermal and Fluids Sciences and Engineering*. Begell House, New York, 1998.
23. Cotta, R.M., and Mikhailov, M.D.: Hybrid methods and symbolic computations. In: Minkowycz, W.J., Sparrow, E.M., Murthy, J.Y. (eds.) *Handbook of Numerical Heat Transfer*, 2nd edn. John Wiley, New York (2006). Chap. 16
24. Cotta, R.M., Naveira-Cotta, C.P., and Knupp, D.C.: Nonlinear eigenvalue problem in the integral transforms solution of convection-diffusion with nonlinear boundary conditions, *Int. J. Num. Meth. Heat & Fluid Flow*, Invited Paper, 25th Anniversary Special Issue, 26:3&4, 767-789 (2016).
25. Cotta, R.M., Knupp, D.C., and Quaresma, J.N.N.: Analytical methods in heat transfer. In: al., F.K. (ed.) *Handbook of Thermal Science and Engineering* vol. 1, pp. 61–126. Springer, Cham (2018). Chap. 2
26. Cotta, R.M., Lisboa, K.M., Curi, M.F., Balabani, S., Quaresma, J.N.N., Pérez Guerrero, J.S., Macêdo, E.N., and Amorim, N.S.: A review of hybrid integral transform solutions in fluid flow problems with heat or mass transfer and under Navier-Stokes equations formulations. *Num. Heat Transfer, Part B* 76, 60–87 (2019).
27. Almeida, A.R. and R.M. Cotta, "Integral Transform Methodology for Convection-Diffusion Problems in Petroleum Reservoir Engineering", *Int. J. Heat & Mass Transfer*, V.38, no.18, pp.3359-3367, 1995.
28. Almeida, A.R. and R.M. Cotta, "Analytical Solution of the Tracer Equation for the Homogeneous Five-Spot Problem", *SPE Journal*, Soc. Petroleum Eng., Paper # 29218, V. 1, no. 1, pp.31-38, 1996.
29. Deucher, R.H., Couto, P., and Bodstein, G.C.R.: Transient solution for the energy balance in porous media considering viscous dissipation and expansion/compression effects using integral transforms. *Transport in Porous Media* 116, 753–775 (2017)
30. Cotta, R.M., Lisboa, K.M., and Zotin, J.L.Z.: Integral transforms for flow and transport in discrete and continuum models of fractured heterogeneous porous media. *Advances in Water Resources* 142, 103621 (2020b)
31. Cotta, R.M., Naveira-Cotta, C.P., Genuchten, M.T., Su, J., and Quaresma, J.N.: Integral transform analysis of radionuclide transport in variably saturated media using a physical

- non-equilibrium model: application to solid waste leaching at a uranium mining installation. *Anais da Academia Brasileira de Ciências* 92, 20190427 (2020c)
32. Silva, E.M., Quaresma, J.N., Macêdo, E.N., and Cotta, R.M.: Integral transforms for three-dimensional pumping in confined, leaky, and unconfined aquifers. *Journal of Hydrology and Hydromechanics* 69(3), 319–331 (2021)
  33. Pelisoli, L.O., Cotta, R.M., Naveira-Cotta, C.P., and Couto, P.: Three-dimensional transient flow in heterogeneous reservoirs: Integral transform solution of a generalized point-source problem. *Journal of Petroleum Science and Engineering* 211, 109976 (2022)
  34. Pontedeiro, A.C., Cotta, R.M., and Su, J.: Thermal analysis of high burn-up nuclear fuel rod using generalized integral transform technique, 2007 International Nuclear Atlantic Conference - INAC 2007, pp.1-10, Santos, SP, Brazil, September 30-October 5 (2007).
  35. Cotta, R.M., Quaresma, J.N.N., Lima, J.A., and Lisboa, K.M.: Integral transforms in nonlinear convection, *Turbulência*, 11, pp.163-211, ABCM, Eds. H.F. Meier and J. Utzig, Edifurb, Blumenau, SC, Brazil (2022).
  36. Lenormand, R., Lorentzen, K., Maas, J.G., and Ruth, D.: Comparison of four numerical simulators for SCAL experiments. *Petrophysics* 58(01), 48–56 (2017)
  37. Skjaeveland, S.M., Siqveland, L.M., Kjosavik, A., Hammervold, W.L., Virnovsky, G.A., 1998. Capillary pressure correlation for mixed-wet reservoirs. In *SPE India Oil and Gas Conference and Exhibition*, New Delhi, India, no. SPE 39497, 17-19 February (1998).
  38. Berg, S., Dijk, H., Unsal, E., Hofmann, R., Zhao, B., and Ahuja, V.R.: Simultaneous determination of relative permeability and capillary pressure from an unsteady-state core flooding experiment? *Computers and Geotechnics* 168, 106091 (2024)
  39. Berg, S., Unsal, E., and Dijk, H.: Non-uniqueness and uncertainty quantification of relative permeability measurements by inverse modelling. *Computers and Geotechnics* 132, 103964 (2021)
  40. IMSL® Fortran Numerical Library Version 2024. IMSL by Perforce © 2024 Perforce Software, Inc (2024)
  41. Cotta, R.M., and Mikhailov, M.D.: Semi-analytical evaluation of integrals for the generalized integral transform technique, *Proc. of the 4th Workshop on Integral Transforms and Benchmark Problems – IV WIT*, pp.1-10, CNEN, Rio de Janeiro, RJ, August (2005).
  42. Sphaier, L.A., Cotta, R.M., Naveira-Cotta, C.P., and Quaresma, J.N.N.: The UNIT algorithm for solving one-dimensional convection-diffusion problems via integral transforms, *Int. Comm. Heat & Mass Transfer*, 38:5, 565-571 (2011).
  43. Jennen, H., Alves, R.J., Eler, F.M., and Müller, S.C., Handling Multimodality with Ensemble MCMC for Uncertainty Quantification of Relative Permeability and Capillary Pressure, in: *OTC Brasil*. Presented at the OTC Brasil, OTC, Rio de Janeiro, Brazil, p. D011S005R002 (2025).