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# Fractional-Order Framework for Gambling Dynamics with Delay and Control Strategies

**Poonam Yadav**
Birla Institute of Technology and Science, Pilani
**Komal Bansal**
komal@vitap.ac.in

VIT-AP University
**Trilok Mathur**
Birla Institute of Technology and Science, Pilani
**Shivi Agarwal**
Birla Institute of Technology and Science, Pilani

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# Fractional-Order Framework for Gambling Dynamics with Delay and Control Strategies

**Abstract**

This work develops a fractional-order mathematical model to investigate the dynamics of gambling behavior within a population, explicitly accounting for memory effects. Individuals are categorized into non-gamblers, occasional gamblers, addicted gamblers, and recovered individuals, with transitions driven by behavioral interactions and recovery mechanisms. The model utilizes the Caputo fractional derivative to represent nonlocal and hereditary features inherent in human decision-making. Analytical results establish existence, uniqueness, and boundedness of the solutions, and the basic reproduction number $\mathcal{R}_0$ is determined via the next-generation matrix approach. Stability analysis indicates that the gambler-free equilibrium is locally asymptotically stable when $\mathcal{R}_0 < 1$, whereas the gambler-persistence equilibrium remains stable for $\mathcal{R}_0 > 1$. A fractional-order delay extension is formulated to incorporate relapse behavior, and conditions for Hopf bifurcation are derived. Numerical simulations confirm the theoretical findings, highlighting the influence of fractional-order memory and time-delay effects on the persistence of gambling addiction. Additionally, the convergence behavior of the proposed numerical scheme for varying fractional orders $\eta$ is examined. This study provides a rigorous quantitative framework for designing behavioral intervention strategies and underscores the significance of fractional dynamics in modeling addiction processes.

**Keywords:** Caputo derivative; Gambling dynamics; Convergence Analysis; Delayed model; Hopf bifurcation.

## 1 Introduction

Gambling can be comprehensively defined as the practice of staking money or any valuable asset on an event whose outcome is uncertain, with the primary intent of securing additional financial or material gain [1]. It fundamentally operates on the principles of chance, uncertainty, and probability, distinguishing it from activities governed by certainty or predictable logic. Every form of gambling is characterized by three core elements: consideration, which denotes the value or amount wagered; risk, reflecting the unpredictability of the outcome; and reward, representing the potential profit or loss [2]. Individuals engage in gambling for varied reasons—some for excitement and leisure, others in pursuit of quick financial gain, and a few due to habitual or psychological influences. Nevertheless, despite its entertainment appeal, gambling can lead to serious social and economic repercussions, such as financial distress, addiction, and emotional instability, when not practiced responsibly [3].

In its broader sense, gambling encompasses several distinct forms, each varying in rules, intent, and degree of risk. One of the most prominent is casino gambling, which features games like roulette, blackjack, baccarat, poker, and slot machines. While many of these depend largely on luck, others integrate elements of strategy and probability, allowing skill to influence the outcome [4]. Over time, casinos have evolved from simple gambling houses into large-scale entertainment hubs that contribute significantly to tourism, employment, and national economies.

Another major category is sports betting, which involves predicting the results of sporting events such as cricket, football, horse racing, basketball, or tennis. Its widespread popularity stems from the thrill of competition and the opportunity to earn substantial rewards [5]. Modern digital platforms and mobile applications have further expanded their reach, allowing individuals to place bets conveniently and securely through online bookmakers.

Lotteries form a more traditional and socially accepted version of gambling, where participants buy

tickets for a random draw with hopes of winning cash or valuable prizes. Governments and charitable institutions often organize lotteries to raise funds for public welfare, education, and health initiatives. Similarly, bingo and raffles are commonly used for community entertainment or charitable causes rather than personal profit [6]. With rapid technological advancement, online gambling has become an increasingly dominant form, offering easy access to virtual casinos, poker rooms, and sports betting platforms [7]. Although this form enhances convenience and global participation, it also raises concerns regarding addiction and unregulated gaming practices. Additionally, financial gambling, though not always legally classified as such, includes speculative activities like stock trading, cryptocurrency investment, and forex trading—where individuals risk capital in pursuit of uncertain financial returns [8]. These activities mirror gambling in essence, as they revolve around risk-taking under uncertainty.

Beyond its recreational and economic dimensions, gambling exhibits profound connections with human cognition, learning, and memory. The act of gambling stimulates multiple neural and cognitive processes, especially those governing reward anticipation, reinforcement, and memory recall. When a gambler experiences a win, the dopaminergic reward system in the brain encodes this event as a positive memory, reinforcing future gambling behavior [9]. This process, known as reinforcement learning, explains why individuals tend to remember wins more vividly than losses, thereby sustaining optimism and overconfidence [10]. Such selective memory and the illusion of control distort rational judgment, encouraging repetitive gambling despite adverse outcomes [11]. Hence, exploring the intricate relationship between memory processes and gambling behavior is essential for understanding the cognitive mechanisms underlying gambling addiction and for designing effective psychological and behavioral interventions.

Hence, investigating the intricate interplay between memory processes and gambling behavior is crucial for comprehending the cognitive foundations of gambling addiction and for developing effective psychological and behavioral interventions. To systematically examine these behavioral dynamics, several scholars have employed mathematical modeling techniques. Such models seek to represent the temporal progression of gambling participation and addiction across populations. For instance, Do and Lee [12] proposed a deterministic framework based on classical differential equations to describe the transition dynamics between youth gamblers and non-gamblers. Their findings indicated that enhanced prevention strategies and treatment programs can markedly reduce the prevalence of problem gambling over time. Similarly, Lee [13] introduced a mathematical formulation for older adult gambling that incorporated awareness and relapse parameters to explore long-term behavioral evolution. Collectively, these integer-order models have provided valuable insights into the deterministic structure of gambling dynamics and have identified critical factors influencing the expansion or decline of gambling prevalence.

However, a fundamental limitation of these traditional models lies in their assumption that the rate of change in gambling populations depends solely on the current state, thereby disregarding the influence of past experiences and memory effects. In reality, gambling behavior is profoundly shaped by prior outcomes such as previous wins, losses, and emotional responses which significantly affect future decision-making. This shortcoming underscores the need to extend classical integer-order models toward a fractional-order framework, capable of incorporating memory and hereditary characteristics with greater fidelity and realism.

To capture the inherent behavioral complexities of gambling addiction, researchers have proposed delay differential models that account for the influence of past states on present system dynamics. Such models are particularly effective in describing behavioral processes where decision-making and recovery exhibit temporal lags rather than instantaneous responses. For instance, the time interval between experiencing a financial loss and choosing to refrain from further gambling can critically determine the persistence of addictive tendencies. By integrating these time delays, the models can more accurately characterize feedback mechanisms, relapse patterns, and sustained behavioral oscillations [14, 15].

Concurrently, optimal control theory has emerged as a robust mathematical framework for the analysis and regulation of complex behavioral systems influenced by social and psychological factors. Within this context, control variables may represent awareness initiatives, rehabilitation efforts, taxation schemes, or intervention policies designed to minimize addiction prevalence. Optimal control frameworks aim to identify the most effective combination of such strategies to curb undesirable behaviors while balancing economic feasibility and social welfare [16, 17]. This approach provides a rigorous quantitative foundation for policy assessment and efficient resource allocation toward sustainable behavioral management.

Nevertheless, both delay and optimal control models have been predominantly constructed within an integer-order framework, implicitly assuming that behavioral responses depend solely on immediate stimuli and decay exponentially over time. However, empirical evidence indicates that human behavior and decision-making often involve memory and hereditary effects that transcend these assumptions. This limitation has inspired the formulation of fractional-order models, which enrich the traditional deterministic paradigm by incorporating nonlocal and memory-dependent dynamics [18].

When fractional-order dynamics are integrated with delay structures and optimal control principles, they yield a more comprehensive and realistic representation of behavioral evolution. This hybrid modeling approach captures the intricate interplay among past experiences, delayed decisions, and control interventions, thereby offering deeper insight into the persistence of addictive behavior. Ultimately, such memory-driven frameworks hold significant promise for designing more effective prevention and treatment strategies grounded in a nuanced understanding of human behavioral memory.

The structure of this paper is outlined as follows: In Section 2, the proposed mathematical model is formulated and developed. Section 3 addresses the well-posedness of the model by proving the existence and uniqueness of its solution. The equilibrium points and the basic reproduction number are derived in Section 4, while their stability properties are analyzed in Section 5. A sensitivity analysis of key parameters is presented in Section 6. Numerical simulations that validate the analytical findings are discussed in Section 7, and the formulation and analysis of the delayed model are detailed in Section 8. Policy suggestions are provided in Section 9, and concluding remarks are presented in Section 10.

## 1.1 Preliminaries

Fractional calculus serves as a fundamental mathematical framework for modeling systems that exhibit memory and hereditary effects [19, 20]. By extending the concept of differentiation and integration to non-integer orders, it allows the dynamics of complex processes to be described more accurately than classical integer-order models. Several formulations of fractional operators have been proposed in the literature, among which the Riemann–Liouville and Caputo definitions are the most commonly employed. In the present study, the Caputo fractional derivative is adopted owing to its ability to incorporate physically meaningful initial conditions and its suitability for real-world modeling.

### Caputo Fractional Derivative

The Caputo fractional derivative of order ($\eta > 0$) is defined as

$$
{}^{C}D^{\eta} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\eta)} \displaystyle\int_0^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\eta+1-n}}\, d\xi, & n-1 < \eta < n \\ \dfrac{d^n f(t)}{dt^n}, & \eta = n \end{cases}
$$

where ($t > 0$) and ($n \in \mathbb{N}$). Unlike the Riemann–Liouville derivative, the Caputo definition satisfies the property that the derivative of a constant is zero and permits the specification of initial conditions in terms of classical integer-order derivatives. These features make it particularly suitable for modeling real-world phenomena and physical systems [21]. In this study, the notation ${}^{C}D_{t}^{\eta} f(t)$ is employed to denote the fractional derivative $\dfrac{d^{\eta} f(t)}{dt^{\eta}}$.

### Mittag–Leffler Function

The Mittag–Leffler function, introduced by G. M. Mittag-Leffler, is defined as [22]

$$\mathbb{E}_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + 1)}.$$

Its two-parameter generalization is given by [23]

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)}, \text{ where } \alpha > 0 \text{ and } \beta > 0.$$

The Mittag–Leffler function plays a central role in fractional calculus, as it naturally arises in the solutions of fractional differential equations and generalizes the exponential function to systems of non-integer order.

## 2 Mathematical Formulation of the Gambling Model

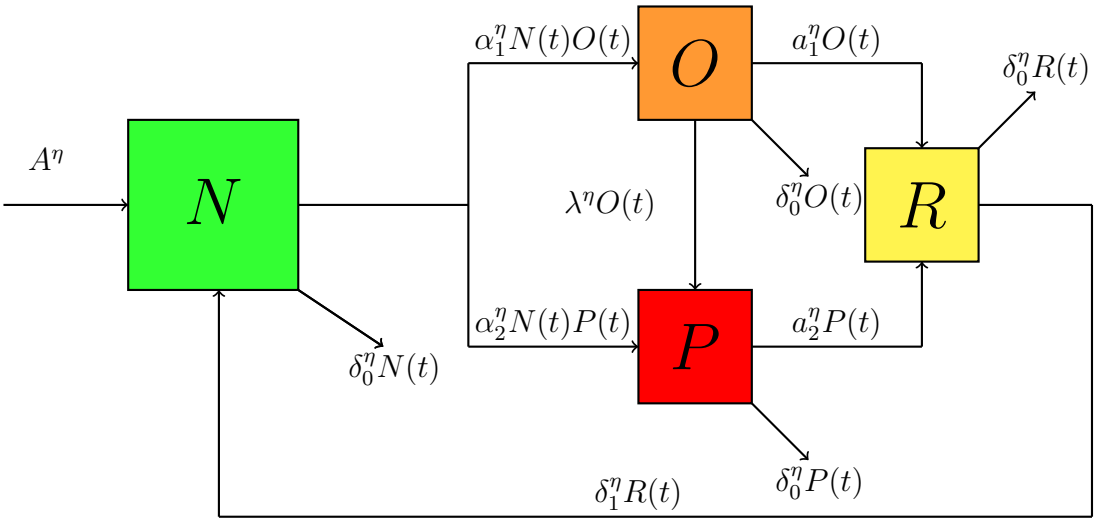



Figure 1: Flow diagram of the proposed model.

To describe the temporal evolution of gambling behavior within a population, the total population at time ($t$), denoted by $K(t) = N(t) + O(t) + P(t) + R(t)$, is partitioned into four mutually exclusive and collectively exhaustive compartments: the non-gambler population ($N(t)$), the occasional gambler population ($O(t)$), the addicted gambler population ($P(t)$), and the recovered population ($R(t)$). Individuals are recruited into the non-gambler class at a constant rate $A$, while all compartments experience a natural death rate represented by $\delta_0$. The behavioral transitions between these subpopulations are regulated by defined

interaction mechanisms. Specifically, non-gamblers are influenced by social exposure and interaction to transition into the occasional gambler class at rates $\alpha_1$ and $\alpha_2$. Occasional gamblers may progress to the addicted gambler class at a rate $\lambda$ or recover directly at a rate $a_1$. Addicted gamblers move to the recovered class at a rate $a_2$, while recovered individuals return to the non-gambler population at a rate of $\delta_1$.

To incorporate the effects of memory and hereditary influences inherent in human behavioral patterns, the model employs the Caputo fractional derivative of order $\eta \in (0, 1]$, which effectively captures the nonlocal and history-dependent characteristics of the system. Accordingly, the fractional-order model governing the transitions among the four behavioral classes is formulated as follows:

$$\begin{aligned}
\frac{d^\eta N(t)}{dt^\eta} =& A^\eta - \alpha_1^\eta N(t)O(t) - \alpha_2^\eta N(t)P(t) - \delta_0^\eta N(t) + \delta_1^\eta R(t),\\
\frac{d^\eta O(t)}{dt^\eta} =& \alpha_1^\eta N(t)O(t) + \alpha_2^\eta N(t)P(t) - \lambda^\eta O(t) - a_1^\eta O(t) - \delta_0^\eta O(t),\\
\frac{d^\eta P(t)}{dt^\eta} =& \lambda^\eta O(t) - a_2^\eta P(t) - \delta_0^\eta P(t),\\
\frac{d^\eta R(t)}{dt^\eta} =& a_1^\eta O(t) + a_2^\eta P(t) - \delta_1^\eta R(t) - \delta_0^\eta R(t).
\end{aligned} \tag{1}$$

$$N(0) = N_0,\ O(0) = O_0,\ P(0) = P_0,\ R(0) = R_0. \tag{2}$$

All parameters in the above system are assumed to be positive, and the fractional order ($\eta$) acts as a crucial measure of memory strength within the behavioral dynamics. It signifies that an individual's present gambling decisions are not purely instantaneous but are substantially influenced by past experiences, exposure patterns, and psychological tendencies. Furthermore, each parameter is raised to the power of $\eta$ to ensure dimensional homogeneity and maintain consistency across the fractional-order system.

Table 1: Parameter Description

| **Parameter** | **Description** |
|---|---|
| A | Recruitment rate of individuals into the non-gambler population |
| $\alpha_1$ | Rate at which non-gamblers become occasional gamblers due to social interaction |
| $\alpha_2$ | Rate at which non-gamblers become occasional gamblers due to exposure to addicted gamblers |
| $\lambda$ | Transition rate of occasional gamblers to the addicted gambler class |
| $a_1$ | Recovery rate of occasional gamblers without progressing to addiction |
| $a_2$ | Recovery rate of addicted gamblers moving to the recovered class |
| $\delta_0$ | Natural death rate |
| $\delta_1$ | Recovered individuals return to the non-gambler population |

# 3 Model Analysis

In this section, it is established that the proposed system (1) admits a unique, bounded, and positive solution within the domain $\Omega \subset \mathbb{R}^4_+$, defined as

$$\Omega = \left\{ (N, O, P, R) \in \mathbb{R}^4_+ : N + O + P + R \leq \frac{A^\eta}{\delta_0^\eta} \right\}.$$

### 3.1 Positively Invariant Region

**Theorem 1.** *The region $\Omega \subset \mathbb{R}^4_+$ acts as a positively invariant domain for the proposed model (1), guaranteeing that all initial conditions remain non-negative for all time.*

*Proof.* Let $K(t)$ represent the total population at time $t$, defined as $K(t) = N(t) + O(t) + P(t) + R(t)$. Then,

$$\begin{aligned} \frac{d^\eta K(t)}{dt^\eta} &= \frac{d^\eta N(t)}{dt^\eta} + \frac{d^\eta O(t)}{dt^\eta} + \frac{d^\eta P(t)}{dt^\eta} + \frac{d^\eta R(t)}{dt^\eta}, \\ \frac{d^\eta K(t)}{dt^\eta} &= A^\eta - \delta_0^\eta K(t). \end{aligned} \tag{3}$$

Using the Laplace transformation [*]:

$$X(s) = \frac{\dfrac{A^\eta}{s} + s^{\eta-1}X(0)}{s^\eta + \delta_0^\eta}.$$

Now, using the inverse Laplace transformation[†]:

$$X(t) = \frac{A^\eta}{\delta_0^\eta}[1 - \mathbb{E}_\eta(-\delta_0^\eta t^\eta)] + X(0)\mathbb{E}_\eta(-\delta_0^\eta t^\eta).$$

as $0 \leq \mathbb{E}_\eta(-\delta_0^\eta t^\eta) \leq 1$, $\mathbb{E}_\eta(-x)$ is completely monotonic for $x > 0$ if $0 < \eta \leq 1$ [22, 24].

$$\implies X(t) \leq \frac{A^\eta}{\delta_0^\eta}. \tag{4}$$

To establish the non-negativity of solutions for the proposed model (1) the following analysis is presented:

$$\begin{aligned} \left.\frac{d^\eta N(t)}{dt^\eta}\right|_{N(t_0)=0} &= A^\eta + \delta_1^\eta R(t) \geq 0, \\ \left.\frac{d^\eta O(t)}{dt^\eta}\right|_{O(t_0)=0} &= \alpha_2^\eta N(t)P(t) \geq 0, \\ \left.\frac{d^\eta P(t)}{dt^\eta}\right|_{P(t_0)=0} &= \lambda^\eta O(t) \geq 0, \\ \left.\frac{d^\eta R(t)}{dt^\eta}\right|_{R(t_0)=0} &= a_1^\eta O(t) + a_2^\eta P(t) \geq 0. \end{aligned}$$

Now, using the generalized mean value theorem [‡] , the solution of the proposed model 1 is non-negative.

[*] $$\mathcal{L}[D^\eta f(t)] = s^\eta F(s) - \sum_{k=1}^{n} s^{\eta-k} f^{(k-1)}(0) \quad \text{where } n-1 < \eta \leq n.$$

[†] $$\mathcal{L}[t^{\gamma-1}\mathbb{E}_{\eta,\gamma}(at^\eta)] = \frac{s^{\eta-\gamma}}{s^\eta - a}.$$

[‡] **Generalized Mean Value Theorem** Consider $f(t) \in \mathbf{C}[0, b]$ and $\frac{d^\eta f(t)}{dt^\eta} \in \mathbf{C}[0, b]$ for $0 < \eta \leq 1$. Then, if

It is demonstrated that the solution of the proposed model remains non-negative for all time. Hence, under non-negative initial conditions, the solution persists as positive and remains bounded within the region $\Omega$. This confirms that $\Omega$ serves as a positively invariant region, ensuring that all trajectories of the model are ultimately confined within it. □

### 3.2 Existence and Uniqueness Analysis

**Theorem 2.** *The proposed model (1) admits a unique solution for non-negative initial conditions when $t \geq 0$, and furthermore, all such solutions remain bounded for all time.*

*Proof.* The right-hand side of the model (1) is continuous and bounded, being represented in polynomial form; therefore, it satisfies the Lipschitz condition. Since polynomial functions are inherently continuous and Lipschitz continuous over any closed and bounded interval, this ensures the existence of a finite Lipschitz constant. Moreover, the boundedness of the right-hand side implies that the Lipschitz constant can be evaluated as the maximum absolute value of the polynomial's derivative within the domain of interest. Therefore, based on Lemma 2 [§], it can be concluded that the solution of the model (1), under appropriate initial conditions, not only exists but is also unique and bounded for all $t \geq 0$. □

## 4 Analysis of Equilibrium Points and Basic Reproduction Number

To determine the equilibrium points, the right-hand sides of the equations in the proposed model (1) are set to zero. Consequently, the system admits two distinct equilibrium points corresponding to the model (1).

- Gambler-free equilibrium exists when the gambler population reduces to zero, i.e.

$$E_1 = \left( \frac{A^\eta}{\delta_0^\eta}, 0, 0, 0 \right).$$

- **Basic Reproduction Number:** The basic reproduction number, denoted by $\mathcal{R}_0$, represents the expected number of secondary infections generated by a single infected individual in a completely susceptible population [26]. The value of $\mathcal{R}_0$ is computed using the next-generation matrix approach evaluated at the gambler-free equilibrium point. Let

$$D^\eta X = f(X) - v(X),$$

$$X = [O, P].$$

  The $v(X)$ represents the terms associated with the inflow and outflow of individuals within a given class, whereas $u(X)$ denotes the terms corresponding to the generation of new gamblers.

- $\frac{d^\eta f(t)}{dt^\eta} \geq 0 \; \forall \; t \in (0, b]$, then $f(t)$ is non-decreasing,
- $\frac{d^\eta f(t)}{dt^\eta} \leq 0 \; \forall \; t \in (0, b]$, then $f(t)$ is non-increasing [25].

[§] **Lemma 2** Consider the fractional differential equation with the Caputo derivative $\frac{d^\eta f(t)}{dt^\eta} = y(t, x), t > t_0, 0 < \eta \leq 1$ and $y : [t_0, \infty] \times \Omega \to R^n, \Omega \in R^n$. Then there exists a unique solution on $[t_0, \infty] \times \Omega$ if $y(t, x)$ is continuous and bounded and assumes that it satisfies a Lipschitz condition with respect to the second variable $x$ [18].

The basic reproduction number, is determined as the dominant (largest) eigenvalue of the matrix product $UV^{-1}$, where

$$U = \left(\frac{\partial u}{\partial X}\right)_{E_1}, \; V = \left(\frac{\partial v}{\partial X}\right)_{E_1}.$$

The Jacobian matrix of $U$ at $E_1$

$$U = \begin{bmatrix} \frac{\alpha_1^\eta A^\eta}{\delta_0^\eta} & \frac{\alpha_2^\eta A^\eta}{\delta_0^\eta} \\ 0 & 0 \end{bmatrix}.$$

The Jacobian matrix of $V$ at $E_1$

$$V = \begin{bmatrix} \lambda^\eta + a_1^\eta + \delta_0^\eta & 0 \\ -\lambda^\eta & a_2^\eta + \delta_0^\eta \end{bmatrix}.$$

The basic reproduction number is formulated as follows:

$$\mathcal{R}_0 = \frac{A^\eta[\alpha_1^\eta(a_2^\eta + \delta_0^\eta) + \alpha_2^\eta\lambda^\eta]}{\delta_0^\eta(a_1^\eta + \delta_0^\eta + \lambda^\eta)(a_2^\eta + \delta_0^\eta)}. \tag{5}$$

- A Gambler-persistence equilibrium point exists whenever the gambler population is non-zero, i.e.

$$E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R}).$$

Where

$$\tilde{N} = \frac{(\lambda^\eta + a_1^\eta + \delta_0^\eta)(a_2^\eta + \delta_0^\eta)}{\alpha_1^\eta(a_2^\eta + \delta_0^\eta) + \alpha_2^\eta\lambda^\eta}, \; \tilde{P} = \frac{\lambda^\eta\tilde{O}}{a_2^\eta + \delta_0^\eta},$$

$$\tilde{R} = \frac{[a_1^\eta(a_2^\eta + \delta_0^\eta) + a_2^\eta\lambda^\eta]\tilde{O}}{(a_2^\eta + \delta_0^\eta)(\delta_0^\eta + \delta_1^\eta)},$$

and

$$\tilde{O} = \frac{(\delta_0^\eta + \delta_1^\eta)(a_2^\eta + \delta_0^\eta)^2(\lambda^\eta + a_1^\eta + \delta_0^\eta)(\mathcal{R}_0 - 1)}{(\lambda^\eta + a_1^\eta + \delta_0^\eta)(a_2^\eta + \delta_0^\eta) + \delta_1^\eta(a_2^\eta + \delta_0^\eta) + \delta_1^\eta\lambda^\eta}.$$

Thus, the Gambler-persistence equilibrium point exists whenever $\mathcal{R}_0 > 1$.

# 5 Stability Analysis

**Theorem 3.** *The Gambler-free equilibrium point* $E_1 = \left(\frac{A^\eta}{\delta_0^\eta}, 0, 0, 0\right)$ *of the proposed model (1) is locally asymptotically stable whenever* $\mathcal{R}_0 < 1$.

*Proof.* The Jacobian matrix at $E_1$ is

$$J_{E_1} = \begin{bmatrix} -\delta_0^\eta & -\frac{\alpha_1^\eta A^\eta}{\delta_0^\eta} & -\frac{\alpha_2^\eta A^\eta}{\delta_0^\eta} & \delta_1^\eta \\ 0 & \frac{\alpha_1^\eta A^\eta}{\delta_0^\eta} - (\lambda^\eta + \delta_0^\eta + a_1^\eta) & \frac{\alpha_2^\eta A^\eta}{\delta_0^\eta} & 0 \\ 0 & \lambda^\eta & -a_2^\eta - \delta_0^\eta & 0 \\ 0 & a_1^\eta & a_2^\eta & -\delta_0^\eta - \delta_1^\eta \end{bmatrix}.$$

The two eigenvalues, $-\delta_0^\eta$ and $-\delta_0^\eta - \delta_1^\eta$, are both strictly negative. Now, for the remaining eigenvalue:

$$J^*_{E_1} = \begin{bmatrix} \frac{\alpha_1^\eta A^\eta}{\delta_0^\eta} - (\lambda^\eta + \delta_0^\eta + a_1^\eta) & \frac{\alpha_2^\eta A^\eta}{\delta_0^\eta} \\ \lambda^\eta & -a_2^\eta - \delta_0^\eta \end{bmatrix}.$$

The characteristic equation for the matrix $J^*_{E_1}$ is expressed as follows:

$$\Lambda^2 + l_1\Lambda + l_2 = 0 \tag{6}$$

where

$$l_1 = (\lambda^\eta + \delta_0^\eta + a_1^\eta)(1 - \mathcal{R}_0) + (a_2^\eta + \delta_0^\eta) + \frac{\alpha_2^\eta A^\eta \lambda^\eta}{\delta_0^\eta (a_2^\eta + \delta_0^\eta)},$$
$$l_2 = (\lambda^\eta + \delta_0^\eta + a_1^\eta)(a_2^\eta + \delta_0^\eta)(1 - \mathcal{R}_0).$$

According to the Matignon stability criterion,[¶], if $l_1$, and $l_2$ are all positive, then all eigenvalues of the system possess negative real parts. Therefore, the proposed model (1) remains stable whenever $\mathcal{R}_0 < 1$. □

**Theorem 4.** *The Gambler-persistence equilibrium point $E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$ in the proposed model (1) is locally asymptotically stable whenever $\mathcal{R}_0 > 1$.*

*Proof.* The Jacobian matrix at $E_2$ is

$$J_{E_2} = \begin{bmatrix} -\alpha_1^\eta \tilde{O} - \alpha_2^\eta \tilde{P} - \delta_0^\eta & -\alpha_1^\eta \tilde{N} & -\alpha_2^\eta \tilde{N} & \delta_1^\eta \\ \alpha_1^\eta \tilde{O} + \alpha_2^\eta \tilde{P} & \alpha^\eta \tilde{N} - \lambda^\eta - a_1^\eta - \delta_0^\eta & \alpha_2^\eta \tilde{N} & 0 \\ 0 & \lambda^\eta & -a_2^\eta - \delta_0^\eta & 0 \\ 0 & a_1^\eta & a_2^\eta & -\delta_1^\eta - \delta_0^\eta \end{bmatrix}.$$

The characteristic equation for the matrix $J_{E_2}$ is expressed as follows:

$$\Lambda^4 + m_1\Lambda^3 + m_2\Lambda^2 + m_3\Lambda + m_4 = 0. \tag{7}$$

For the equilibrium point $E_2$ to be locally asymptotically stable, the Matignon criterion must be fulfilled, which demands that all Hurwitz determinants ($H_i$) (for ($i = 1, 2, \ldots, 4$)) are positive [27]. Hence, the Gambler-persistence equilibrium becomes stable provided the following conditions are satisfied:

$$m_i > 0 \; i = 1, 2, 3, 4, \; m_1 m_2 m_3 > m_3^2 + m_1^2 m_4. \tag{8}$$

These criteria guarantee that all eigenvalues of the system possess strictly negative real parts, and conse-

[¶]**Matignon Criteria [27]** Consider our model described by the following fractional differential equation:

$$\frac{d^\eta X(t)}{dt^\eta} = F(X(t)),$$

where $X(t) = (x_1(t); x_2(t); \ldots; x_n(t))^T$ and $F(X(t)) = (f_1; f_2; \ldots; f_n)^T$. The equilibrium point is denoted by $E^* = (x_1^*; x_2^*; \ldots; x_n^*)^T$. For this fractional-order system, the equilibrium points are considered asymptotically stable if all eigenvalues ($\lambda_j$) of the Jacobian matrix $J$ evaluated at equilibrium $E^*$ satisfy the criterion:

$$|\arg(\text{eig}(J))| = |\arg(\lambda_j)| > \frac{\pi}{2}\eta,$$

where $j = 1, 2, \ldots, n$, $J$ is the Jacobian matrix of the system at the equilibrium point $E^*$, and $\lambda_j$ is eigenvalues of the Jacobian matrix.

quently, the equilibrium point is locally asymptotically stable. □

# 6 Sensitivity Analysis

This study investigates how the parameters embedded within the proposed model (1) affect the behavioural dynamics of individuals. In particular, the threshold parameter $\mathcal{R}_0$ plays a pivotal role in determining the persistence of gambling behaviour. To quantify its sensitivity with respect to each governing parameter, the partial derivatives of $\mathcal{R}_0$ known as elasticities are computed accordingly.

Table 2: Sensitivity analysis of $\mathcal{R}_0$ for all parameters

| **Metric** | $\frac{\partial \mathcal{R}_0}{\partial A}$ | $\frac{\partial \mathcal{R}_0}{\partial \alpha_1}$ | $\frac{\partial \mathcal{R}_0}{\partial \alpha_2}$ | $\frac{\partial \mathcal{R}_0}{\partial a_1}$ | $\frac{\partial \mathcal{R}_0}{\partial a_2}$ | $\frac{\partial \mathcal{R}_0}{\partial \lambda}$ | $\frac{\partial \mathcal{R}_0}{\partial \delta_0}$ |
|---|---|---|---|---|---|---|---|
| **Sensitivity Index Sign** | Positive | Positive | Positive | Negative | Negative | Negative | Negative |

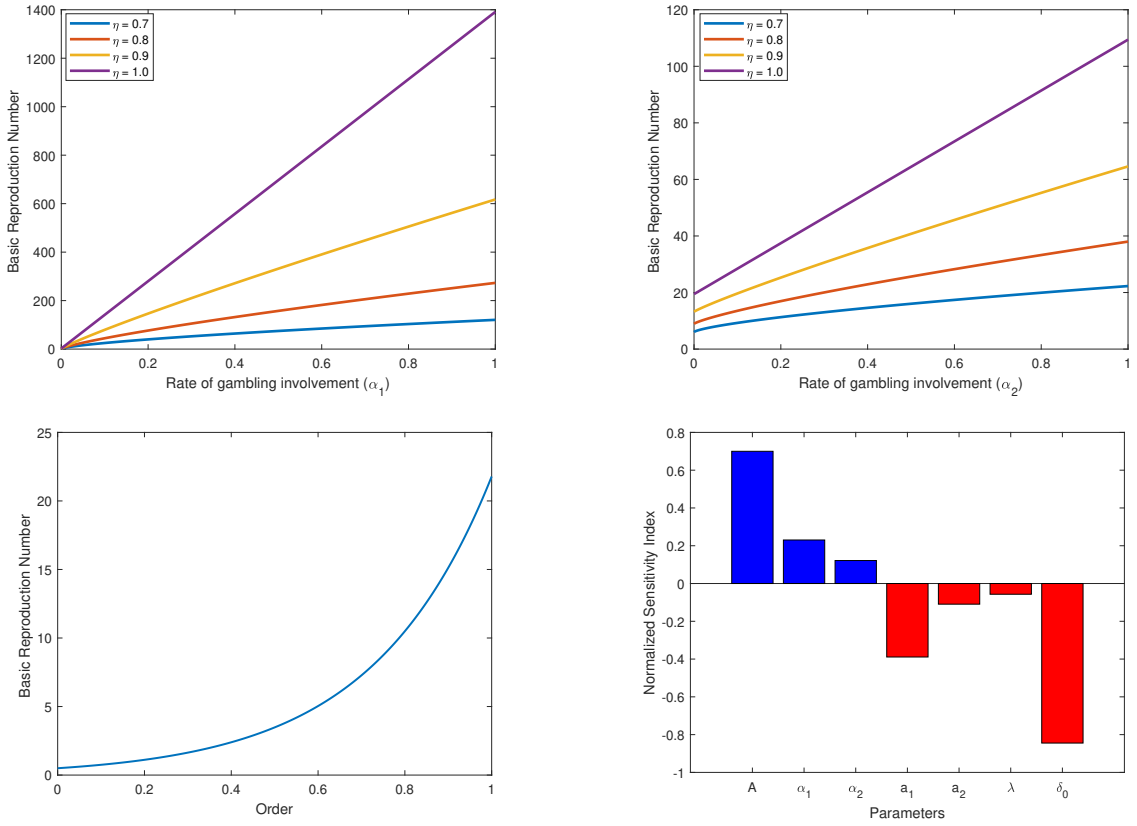



Figure 2: Sensitivity of the basic reproduction number $\mathcal{R}_0$ with respect to key model parameters.

$$
\begin{aligned}
\Phi_H =& \frac{H}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial H}, \\
\Phi_A =& \frac{A}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial A} = \eta, \\
\Phi_{\alpha_1} =& \frac{\alpha_1}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial \alpha_1} = \frac{\eta \alpha_1^\eta (a_2^\eta + \delta_0^\eta)}{a_2^\eta \alpha_1^\eta + \delta_0^\eta \alpha_1^\eta + \lambda^\eta \alpha_2^\eta}, \\
\Phi_{\alpha_2} =& \frac{\alpha_2}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial \alpha_2} = \frac{\eta \alpha_2^\eta \lambda^\eta}{a_2^\eta \alpha_1^\eta + \delta_0^\eta \alpha_1^\eta + \lambda^\eta \alpha_2^\eta}, \\
\Phi_{a_1} =& \frac{a_1}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial a_1} = -\frac{\eta a_1^\eta}{a_1^\eta + \delta_0^\eta + \lambda^\eta}, \\
\Phi_{a_2} =& \frac{a_2}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial a_2} = -\frac{\eta a_2^\eta \alpha_2^\eta \lambda^\eta}{(a_2^\eta + \delta_0^\eta)(a_2^\eta \alpha_1^\eta + \delta_0^\eta \alpha_1^\eta + \alpha_2^\eta \lambda^\eta)}, \\
\Phi_\lambda =& \frac{\lambda}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial \lambda} = \frac{\eta \lambda^\eta (a_1^\eta \alpha_2^\eta - a_2^\eta \alpha_1^\eta - \alpha_1^\eta \delta_0^\eta + \alpha_2^\eta \delta_0^\eta)}{(a_2^\eta \alpha_1^\eta + \delta_0^\eta \alpha_1^\eta + \lambda^\eta \alpha_2^\eta)(a_1^\eta + \lambda^\eta + \delta_0^\eta)}, \\
\Phi_{\delta_0} =& \frac{\delta_0}{\mathcal{R}_0} \cdot \frac{\partial \mathcal{R}_0}{\partial \delta} = -\frac{\eta \delta_0^\eta}{a_1^\eta + \delta_0^\eta + \lambda^\eta} + \frac{\eta \alpha_1^\eta \delta_0^\eta}{\alpha_1^\eta (a_2^\eta + \delta_0^\eta) + \alpha_2^\eta \lambda^\eta} - \frac{\eta \delta_0^\eta}{a_2^\eta + \delta_0^\eta} - \eta.
\end{aligned}
$$

Figure 2 and Table 2 illustrates the normalized sensitivity indices of the basic reproduction number with respect to all model parameters. The parameters $A, \alpha_1, \alpha_2$ possess positive sensitivity values, indicating that increasing recruitment into the system or intensifying social and addictive influences leads to a rise in gambling prevalence. Conversely, the recovery parameters $a_1, a_2$, transition rate $\lambda$, and the natural death rate $\delta_0$ display negative sensitivity indices, showing that improvements in recovery or removal of gamblers substantially reduce $\mathcal{R}_0$. This figure highlights the parameters to prioritize for intervention strategies especially those with the highest magnitude of sensitivity.

# 7 Numerical simulations

This section presents numerical simulations to validate the theoretical results and emphasize the significance of the $\mathcal{R}_0$ threshold value. These simulations enhance understanding of the interactions among different groups of individuals, as well as the characteristics and dynamics of gambling at the population level. Table 3 lists the variables and parameters used for the evaluation [12, 13, 28–30]. The model is solved using predictor-corrector method of Adams-Bashforth-Moulton [31].

Table 3: Parameter Description

| **Parameter** | **Description** | **Value** |
|---|---|---|
| $A$ | Recruitment rate of individuals into the non-gambler population | 2.25 |
| $\alpha_1$ | Rate at which non-gamblers become occasional gamblers | 0.014 |
| $\alpha_2$ | Rate at which non-gamblers become occasional gamblers | 0.026 |
| $\lambda$ | Transition rate of occasional gamblers to the addicted gambler class | 0.023 |
| $a_1$ | Recovery rate of occasional gamblers without progressing to ad- diction | 0.070 |
| $a_2$ | Recovery rate of addicted gamblers moving to the recovered class | 0.34 |
| $\delta_0$ | Natural death rate | 0.015 |
| $\delta_1$ | Recovered individuals return to the non-gambler population | 0.21 |

Figure 3 presents the simulation results of the gambling system for four different fractional orders: $\eta = 0.7$, $0.8$, $0.9$, and $1.0$. As the fractional order decreases, the memory effect in the system becomes stronger, which delays the response of the behavioural dynamics. For $\eta < 1$, both the occasional gambler population $O(t)$ and the addicted gambler population $P(t)$ exhibit slower growth and more gradual convergence toward equilibrium. In contrast, the classical integer-order case $\eta = 1$ shows the fastest dynamics, where all populations stabilize more rapidly. These observations confirm the significance of fractional-order memory in moderating behavioral transitions: stronger memory effects slow down both addiction progression and recovery processes, resulting in more realistic temporal behavior compared to the integer-order model.

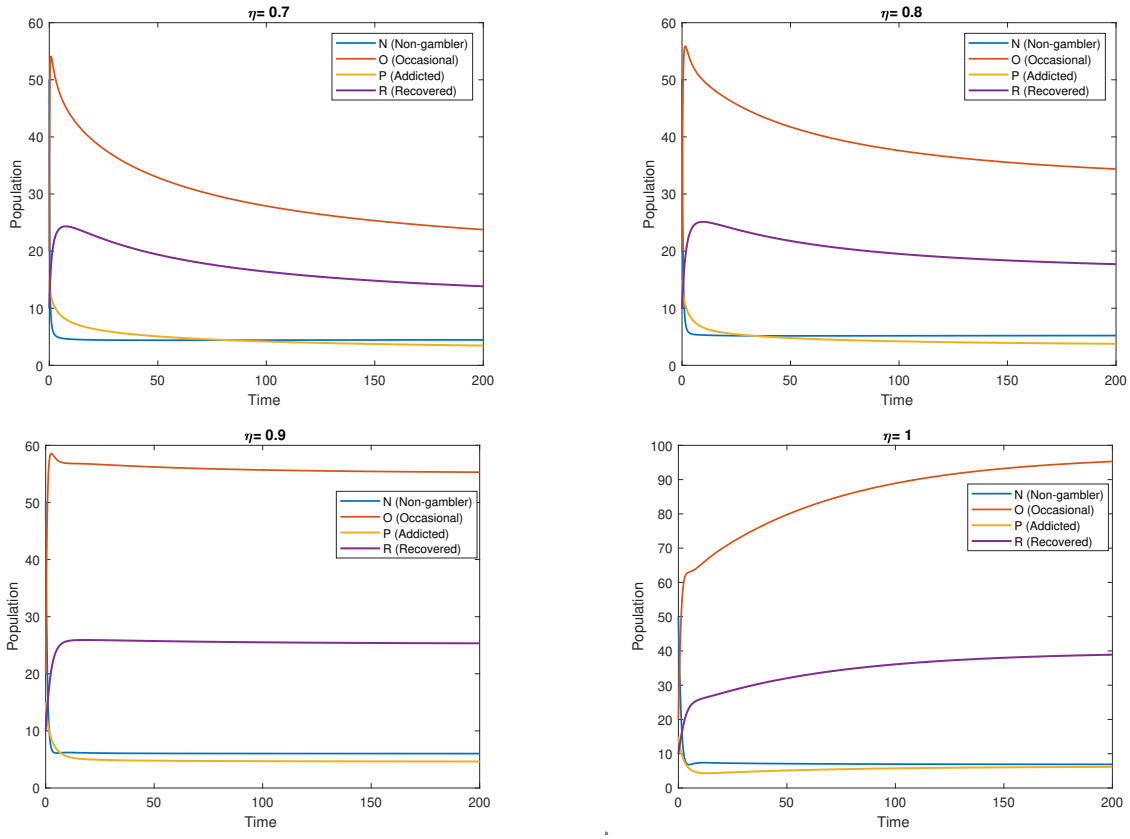



Figure 3: Time evolution of the four compartments $N(t), O(t), P(t), R(t)$ for different values of the fractional order $\eta$. Lower values of $\eta$ reflect stronger memory effects.

## 7.1 Error and Convergence Analysis

To ensure the reliability of the numerical simulations presented in this study, an error and convergence analysis is carried out for the proposed fractional-order model. This analysis examines how the numerical solution behaves under mesh refinement and verifies whether the employed numerical scheme achieves the expected rate of convergence. By comparing the $L_2$ errors for successive step sizes, the consistency and accuracy of the method can be quantitatively assessed.

Figure 4 shows the convergence pattern of the numerical method used for solving the fractional-order system. In each panel, the $L_2$ error decreases steadily as the mesh size $h$ becomes smaller. The slopes of the curves confirm that the numerical method achieves an approximate second-order rate of convergence for all

fractional orders. This validates the accuracy and stability of the computational technique and supports the reliability of the numerical simulations presented.

Table 4: Convergence results and rate of convergence

| $h$ | $L_2$ Error | Rate of Convergence $p$ |
|---|---|---|
| 0.05000 | $2.446827 \times 10^{-1}$ | – |
| 0.02500 | $6.281313 \times 10^{-2}$ | 1.961 |
| 0.01250 | $1.753634 \times 10^{-2}$ | 1.843 |
| 0.00625 | $5.105756 \times 10^{-3}$ | 1.782 |

Table 5: Convergence results and rate of convergence

| $h$ | $L_2$ Error | Rate of Convergence $p$ |
|---|---|---|
| 0.05000 | $7.399777 \times 10^{-2}$ | – |
| 0.02500 | $1.957478 \times 10^{-2}$ | 1.918 |
| 0.01250 | $5.374395 \times 10^{-3}$ | 1.864 |
| 0.00625 | $1.504964 \times 10^{-3}$ | 1.835 |

Table 6: Convergence results and rate of convergence

| $h$ | $L_2$ Error | Rate of Convergence $p$ |
|---|---|---|
| 0.05000 | $2.728630 \times 10^{-2}$ | – |
| 0.02500 | $7.020625 \times 10^{-3}$ | 1.956 |
| 0.01250 | $1.840521 \times 10^{-3}$ | 1.931 |
| 0.00625 | $4.872397 \times 10^{-4}$ | 1.918 |

Table 7: Convergence results and rate of convergence

| $h$ | $L_2$ Error | Rate of Convergence $p$ |
|---|---|---|
| 0.05000 | $1.109133 \times 10^{-2}$ | – |
| 0.02500 | $2.720748 \times 10^{-3}$ | 2.018 |
| 0.01250 | $6.738371 \times 10^{-4}$ | 2.013 |
| 0.00625 | $1.676753 \times 10^{-4}$ | 2.007 |

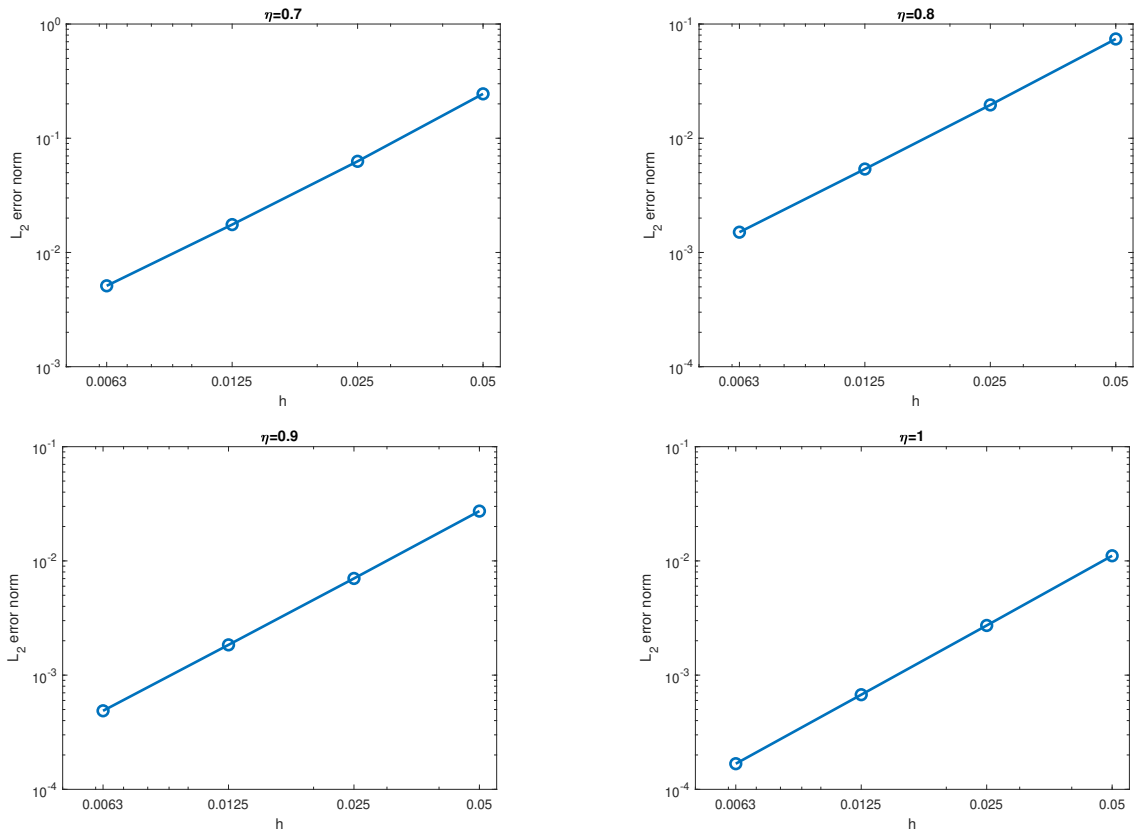



Figure 4: Convergence behaviour of the proposed numerical method for different fractional orders $\eta$. The $L_2$ error decreases with decreasing step size $h$, confirming second-order convergence.

# 8 Delayed Model

The proposed model is a delayed fractional-order compartmental system that describes the evolution of gambling behaviour in a population. A time delay ($\tau$) is introduced in the transition from addiction to recovery, reflecting the realistic duration of rehabilitation programs and behavioural therapy. The model captures the influence of social interactions, relapse possibility, gradual addiction progression, and delayed recovery, providing a realistic framework for analysing gambling addiction and evaluating intervention strategies.

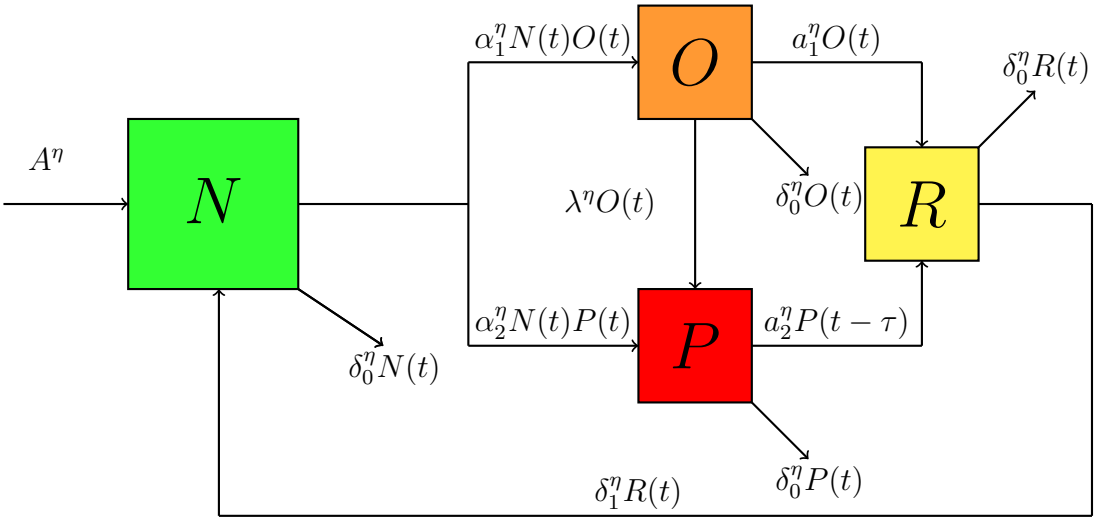



Figure 5: Flowchart depicting the dynamics of the delayed model.

$$
\begin{aligned}
\frac{d^\eta N(t)}{dt^\eta} &= A^\eta - \alpha_1^\eta N(t)O(t) - \alpha_2^\eta N(t)P(t) - \delta_0^\eta N(t) + \delta_1^\eta R(t),\\
\frac{d^\eta O(t)}{dt^\eta} &= \alpha_1^\eta N(t)O(t) + \alpha_2^\eta N(t)P(t) - \lambda^\eta O(t) - a_1^\eta O(t) - \delta_0^\eta O(t),\\
\frac{d^\eta P(t)}{dt^\eta} &= \lambda^\eta O(t) - a_2^\eta P(t-\tau) - \delta_0^\eta P(t),\\
\frac{d^\eta R(t)}{dt^\eta} &= a_1^\eta O(t) + a_2^\eta P(t-\tau) - \delta_1^\eta R(t) - \delta_0^\eta R(t).
\end{aligned}
\tag{9}
$$

## 8.1 Stability Analysis of the Fractional-Order Delayed Model

The persistence equilibrium of the system (9) is represented by $E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$, where the equilibrium components $(\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$ are obtained from section 4. In this subsection, we investigate how the stability of the system varies with respect to the increasing time delay. Accordingly, the characteristic equation of the system (9), evaluated at the equilibrium $E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$, is given by:

$$
|J_{\tau_0} + e^{-\Lambda\tau} J_{\tau_1} - \Lambda I| = 0. \tag{10}
$$

$$
J_{\tau_0} = \begin{bmatrix}
-\alpha_1^\eta \tilde{O} - \alpha_2^\eta \tilde{P} - \delta_0^\eta & -\alpha_1^\eta \tilde{N} & -\alpha_2^\eta \tilde{N} & \delta_1^\eta \\
\alpha_1^\eta \tilde{O} + \alpha_2^\eta \tilde{P} & \alpha^\eta \tilde{N} - \lambda^\eta - a_1^\eta - \delta_0^\eta & \alpha_2^\eta \tilde{N} & 0 \\
0 & \lambda^\eta & -\delta_0^\eta & 0 \\
0 & a_1^\eta & 0 & -\delta_0^\eta - \delta_1^\eta
\end{bmatrix},
$$

$$J_{\tau_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2^\eta & 0 \\ 0 & 0 & a_2^\eta & 0 \end{bmatrix},$$

Upon solving, we obtain

$$\Lambda^4 + D_1\Lambda^3 + D_2\Lambda^2 + D_3\Lambda + D_4 + e^{-\Lambda\tau}(D_5\Lambda^3 + D_6\Lambda^2 + D_7\Lambda + D_8) = 0. \tag{11}$$

where

$$\begin{aligned}
D_1 =& a_1^\eta + 4\,\delta_0^\eta + \delta_1^\eta + \lambda^\eta - \tilde{N}\,\alpha_1^\eta + \tilde{O}\,\alpha_1^\eta + \tilde{P}\,\alpha_2^\eta,\\
D_2 =& 6\,\delta_0^{2\eta} + 3\,a_1^\eta\delta_0^\eta + a_1^\eta\delta_1^\eta + 3\,\delta_0^\eta\delta_1^\eta + 3\,\delta_0^\eta\lambda^\eta + \delta_1^\eta\lambda^\eta - \tilde{N}\,\alpha_2^\eta\lambda^\eta + \tilde{O}\,\alpha_1^\eta\lambda^\eta + \tilde{P}\,\alpha_2^\eta\lambda^\eta ...\\
&+ \tilde{O}\,a_1^\eta\alpha_1^\eta + \tilde{P}\,a_1^\eta\alpha_2^\eta - 3\,\tilde{N}\,\alpha_1^\eta\delta_0^\eta - \tilde{N}\,\alpha_1^\eta\delta_1^\eta + 3\,\tilde{O}\,\alpha_1^\eta\delta_0^\eta + \tilde{O}\,\alpha_1^\eta\delta_1^\eta + 3\,\tilde{P}\,\alpha_2^\eta\delta_0^\eta + \tilde{P}\,\alpha_2^\eta\delta_1^\eta,\\
D_3 =& 3\,a_1^\eta\delta_0^{2\eta} + 3\,\delta_0^{2\eta}\delta_1^\eta + 3\,\delta_0^{2\eta}\lambda^\eta + 4\,\delta_0^{3\eta} - 3\,\tilde{N}\,\alpha_1^\eta\delta_0^{2\eta} + 3\,\tilde{O}\,\alpha_1^\eta\delta_0^{2\eta} + 3\,\tilde{P}\,\alpha_2^\eta\delta_0^{2\eta} + 2\,a_1^\eta\delta_0^\eta\delta_1^\eta ...\\
&+ 2\,\delta_0^\eta\delta_1^\eta\lambda^\eta + 2\,\tilde{O}\,a_1^\eta\alpha_1^\eta\delta_0^\eta + 2\,\tilde{P}\,a_1^\eta\alpha_2^\eta\delta_0^\eta - 2\,\tilde{N}\,\alpha_1^\eta\delta_0^\eta\delta_1^\eta + 2\,\tilde{O}\,\alpha_1^\eta\delta_0^\eta\delta_1^\eta + 2\,\tilde{P}\,\alpha_2^\eta\delta_0^\eta\delta_1^\eta ...\\
&- 2\,\tilde{N}\,\alpha_2^\eta\delta_0^\eta\lambda^\eta - \tilde{N}\,\alpha_2^\eta\delta_1^\eta\lambda^\eta + 2\,\tilde{O}\,\alpha_1^\eta\delta_0^\eta\lambda^\eta + \tilde{O}\,\alpha_1^\eta\delta_1^\eta\lambda^\eta + 2\,\tilde{P}\,\alpha_2^\eta\delta_0^\eta\lambda^\eta + \tilde{P}\,\alpha_2^\eta\delta_1^\eta\lambda^\eta,\\
D_4 =& a_1^\eta\delta_0^{3\eta} + \delta_0^{3\eta}\delta_1^\eta + \delta_0^{3\eta}\lambda^\eta + \delta_0^{4\eta} + a_1^\eta\delta_0^{2\eta}\delta_1^\eta + \delta_0^{2\eta}\delta_1^\eta\lambda^\eta - \tilde{N}\alpha_1^\eta\delta_0^{3\eta} + \tilde{O}\alpha_1^\eta\delta_0^{3\eta} + \tilde{P}\alpha_2^\eta\delta_0^{3\eta} ...\\
&+ \tilde{O}a_1^\eta\alpha_1^\eta\delta_0^{2\eta} + \tilde{P}a_1^\eta\alpha_2^\eta\delta_0^{2\eta} - \tilde{N}\alpha_1^\eta\delta_0^{2\eta}\delta_1^\eta + \tilde{O}\alpha_1^\eta\delta_0^{2\eta}\delta_1^\eta + \tilde{P}\alpha_2^\eta\delta_0^{2\eta}\delta_1^\eta - \tilde{N}\alpha_2^\eta\delta_0^{2\eta}\lambda^\eta ...\\
&+ \tilde{O}\alpha_1^\eta\delta_0^{2\eta}\lambda^\eta + \tilde{P}\alpha_2^\eta\delta_0^{2\eta}\lambda^\eta - \tilde{N}\alpha_2^\eta\delta_0^\eta\delta_1^\eta\lambda^\eta + \tilde{O}\alpha_1^\eta\delta_0^\eta\delta_1^\eta\lambda^\eta + \tilde{P}\alpha_2^\eta\delta_0^\eta\delta_1^\eta\lambda^\eta,\\
D_5 =& a_2^\eta,\\
D_6 =& a_1^\eta a_2^\eta + 3\,a_2^\eta\delta_0^\eta + a_2^\eta\delta_1^\eta + a_2^\eta\lambda^\eta - \tilde{N}\,a_2^\eta\alpha_1^\eta + \tilde{O}\,a_2^\eta\alpha_1^\eta + \tilde{P}\,a_2^\eta\alpha_2^\eta,\\
D_7 =& 3\,a_2^\eta\delta_0^{2\eta} + 2\,a_1^\eta a_2^\eta\delta_0^\eta + a_1^\eta a_2^\eta\delta_1^\eta + 2\,a_2^\eta\delta_0^\eta\delta_1^\eta + 2\,a_2^\eta\delta_0^\eta\lambda^\eta + a_2^\eta\delta_1^\eta\lambda^\eta + \tilde{O}\,a_1^\eta a_2^\eta\alpha_1^\eta + \tilde{P}\,a_1^\eta a_2^\eta\alpha_2^\eta ...\\
&- 2\,\tilde{N}\,a_2^\eta\alpha_1^\eta\delta_0^\eta - \tilde{N}\,a_2^\eta\alpha_1^\eta\delta_1^\eta + 2\,\tilde{O}\,a_2^\eta\alpha_1^\eta\delta_0^\eta + \tilde{O}\,a_2^\eta\alpha_1^\eta\delta_1^\eta + 2\,\tilde{P}\,a_2^\eta\alpha_2^\eta\delta_0^\eta + \tilde{P}\,a_2^\eta\alpha_2^\eta\delta_1^\eta ...\\
&+ \tilde{O}\,a_2^\eta\alpha_1^\eta\lambda^\eta + \tilde{P}\,a_2^\eta\alpha_2^\eta\lambda^\eta,\\
D_8 =& a_2^\eta\delta_0^{3\eta} + a_1^\eta a_2^\eta\delta_0^{2\eta} + a_2^\eta\delta_0^{2\eta}\delta_1^\eta + a_2^\eta\delta_0^{2\eta}\lambda^\eta + a_2^\eta\delta_0^\eta\delta_1^\eta\lambda^\eta - \tilde{N}\,a_2^\eta\alpha_1^\eta\delta_0^{2\eta} + \tilde{O}\,a_2^\eta\alpha_1^\eta\delta_0^{2\eta} + \tilde{P}\,a_2^\eta\alpha_2^\eta\delta_0^{2\eta} ...\\
&+ a_1^\eta a_2^\eta\delta_0^\eta\delta_1^\eta + \tilde{O}\,a_1^\eta a_2^\eta\alpha_1^\eta\delta_0^\eta + \tilde{P}\,a_1^\eta a_2^\eta\alpha_2^\eta\delta_0^\eta - \tilde{N}\,a_2^\eta\alpha_1^\eta\delta_0^\eta\delta_1^\eta + \tilde{O}\,a_2^\eta\alpha_1^\eta\delta_0^\eta\delta_1^\eta + \tilde{P}\,a_2^\eta\alpha_2^\eta\delta_0^\eta\delta_1^\eta ...\\
&+ \tilde{O}\,a_2^\eta\alpha_1^\eta\delta_0^\eta\lambda^\eta + \tilde{P}\,a_2^\eta\alpha_2^\eta\delta_0^\eta\lambda^\eta.
\end{aligned}$$

It is well-known that the transcendental characteristic equation admits infinitely many complex roots. When the time delay ($\tau$) is involved, determining the sign of these roots becomes highly challenging. Therefore, we begin our analysis by considering the case $\tau = 0$, and subsequently derive the stability conditions for $\tau > 0$. In the absence of delay $\tau = 0$, equation (11) reduces to equation (7), for which the corresponding stability conditions have already been established.

**Theorem 5.** *The equilibrium point $E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$ of system (9) is locally asymptotically stable at $\tau = 0$ if and only if condition (8) is satisfied.*

For $\tau > 0$, the characteristic equation (11) possesses infinitely many roots. A necessary criterion for a change in the stability of $E_2$ is the existence of purely imaginary roots of equation (11). Let $i\omega$ be such a root of equation (11). Then, we obtain:

$$\begin{aligned}
(D_7\omega - D_5\omega^3)Sin\tau\omega + (D_8 - D_6\omega^2)Cos\tau\omega =& D_2\omega^2 - \omega^4 - D_4,\\
(D_7\omega - D_5\omega^3)Cos\tau\omega - (D_8 - D_6\omega^2)Sin\tau\omega =& D_1\omega^3 - D_3\omega.
\end{aligned} \tag{12}$$

Hence, the following is obtained

$$\omega^8 + d_1\omega^6 + d_2\omega^4 + d_3\omega^2 + d_4 = 0, \tag{13}$$

where

$$\begin{aligned} d_1 =& D_1^2 - 2D_2 - D_5^2, \\ d_2 =& D_2^2 + 2D_4 - 2D_3D_1 - D_6^2 + 2D_7D_5, \\ d_3 =& D_3^2 - 2D_4D_2 - D_8^2 + 2D_8D_6, \\ d_4 =& D_4^2 - D_8^2. \end{aligned}$$

If $d_4 < 0$, then equation (13) admits at least one positive real root, and equation (12) possesses two purely imaginary roots $i\omega$. The corresponding value of $\tau$ is subsequently determined from equation (12).

$$\tau_k = \frac{1}{\omega} Cos^{-1}\left[\frac{(D_6\omega^2 - D_8)(\omega^4 - D_2\omega^2 + D_4) - \omega^2(D_1\omega^2 - D_3)(D_5\omega^2 - D_7)}{(D_6\omega^2 - D_8)^2 + (D_5\omega^3 - D_7\omega)^2}\right] + \frac{2k\pi}{\omega}. \tag{14}$$

Furthermore, differentiating equation (11) with respect to $\tau$

$$\frac{d\Lambda}{d\tau} = \frac{\Lambda}{-\left(\dfrac{4\Lambda^3 + 3D_1\Lambda^2 + 2D_2\Lambda + D_3}{\Lambda^4 + D_1\Lambda^3 + D_2\Lambda^2 + D_3\Lambda + D_4}\right) + \left(\dfrac{3D_5\Lambda^2 + 2D_6\Lambda + D_7}{D_5\Lambda^3 + D_6\Lambda^2 + D_7\Lambda + D_8}\right) - \tau},$$

i.e.

$$\left(\frac{d\Lambda}{d\tau}\right)^{-1} = -\frac{1}{\Lambda}\left(\frac{4\Lambda^3 + 3D_1\Lambda^2 + 2D_2\Lambda + D_3}{\Lambda^4 + D_1\Lambda^3 + D_2\Lambda^2 + D_3\Lambda + D_4}\right) + \frac{1}{\Lambda}\left(\frac{3D_5\Lambda^2 + 2D_6\Lambda + D_7}{D_5\Lambda^3 + D_6\Lambda^2 + D_7\Lambda + D_8}\right) - \frac{\tau}{\Lambda}. \tag{15}$$

Suppose that

$$\begin{aligned} \left[Re\left(\frac{d\Lambda}{d\tau}\right)^{-1}\right] =& \left[Re\left(-\frac{4\Lambda^3 + 3D_1\Lambda^2 + 2D_2\Lambda + D_3}{\Lambda(\Lambda^4 + D_1\Lambda^3 + D_2\Lambda^2 + D_3\Lambda + D_4)}\right)\right]_{\Lambda=i\omega} \cdots \\ &+ \left[Re\left(\frac{3D_5\Lambda^2 + 2D_6\Lambda + D_7}{\Lambda(D_5\Lambda^3 + D_6\Lambda^2 + D_7\Lambda + D_8)} - \frac{\tau}{\Lambda}\right)\right]_{\Lambda=i\omega} \\ =& V_1 + V_2, \end{aligned} \tag{16}$$

where

$$\begin{aligned} V_1 =& \frac{4\omega^3 + \omega^6(3D_1^2 - 2D_2) + \omega^4(4D_4 - 2D_3D_1 + 2D_2^2) + \omega^2(D_3^2 - 2D_2D_4)}{(D_3\omega^2 - D_1\omega^4)^2 + (D_1\omega^3 - D_4\omega - \omega^2)^2}, \\ V_2 =& \frac{\omega^4(D_5^2 - 2D_6^2 + 3D_5D_7) + \omega^2(2D_6D_8 - D_7D_5) - 3\omega^6D_5^2}{(D_5\omega^4 - D_7\omega^2)^2 + (D_8\omega - D_6\omega^3)^2}. \end{aligned}$$

If $V_1$ and $V_2$ are positive, then

$$\left[Re\left(\frac{d\Lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau^*} = \left[Re\left(\frac{d\Lambda}{d\tau}\right)^{-1}\right]_{\tau=i\omega} > 0,$$

Hence, a branch of periodic solutions bifurcates from the equilibrium point $E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$ in a neighbourhood of $\tau = \tau^*$.

**Theorem 6.** *Assume that $H_1$ and $H_2$ are positive for system (9). Then, the Gambler-persistence equilibrium $E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$ remains locally asymptotically stable for $\tau \in [0, \tau^*)$, and becomes unstable when $\tau > \tau^*$. Moreover, a Hopf bifurcation occurs at the equilibrium point $E_2 = (\tilde{N}, \tilde{O}, \tilde{P}, \tilde{R})$ when $\tau = \tau^*$.*

Table 8: Change in delay with respect to the derivative order

| $\eta$ | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|
| $\tau$ | 5.75 | 5.3 | 4.8 | 4.3 |

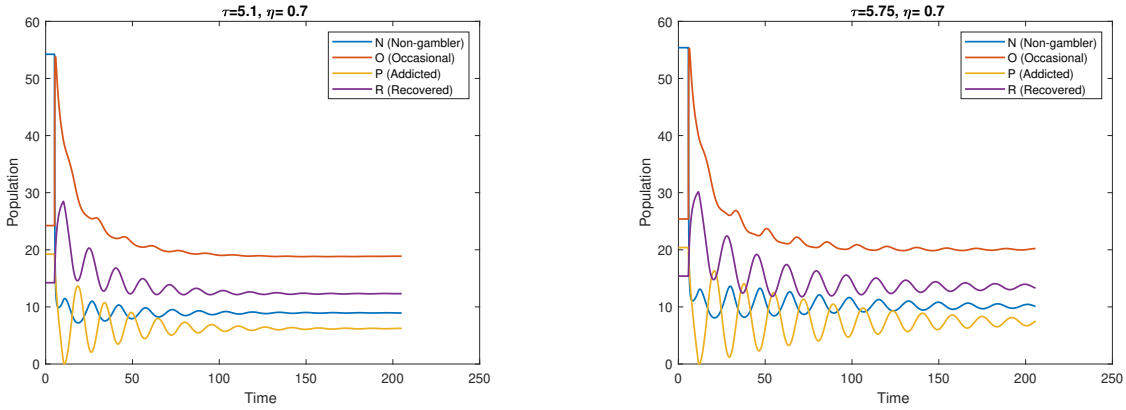



Figure 6: Behaviour of the delayed fractional-order system for different delays $\tau$ when $\eta = 0.7$. Increasing delay leads to oscillations and Hopf bifurcation.

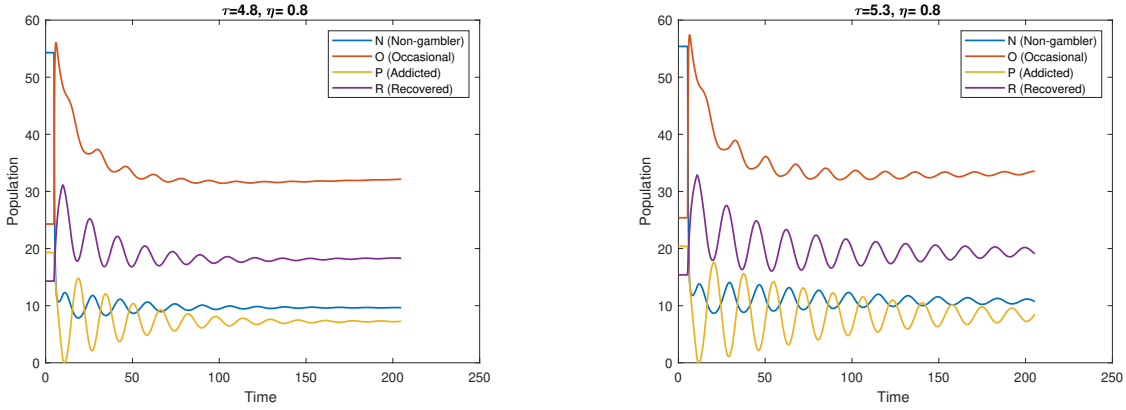



Figure 7: Dynamics of the delayed system for $\eta = 0.8$. Delay beyond the critical threshold generates periodic oscillations.

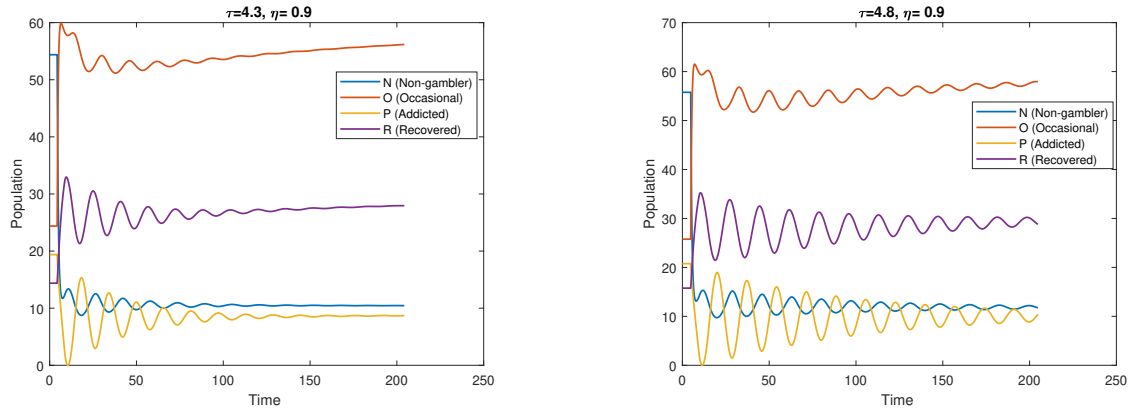



Figure 8: Oscillatory behaviour of the delayed system for $\eta = 0.9$. The amplitude of oscillations increases as delay grows.

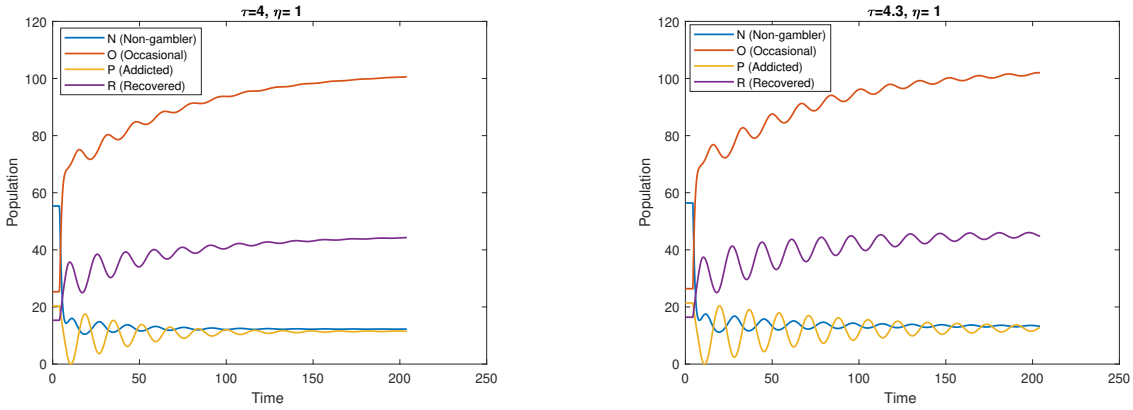



Figure 9: Dynamics of the delayed system for the integer-order case $\eta = 1$. Hopf bifurcation is observed when the delay exceeds the critical threshold.

Figures 6 - 9 display the dynamical behaviour of the delayed system for different values of delay $\tau$ and fractional order $\eta$. For smaller delays, the system stabilizes at the endemic equilibrium, indicating persistence of gambling behaviour. However, as the delay increases beyond a critical threshold $\tau^*$, sustained oscillations appear in the populations, signifying a Hopf bifurcation. The emergence of oscillations reflects relapse cycles in gambling addiction—delayed transition from addiction to recovery can create recurrent waves of addictive behaviour. Additionally, for lower fractional orders, the amplitude of oscillations is more pronounced due to stronger memory effects.

# 9 Policy Suggestions

The findings of the present study offer several practical insights for designing effective gambling intervention policies. The sensitivity analysis highlights that parameters governing recruitment into gambling and exposure-driven transitions significantly increase the basic reproduction number $\mathcal{R}_0$, while recovery-related parameters have a strong suppressive effect. These observations suggest that policies reducing social exposure to gambling environments and enhancing recovery rates can substantially mitigate addiction levels.

Based on the model dynamics, especially the influence of memory effects and time-delay–induced relapse, the following policy strategies are recommended:

- **Early-Stage Prevention:** Since the transition of non-gamblers to occasional gamblers strongly influences $\mathcal{R}_0$, preventive programs targeting youth and first-time gamblers should be prioritized. Community awareness campaigns, educational modules, and restrictions on advertisement of gambling platforms can reduce initial exposure.
- **Strengthening Recovery Mechanisms:** The model shows that increasing recovery rates $(a_1, a_2)$ accelerates movement towards the recovered class and reduces long-term addiction. Policies should expand access to counselling, behavioral therapy, and rehabilitation centers, ensuring timely and affordable treatment.
- **Reduction of Environmental and Social Exposure:** As social interaction parameters $(\alpha_1, \alpha_2)$ positively affect $\mathcal{R}_0$, policies should regulate the density of gambling venues, limit online gambling promotions, and introduce digital behavioural nudges that discourage excessive betting.
- **Targeting Relapse Prevention:** The delay analysis indicates that prolonged relapse duration leads to oscillatory behaviour, reflecting repeated addiction cycles. Incorporating long-term follow-up, post-recovery monitoring, and relapse-prevention counselling is essential for maintaining stability in gambler populations.

Overall, the mathematical analysis underscores the importance of multi-dimensional policy interventions that simultaneously reduce exposure, enhance recovery, and prevent relapse. Integrating these insights into public health planning can significantly weaken the persistence of gambling addiction within the population.

# 10 Conclusion

In this work, a fractional-order mathematical model has been developed to investigate the dynamics of gambling behavior within a population, explicitly incorporating memory effects via the Caputo derivative. The population is classified into non-gamblers, occasional gamblers, addicted gamblers, and recovered individuals, allowing the model to capture complex transitions driven by behavioral interactions and recovery mechanisms. Analytical results establish the existence, uniqueness, and boundedness of solutions, while the basic reproduction number $\mathcal{R}_0$ provides a critical threshold for determining the stability of both gambler-free and gambler-persistent equilibria. The model is further extended to include fractional-order delays to account for relapse phenomena, and conditions for Hopf bifurcation are derived, highlighting the possibility of oscillatory dynamics. Numerical simulations corroborate the theoretical findings, emphasizing the significant role of fractional-order memory and time-delay effects in the persistence of gambling addiction. Additionally, a detailed error and convergence analysis confirms that the proposed numerical scheme is stable, consistent, and achieves an approximate second-order convergence rate, ensuring the reliability and accuracy of computational results. This study provides a robust quantitative framework for understanding gambling behavior and offers a foundation for designing effective behavioral interventions.

**Key Contributions:**

- Developed a fractional-order model incorporating memory effects to realistically describe human decision-making in gambling behavior.

- Established analytical results on existence, uniqueness, and boundedness of solutions, ensuring mathematical rigor.
- Determined stability conditions: gambler-free equilibrium is stable for $\mathcal{R}_0 < 1$; gambler-persistent equilibrium is stable for $\mathcal{R}_0 > 1$.
- Extended the model with fractional-order delays to include relapse behavior and derived Hopf bifurcation conditions.
- Conducted numerical simulations validating theoretical predictions and illustrating the impact of memory and delay on addiction dynamics.
- Performed convergence and error analysis confirming the second-order accuracy and reliability of the numerical scheme.

## Data Availability

Data sharing not applicable – no new data generated.

## Future Scope

The study highlights the potential to enhance the current model by incorporating additional factors that influence gambling behavior and by validating the outcomes against empirical observations. Future research may investigate optimal control strategies to develop targeted interventions that effectively reduce addictive tendencies over time. Furthermore, extending the framework to partial differential equations (PDEs) would enable the analysis of spatial-temporal variations in gambling behavior across diverse populations or regions. Integrating these models with game-theoretic approaches also offers opportunities to advance artificial intelligence in competitive and uncertain settings, facilitating more realistic simulations of strategic decision-making and the evolution of behavior.

## Conflict of interest

The authors confirm that they have no conflicts of interest related to the publication of this manuscript.

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