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Mittag–Leffler Synchronization of Fractional–Order Lü Chaotic Systems Using Hybrid Control and Linear Matrix Inequalities

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Abstract

This paper studies the Mittag-Leffler synchronization of fractional-order Lü chaotic systems using a hybrid control approach. The method combines state-feedback, nonlinear compensation, and impulsive control in a single framework so that these components can work together and improve robustness. By using a Lyapunov function and linear matrix inequality techniques, sufficient conditions are obtained to guarantee global Mittag-Leffler synchronization, and the control gains can be computed directly. The results also include several special cases that follow from the main theorem. Numerical simulations are presented to support the theoretical results and show that the proposed method works effectively. Compared with existing approaches, the method is more flexible and less restrictive, and it can be useful in applications such as secure communication and complex dynamical systems.

Keywords: Mittag-Leffler synchronization, fractional-order systems, Lü chaotic system, hybrid control, impulsive control, Lyapunov stability, linear matrix inequalities.
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1. Introduction

Fractional calculus has received much attention because it can describe systems with memory and hereditary properties more effectively than classical integer-order calculus. By allowing derivatives and integrals of non-integer order, fractional differential equations (FDEs) offer a flexible framework for modeling complex physical and engineering processes. Applications of fractional calculus can be found in many areas such as viscoelasticity, electromagnetism, signal processing, neural networks, and chaotic systems [1]. In particular, fractional-order models are useful for describing phenomena such as anomalous diffusion and long-term memory, which are difficult to capture using integer-order models.

Among the different definitions of fractional derivatives, the Caputo derivative is widely used because it allows the use of standard initial conditions and has clear physical meaning [2]. In addition, the Mittag-Leffler function plays an important role in fractional-order systems, as it extends the exponential function and describes the behavior of solutions of FDEs. Based on this, Mittag-Leffler stability has been introduced as a natural extension of exponential stability and is now commonly used to study the long-term behavior of fractional-order systems [3, 4, 5, 6].

Chaotic systems are nonlinear systems that show complex behavior such as sensitivity to initial conditions and the presence of strange attractors. Since the work of Lorenz, many chaotic systems, including the Chen and Lü systems, have been studied due to their theoretical interest and practical applications in areas such as secure communication, cryptography, and engineering [7, 8]. The Lü system lies between the Lorenz and Chen systems and exhibits rich dynamics, which makes it an important model in chaos theory.

Synchronization of chaotic systems has become an important topic in nonlinear dynamics because of its wide applications, especially in secure communication [9]. Different synchronization methods have been proposed, such

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as complete synchronization, anti-synchronization, finite-time synchronization, and impulsive synchronization [10, 11]. Recently, Mittag-Leffler synchronization has attracted attention as it provides a more suitable framework for fractional-order systems, where exponential convergence is generally not expected due to memory effects [12, 13, 14]. Several studies on Lü systems and fractional-order chaotic dynamics have been reported in [15, 16, 17]. Moreover, the theory of impulsive systems and fractional-order stability has been developed in [18, 19, 20]. Recent work on synchronization and stabilization of complex systems can also be found in [21, 22, 23, 24, 25].

In the literature, many control methods have been proposed for synchronization and stabilization of fractional-order chaotic systems. Impulsive control is widely used because it can handle discontinuities and reduce control effort [26, 27]. State feedback and nonlinear control methods have also been applied to stabilize such systems [28]. In addition, synchronization of fractional-order neural networks and complex systems using impulsive or adaptive control has been studied in [29, 30]. However, most existing works consider these control methods separately. As a result, they may not fully utilize the advantages of combining different control strategies. For highly nonlinear fractional systems, a single control method may not be enough to ensure robustness and fast convergence.

To overcome these limitations, hybrid control has been introduced as a way to combine different control methods within a single framework. For example, hybrid control has been used in fractional-order neural networks and coupled systems [31]. However, its application to fractional-order chaotic systems, especially for Mittag-Leffler synchronization, is still limited. In addition, the interaction between different control components has not been fully studied. A careful examination of the existing literature reveals several research gaps. Most studies on Mittag-Leffler synchronization focus on either impulsive or continuous control methods, without considering their combined use. Hybrid control has rarely been applied to fractional-order chaotic systems, especially Lü-type systems. Moreover, many results assume that control components work independently, ignoring the benefits of their interaction. In addition, the use of linear matrix inequality (LMI) techniques for designing hybrid controllers in fractional-order systems is still limited.

Motivated by these observations, this paper proposes a hybrid control framework for Mittag-Leffler synchronization of fractional-order chaotic systems. In particular, we study the fractional-order Lü system and design a controller that combines nonlinear feedback, state feedback, and impulsive control. The proposed controller allows these components to work together, so that if one part becomes less effective, the others can still maintain synchronization. This improves the overall stability and robustness of the system.

The main contributions of this work can be summarized as follows:

- A novel hybrid control framework is proposed for fractional-order Lü chaotic systems, integrating nonlinear compensation, state-feedback, and impulsive control in a unified structure.
- The proposed control design incorporates interdependence among different control components, improving robustness and synchronization performance compared to independent strategies.
- New sufficient conditions for global Mittag-Leffler synchronization are derived using linear matrix inequality (LMI) techniques, which allow systematic computation of control gains.
- The proposed approach extends existing results on fractional-order and impulsive control systems [4, 28], and its effectiveness is confirmed through numerical simulations.

The objective of this paper is to study global Mittag-Leffler synchronization of fractional-order nonlinear chaotic systems under hybrid control. The problem is formulated using an error system, and Lyapunov methods together with LMI techniques are used to derive sufficient conditions for synchronization.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries on fractional calculus. Section 3 formulates the synchronization problem for the fractional-order Lü system. Section 4 provides the main theoretical results based on LMI conditions. Section 5 gives numerical simulations to support the results. Finally, Section 6 concludes the paper.

2. Notation and Preliminaries

In this section, we present the fundamental concepts, definitions, and mathematical tools from fractional calculus and stability theory that are essential for the development of the main results.

Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimensional Euclidean space and the set of all real $n \times n$ matrices, respectively. Let $\mathbb{R}^+ = (0, \infty)$ be the set of positive real numbers. The transpose of a vector or matrix y is denoted by y^T. The

Euclidean norm of a vector $u \in \mathbb{R}^n$ is defined by

$$\|u\| = (u^T u)^{1/2}.$$

Let $\Upsilon \subset \mathbb{R}^n$ be an open set, and denote $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. A matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite if $x^T P x > 0$ for all $x \neq 0$.

Fractional calculus generalizes classical differentiation and integration to non-integer orders. Among several definitions, the Caputo derivative is widely used due to its compatibility with classical initial conditions.

Definition 2.1. [1] The fractional integral of order $\delta > 0$ of a function $g(\ell)$ is defined as

$$ {}_{\ell_0}\mathbb{I}^{\delta}_{\ell} g(\ell) = \frac{1}{\Gamma(\delta)} \int_{\ell_0}^{\ell} (\ell - s)^{\delta-1} g(s)\, ds, \tag{1}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2. [1] The Caputo fractional derivative of order $\delta \in (0, 1)$ of a function $g(\ell)$ is defined by

$$ {}^{C}_{\ell_0}\mathbb{D}^{\delta}_{\ell} g(\ell) = \frac{1}{\Gamma(1-\delta)} \int_{\ell_0}^{\ell} (\ell - s)^{-\delta} g'(s)\, ds. \tag{2}$$

Remark 2.1. The Caputo derivative is particularly suitable for physical and engineering applications because it allows the use of classical initial conditions expressed in terms of integer-order derivatives.

The Mittag-Leffler function plays a fundamental role in the analysis of fractional-order systems, analogous to the exponential function in integer-order systems.

Definition 2.3. [1] The one-parameter Mittag-Leffler function is defined as

$$\mathbb{E}_{\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + 1)}, \quad \delta > 0. \tag{3}$$

The two-parameter Mittag-Leffler function is defined as

$$\mathbb{E}_{\delta,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + \beta)}, \quad \delta > 0,\ \beta > 0. \tag{4}$$

Remark 2.2. For $\delta = 1$, the Mittag-Leffler function reduces to the exponential function, i.e., $\mathbb{E}_1(z) = e^z$. Hence, it provides a natural generalization for describing the solution behavior of fractional-order systems.

Definition 2.4. [4] A function $\mathbb{V} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to belong to class $\mathbb{V}_0$ if:

1. $\mathbb{V}(\ell, y)$ is continuous in $(\ell_{k-1}, \ell_k) \times \Upsilon$, $k \in \mathbb{Z}_+$, and satisfies

$$\lim_{(\ell,z)\to(\ell_k^+,y)} \mathbb{V}(\ell, z) = \mathbb{V}(\ell_k^+, y), \quad \mathbb{V}(\ell_k^-, y) = \mathbb{V}(\ell_k, y),$$

2. $\mathbb{V}(\ell, y)$ is locally Lipschitz continuous with respect to y.

Definition 2.5. The trivial solution of a fractional-order system is said to be Mittag-Leffler stable if there exist positive constants λ, α, and c such that

$$\|x(\ell)\| \leq c\, \mathbb{E}_{\delta}(-\lambda \ell^{\delta})\|x(0)\|, \quad \ell \geq 0.$$

Remark 2.3. Mittag-Leffler stability generalizes exponential stability and is more suitable for fractional-order systems, where solutions typically exhibit slower decay rates.

The following lemmas are useful for the derivation of the main results.

Lemma 2.1. *[32] Let* $\mathrm{y}(\ell) \in \mathbb{R}^n$ *be a differentiable vector function and let* P *be a positive definite matrix. Then, for any time* $\ell \geq \ell_0$*,*

$$\frac{1}{2}\,{}^{C}_{\ell_0}\mathbb{D}^{\delta}_{\ell}[\mathrm{y}^T(\ell)\mathrm{P}\mathrm{y}(\ell)] \leq \mathrm{y}^T(\ell)\mathrm{P}\,{}^{C}_{\ell_0}\mathbb{D}^{\delta}_{\ell}\mathrm{y}(\ell), \tag{5}$$

$\forall \delta \in (0, 1]$.

Lemma 2.2. *[33] Let* $\varrho > 0$ *and* $\mathrm{y}, \mathrm{z} \in \mathbb{R}^n$*. Then*

$$\mathrm{y}^T\mathrm{z} + \mathrm{z}^T\mathrm{y} \leq \varrho^{-1}\mathrm{y}^T\mathrm{y} + \varrho\mathrm{z}^T\mathrm{z}. \tag{6}$$

Lemma 2.3. *[34] If the function* $\mathrm{h}(\ell, \mathrm{z}) : \mathbb{R}_+ \times \Upsilon \to \mathbb{R}^n$ *is Lipschitz continuous with* $\mathrm{h}(\ell, 0) = 0$*, then there exists a constant* $\omega > 0$ *such that*

$$\| \mathrm{h}(\ell, \mathrm{z}) - \mathrm{h}(\ell, \mathrm{y}) \| \leq \omega \| \mathrm{z} - \mathrm{y} \|, \quad \forall \mathrm{z}, \mathrm{y} \in \mathbb{R}^n. \tag{7}$$

Remark 2.4. The Lipschitz condition ensures the existence and uniqueness of solutions for fractional differential equations and plays a crucial role in stability analysis.

3. Problem formulation

In this section, we formulate the synchronization problem for the fractional-order Lü chaotic system and construct an appropriate hybrid control framework.

The fractional-order Lü chaotic system is given by

$$\begin{cases} {}^{c}\mathbb{D}^{\delta}\mathrm{x}_1 = a(\mathrm{x}_2 - \mathrm{x}_1), \\ {}^{c}\mathbb{D}^{\delta}\mathrm{x}_2 = -\mathrm{x}_1\mathrm{x}_3 + c\mathrm{x}_2, \\ {}^{c}\mathbb{D}^{\delta}\mathrm{x}_3 = \mathrm{x}_1\mathrm{x}_2 - b\mathrm{x}_3, \end{cases} \tag{8}$$

where $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)^T \in \mathbb{R}^3$, $a, b, c > 0$ are parameters, and $\delta \in (0, 1)$. It is observed that the nonlinear coupling terms $\mathrm{x}_1\mathrm{x}_3$ and $\mathrm{x}_1\mathrm{x}_2$ are responsible for the chaotic behavior of the system.

The controlled response system is constructed as

$$\begin{cases} {}^{c}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathrm{y}_1 = a(\mathrm{y}_2 - \mathrm{y}_1) + \mathrm{u}_1, \quad \ell \neq \ell_k, \\ {}^{c}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathrm{y}_2 = -\mathrm{y}_1\mathrm{y}_3 + c\mathrm{y}_2 + \alpha(\ell) + \mathrm{u}_2, \quad \ell \neq \ell_k, \\ {}^{c}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathrm{y}_3 = \mathrm{y}_1\mathrm{y}_2 - b\mathrm{y}_3 + \beta(\ell) + \mathrm{u}_3, \quad \ell \neq \ell_k, \\ \mathrm{y}(\ell) = \mathrm{u}_0(\ell^-) + \mathrm{x}(\ell^-), \quad \ell = \ell_k, \end{cases} \tag{9}$$

where $\mathrm{u}(\ell) = (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3)^T$ and $\mathrm{G}(\ell) = (0, \alpha(\ell), \beta(\ell))^T$. The response system incorporates a hybrid control strategy consisting of continuous control and impulsive control.

Define the synchronization error as

$$e(\ell) = \mathrm{y}(\ell) - \mathrm{x}(\ell) = (e_1, e_2, e_3)^T.$$

and the control laws are chosen as

$$\mathrm{u}(\ell) = \psi e(\ell), \quad \mathrm{u}_0 = \phi e(\ell_k^-),$$

where $\psi, \phi \in \mathbb{R}^{3\times3}$.

Then, the error system is given by

$$\begin{cases} {}^{c}_{\ell_k}\mathbb{D}^{\delta}_{\ell} e(\ell) = \mathrm{A}e(\ell) + \mathrm{h}(\ell, e) + \mathrm{B}\mathrm{u}(\ell) + \mathrm{G}(\ell), & \ell \neq \ell_k, \\ e(\ell) = \phi e(\ell_k^-), & \ell = \ell_k, \end{cases} \tag{10}$$

with initial condition $e(0) = e_0$.

Here,

$$\mathrm{A} = \begin{pmatrix} -a & a & 0 \\ 0 & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad \mathrm{B} = \mathrm{I}_3,$$

and

$$\mathrm{h}(\ell, e) = \begin{pmatrix} 0 \\ \mathrm{y}_1 e_3 - \mathrm{x}_3 e_1 \\ \mathrm{y}_1 e_2 + \mathrm{x}_2 e_1 \end{pmatrix}.$$

The secondary control is designed as

$$\mathrm{G}(\ell) = \begin{pmatrix} 0 \\ \alpha(\ell) \\ \beta(\ell) \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathrm{y}_1 e_3 + \mathrm{x}_3 e_1 \\ -\mathrm{y}_1 e_2 - \mathrm{x}_2 e_1 \end{pmatrix}. \tag{11}$$

Remark 3.1. This control cancels the nonlinear terms in the error system, simplifying stability analysis.

The impulsive instants satisfy

$$0 = \ell_0 < \ell_1 < \cdots < \ell_k < \ell_{k+1} < \cdots, \quad \lim_{k\to\infty} \ell_k = \infty.$$

Definition 3.1. [5] The systems (8) and (9) are said to be globally Mittag-Leffler synchronized if

$$\|\mathrm{y}(\ell) - \mathrm{x}(\ell)\| \leq \left\{ m(e(\ell_0)) \prod_{i=1}^{k} \mathbb{E}_{\delta}[-\eta(\ell_i - \ell_{i-1})^{\delta}] \mathbb{E}_{\delta}[-\eta(\ell - \ell_k)^{\delta}] \right\}^{b}. \tag{12}$$

This definition extends exponential synchronization to fractional-order systems using Mittag-Leffler decay.

Remark 3.2. [5, 26] Two approaches are used for Caputo fractional derivatives in impulsive systems:

1. Fixed lower limit ℓ_0 over the entire interval.
2. Piecewise lower limit ℓ_k over each interval (ℓ_k, ℓ_{k+1}).

In this paper, we adopt the second approach:

$${}^{C}_{\ell_0}\mathbb{D}^{\delta}_{\ell} \mathrm{x}(\ell) \neq {}^{C}_{\ell_k}\mathbb{D}^{\delta}_{\ell} \mathrm{x}(\ell).$$

This formulation is more suitable for hybrid impulsive control since it allows resetting the memory at each impulsive instant.

4. Main results

In this section, we establish sufficient conditions for the global Mittag-Leffler synchronization of the fractional-order Lü chaotic system under the proposed hybrid control law. The analysis is carried out by combining a quadratic Lyapunov function with linear matrix inequality (LMI) techniques. The main advantage of the proposed approach lies in the fact that the synchronization criteria can be expressed in a computationally tractable form, from which the state-feedback gain and impulsive gain can be explicitly determined.

We first study the complete hybrid control case corresponding to the error system (10), where the continuous state-feedback control, nonlinear compensation control, and impulsive control act simultaneously. Then, by suitable specialization of the control structure, we derive several reduced-order results as direct corollaries of the main theorem.

Theorem 4.1. *Assume that there exist a symmetric positive definite matrix* $\mathrm{P} \in \mathbb{R}^{3\times3}$*, matrices* $\mathrm{W}, \mathrm{W}_0 \in \mathbb{R}^{3\times3}$*, and constants* $\mu > 0$*,* $q > 0$*, and* $\eta > 0$ *such that*

$$\begin{bmatrix} \mathrm{A}^T\mathrm{P} + \mathrm{W}^T + \mathrm{PA} + \mathrm{W} + \eta\mathrm{P} & 0 \\ 0 & -\mathrm{P} \end{bmatrix} \leq 0, \tag{13}$$

and

$$\begin{bmatrix} -e^{-\mu}\mathrm{P} & \mathrm{W}_0^T \\ * & -\mathrm{P} \end{bmatrix} \leq 0. \tag{14}$$

If the control gains are chosen as

$$\mathrm{B}\psi = \mathrm{P}^{-1}\mathrm{W}, \qquad \mathrm{B}\phi = \mathrm{P}^{-1}\mathrm{W}_0,$$

then the drive system (8) *and the response system* (9) *are globally Mittag-Leffler synchronized.*

PROOF. *Consider the Lyapunov candidate function*

$$\mathbb{V}(\ell) = e^T(\ell)\mathrm{P}e(\ell),$$

where P *is symmetric positive definite. Hence, there exist constants* $\lambda_{\min}(\mathrm{P}) > 0$ *and* $\lambda_{\max}(\mathrm{P}) > 0$ *such that*

$$\lambda_{\min}(\mathrm{P})\|e(\ell)\|^2 \leq \mathbb{V}(\ell) \leq \lambda_{\max}(\mathrm{P})\|e(\ell)\|^2. \tag{15}$$

For $\ell \neq \ell_k$*, by the design of the nonlinear compensation control* $\mathrm{G}(\ell)$ *in* (11)*, the nonlinear terms in the error system are canceled. Therefore, the closed-loop error system reduces to*

$${}^{\mathrm{C}}_{\ell_k}\mathbb{D}^{\delta}_{\ell}e(\ell) = (\mathrm{A} + \mathrm{B}\psi)e(\ell), \qquad \ell \neq \ell_k.$$

Using Lemma 2.1 and the above dynamics, we obtain

$$\begin{aligned} {}^{\mathrm{C}}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) &\leq e^T(\ell)\Big((\mathrm{A} + \mathrm{B}\psi)^T\mathrm{P} + \mathrm{P}(\mathrm{A} + \mathrm{B}\psi)\Big)e(\ell) \\ &= e^T(\ell)\Big(\mathrm{A}^T\mathrm{P} + \psi^T\mathrm{B}^T\mathrm{P} + \mathrm{PA} + \mathrm{PBK}\Big)e(\ell). \end{aligned} \tag{16}$$

Since $\mathrm{B}\psi = \mathrm{P}^{-1}\mathrm{W}$*, it follows that* $\mathrm{PBK} = \mathrm{W}$ *and* $\psi^T\mathrm{B}^T\mathrm{P} = \mathrm{W}^T$*. Hence,* (16) *becomes*

$${}^{\mathrm{C}}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) \leq e^T(\ell)\Big(\mathrm{A}^T\mathrm{P} + \mathrm{W}^T + \mathrm{PA} + \mathrm{W}\Big)e(\ell).$$

From (13)*, we deduce that*

$$\mathrm{A}^T\mathrm{P} + \mathrm{W}^T + \mathrm{PA} + \mathrm{W} + \eta\mathrm{P} \leq 0,$$

and therefore

$${}^{\mathrm{C}}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) \leq -\eta\mathbb{V}(\ell), \qquad \ell \neq \ell_k. \tag{17}$$

Next, for the impulsive instant $\ell = \ell_k$*, from* (10) *we have*

$$e(\ell_k) = \phi e(\ell_k^-).$$

Thus,

$$\begin{aligned} \mathbb{V}(\ell_k) = e^T(\ell_k)\mathrm{P}e(\ell_k) &= \Big(\phi e(\ell_k^-)\Big)^T\mathrm{P}\Big(\phi e(\ell_k^-)\Big) \\ &= e^T(\ell_k^-)\phi^T\mathrm{P}\phi e(\ell_k^-). \end{aligned} \tag{18}$$

By the Schur complement, condition (14) *implies*

$$\phi^T\mathrm{P}\phi \leq e^{-\mu}\mathrm{P},$$

which, together with (18)*, yields*

$$\mathbb{V}(\ell_k) \le e^{-\mu}\mathbb{V}(\ell_k^-). \tag{19}$$

Combining (17) *and* (19)*, we obtain the following fractional impulsive comparison system:*

$$\begin{cases} {}^{C}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) \le -\eta\mathbb{V}(\ell), & \ell \ne \ell_k, \\ \mathbb{V}(\ell_k) \le e^{-\mu}\mathbb{V}(\ell_k^-), & \ell = \ell_k. \end{cases}$$

Applying the comparison principle for impulsive fractional-order systems and using the Mittag-Leffler estimate, we obtain successively:

$$\mathbb{V}(\ell) \le \mathbb{V}(\ell_0)\mathbb{E}_{\delta}\Big(-\eta(\ell-\ell_0)^{\delta}\Big), \qquad \ell \in (\ell_0, \ell_1],$$

$$\mathbb{V}(\ell) \le \mathbb{V}(\ell_0)e^{-\mu}\mathbb{E}_{\delta}\Big(-\eta(\ell_1-\ell_0)^{\delta}\Big)\mathbb{E}_{\delta}\Big(-\eta(\ell-\ell_1)^{\delta}\Big), \qquad \ell \in (\ell_1, \ell_2],$$

and, inductively,

$$\mathbb{V}(\ell) \le \mathbb{V}(\ell_0)e^{-k\mu}\prod_{i=1}^{k}\mathbb{E}_{\delta}\Big(-\eta(\ell_i-\ell_{i-1})^{\delta}\Big)\mathbb{E}_{\delta}\Big(-\eta(\ell-\ell_k)^{\delta}\Big), \qquad \ell \in (\ell_k, \ell_{k+1}]. \tag{20}$$

Using (15)*, we derive*

$$\begin{aligned} \|e(\ell)\| &\le \left(\frac{\mathbb{V}(\ell)}{\lambda_{\min}(\mathrm{P})}\right)^{1/2} \\ &\le \left(\frac{\mathbb{V}(\ell_0)}{\lambda_{\min}(\mathrm{P})}e^{-k\mu}\prod_{i=1}^{k}\mathbb{E}_{\delta}\Big(-\eta(\ell_i-\ell_{i-1})^{\delta}\Big)\mathbb{E}_{\delta}\Big(-\eta(\ell-\ell_k)^{\delta}\Big)\right)^{1/2}. \end{aligned} \tag{21}$$

Hence, by Definition 3.1, the drive and response systems are globally Mittag-Leffler synchronized. This completes the proof.

Remark 4.1. Theorem 4.1 provides the principal synchronization criterion for the proposed hybrid controller. The nonlinear compensation term eliminates the nonlinear couplings in the error dynamics, the state-feedback term stabilizes the continuous-time dynamics between impulses, and the impulsive control further reduces the synchronization error at the prescribed impulsive instants.

Remark 4.2. The LMI conditions (13)–(14) are convenient from a computational perspective. Once feasible matrices P, W, and W_0 are obtained, the gains ψ and ϕ can be recovered directly. This makes the proposed synchronization criterion suitable for numerical implementation via standard LMI solvers.

We now derive several reduced synchronization criteria as particular cases of Theorem 4.1.

Corollary 4.1. *Assume that the nonlinear compensation control is removed, that is,* $\mathrm{G}(\ell) \equiv 0$*. Then the error system becomes*

$$\begin{cases} {}^{C}_{\ell_k}\mathbb{D}^{\delta}_{\ell}e(\ell) = \mathrm{A}e(\ell) + \mathrm{h}(\ell, e) + \mathrm{B}\mathrm{u}(\ell), & \ell \ne \ell_k, \\ e(\ell) = \phi e(\ell_k^-), & \ell = \ell_k. \end{cases} \tag{22}$$

Suppose there exist a symmetric positive definite matrix $\mathrm{P} \in \mathbb{R}^{3\times3}$*, matrices* $\mathrm{W}, \mathrm{W}_0 \in \mathbb{R}^{3\times3}$*, constants* $\mu > 0$*,* $q > 0$*,* $\eta > 0$*,* $\varrho > 0$*, and a Lipschitz constant* $\omega > 0$ *such that*

$$\begin{bmatrix} \mathrm{A}^T\mathrm{P} + \mathrm{W}^T + \mathrm{P}\mathrm{A} + \mathrm{W} + \mathrm{P}\varrho^{-1} + \omega^2\mathrm{P}\varrho + \eta\mathrm{P} & 0 \\ 0 & -\mathrm{P} \end{bmatrix} \le 0, \tag{23}$$

and

$$\begin{bmatrix} -e^{-\mu}\mathrm{P} & \mathrm{W}_0^T \\ * & -\mathrm{P} \end{bmatrix} \le 0. \tag{24}$$

If $\mathrm{B}\psi = \mathrm{P}^{-1}\mathrm{W}$ *and* $\mathrm{B}\phi = \mathrm{P}^{-1}\mathrm{W}_0$*, then the zero solution of* (22) *is globally Mittag-Leffler synchronized.*

PROOF. *Since* $G(\ell) \equiv 0$, *the nonlinear term* $h(\ell, e)$ *is no longer canceled and must be estimated. Consider again*

$$\mathbb{V}(\ell) = e^T(\ell)Pe(\ell).$$

For $\ell \neq \ell_k$, *by Lemma 2.1,*

$$\overset{C}{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) \leq e^T(\ell)\Big((A + B\psi)^T P + P(A + B\psi)\Big)e(\ell) + 2e^T(\ell)Ph(\ell, e). \tag{25}$$

Using Lemma 2 with parameter $\varrho > 0$, *we estimate the cross term as*

$$2e^T(\ell)Ph(\ell, e) \leq \varrho^{-1}e^T(\ell)Pe(\ell) + \varrho h^T(\ell, e)Ph(\ell, e).$$

Since $h(\ell, e)$ *is Lipschitz, there exists* $\omega > 0$ *such that*

$$\|h(\ell, e)\| \leq \omega\|e(\ell)\|,$$

and thus

$$h^T(\ell, e)Ph(\ell, e) \leq \omega^2 e^T(\ell)Pe(\ell).$$

Substituting these estimates into (25), *we get*

$$\overset{C}{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) \leq e^T(\ell)\Big(A^T P + W^T + PA + W + P\varrho^{-1} + \omega^2 P\varrho\Big)e(\ell).$$

Condition (23) *implies*

$$\overset{C}{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) \leq -\eta\mathbb{V}(\ell), \qquad \ell \neq \ell_k.$$

At the impulsive instants, condition (24) *yields*

$$\mathbb{V}(\ell_k) \leq e^{-\mu}\mathbb{V}(\ell_k^-).$$

The rest of the proof follows exactly as in Theorem 4.1, and therefore

$$\|e(\ell)\| \leq \left(\frac{\mathbb{V}(\ell_0)}{\lambda_{\min}(P)}e^{-k\mu}\prod_{i=1}^{k}\mathbb{E}_{\delta}\Big(-\eta(\ell_i - \ell_{i-1})^{\delta}\Big)\mathbb{E}_{\delta}\Big(-\eta(\ell - \ell_k)^{\delta}\Big)\right)^{1/2}.$$

Hence, the zero solution of (22) *is globally Mittag-Leffler synchronized.*

Remark 4.3. Corollary 4.1 shows that even in the absence of nonlinear compensation, synchronization can still be achieved provided that the combined effect of the state-feedback and impulsive control is strong enough to dominate the nonlinear term through the Lipschitz estimate.

Corollary 4.2. *Assume that both the nonlinear compensation control and the continuous state-feedback control are absent, i.e.,* $G(\ell) \equiv 0$ *and* $u(\ell) \equiv 0$. *Then the error system reduces to*

$$\begin{cases} \overset{C}{\ell_k}\mathbb{D}^{\delta}_{\ell}e(\ell) = Ae(\ell) + h(\ell, e), & \ell \neq \ell_k, \\ e(\ell) = \phi e(\ell_k^-), & \ell = \ell_k. \end{cases} \tag{26}$$

If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{3\times3}$, *a matrix* $W_0 \in \mathbb{R}^{3\times3}$, *and constants* $\mu > 0$, $q > 0$, $\eta > 0$, $\varrho > 0$, *and* $\omega > 0$ *such that*

$$\begin{bmatrix} A^T P + PA + P\varrho^{-1} + \omega^2 P\varrho + \eta P & 0 \\ 0 & -P \end{bmatrix} \leq 0, \tag{27}$$

and

$$\begin{bmatrix} -e^{-\mu}P & W_0^T \\ * & -P \end{bmatrix} \leq 0, \tag{28}$$

with $B\phi = P^{-1}W_0$, *then the zero solution of* (26) *is globally Mittag-Leffler synchronized.*

PROOF. *This is the special case of Corollary 4.1 obtained by setting* $\mathrm{W} = 0$*, equivalently* $\mathrm{u}(\ell) \equiv 0$*. Therefore, the same Lyapunov analysis applies, and the desired conclusion follows immediately.*

Corollary 4.3. *Assume that the continuous state-feedback control is absent, i.e.,* $\mathrm{u}(\ell) \equiv 0$*, while the nonlinear compensation control remains active. Then the error system becomes*

$$\begin{cases} {}^{C}_{\ell_k}\mathbb{D}^{\delta}_{\ell} e(\ell) = \mathrm{A}e(\ell), & \ell \neq \ell_k, \\ e(\ell) = \phi e(\ell_k^-), & \ell = \ell_k. \end{cases} \tag{29}$$

If there exist a symmetric positive definite matrix $\mathrm{P} \in \mathbb{R}^{3\times 3}$*, a matrix* $\mathrm{W}_0 \in \mathbb{R}^{3\times 3}$*, and constants* $\mu > 0$*,* $q > 0$*, and* $\eta > 0$ *such that*

$$\begin{bmatrix} \mathrm{A}^T\mathrm{P} + \mathrm{P}\mathrm{A} + \eta\mathrm{P} & 0 \\ 0 & -\mathrm{P} \end{bmatrix} \leq 0, \tag{30}$$

and

$$\begin{bmatrix} -e^{-\mu}\mathrm{P} & \mathrm{W}_0^T \\ * & -\mathrm{P} \end{bmatrix} \leq 0, \tag{31}$$

with $BK_0 = \mathrm{P}^{-1}\mathrm{W}_0$*, then the zero solution of* (29) *is globally Mittag-Leffler synchronized.*

PROOF. *This result is a direct particular case of Theorem 4.1 obtained by setting* $\mathrm{W} = 0$ *and retaining the nonlinear compensation control* $\mathrm{G}(\ell)$ *so that the nonlinear term is canceled. The resulting continuous-time error dynamics is linear:*

$${}^{C}_{\ell_k}\mathbb{D}^{\delta}_{\ell} e(\ell) = \mathrm{A}e(\ell), \qquad \ell \neq \ell_k.$$

Hence,

$${}^{C}_{\ell_k}\mathbb{D}^{\delta}_{\ell}\mathbb{V}(\ell) \leq e^T(\ell)(\mathrm{A}^T\mathrm{P} + \mathrm{P}\mathrm{A})e(\ell) \leq -\eta\mathbb{V}(\ell),$$

and the jump condition is handled exactly as in Theorem 4.1. The conclusion follows immediately.

Remark 4.4. Corollaries 4.1–4.3 demonstrate the flexibility of the proposed framework. In particular, the full hybrid controller in Theorem 4.1 contains several practically relevant synchronization schemes as special cases. This also highlights the robustness of the proposed strategy: if one control channel becomes unavailable, synchronization may still be preserved under suitable reduced-order conditions.

5. Examples

In this section, we validate the theoretical results by considering the fractional-order Lü system with parameters

$$a = 36, \qquad b = 3, \qquad c = 20.$$

Accordingly,

$$\mathrm{A} = \begin{pmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Figure 1 illustrates that the uncontrolled fractional-order Lü system exhibits chaotic behavior.

Proposition 5.1. *For* $\delta = 0.85$ *and initial condition* $e(0) = [0.1,\ 0.2,\ 0.3]^T$*, the error system* (10) *is globally Mittag-Leffler synchronized under the conditions of Theorem 4.1.*

PROOF. Choose $\eta = 48$ and $\mu = 4$. Solving the LMIs in Theorem 4.1, one feasible set of gains is

$$\psi = \begin{pmatrix} 56.1081 & 8.9234 & 5.0340 \\ -3.3677 & -0.0737 & -1.4512 \\ -0.8328 & 3.5571 & 23.3889 \end{pmatrix}, \qquad \phi = \begin{pmatrix} 0.0435 & 0.1406 & 0.0565 \\ 0.0118 & 0.0382 & 0.0155 \\ 0.0327 & 0.1038 & 0.0424 \end{pmatrix}.$$

Hence, all the conditions of Theorem 4.1 are satisfied, and the error system (10) is globally Mittag-Leffler synchronized. The corresponding trajectories are shown in Figure 2.

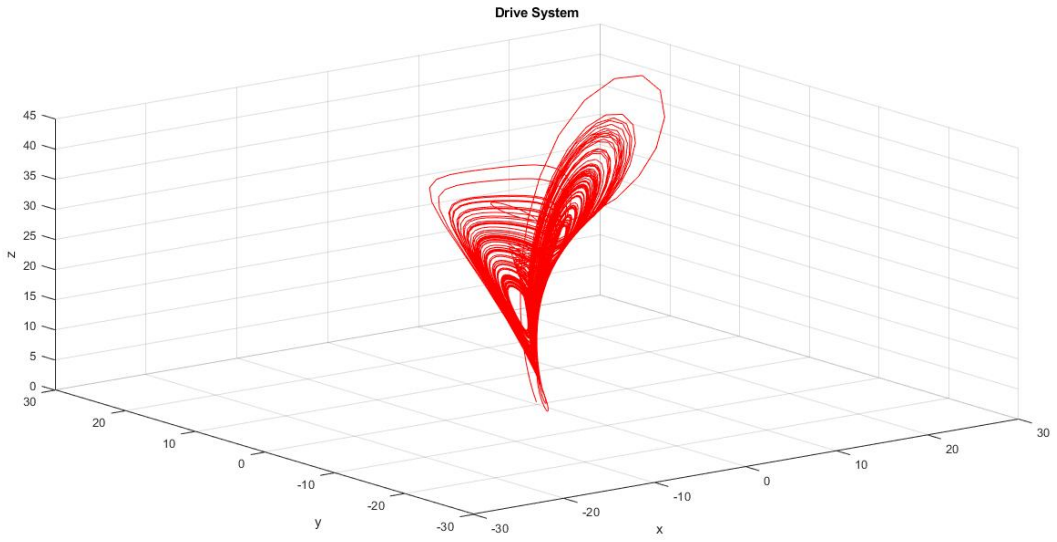


Figure 1: Fractional order chaotic Lü system.

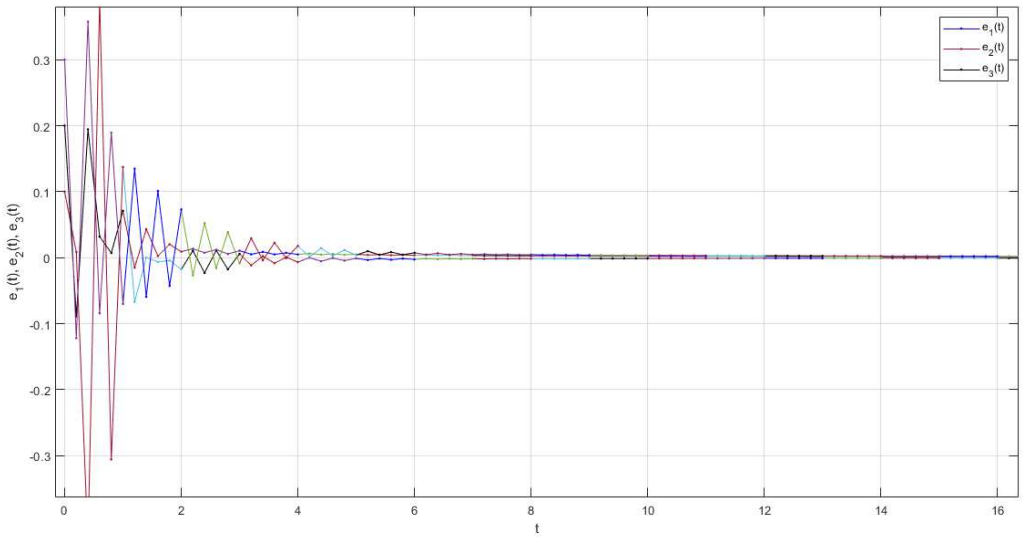


Figure 2: The error system trajectories with hybrid controllers.

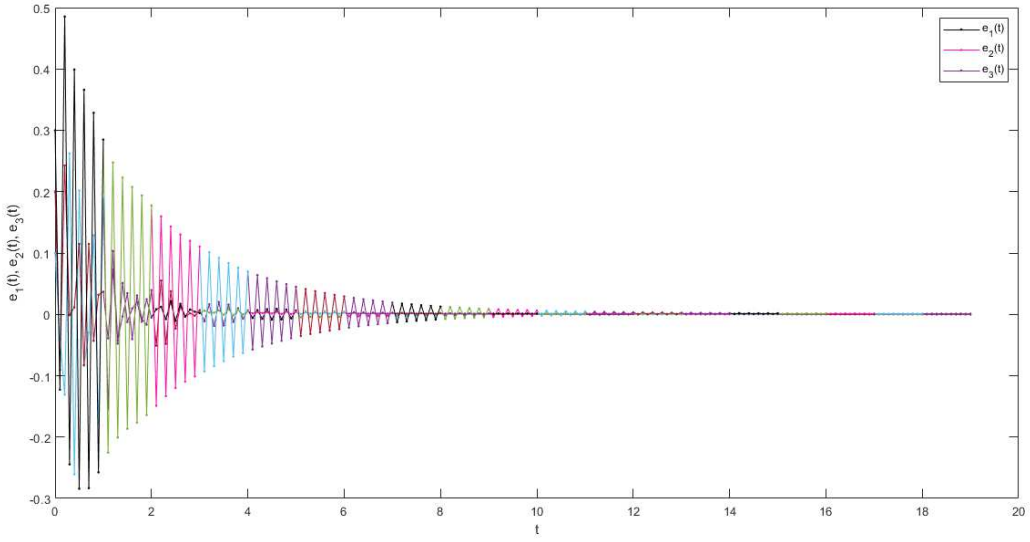


Figure 3: The error system trajectories with G = 0.

Proposition 5.2. *For* $\delta = 0.85$ *and initial condition* $e(0) = [0.1, 0.2, 0.3]^T$*, the reduced error system* (22) *is globally Mittag-Leffler synchronized under the conditions of Corollary 4.1.*

PROOF. Choose $\eta = 0.5$ and $\mu = 0.12$. Solving the LMIs in Corollary 4.1, one obtains the feasible gains

$$\psi = \begin{pmatrix} 65.0280 & -6.9957 & 0 \\ -6.1649 & 17.1499 & 0 \\ 0 & 0 & 38.5120 \end{pmatrix}, \qquad \phi = \begin{pmatrix} 0.5670 & -0.1981 & 0 \\ -0.0538 & 0.4857 & 0 \\ 0 & 0 & 0.4875 \end{pmatrix}.$$

Therefore, the conditions of Corollary 4.1 hold, and (22) is globally Mittag-Leffler synchronized. See Figure 3.

Proposition 5.3. *For* $\delta = 0.85$ *and initial condition* $e(0) = [0.1, 0.2, 0.3]^T$*, the error system* (26) *is globally Mittag-Leffler synchronized under the conditions of Corollary 4.2.*

PROOF. Choose $\eta = 1$ and $\mu = 2$. From the LMIs in Corollary 4.2, one admissible impulsive gain is

$$\phi = \begin{pmatrix} 0.2472 & -0.0926 & 0 \\ -0.0371 & 0.2262 & 0 \\ 0 & 0 & 0.2417 \end{pmatrix}.$$

Hence, (26) is globally Mittag-Leffler synchronized. The corresponding behavior is illustrated in Figure 4.

Proposition 5.4. *For* $\delta = 0.85$ *and initial condition* $e(0) = [0.1, 0.2, 0.3]^T$*, the error system* (29) *is globally Mittag-Leffler synchronized under the conditions of Corollary 4.3.*

PROOF. Choose $\eta = 55$ and $\mu = 2$. Solving the LMIs in Corollary 4.3, one obtains

$$\phi = \begin{pmatrix} 0.2049 & -0.2180 & 0 \\ -0.0225 & 0.2001 & 0 \\ 0 & 0 & 0.2198 \end{pmatrix}.$$

Therefore, the reduced system (29) is globally Mittag-Leffler synchronized. See Figure 5.

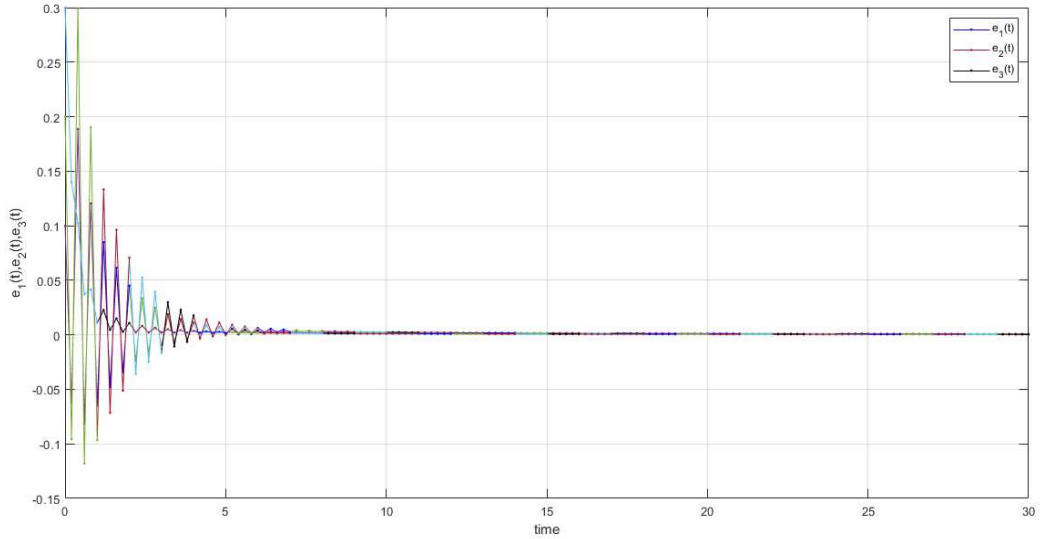


Figure 4: The error system trajectories with $u = 0$ and G = 0.

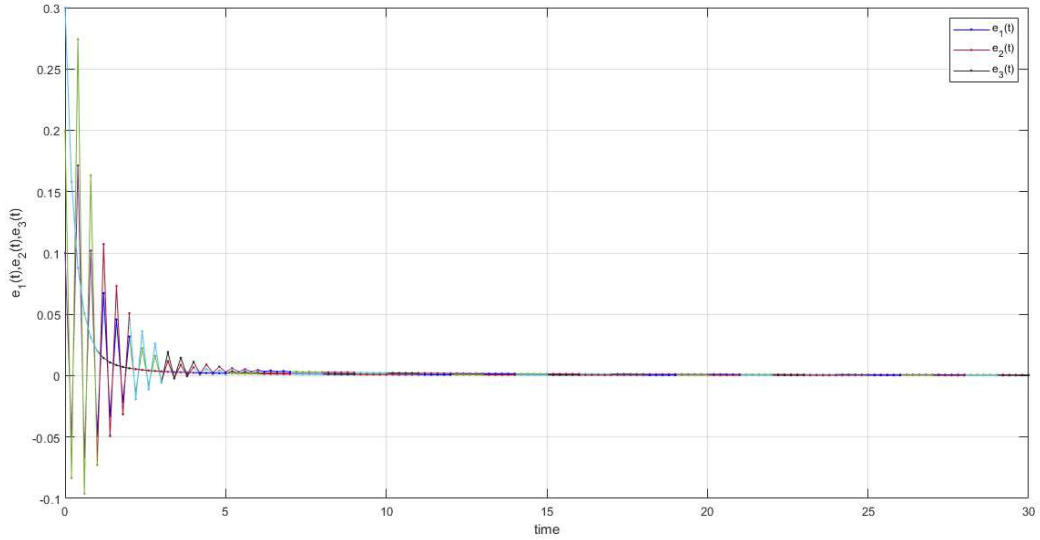


Figure 5: The error system trajectories with B = 0.

Remark 5.1. Figure 2 shows the synchronization performance of the complete hybrid controller. Figure 3 corresponds to the case where the nonlinear compensation is removed, while Figures 4 and 5 illustrate two further reduced control configurations. These simulations confirm that the full hybrid strategy offers the most comprehensive synchronization mechanism, whereas the corollaries describe meaningful special cases that remain effective under reduced control action.

6. Conclusion

In this paper, the Mittag-Leffler synchronization of a fractional-order Lü chaotic system has been studied using a hybrid control approach. By combining nonlinear compensation, state-feedback control, and impulsive control, new synchronization conditions were obtained using Lyapunov theory and linear matrix inequalities. The proposed method provides a clear and systematic way to design suitable control gains.

The main idea of this work is to use a unified control framework in which different control components act together. Unlike many existing results where control methods are applied separately, the proposed approach allows them to work cooperatively, which improves the overall robustness of the system. Even if one control part becomes less effective, the others can still maintain synchronization. The results also show that the derived conditions are easy to verify numerically using LMI tools. The general result includes several special cases, which helps to better understand the role of each control component. Numerical simulations confirm that the proposed method achieves synchronization effectively under different control settings. The obtained results may be useful in applications such as secure communication and complex systems where fractional-order chaotic models are used. The hybrid control structure is especially suitable in situations where continuous control needs to be supported by impulsive actions.

Future work may include extending the results to systems with delays, uncertainties, or disturbances. It would also be interesting to study adaptive or observer-based hybrid control methods, as well as applications to complex networks and neural systems.

Declarations

Conflict of interest: The authors declare that they have no conflicts of interest with respect to the research, authorship, and/or publication of this article.
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