Supplementary Information for: HexagonalWarriorMamba: Superior Threshold-Dependent Multi-label Classification of 12-Lead ECG Cardiac Abnormalities

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ABSTRACT

This document provides the detailed mathematical derivation and proof for the discretization process of the State Space Model (SSM) utilized in the HexagonalWarriorMamba framework. Additionally, it defines the nomenclature and variables essential to this process as summarized in Table S1 and details the abbreviations for each diagnosis label in Table S2.

Derivation of SSM Discretization

The Mamba architecture transforms the continuous-time State Space Model (SSM) into a discrete-time formulation suitable for deep learning sequences.

Consider the continuous-time system:

$$h'(t) = \mathbf{A}h(t) + \mathbf{B}x(t) \tag{S1}$$

$$y(t) = \mathbf{C}h(t) \tag{S2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant matrices.

Step 1: General Solution via Laplace Transform

Since the state equation (S1) is a first-order differential equation, the Laplace transform is applied. For a function $f(t)$, the transform is defined as:

$$\mathscr{L}\{f(t)\} = H(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{S3}$$

The transform of the derivative $h'(t)$ is derived using integration by parts:

$$\mathscr{L}\{h'(t)\} = \int_0^\infty h'(t)e^{-st}\,dt \tag{S4}$$

Let $u = e^{-st}$ and $dv = h'(t)dt$. Then $du = -se^{-st}dt$ and $v = h(t)$. Applying the formula $\int u\,dv = uv - \int v\,du$:

$$\int_0^\infty h'(t)e^{-st}\,dt = \left[h(t)e^{-st}\right]_0^\infty + s\int_0^\infty h(t)e^{-st}\,dt \tag{S5}$$

Assuming system stability (where $\lim_{t\to\infty} h(t)e^{-st} = 0$), the boundary term simplifies to $-h(0)$. Substituting the definition of $H(s)$:

$$\mathscr{L}\{h'(t)\} = sH(s) - h(0) \tag{S6}$$

Next, the linearity of the Laplace transform is applied to the right-hand side of Eq. (S1):

$$\mathscr{L}\{\mathbf{A}h(t) + \mathbf{B}x(t)\} = \mathbf{A}H(s) + \mathbf{B}X(s) \tag{S7}$$

Equating the transformed sides:

$$sH(s) - h(0) = \mathbf{A}H(s) + \mathbf{B}X(s) \tag{S8}$$

Rearranging to solve for $H(s)$ (using the identity matrix $\mathbf{I}$):

$$(s\mathbf{I} - \mathbf{A})H(s) = h(0) + \mathbf{B}X(s) \tag{S9}$$

$$H(s) = (s\mathbf{I} - \mathbf{A})^{-1}h(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}X(s) \tag{S10}$$

The time-domain solution $h(t)$ is recovered via the inverse Laplace transform:

1. **First term:** Recognized as the matrix exponential: $\mathscr{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = e^{\mathbf{A}t}$.
2. **Second term:** Handled via the Convolution Theorem, $\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$.

Thus, the general solution is:

$$h(t) = e^{\mathbf{A}t}h(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}x(\tau)\,d\tau \tag{S11}$$

Generalizing the initial time from 0 to an arbitrary t_0 yields the Variation of Constants formula:

$$h(t) = e^{\mathbf{A}(t-t_0)}h(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}x(\tau)\,d\tau \tag{S12}$$

Step 2: Discrete-Time Formulation

To discretize the system, let the time steps be defined by a fixed step size Δ, such that:

$$t_0 = t_{k-1}, \quad t = t_k, \quad \text{and} \quad \Delta = t_k - t_{k-1} \tag{S13}$$

Substituting these into Eq. (S12):

$$h(t_k) = e^{\mathbf{A}\Delta}h(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{\mathbf{A}(t_k-\tau)}\mathbf{B}x(\tau)\,d\tau \tag{S14}$$

Step 3: Zero-Order Hold Discretization

The analytical evaluation of the integral requires knowledge of $x(\tau)$ between discrete steps. The Zero-Order Hold assumption is applied, where the input is assumed constant over the interval $[t_{k-1}, t_k]$:

$$x(\tau) \approx x_k \quad \text{for} \quad \tau \in [t_{k-1}, t_k] \tag{S15}$$

Since x_k and $\mathbf{B}$ are constant with respect to τ, they are extracted from the integral:

$$h_k = e^{\mathbf{A}\Delta}h_{k-1} + \left(\int_{t_{k-1}}^{t_k} e^{\mathbf{A}(t_k-\tau)}\,d\tau\right)\mathbf{B}x_k \tag{S16}$$

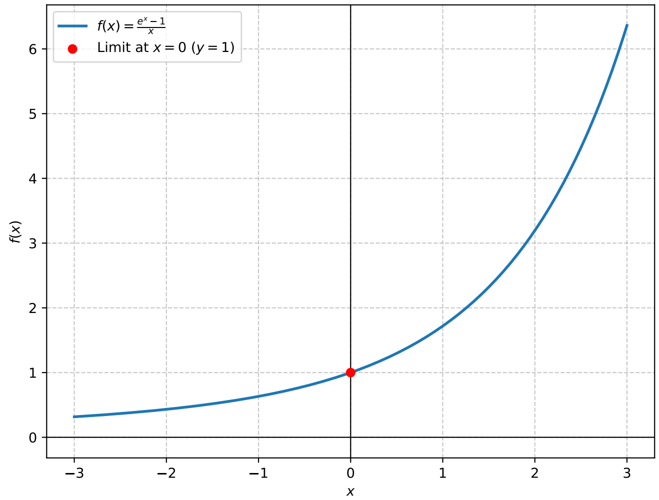

Figure S1. Graph of the Function $f(x) = \frac{e^x - 1}{x}$

Step 4: Evaluation of the Integral

The integral term is solved using the change of variables $u = t_k - \tau$, implying $du = -d\tau$. The limits change from $[t_{k-1}, t_k]$ to $[\Delta, 0]$:

$$\int_{t_{k-1}}^{t_k} e^{\mathbf{A}(t_k - \tau)}\, d\tau = \int_{\Delta}^{0} e^{\mathbf{A}u}(-du) = \int_{0}^{\Delta} e^{\mathbf{A}u}\, du \tag{S17}$$

Assuming $\mathbf{A}$ is invertible, the integral of the matrix exponential is:

$$\left[\mathbf{A}^{-1} e^{\mathbf{A}u}\right]_0^{\Delta} = \mathbf{A}^{-1}(e^{\mathbf{A}\Delta} - \mathbf{I}) \tag{S18}$$

Final Discrete Recurrence

Substituting the integral result back into the state equation yields the discrete recurrence rule:

$$h_k = \underbrace{e^{\mathbf{A}\Delta}}_{\overline{\mathbf{A}}} h_{k-1} + \underbrace{\mathbf{A}^{-1}(e^{\mathbf{A}\Delta} - \mathbf{I})\mathbf{B}}_{\overline{\mathbf{B}}} x_k \tag{S19}$$

The discretized system parameters $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ are formally defined as:

$$\overline{\mathbf{A}} = \exp(\mathbf{A}\Delta) \tag{S20}$$

$$\overline{\mathbf{B}} = \mathbf{A}^{-1}(\exp(\mathbf{A}\Delta) - \mathbf{I}) \cdot \mathbf{B} \tag{S21}$$

resulting in the discrete linear recurrence:

$$h_k = \overline{\mathbf{A}} h_{k-1} + \overline{\mathbf{B}} x_k \tag{S22}$$

Equation (S21) implies an inversion of $\mathbf{A}$, which can be conceptually related to the scalar function:

$$f(x) = \frac{e^x - 1}{x} \tag{S23}$$

As illustrated in Figure S1, this function $f(x)$ remains strictly positive for all real x. However, regarding the matrix term $\mathbf{A}^{-1}$, if $\mathbf{A}$ possesses eigenvalues close to zero (singular or ill-conditioned), direct computation of the inverse leads to numerical instability or undefined values (NaN).

To resolve this, the expression is reformulated to avoid direct inversion of small values. The term $(\mathbf{\Delta A})^{-1}(\exp(\mathbf{\Delta A}) - \mathbf{I})$ corresponds to the scalar function $f(x)$ where $x = \mathbf{\Delta A}$. Since $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$, this form remains numerically stable even as $\mathbf{A} \to 0$.

By multiplying the numerator and denominator by $\mathbf{\Delta}$, the discretization is rewritten in its robust form:

$$\overline{\mathbf{A}} = \exp(\mathbf{\Delta A}) \tag{S24}$$

$$\overline{\mathbf{B}} = (\mathbf{\Delta A})^{-1}(\exp(\mathbf{\Delta A}) - \mathbf{I}) \cdot (\mathbf{\Delta B}) \tag{S25}$$

Table S1. Nomenclature and Variable Definitions for SSM Discretization

Symbol	**Description**
Continuous-Time System Parameters	
$\mathbf{A}$	Continuous-time state transition matrix
$\mathbf{B}$	Continuous-time input projection matrix
$\mathbf{C}$	Output projection matrix
$\mathbf{I}$	Identity matrix
Time-Domain Variables	
t	Continuous time variable
$h(t)$	Continuous-time hidden state vector
$h'(t)$	Time derivative of the hidden state
$x(t)$	Continuous-time input signal
$y(t)$	Output signal
t_0	Arbitrary initial time
τ	Integration variable (dummy time variable)
Laplace Domain	
s	Complex frequency variable in the Laplace domain
$\mathscr{L}\{\cdot\}$	Laplace transform operator
$H(s)$	Laplace transform of the hidden state $h(t)$
$X(s)$	Laplace transform of the input $x(t)$
Discretization Parameters	
Δ	Discrete time step size (sampling interval)
k	Discrete time step index
t_k	Discrete time point at step k
h_k	Discrete hidden state vector at step k
x_k	Discrete input vector at step k (Zero-Order Hold)
$\overline{\mathbf{A}}$	Discretized state transition matrix ($\exp(\mathbf{A}\Delta)$)
$\overline{\mathbf{B}}$	Discretized input matrix
$f(x)$	Scalar function $\frac{e^x-1}{x}$ used for numerical stability

Class-Wise Performance Analysis

Table S2. PhysioNet/Computing in Cardiology Challenge 2021 Scored Diagnoses

Abbreviation	Diagnosis Label
AF	Atrial Fibrillation
AFL	Atrial Flutter
BBB	Bundle Branch Block
Brady	Bradycardia
CLBBB	Complete Left Bundle Branch Block
CRBBB	Complete Right Bundle Branch Block
1AVB	1st Degree Atrioventricular Block
IRBBB	Incomplete Right Bundle Branch Block
LAD	Left Axis Deviation
LAnFB	Left Anterior Fascicular Block
LPR	Prolonged PR Interval
LQRSV	Low QRS Voltages
LQT	Prolonged QT Interval
NSIVCD	Nonspecific Intraventricular Conduction Disorder
NSR	Sinus Rhythm
PAC	Premature Atrial Contraction
PR	Pacing Rhythm
PRWP	Poor R Wave Progression
PVC	Premature Ventricular Contractions
QAb	Q Wave Abnormal
RAD	Right Axis Deviation
SA	Sinus Arrhythmia
SB	Sinus Bradycardia
STach	Sinus Tachycardia
TAb	T Wave Abnormal
TInv	T Wave Inversion