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Robustly self-testing all maximally entangled states in every finite dimension

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We establish a device-independent, noise-tolerant certification of maximally entangled states in every finite dimension d . The core ingredient is a d -input, d -outcome Bell experiment that recovers the genuinely qudit Heisenberg–Weyl structure. Yielding the Tsirelson bound in closed form, the associated Bell operator has an exact sum-of-positive-operators decomposition for $d \geq 3$ prime, from which we reconstruct the Heisenberg–Weyl commutation relations on the support of the state. Representation-theoretically, once the twisted-commutation relations are established, the finite Stone–von Neumann theorem identifies the realization with the canonical Heisenberg–Weyl model; in addition, we provide an explicit SWAP-type local isometry that yields the robustness constants. This gives a genuinely qudit certification: rather than reducing to qubit subtests, the higher-dimensional Heisenberg–Weyl algebra itself is recovered. We show that any ϵ -near-optimal strategy is, up to local isometries, within trace distance $\delta = \mathcal{O}(\sqrt{\epsilon})$ of the ideal maximally entangled state (with dimension-explicit constants), and the implemented measurements are correspondingly close to the target observables. Via a tensor-factor argument, the prime-dimensional result extends the self-testing protocol to every composite dimension d , thereby yielding a fully rigorous, robust, and genuinely qudit self-test of maximally entangled states in all finite dimensions.

Introduction. Bell’s theorem reshaped our understanding of nature by demonstrating that quantum predictions fundamentally diverge from classical expectations, ruling out any explanation via local-hidden-variable (LHV) models [1, 2]. Today, Bell experiments not only serve as powerful tools to demonstrate nonlocality but also form the foundation for device-independent self-testing, enabling certification of quantum states and measurements purely from observed correlations [3, 4]. In particular, self-testing has emerged as a pivotal technique in quantum information science, with crucial implications for secure quantum communication [5–7], delegated quantum computation [8] and computational complexity [9].

While self-testing is well-developed and robustly characterized for qubit systems, extending such certification protocols to higher-dimensional quantum systems remains a major challenge. Qudits enable stronger dimension witnesses and richer forms of multipartite entanglement [10–13], exhibit novel nonlocal features [14–22], and support a variety of cryptographic and communication tasks [23–29]. Experimentally, high-quality qudits have already been realized in photonic path, time-bin, and orbital-angular-momentum degrees of freedom [30, 31]. Yet existing self-testing protocols for higher-dimensional systems either rely on decompositions into qubit subspaces [32–34] or lack complete robustness analyses [35–38].

A central open problem is therefore to obtain genuinely qudit self-tests that are both fully rigorous and robust. By “genuinely qudit” we mean that the certified structure is the higher-dimensional Heisenberg–Weyl algebra itself, rather than a decomposition into qubit subtests or a dimension-specific construction. What makes

this difficult is precisely that, beyond qubits, the relevant algebra is no longer governed by pairs of binary observables: standard CHSH-type anticommutation arguments do not transfer directly, there is no canonical reduction to 2×2 blocks, and the higher-dimensional operator structure must be controlled through genuinely d -dimensional twisted-commutation relations. Previous high-dimensional constructions showed that maximally entangled states can in principle be certified in dimension-independent or constant-size scenarios, but either lacked a full unconditional robustness proof or relied on structures not amenable to the standard robust estimates based on triangle and Cauchy-Schwarz inequalities.

In this work we overcome that obstacle by giving a fully analytic route to robust high-dimensional self-testing. For odd prime d , we introduce a Bell test built from non-diagonal Heisenberg–Weyl observables and show that maximal, and more generally near-maximal, violation rigidly enforces the canonical Heisenberg–Weyl relations on the support of the state. The proof derives a closed-form sum-of-positive-operators decomposition for the Bell operator, uses an explicit rank/orthogonality argument in place of semidefinite programming, and completes the identification via the finite Stone–von Neumann theorem. A tensor-factor lift then extends the result from prime to arbitrary finite local dimension. This yields, to our knowledge, the first *fully rigorous, unconditionally robust, genuinely qudit* self-test of maximally entangled states in all finite dimensions: if the observed Bell value is within ϵ of the optimum, then the realized state and measurements are, up to local isometries, within $\mathcal{O}(\sqrt{\epsilon})$ of the canonical maximally entangled state and Heisenberg–Weyl observables, with explicit dimension-dependent constants.

We achieve this by using a Bell experiment genuinely specific for prime-dimensional qudits with non-diagonal Heisenberg–Weyl observables, and building upon recent

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proposals introduced by the authors in previous work [39]. By a tensor-factor argument, the protocol extends to every composite dimension. We explicitly derive analytic robustness bounds, showing that correlations approaching the Tsirelson bound [40] enforce strict approximations of the ideal maximally entangled state and canonical Heisenberg–Weyl measurement operators.

Our route is a fully analytic, Heisenberg–Weyl-only approach. We derive a closed-form SOPO for a non-diagonal HW Bell operator with explicit coefficients, avoid SDPs using a rank/orthogonality argument, and complete the proof with the finite Stone–von Neumann theorem. A prime-factor lift then extends the result to all finite d .

Related work and comparison. High-dimensional self-testing has progressed along several complementary directions. Early approaches relied on tailored Bell inequalities, multiple correlation constraints, or reductions to lower-dimensional substructures [32–34]. A major conceptual step was the observation of Sarkar et al. that a maximally entangled state in arbitrary local dimension can, in principle, be certified in a single two-setting, d -outcome Bell scenario [38]; a related two-setting construction for local dimension 2^d was later given in [41]. These works show that high-dimensional self-testing need not require a growing number of settings, but they were not accompanied by a full unconditional robustness analysis. Moreover, as noted in [42], their algebraic structure does not admit a straightforward robust treatment by the standard triangle-inequality and Cauchy-Schwarz methods that succeed in many qubit settings.

A different line of work has established constant-size self-tests, in which the number of measurements is independent of the local dimension. Fu constructed explicit fixed-size correlations self-testing an infinite family of maximally entangled states via number-theoretic methods [43]. Mančinska et al. then developed a fully robust algebraic scheme for every odd dimension, as well as arbitrarily high-rank two-outcome measurements, using four binary settings per party [44], and Volčič subsequently removed the parity restriction, extending this constant-size paradigm to all finite dimensions [45]. In parallel, Chen et al. introduced a post-hoc iterative method that certifies additional observables on a previously self-tested high-dimensional maximally entangled state, enabling device-independent verification of essentially any real projective measurement on that state [46].

These advances address different aspects of the problem rather than directly competing. The constant-size results optimize the number of settings, while the two-setting constructions show that dimension-independent scenarios are possible in principle. Our aim is different: we seek a single, fully analytic, and unconditionally robust protocol that certifies the genuinely qudit Heisenberg–Weyl structure itself and extends uniformly to all finite dimensions. Accordingly, the distinguishing feature of our result is not the number of settings, but the combination of four properties in one theorem: it is genuinely qudit, in that it reconstructs the higher-dimensional Heisenberg–Weyl alge-

bra rather than reducing to qubit subtests; fully rigorous, in that the argument is analytic throughout; unconditionally robust, in that the conclusion follows directly from the Bell-value deficit without auxiliary assumptions; and universal, in that the same construction extends from odd primes to arbitrary finite d .

In this sense, our work fills a different gap in the high-dimensional self-testing landscape. Compared with the two-setting schemes of [38, 41], it provides a complete robustness proof. Compared with the constant-size protocols of [43–45], it is built from an explicit universal family of non-diagonal Heisenberg–Weyl observables whose maximal violation admits a closed-form SOPO decomposition and a representation-theoretic completion via the finite Stone–von Neumann theorem. This explicit analytic structure is what allows us to derive a direct robustness argument and to extend the certification uniformly across all finite dimensions.

Nonlocality & self-testing. A bipartite Bell experiment involves two parties, each receiving an input (x and y) and producing an output (a or b) from finite sets. After many repetitions, this process leads to an empirical distribution $p(a, b|x, y)$. We assume the no-signaling condition, meaning that the marginal distribution of one party is independent of the other’s input $\sum_a p(a, b|x, y) = \sum_a p(a, b|x', y)$ and $\sum_b p(a, b|x, y) = \sum_b p(a, b|x', y)$ for all a, b, x, x', y, y' . Nonlocality refers to the impossibility of reproducing such a distribution with any LHV model of the form $p(a, b|x, y) = \sum_\lambda p(\lambda) p(a|x, \lambda) p(b|y, \lambda)$. Certain correlations arising from local measurements on entangled quantum states violate this classical description and are therefore nonlocal [47]. The distinction from classical models is typically certified by a Bell inequality, i.e. a bound on the expectation value of a Bell operator under LHV models.

Some quantum correlations go further: they are not only nonlocal but also unique, in the sense that no alternative Bell experiment can reproduce them, except up to irrelevant degrees of freedom such as local unitaries or actions on auxiliary subsystems. When all experimental realizations of the observed correlations can be shown to be equivalent in this sense to a known reference experiment, one can conclude that the underlying state and measurements are uniquely characterized. This property is called *self-testing* [4].

Qudit Bell experiment. We work with qudits of odd prime local dimension d . The detailed experiment, definitions and notation are in the Appendix A. Measurements are taken in eigenbases of Heisenberg–Weyl (HW) displacement operators T_u , yielding a scenario with d inputs and d outputs per party. The shared state is the maximally entangled pair rotated by a diagonal non-Clifford unitary,

$$|\Phi\rangle = \frac{1}{\sqrt{d}} (U_\nu \otimes \mathbb{1}) \sum_{k \in \mathbb{Z}_d} |kk\rangle, \quad (1)$$

where U_ν is a computational-basis phase. Omitting the Z -basis measurements eliminates purely stabilizer correlations and leaves a Bell operator $\mathcal{B}_d$ built only from non-diagonal HW terms:

$$\tilde{\mathcal{B}}_d := \frac{1}{d(d-1)} \sum_{n \in \mathbb{Z}_d^*} \sum_{j,k \in \mathbb{Z}_d} g(j,k,n) A_j^n \otimes B_k^n, \quad (2)$$

where the coefficients $g(j,k,n)$ are given explicitly in the appendix, Eq. (A12). For the target state $|\Phi\rangle$, one has

$$\langle \Phi | \tilde{\mathcal{B}}_d | \Phi \rangle = 1. \quad (3)$$

A key structural fact is an exact sum-of-positive-operators (SOPO) decomposition

$$\mathbb{1} - \tilde{\mathcal{B}}_d = \frac{1}{d(d-1)} \sum_{n=1}^{(d-1)/2} \sum_{j \in \mathbb{Z}_d} C_{n,j}^\dagger C_{n,j}, \quad (4)$$

which pins down the optimal correlations and underlies our robustness analysis. The gap to the optimum is the starting point of both the ideal and robust analyses.

For $d = 3$ we use a tailored 9-term operator $\tilde{\mathcal{B}}_3$ given explicitly in the appendix, Eq. (A14). The qutrit Bell state satisfies $\langle \Phi | \tilde{\mathcal{B}}_3 | \Phi \rangle = 1$, while any local-hidden-variable model attains a strictly smaller value.

Main result The maximally entangled qudit state and the canonical Heisenberg–Weyl measurements can be robustly self-tested from the maximal violation of our Bell inequality.

Theorem 1. *Given unitary operators $A_j \in \mathcal{H}_A$, $B_j \in \mathcal{H}_B$ for $j = 0, \dots, d-1$ that constitute the operator $\tilde{\mathcal{B}}_d$ in Eq. (A11) when $d > 3$, and the operator $\tilde{\mathcal{B}}_3$ in Eq. (A14) if $d = 3$. Suppose further that $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a bipartite state such that $|\langle \psi | \tilde{\mathcal{B}}_d | \psi \rangle - 1| \leq \epsilon$. Then, there exist local isometries $V_A : \mathcal{H}_A \rightarrow \mathbb{C}^d \otimes \mathcal{H}_{A'}$ and $V_B : \mathcal{H}_B \rightarrow \mathbb{C}^d \otimes \mathcal{H}_{B'}$ together with an error function $\delta(\epsilon) = \sqrt{\epsilon}(\mu_d(4 + 1/d) + 1)$ such that*

$$\|(V_A \otimes V_B)(|\psi\rangle) - |\Phi\rangle \otimes |aux\rangle\| \leq \delta, \quad (5)$$

where $|\Phi\rangle$ is the maximally entangled state in Eq. (1), $|aux\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$ is an auxiliary state, $\mu_3 = 9\sqrt{\epsilon}(\sqrt{3}+2)$, and $\mu_d = \sqrt{d}(\sqrt{d}+2)$ when $d > 3$. For $d = 3$, we further assume that $\epsilon < (72(\sqrt{3}+2))^{-2}$. Moreover, under these isometries the observables are mapped as $V_A(A_j) \rightarrow T_{(1,j)} \otimes \mathbb{1}_{A'}$, $V_B(B_j) \rightarrow T_{(1,j)} \otimes \mathbb{1}_{B'}$, where $T_{(1,j)}$ are the canonical Heisenberg–Weyl operators. The auxiliary spaces $\mathcal{H}_{A'}$ and $\mathcal{H}_{B'}$ may have arbitrary finite dimensions.

We first present the proof in the ideal case, $\delta = \epsilon = 0$, where the argument is somewhat cleaner. The robustness, given in the appendix, follows the same structure: each algebraic step is reproduced with error bounds, leading to the stated dependence on ϵ . Both the ideal and robust proofs have a two-step structure:

1. *Twisted commutation relations.* We show that the optimal violation forces the measurement operators to satisfy twisted commutation relations, characteristic of the HW algebra (Lemma 1 for the ideal case).
2. *Isometry.* Given these relations, we use the finite Stone-von Neumann theorem [48, 49], which states that any irreducible unitary representation of the HW group with a fixed central character, here ω , is d -dimensional and unique up to unitary conjugation to the canonical Heisenberg–Weyl representation. In addition, we explicitly construct a local isometry that extracts the maximally entangled state and maps the physical observables to their canonical HW counterparts.

Lemma 1. *Let A_j , B_j and $|\psi\rangle$ be as in Theorem 1, and assume $\langle \psi | \tilde{\mathcal{B}}_d | \psi \rangle = 1$. Then for all $j, k \in \mathbb{Z}_d$ the following twisted commutation relations hold:*

$$\begin{aligned} (A_j A_k - \omega^{j-k} A_k A_j) |\psi\rangle &= 0, \\ (B_j B_k - \omega^{j-k} B_k B_j) |\psi\rangle &= 0. \end{aligned} \quad (6)$$

The full proof is in Appendix D. We use the SOPO decomposition (A13) to obtain a relation between the A_j and B_k with respect to their action on the state $|\psi\rangle$. And then deduce the commutation relation as the only solution to the restrictions $A_j^2 = (A_j)^2$ for all $j \in \mathbb{Z}_d$.

Proof sketch of Lemma 1 (odd prime, $d > 3$). At the optimum, $\langle \psi | \mathbb{1} - \tilde{\mathcal{B}}_d | \psi \rangle = 0$. By the SOPO decomposition (A13), every summand $\langle \psi | C_{n,j}^\dagger C_{n,j} | \psi \rangle$ must vanish, hence $C_{n,j} |\psi\rangle = 0$ for all $n \in \mathbb{Z}_d^*$ and $j \in \mathbb{Z}_d$. This yields the linear relations between Alice’s and Bob’s operators,

$$A_j^n |\psi\rangle = \sum_{k \in \mathbb{Z}_d} g(j,k,n)^* (B_k^n)^\dagger |\psi\rangle \quad (\text{cf. (D1)}),$$

Using the closure $A_j^n A_j^{n'} = A_j^{n+n'}$ (in particular $A_j^2 = A_j A_j$), we eliminate the A_j to obtain a linear system for the grade-2 vectors $\{B_k B_l |\psi\rangle\}_{k,l}$, see Eq. (D3). Because $g(j,k,1)$ are Fourier coefficients of a non-linear phase (our choice of ν has $\deg(\nu) \geq 3$), the associated matrix has full row rank; the unique solution (up to a global phase) enforces the HW structure on the support, implying

$$B_k B_l |\psi\rangle = \omega^{k-l} B_l B_k |\psi\rangle, \quad \forall k, l \in \mathbb{Z}_d.$$

By symmetry (or by repeating the argument with the swapped SOPO), the same twisted-commutation holds for the A ’s. $\square$

Proof sketch of Lemma 1 ($d = 3$). Because $2 \equiv -1 \pmod{3}$, the $d > 3$ elimination renders (D3) under-determined. We instead use the SOPO for $d = 3$ to get $C_j |\psi\rangle = 0$ and $A_j^\dagger = A_j A_j$, which yield symmetric constraints. From these, the pairwise commutators $Q = B_0 B_1 B_0^\dagger B_1^\dagger$, $Q' = B_2 B_0 B_2^\dagger B_0^\dagger$, $Q'' = B_1 B_2 B_1^\dagger B_2^\dagger$ are central and equal to a

nontrivial cube root of unity. Fixing an orientation gives $Q = Q' = Q'' = \omega \mathbb{1}$ and hence

$$B_0 B_1 = \omega B_1 B_0, \quad B_1 B_2 = \omega B_2 B_1, \quad B_2 B_0 = \omega B_0 B_2,$$

i.e., the HW algebra in dimension $d = 3$; the same follows for the A_j 's. $\square$

Proof Sketch of Theorem 1. Using the SOPO decomposition (A13), Lemma 1 yields the twisted-commutation relations (6). These relations realize a projective representation of the finite Heisenberg group with the canonical central character; by the finite Stone–von Neumann theorem there exists a local isometry identifying it with the standard HW model on a qudit factor. This completes the ideal case.

Specifically, let $\mathcal{H}_d = \langle X, Z : ZX = \omega XZ, X^d = Z^d = \mathbb{1} \rangle$ with projective representations generated on the cyclic subspaces of $|\psi\rangle$ by $\{T_{(x,z)}\}$ on Alice's and Bob's system. Here, $X \sim A_0 (B_0)$ and $Z \sim \omega^{2^{-1}} A_0^\dagger A_1 (\omega^{2^{-1}} B_0^\dagger B_1)$. The relations (6) fix the central character to be the canonical one ω . By the finite Stone–von Neumann theorem, any such representation is (unitarily) a direct sum of copies of the unique d -dimensional irreducible representations. Equivalently, there exist local isometries

$$V_A : \mathcal{H}_A \rightarrow \mathbb{C}^d \otimes \mathcal{K}_A, \quad V_B : \mathcal{H}_B \rightarrow \mathbb{C}^d \otimes \mathcal{K}_B,$$

such that for all $j \in \mathbb{Z}_d^2$

$$V_A A_j V_A^\dagger = T_{(1,j)} \otimes \mathbb{1}_{\mathcal{H}_{A'}}, \quad V_B B_j V_B^\dagger = T_{(1,j)} \otimes \mathbb{1}_{\mathcal{H}_{B'}},$$

i.e., the realized projective representations are unitarily equivalent to the canonical HW one up to trivial action on some supspace. In particular, for some local unitary $U_A \otimes U_B$ (acting within the multiplicity spaces) we obtain the state factorization

$$(V_A \otimes V_B)|\psi\rangle = |\Phi_d\rangle \otimes |\text{junk}\rangle,$$

and the operators act canonically on the $\mathbb{C}^d$. This is precisely the decomposition–and–isometry content provided by the finite Stone–von Neumann theorem.

While the finite Stone–von Neumann theorem guarantees the existence of a local isometry abstractly, we explicitly construct one that yields the stated robustness constants under near-optimal violations. The isometry is $V_A \otimes V_B$ where each V_A, V_B is defined by the circuit in Fig. 1. For $X := A_0^\dagger, Z := \omega^{-2} A_0^\dagger A_1$ for V_A , and $X := B_0^\dagger, Z := \omega^{-2} B_0^\dagger B_1$ for V_B . Using Eq. (6) and Eq. (D1), we directly calculate that this isometry maps the state $|\psi\rangle$ to the rotated Bell state $|\Phi\rangle$, and the operators A_j and B_j to the canonical Heisenberg-Weyl operators $T_{(1,j)}$, up to rotational degrees of freedom and some auxiliary space that the experiment acts on trivially.

If $\langle \mathcal{B}_d \rangle \geq 1 - \varepsilon$, each SOPO term obeys $\|C_{n,j} |\psi\rangle\|_2^2 = O(\varepsilon)$; propagating these defects through our constructive SWAP-type yields the stated $O(\sqrt{\varepsilon})$ bounds with the dimension-explicit constants in Theorem 1. $\square$

Appendices E–H contain the full proof, derivations and explicit constants.

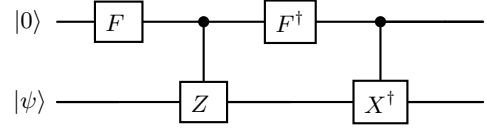


FIG. 1: Circuit description of a SWAP gate between an input state $|\psi\rangle$ and the state $|0\rangle$ in qudits with the Fourier operation (F , a generalized Hadamard matrix), $F = \frac{1}{\sqrt{d}} \sum_{i,j \in \mathbb{Z}_d} \omega^{jk} |j\rangle \langle k|$, and Heisenberg-Weyl operators $X = T_{(1,0)}, Z = T_{(0,1)}$.

Universality across finite dimensions and maximally entangled states. The prime-dimensional certification extends canonically to every finite local dimension. Writing

$$d = \prod_j p_j^{k_j},$$

the local Hilbert space decomposes into prime-power factors, and the corresponding Heisenberg-Weyl structure splits accordingly. Our prime-dimensional result therefore certifies each factor independently, while the finite Stone–von Neumann theorem identifies each realized block with the corresponding canonical Heisenberg-Weyl representation [48, 49]. Combining these blocks yields a certification of the full composite-dimensional state and measurement algebra, with robustness degrading at most linearly in the number of factors.

This universality is not merely a formal corollary of the prime case. Rather, it shows that prime dimensions provide the rigid building blocks from which all finite-dimensional qudit self-tests can be assembled. Once the prime-dimensional Heisenberg-Weyl relations are recovered on each factor, the full composite-dimensional representation is fixed up to local isometries and auxiliary multiplicities. Thus the same protocol yields robust certification in arbitrary finite dimension, not only for primes.

The conclusion also extends from one canonical Bell pair to all maximally entangled states. Indeed, the self-testing statement is invariant under local basis changes: replacing $|\Phi\rangle$ by

$$(U_A \otimes U_B)|\Phi\rangle$$

simply conjugates the certified Heisenberg-Weyl observables by the same local unitaries [4]. Equivalently, after an allowed local rotation, the extracted state is the canonical Bell pair and the certified measurement content is the Heisenberg-Weyl algebra up to the corresponding conjugation. Hence our result gives a robust self-test of maximally entangled states in every finite dimension.

Discussion. The main conceptual advance of this work is a fully analytic and unconditionally robust route from Bell-optimal correlations to the canonical higher-dimensional Heisenberg-Weyl model. Rather than reducing the problem to qubit subtests, the argument reconstructs the genuinely qudit twisted-commutation relations

and thereby certifies not only a maximally entangled state, but the underlying finite Heisenberg representation itself.

For prime d , this route starts from a non-diagonal Heisenberg-Weyl Bell test introduced in [39], derives a closed-form SOPO decomposition, and uses it to force the finite Heisenberg relations directly on the support of the state. The uniqueness of the target model then follows from the finite Stone-von Neumann theorem, which identifies the realized observables and state, up to local isometries, with the canonical Heisenberg-Weyl model [48, 49]. A tensor-factor lift extends the result from primes to arbitrary finite d , showing that the same mechanism yields a genuinely qudit self-test in all finite dimensions.

Our explicit SWAP-type construction complements this representation-theoretic argument. While the finite Stone-von Neumann theorem gives the abstract identification of the target model, the constructive isometry provides a concrete intertwiner and yields the quantitative robustness bounds. The same two-layer structure is familiar in qubit self-testing, where algebraic rigidity and explicit extraction maps play complementary roles [50–53].

The result also has an immediate nonlocality consequence. For the Bell operators considered here, maximal quantum violation enforces genuinely nonclassical commutation relations and therefore cannot be reproduced by local-hidden-variable models [1, 2]. In particular, the

SOPO decomposition gives an analytic certificate of non-locality for the rotated Bell-state correlations associated with our construction, beyond the numerical evidence previously available in small dimensions [39].

Several directions remain open. A natural question is whether the present Heisenberg-Weyl route can be compressed to fewer settings while preserving its fully analytic and unconditionally robust character, in the spirit of recent constant-size self-tests [43–45]. More broadly, it would be interesting to understand whether the key mechanism is the recovery of a sufficiently large part of the operator algebra from Bell-optimal correlations, and whether analogous rigidity phenomena persist for algebraic structures beyond the Heisenberg-Weyl case. Another promising direction is parallel self-testing [51, 54]; in particular, the multi-copy qudit scenarios considered in [39] suggest that stronger algebraic constraints may lead to enhanced rigidity and possibly improved robustness.

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Appendix A: Qudit Bell experiment

Qudit preliminaries. We consider finite-dimensional systems of odd prime dimension d (qudits). Qudits generalize qubits ($d = 2$) to higher dimensions. The qudit Pauli operators are the generalized shift X and phase Z , acting as

$$X|k\rangle = |(k+1) \bmod d\rangle, \quad Z|k\rangle = \omega^k|k\rangle,$$

where $\omega = \exp(2\pi i/d)$ is a primitive d^{th} root of unity. These are unitary with $X^{-1} = X^{d-1} = X^\dagger$ and $Z^{-1} = Z^{d-1} = Z^\dagger$, and satisfy the twisted commutation relations

$$Z^z X^x = \omega^{xz} X^x Z^z \quad \text{for } x, z \in \mathbb{Z}_d,$$

where, from here on, integers and arithmetic are taken over the finite field $\mathbb{Z}_d$. From X and Z we form the HW displacement operators

$$T_{(x,z)} = \omega^{2^{-1}xz} X^x Z^z, \tag{A1}$$

which constitute the generalized qudit Pauli operators. Every qudit state ρ admits a unique HW expansion; its characteristic function (the Fourier transform of Gross’ Wigner function [55]) is

$$\chi[\rho]_{(u_x, u_z)} = \text{Tr} \left(T_{(u_x, u_z)}^\dagger \rho \right). \tag{A2}$$

In the main text we use normalized Bell operators so that the optimal quantum value is 1. For the derivations in the appendices, however, it is more convenient to work with the unnormalized operators, which avoid repeated factors.

For $d > 3$, we therefore write

$$\mathcal{B}_d = d(d-1)\tilde{\mathcal{B}}_d. \quad (\text{A3})$$

Likewise, in the qutrit case we write

$$\mathcal{B}_3 = 6\tilde{\mathcal{B}}_3. \quad (\text{A4})$$

Previous Bell experiment. A quantum realization measures in eigenbases of HW operators T_u , $u \in \mathbb{Z}_d^2$. Each input selects one of the mutually unbiased bases associated with the subgroups

$$\mathcal{G}_k = \{T_{(n,nk)} : n \in \mathbb{Z}_d\}, \quad k \in \mathbb{Z}_d, \quad \mathcal{G}_d = \{T_{(0,n)} : n \in \mathbb{Z}_d\}. \quad (\text{A5})$$

The initial state is the rotated maximally entangled state

$$|\Phi\rangle = \frac{1}{\sqrt{d}} (U \otimes \mathbb{1}) \sum_{k \in \mathbb{Z}_d} |kk\rangle. \quad (\text{A6})$$

The presence of U is essential: it injects non-classical features into an otherwise stabilizer-only experiment [56–58]. Operationally, one may view this as measuring HW operators in rotated bases on the standard Bell state. We use the diagonal family

$$U_\nu = \sum_{k \in \mathbb{Z}_d} \omega^{\nu k} |k\rangle\langle k|, \quad (\text{A7})$$

with ν generalizing the qubit $\pi/8$ gate [59]; for $d > 3$, ν can be chosen as a polynomial of degree $\deg(\nu) \geq 3$. The corresponding Bell operator is

$$\mathcal{B}_d^{(\nu)} = \sum_{u,v \in \mathbb{Z}_d^2} \chi[(U_\nu \otimes \mathbb{1})|\Phi\rangle]_{(u,v)} T_u \otimes T_v, \quad (\text{A8})$$

with characteristic function

$$\chi[(U \otimes \mathbb{1})|\Phi\rangle]_{(x_1, z_1), (x_2, z_2)} = \frac{1}{d} \delta_{x_1=x_2} \sum_{s \in \mathbb{Z}_d} \omega^{(z_1+z_2)s + \nu_{s+2^{-1}x_1} - \nu_{s-2^{-1}x_1}}. \quad (\text{A9})$$

Then $\langle \Phi | \mathcal{B}_d^{(\nu)} | \Phi \rangle = d^2$. Although our analysis allows general ν , a minimal degree-three family suffices [59]; see Appendix B for this special case. In [39] we conjectured (and verified numerically for $5 \leq d \leq 23$) that any LHV model satisfies $\langle \mathcal{B}_d^{(\nu)} \rangle_{\text{lhv}} < d^2$ for all odd primes d .

Present Bell experiment (non-diagonal reduction). Because U_ν is diagonal in the Z basis [59], including $\mathcal{G}_d$ would constrain the state to a stabilizer subspace: $Z \otimes Z^\dagger$ remains a stabilizer of (1). Such measurements are classically simulable [60] and do not aid nonlocality or self-testing. We therefore omit the Z -basis inputs and rewrite $T_{(0,z)}$ as $\omega^{-2^{-1}z} T_{(-1,0)} T_{(1,z)}$ to absorb diagonal pieces. The Bell operator $\mathcal{B}_d^{(\nu)}$ splits into

$$\mathcal{B}_d^{(\nu)} = \mathbb{1} + \mathcal{S} + \mathcal{B}_d,$$

where the (purely diagonal) stabilizer block is

$$\mathcal{S} = \sum_{j \in \mathbb{Z}_d^*} A_{0,j} \otimes B_{0,-j}, \quad (\text{A10})$$

and the non-diagonal part driving nonlocality is

$$\mathcal{B}_d = \sum_{n \in \mathbb{Z}_d^*} \sum_{j,k \in \mathbb{Z}_d} g(j,k,n) A_j^n \otimes B_k^n, \quad (\text{A11})$$

with coefficients

$$g(j,k,n) = \frac{1}{d} \sum_{s \in \mathbb{Z}_d} \omega^{n(j+k)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}}. \quad (\text{A12})$$

On the state (1), $\langle \Phi | \mathcal{B}_d | \Phi \rangle = d(d-1)$. Moreover,

$$d(d-1)\mathbb{1} - \mathcal{B}_d = \sum_{n=1}^{\frac{d-1}{2}} \sum_{j \in \mathbb{Z}_d} C_{n,j}^\dagger C_{n,j}, \quad (\text{A13})$$

with

$$C_{n,j} := A_j^n - \sum_{k \in \mathbb{Z}_d} g(j, k, n)^* (B_k^n)^\dagger, \quad C_{n,j}^\dagger = C_{-n,j}.$$

This SOPO decomposition is verified in Appendix B.

For $d = 3$, both the ν and the Bell operator differ. A convenient choice is the 9-term operator

$$\begin{aligned} \mathcal{B}_3 = \frac{1}{\sqrt{3}} & \left(e^{i\phi_1} A_0 \otimes B_0 + e^{i\phi_2} A_0 \otimes B_1 + e^{i\phi_1} A_0 \otimes B_2 \right. \\ & + e^{i\phi_2} A_1 \otimes B_0 + e^{i\phi_1} A_1 \otimes B_1 + e^{i\phi_1} A_1 \otimes B_2 \\ & \left. + e^{i\phi_1} A_2 \otimes B_0 + e^{i\phi_1} A_2 \otimes B_1 + e^{i\phi_2} A_2 \otimes B_2 \right) + \text{h.c.}, \end{aligned} \quad (\text{A14})$$

with phases constrained by

$$\phi_1 - \phi_2 \equiv \frac{2\pi}{3}, \quad 1 - \sqrt{3} e^{3i\phi_1} = \omega, \quad (\text{A15})$$

One can write $\mathcal{B}_3$ in the form of (A11) and define $B_\nu = \mathbb{1} + \mathcal{S}_3 + \mathcal{B}_3$ in the spirit of (A8), where

$$\mathcal{S}_3 = A_3 \otimes B_3 + A_3^\dagger \otimes B_3^\dagger = 2\mathbb{1} - (A_3^\dagger - B_3)(A_3 - B_3^\dagger)$$

is diagonal in the computational basis and included in the present experiment. For the qutrit Bell state, $\langle \Phi | \mathcal{B}_3 | \Phi \rangle = 6$, whereas for a suitable choice (e.g. $\phi_1 = -\pi/18$, $\phi_2 = -13\pi/18$) one finds $\langle \mathcal{B}_3 \rangle_{\text{lhv}} < 5.640$. Finally, there is a SOPO decomposition

$$6\mathbb{1} - \mathcal{B}_3 = C_0^\dagger C_0 + C_1^\dagger C_1 + C_2^\dagger C_2,$$

with

$$C_j := A_j - \frac{1}{\sqrt{3}} \sum_{k=0,1,2} e^{-i\phi_{j,k}} B_k^\dagger, \quad (\text{A16})$$

where $\phi_{0,1} = \phi_{1,0} = \phi_{2,2} = \phi_2$ and $\phi_{j,k} = \phi_1$ otherwise.

The characteristic function used in the Bell operator (A11) is

$$\chi[(U \otimes \mathbb{1}) | \Phi]_{(x_1, z_1), (x_2, z_2)} = \omega^{2^{-1}(x_1 z_1 + x_2 z_2)} \langle \psi | U X^{-x_1} U^\dagger Z^{-z_1} \otimes X^{-x_2} Z^{-z_2} | \psi \rangle \quad (\text{A17})$$

$$= \frac{1}{d} \omega^{2^{-1}(x_1 z_1 + x_2 z_2)} \sum_{j,k \in \mathbb{Z}_d} \langle jj | U^\dagger X^{-x_1} U Z^{-z_1} \otimes X^{-x_2} Z^{-z_2} | kk \rangle \quad (\text{A18})$$

$$= \frac{1}{d} \omega^{2^{-1}(x_1 z_1 + x_2 z_2)} \sum_{j,k \in \mathbb{Z}_d} \omega^{-(z_1 + z_2)k} \langle jj | U^\dagger X^{-x_1} U \otimes X^{-x_2} | kk \rangle \quad (\text{A19})$$

$$= \frac{1}{d} \omega^{2^{-1}(x_1 z_1 + x_2 z_2)} \sum_{j,k \in \mathbb{Z}_d} \omega^{-(z_1 + z_2)k} \omega^{\nu_k - \nu_{k-x_1}} \langle jj | X^{-x_1} \otimes X^{-x_2} | kk \rangle \quad (\text{A20})$$

$$= \frac{1}{d} \omega^{2^{-1}(x_1 z_1 + x_2 z_2)} \sum_{j,k \in \mathbb{Z}_d} \omega^{-(z_1 + z_2)k} \omega^{\nu_k - \nu_{k-x_1}} \langle jj | k - x_1, k - x_2 \rangle \quad (\text{A21})$$

$$= \frac{1}{d} \delta_{x:=x_1=x_2} \sum_{k \in \mathbb{Z}_d} \omega^{(z_1 + z_2)(2^{-1}x - k) + \nu_k - \nu_{k-x}} \quad (\text{A22})$$

$$= \frac{1}{d} \delta_{x:=x_1=x_2} \sum_{k \in \mathbb{Z}_d} \omega^{(z_1 + z_2)k + \nu_{k+2^{-1}x} - \nu_{k-2^{-1}x}}. \quad (\text{A23})$$

In particular, given $\nu_k = 12^{-1}(k - 3k^2 + 2k^3)$ as in [59] for $d > 3$,

$$\chi[(U \otimes \mathbb{1})|\Phi]_{(x_1, z_1), (x_2, z_2)} = \delta_{x:=x_1=x_2} \frac{\epsilon_d}{\sqrt{d}} \left(\frac{-2^{-1}x}{d} \right) \omega^{(2x)^{-1}(z_1+z_2)^2 - 2^{-1}(z_1+z_2) - 24^{-1}(x^3-x)}, \quad (\text{A24})$$

with the Legendre symbol $(\frac{\cdot}{d})$ and $\epsilon_d = \begin{cases} 1, & d \equiv 1 \pmod{4} \\ i, & d \equiv 3 \pmod{4} \end{cases}$.

Appendix B: Sum-of-positive-operators decomposition

For $d > 3$: We show that there are two sum-of-positive-operators (SOPO) decomposition for the Bell operator (A11),

$$d(d-1)\mathbb{1} - \mathcal{B}_d = \sum_{n=1}^{\frac{d-1}{2}} \sum_{j \in \mathbb{Z}_d} C_{n,j}^\dagger C_{n,j} = \sum_{n=1}^{\frac{d-1}{2}} \sum_{j \in \mathbb{Z}_d} D_{n,j}^\dagger D_{n,j}, \quad (\text{B1})$$

with

$$C_{n,j} := A_j^n - \sum_{k \in \mathbb{Z}_d} g(j, k, n)^* (B_k^n)^\dagger, \quad D_{n,j} := B_j^n - \sum_{k \in \mathbb{Z}_d} g(j, k, n)^* (A_k^n)^\dagger, \quad (\text{B2})$$

and

$$g(j, k, n) = \frac{1}{d} \sum_{s \in \mathbb{Z}_d} \omega^{n(j+k)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}}. \quad (\text{B3})$$

For brevity, we write $C_{n,j} = A_j^n - P_{n,j}^\dagger$ with $P_{n,j} = \sum_{k \in \mathbb{Z}_d} g(j, k, n) B_k^n$. Then, we regard the terms

$$P_{n,j} P_{-n,j} = \sum_{k,l \in \mathbb{Z}_d} g(j, k, n) g(j, l, n)^* B_k^n (B_l^n)^\dagger, \quad (\text{B4})$$

and calculate, for $n \neq 0$, that

$$\sum_{j \in \mathbb{Z}_d} g(j, k, n) g(j, l, n)^* = \frac{1}{d^2} \sum_{j, s, t \in \mathbb{Z}_d} \omega^{n(j+k)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n} - n(j+l)t - \nu_{t+2^{-1}n} + \nu_{t-2^{-1}n}} \quad (\text{B5})$$

$$= \frac{1}{d} \sum_{s, t \in \mathbb{Z}_d} \delta_{s,t} \omega^{nks + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n} - nlt - \nu_{t+2^{-1}n} + \nu_{t-2^{-1}n}} \quad (\text{B6})$$

$$= \frac{1}{d} \sum_{s \in \mathbb{Z}_d} \omega^{ns(k-l)} \quad (\text{B7})$$

$$= \delta_{k,l}. \quad (\text{B8})$$

With this, the decomposition is

$$\sum_{n=1}^{\frac{d-1}{2}} \sum_{j \in \mathbb{Z}_d} C_{n,j}^\dagger C_{n,j} = \sum_{n=1}^{\frac{d-1}{2}} \sum_{j \in \mathbb{Z}_d} \left((A_j^n)^\dagger - P_{n,j} \right) \left(A_{n,j} - (P_{n,j})^\dagger \right) \quad (\text{B9})$$

$$= \sum_{n=1}^{\frac{d-1}{2}} \sum_{j \in \mathbb{Z}_d} \left((A_j^n)^\dagger A_j^n - (A_j^n)^\dagger (P_{-n,j})^\dagger - A_j^n P_{n,j} + P_{n,j} (P_{-n,j})^\dagger \right) \quad (\text{B10})$$

$$= \frac{d(d-1)}{2} - \sum_{n \in \mathbb{Z}_d^*} \sum_{j, k \in \mathbb{Z}_d} g(j, k, n) A_j^n B_k^n + \sum_{n=1}^{\frac{d-1}{2}} \sum_{j, k, l \in \mathbb{Z}_d} g(j, k, n) g(j, l, n)^* B_k^n (B_l^n)^\dagger \quad (\text{B11})$$

$$= \frac{d(d-1)}{2} - \mathcal{B}_d + \sum_{n=1}^{\frac{d-1}{2}} \sum_{k, l \in \mathbb{Z}_d} \delta_{k,l} B_k^n (B_l^n)^\dagger \quad (\text{B12})$$

$$= d(d-1)\mathbb{1} - \mathcal{B}_d. \quad (\text{B13})$$

From Eq. (B10) to Eq. (B11), we used $A_j^n (A_j^n)^\dagger = \mathbb{1}$ for all $j, n \in \mathbb{Z}_d$, and Eq. (B4). From Eq. (B11) to Eq. (B12), we used that the cross terms $A_j^n P_{n,j}$ constitute the Bell operator (A11), as well as Eq. (B8). Lastly, $B_k^n (B_l^n)^\dagger = \mathbb{1}$, for all $k, n \in \mathbb{Z}_d$, verifies the decomposition.

Similarly, it is

$$\sum_{n=1}^{\frac{d-1}{2}} \sum_{j \in \mathbb{Z}_d} D_{n,j}^\dagger D_{n,j} = \frac{d(d-1)}{2} - \sum_{n \in \mathbb{Z}_d^*} \sum_{j,k \in \mathbb{Z}_d} g(j,k,n) A_j^n B_k^n + \sum_{n=1}^{\frac{d-1}{2}} \sum_{j,k,l \in \mathbb{Z}_d} g(j,k,n) g(j,l,n)^* A_k^n (A_l^n)^\dagger \quad (\text{B14})$$

$$= \frac{d(d-1)}{2} - \mathcal{B}_d + \sum_{n=1}^{\frac{d-1}{2}} \sum_{k,l \in \mathbb{Z}_d} \delta_{k,l} A_k^n (A_l^n)^\dagger \quad (\text{B15})$$

$$= d(d-1)\mathbb{1} - \mathcal{B}_d. \quad (\text{B16})$$

For $d = 3$: We use the operators Eq. (A16), and calculate

$$C_0^\dagger C_0 + C_1^\dagger C_1 + C_2^\dagger C_2 \quad (\text{B17})$$

$$= \left[A_0^2 - \frac{1}{\sqrt{3}} (e^{i\phi_1} B_0 + e^{i\phi_2} B_1 + e^{i\phi_1} B_2) \right] \left[A_0 - \frac{1}{\sqrt{3}} (e^{-i\phi_1} B_0^2 + e^{-i\phi_2} B_1^2 + e^{-i\phi_1} B_2^2) \right] \\ + \left[A_1^2 - \frac{1}{\sqrt{3}} (e^{i\phi_2} B_0 + e^{i\phi_1} B_1 + e^{i\phi_1} B_2) \right] \left[A_1 - \frac{1}{\sqrt{3}} (e^{-i\phi_2} B_0^2 + e^{-i\phi_1} B_1^2 + e^{-i\phi_1} B_2^2) \right] \\ + \left[A_2^2 - \frac{1}{\sqrt{3}} (e^{i\phi_1} B_0 + e^{i\phi_1} B_1 + e^{i\phi_2} B_2) \right] \left[A_2 - \frac{1}{\sqrt{3}} (e^{-i\phi_1} B_0^2 + e^{-i\phi_1} B_1^2 + e^{-i\phi_2} B_2^2) \right] \quad (\text{B18})$$

$$= 6 - \mathcal{B}_3 \\ + \frac{1}{3} \left(e^{-i(\phi_1-\phi_2)} B_1 B_0^2 + B_2 B_0^2 + e^{i(\phi_1-\phi_2)} B_1^2 B_0 + e^{i(\phi_1-\phi_2)} B_2 B_1^2 + B_0 B_2^2 e^{-i(\phi_1-\phi_2)} + B_1 B_2^2 \right. \\ + e^{i(\phi_1-\phi_2)} B_1 B_0^2 + e^{i(\phi_1-\phi_2)} B_2 B_0^2 + e^{-i(\phi_1-\phi_2)} B_1^2 B_0 + B_2 B_1^2 + e^{-i(\phi_1-\phi_2)} B_0 B_2^2 + B_1 B_2^2 \\ \left. + B_1 B_0^2 + e^{-i(\phi_1-\phi_2)} B_2 B_0^2 + B_1^2 B_0 + e^{-i(\phi_1-\phi_2)} B_2 B_1^2 + e^{i(\phi_1-\phi_2)} B_0 B_2^2 + e^{i(\phi_1-\phi_2)} B_1 B_2^2 \right) \quad (\text{B19})$$

$$= 6 - \mathcal{B}_3 \\ + \frac{1}{3} \left(\omega^2 B_1 B_0^2 + B_2 B_0^2 + \omega B_1^2 B_0 + \omega B_2 B_1^2 + B_0 B_2^2 \omega^2 + B_1 B_2^2 \right. \\ + \omega B_1 B_0^2 + \omega B_2 B_0^2 + \omega^2 B_1^2 B_0 + B_2 B_1^2 + \omega^2 B_0 B_2^2 + B_1 B_2^2 \\ \left. + B_1 B_0^2 + \omega^2 B_2 B_0^2 + B_1^2 B_0 + \omega^2 B_2 B_1^2 + \omega B_0 B_2^2 + \omega B_1 B_2^2 \right) \quad (\text{B20})$$

$$= 6 - \mathcal{B}_3 \quad (\text{B21})$$

In the last line, we used that $1 + \omega + \omega^2 = 0$. Equivalently, we can also show that

$$D_0^\dagger D_0 + D_1^\dagger D_1 + D_2^\dagger D_2 \quad (\text{B22})$$

$$= \left[B_0^2 - \frac{1}{\sqrt{3}} (e^{i\phi_1} A_0 + e^{i\phi_2} A_1 + e^{i\phi_1} A_2) \right] \left[B_0 - \frac{1}{\sqrt{3}} (e^{-i\phi_1} A_0^2 + e^{-i\phi_2} A_1^2 + e^{-i\phi_1} A_2^2) \right] \\ + \left[B_1^2 - \frac{1}{\sqrt{3}} (e^{i\phi_2} A_0 + e^{i\phi_1} A_1 + e^{i\phi_1} A_2) \right] \left[B_1 - \frac{1}{\sqrt{3}} (e^{-i\phi_2} A_0^2 + e^{-i\phi_1} A_1^2 + e^{-i\phi_1} A_2^2) \right] \\ + \left[B_2^2 - \frac{1}{\sqrt{3}} (e^{i\phi_1} A_0 + e^{i\phi_1} A_1 + e^{i\phi_2} A_2) \right] \left[B_2 - \frac{1}{\sqrt{3}} (e^{-i\phi_1} A_0^2 + e^{-i\phi_1} A_1^2 + e^{-i\phi_2} A_2^2) \right] \quad (\text{B23})$$

$$= 6 - \mathcal{B}_3. \quad (\text{B24})$$

Appendix C: Twisted commutation relations

For $d > 3$: For $g(j, k, n)$ from Eq. (A12), we show the *quadratic folding identity*:

$$g(j, k, n + n') = \sum_{r \in \mathbb{Z}_d} g(j, k + n'r, n) g(j, k - nr, n') \omega^{2^{-1}nn'(n+n')r} \quad (\text{C1})$$

by calculating

$$\sum_{r \in \mathbb{Z}_d} g(j, k + n'r, n) g(j, k - nr, n') \omega^{2^{-1}nn'(n+n')r} \quad (\text{C2})$$

$$= \frac{1}{d^2} \sum_{r, s, t \in \mathbb{Z}_d} \omega^{n(j+k+n'r)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}} \omega^{n'(j+k-nr)t + \nu_{t+2^{-1}n'} - \nu_{t-2^{-1}n'}} \omega^{2^{-1}nn'(n+n')r} \quad (\text{C3})$$

$$= \frac{1}{d} \sum_{s, t \in \mathbb{Z}_d} \delta_{nn'(s-t+2^{-1}(n+n'))} \omega^{n(j+k)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}} \omega^{n'(j+k)t + \nu_{t+2^{-1}n'} - \nu_{t-2^{-1}n'}} \quad (\text{C4})$$

$$= \frac{1}{d} \sum_{s \in \mathbb{Z}_d} \omega^{n(j+k)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n} + n'(j+k)(s+2^{-1}(n+n')) + \nu_{s+2^{-1}(n+n')} + 2^{-1}n' - \nu_{s+2^{-1}(n+n')} - 2^{-1}n'} \quad (\text{C5})$$

$$= \frac{1}{d} \sum_{s \in \mathbb{Z}_d} \omega^{(n+n')(j+k)(s+2^{-1}n') - \nu_{s-2^{-1}n} - \nu_{s+2^{-1}(n+2n')}} \quad (\text{C6})$$

$$= \frac{1}{d} \sum_{s \in \mathbb{Z}_d} \omega^{(n+n')(j+k)s + \nu_{s+2^{-1}(n+n')} - \nu_{s-2^{-1}(n+n')}} \quad (\text{C7})$$

$$= g(j, k, n + n'). \quad (\text{C8})$$

Then, since $(A_j^n)^\dagger (A_j^{n'})^\dagger = (A_j^{n+n'})^\dagger$, Eq. (D1) and Eq. (C1) lead to

$$\left[(A_j^n)^\dagger (A_j^{n'})^\dagger - (A_j^{n+n'})^\dagger \right] |\psi\rangle \quad (\text{C9})$$

$$= \sum_{k, l \in \mathbb{Z}_d} g(j, k, n) g(j, l, n') B_k^n B_l^{n'} |\psi\rangle - \sum_{m \in \mathbb{Z}_d} g(j, m, n + n') B_m^{n+n'} |\psi\rangle \quad (\text{C10})$$

$$= \sum_{k, l \in \mathbb{Z}_d} g(j, k, n) g(j, l, n') B_k^n B_l^{n'} |\psi\rangle - \sum_{m, r \in \mathbb{Z}_d} g(j, m + n'r, n) g(j, m - nr, n') \omega^{2^{-1}nn'(n+n')r} B_m^{n+n'} |\psi\rangle \quad (\text{C11})$$

$$= \sum_{k, l \in \mathbb{Z}_d} g(j, k, n) g(j, l, n') \left(B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle \quad (\text{C12})$$

$$\stackrel{!}{=} 0, \quad (\text{C13})$$

using the substitution $m + n'r = k, m - nr = l$ such that $(n' + n)r = k - l, (n + n')m = (nk + n'l)$ from Eq. (C11) to Eq. (C12). Since ν is not a polynomial of degree less than 3, $g(j, k, n) \neq 0$ for any $j, k \in \mathbb{Z}_d$ and $n \in \mathbb{Z}_d^*$. As a result, it is

$$\left(B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle = 0, \quad (\text{C14})$$

for all $k, l \in \mathbb{Z}_d$ and $n, n', n + n' \in \mathbb{Z}_d^*$. In fact one can derive the expression for general n, n' from the case $n = n' = 1$. It will be beneficial however, to use the general case when $\epsilon > 0$ in Lemma 1. Furthermore, the above can be shown for the operators A_j , with the exact same reasoning by swapping the operators B_j and A_j , such that

$$\left(A_k A_l - \omega^{2^{-1}(k-l)} A_{2^{-1}(k+l)}^2 \right) |\psi\rangle = 0, \quad \left(B_k B_l - \omega^{2^{-1}(k-l)} B_{2^{-1}(k+l)}^2 \right) |\psi\rangle = 0, \quad (\text{C15})$$

for all $l, k \in \mathbb{Z}_d$.

For $d = 3$: We complement the proof by the missing steps, i.e. how Eq.(D6) follows from Eq.(D5). From $\langle \psi | 6\mathbb{1} - \mathcal{B}_3 | \psi \rangle = 0$, it follows $C_j | \psi \rangle = 0$ in Eq.(A16) or

$$A_0 | \psi \rangle = \frac{e^{-i\phi_1}}{\sqrt{3}} \left(B_0^\dagger + \omega B_1^\dagger + B_2^\dagger \right) | \psi \rangle, \quad (C16)$$

$$A_1 | \psi \rangle = \frac{e^{-i\phi_1}}{\sqrt{3}} \left(\omega B_0^\dagger + B_1^\dagger + B_2^\dagger \right) | \psi \rangle, \quad (C17)$$

$$A_2 | \psi \rangle = \frac{e^{-i\phi_1}}{\sqrt{3}} \left(B_0^\dagger + B_1^\dagger + \omega B_2^\dagger \right) | \psi \rangle, \quad (C18)$$

where we used the first constraint from Eq.(A15), i.e. $\phi_1 - \phi_2 \equiv 2\pi/3$. Since it must be $A_j^2 = A_j^\dagger$ for $j \in \{0, 1, 2\}$, the above lead to the equations

$$\left[\frac{e^{-2i\phi_1}}{3} \left(B_0^\dagger + \omega B_1^\dagger + B_2^\dagger \right)^2 - \frac{e^{i\phi_1}}{\sqrt{3}} (B_0 + \omega^2 B_1 + B_2) \right] | \psi \rangle = 0, \quad (C19)$$

$$\left[\frac{e^{-2i\phi_1}}{3} \left(\omega B_0^\dagger + B_1^\dagger + B_2^\dagger \right)^2 - \frac{e^{i\phi_1}}{\sqrt{3}} (\omega^2 B_0 + B_1 + B_2) \right] | \psi \rangle = 0, \quad (C20)$$

$$\left[\frac{e^{-2i\phi_1}}{3} \left(B_0^\dagger + B_1^\dagger + \omega B_2^\dagger \right)^2 - \frac{e^{i\phi_1}}{\sqrt{3}} (B_0 + B_1 + \omega^2 B_2) \right] | \psi \rangle = 0. \quad (C21)$$

The second constraint from Eq.(A15), i.e. $1 - \sqrt{3}e^{3i\phi_1} = \omega$ leads to the system of linear equations

$$[\omega\{B_0^2, B_1^2\} + \{B_0^2, B_2^2\} + \omega\{B_1^2, B_2^2\} + \omega B_0 + B_1 + \omega B_2] | \psi \rangle = 0, \quad (C22)$$

$$[\{B_0^2, B_1^2\} + \omega\{B_0^2, B_2^2\} + \omega\{B_1^2, B_2^2\} + B_0 + \omega B_1 + \omega B_2] | \psi \rangle = 0, \quad (C23)$$

$$[\omega\{B_0^2, B_1^2\} + \omega\{B_0^2, B_2^2\} + \{B_1^2, B_2^2\} + \omega B_0 + \omega B_1 + B_2] | \psi \rangle = 0. \quad (C24)$$

Rearranging these leads to Eq.(D5), namely,

$$[\{B_0^2, B_1^2\} + B_2] | \psi \rangle = 0, \quad [\{B_0^2, B_2^2\} + B_1] | \psi \rangle = 0, \quad [\{B_1^2, B_2^2\} + B_0] | \psi \rangle = 0. \quad (C25)$$

In the following, we omit the state $|\psi\rangle$ but note that all identities hold with respect to their action on it. First, one finds that

$$B_0 B_1 B_2 = -B_0 B_0^2 - B_0 B_2 B_1 = -\mathbb{1} - B_0 B_2 B_1 = B_2 B_0 B_1 = B_1 B_2 B_0, \quad (C26)$$

and for the anti-cyclic permutations,

$$B_0 B_2 B_1 = B_2 B_1 B_0 = B_1 B_0 B_2. \quad (C27)$$

It immediately follows that

$$[B_j, B_0 B_1 B_2] = 0, \quad (C28)$$

for all $j \in \{0, 1, 2\}$. Recall the conjugation elements

$$Q := B_0 B_1 B_0^\dagger B_1^\dagger, \quad Q' = B_2 B_0 B_2^\dagger B_0^\dagger, \quad Q'' = B_1 B_2 B_1^\dagger B_2^\dagger. \quad (C29)$$

Multiplying $B_0^\dagger B_1^\dagger + B_1^\dagger B_0^\dagger = -B_2$ from the right by $B_0 B_1$ gives $\mathbb{1} + Q = -B_0 B_1 B_2$. The same hold of Q' and Q'' , i.e. $\mathbb{1} + Q' = -B_2 B_0 B_1$ and $\mathbb{1} + Q'' = -B_1 B_2 B_0$. From Eq.(C26), we find $Q = Q' = Q''$ as well as

$$[B_i, Q] = 0, \quad \text{for } i = 1, 2, 3.$$

With the twisted commutation relations it is straightforward to calculate $Q^3 = \mathbb{1}$. As a result, Q commutes with the whole algebra and must be a scalar multiple of the identity:

$$Q = \lambda \mathbb{1}, \quad \lambda^3 = 1, \quad \lambda \neq 1. \quad (C30)$$

Choosing the orientation $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ sets $\lambda = \omega$ (the opposite orientation would give ω^2). With $Q = \omega \mathbb{1}$, the defining equation becomes the ω -commutation rules

$$B_0 B_1 = \omega B_1 B_0, \quad B_1 B_2 = \omega B_2 B_1, \quad B_2 B_0 = \omega B_0 B_2. \quad (C31)$$

from Eq.(D6).

Appendix D: Proof of Lemma 1

Proof of Lemma 1 for $d > 3$. Assume unitary A_j, B_j as in (A11) and a state $|\psi\rangle$ with $\langle\psi|\mathcal{B}_d|\psi\rangle = d(d-1)$. Then $\langle\psi|d(d-1) - \mathcal{B}_d|\psi\rangle = 0$. By the SOPO (A13), each summand must vanish, so $C_{n,j}|\psi\rangle = 0$ for all $n \in \mathbb{Z}_d^*$, $j \in \mathbb{Z}_d$, i.e.,

$$A_j^n |\psi\rangle = \sum_{k \in \mathbb{Z}_d} g(j, k, n)^* (B_k^n)^\dagger |\psi\rangle. \quad (\text{D1})$$

Using $A_j^n A_j^{n'} = A_j^{n+n'}$ for all n, n', j (in particular $A_j^2 = A_j A_j$) and taking adjoints of A_j^2 ,

$$A_j A_j = A_j^2 \quad \Rightarrow \quad (A_j)^\dagger (A_j)^\dagger = (A_j^2)^\dagger, \quad (\text{D2})$$

we combine (D1) at $n = 1, 2$ with (D2) to eliminate A and obtain, for each $j \in \mathbb{Z}_d$,

$$\sum_{k, l \in \mathbb{Z}_d} g(j, k, 1) g(j, l, 1) B_k B_l |\psi\rangle = \sum_{k \in \mathbb{Z}_d} g(j, k, 2) B_k^2 |\psi\rangle. \quad (\text{D3})$$

Write every grade-2 vector in the orthonormal basis $\{B_m^2 |\psi\rangle\}_{m \in \mathbb{Z}_d}$ as $B_k B_l |\psi\rangle = \sum_{m \in \mathbb{Z}_d} C_{kl}^m B_m^2 |\psi\rangle$. Projecting (D3) with $\langle\psi|(B_n^2)^\dagger$ yields, for all $j, n \in \mathbb{Z}_d$,

$$\sum_{k, l \in \mathbb{Z}_d} C_{kl}^n g(j, k, 1) g(j, l, 1) = g(j, n, 2). \quad (\text{D4})$$

Collect the C_{kl}^n as a length- d^2 vector $\mathbf{c}^{(n)}$, and let M be the $d \times d^2$ matrix whose j -th row consists of $\{g(j, k, 1) g(j, l, 1)\}_{k, l}$; let $\mathbf{g}_n^{(2)}$ have entries $\{g(j, n, 2)\}_j$. Then (D4) is $M \mathbf{c}^{(n)} = \mathbf{g}_n^{(2)}$.

Because $g(j, k, 1)$ are the $(j+k)$ -th Fourier coefficients of a non-linear phase (our ν is not linear or quadratic over $\mathbb{Z}_d$), one has row orthogonality

$$\sum_{j \in \mathbb{Z}_d} g(j, k, 1) g(j, l, 1)^* = d \delta_{k, l},$$

so M has full row rank d . As a result the solution for each n is unique (up to an overall phase convention). Solving gives

$$C_{kl}^n = \omega^{2^{-1}(k-l)} \delta_{n, 2^{-1}(k+l)},$$

which implies

$$B_k B_l |\psi\rangle = \omega^{2^{-1}(k-l)} B_{2^{-1}(k+l)}^2 |\psi\rangle.$$

From these identities one derives the twisted-commutation on the support,

$$B_k B_l |\psi\rangle = \omega^{k-l} B_l B_k |\psi\rangle, \quad \forall k, l \in \mathbb{Z}_d.$$

By symmetry of (A11) under exchanging parties - or by repeating the argument with the alternative SOPO $d(d-1)\mathbb{1} - \mathcal{B}_d = \sum_{n=1}^{(d-1)/2} \sum_{j \in \mathbb{Z}_d} D_{n,j}^\dagger D_{n,j}$ where $D_{n,j} := B_j^n - \sum_{k \in \mathbb{Z}_d} g(j, k, n) (A_k^n)^\dagger$, - the same relations hold for all A_k . $\square$

Proof of Lemma 1 for $d = 3$. Here $2 \equiv -1 \pmod{3}$, so $A_j^2 = A_j A_j = A_j^\dagger$ and the relations used to derive (D3) are not independent; the system becomes under-determined. Using the $d = 3$, SOPO, $\langle\psi|6\mathbb{1} - \mathcal{B}_3|\psi\rangle = 0$ implies $C_j |\psi\rangle = 0$ (cf. Eq. (A16)), relating A_j to B_k . Together with $A_j^\dagger = A_j A_j$, one obtains the symmetric constraints

$$\begin{aligned} (\{B_0^2, B_1^2\} + B_2) |\psi\rangle &= 0, \\ (\{B_0^2, B_2^2\} + B_1) |\psi\rangle &= 0, \\ (\{B_1^2, B_2^2\} + B_0) |\psi\rangle &= 0, \end{aligned} \quad (\text{D5})$$

with $\{X, Y\} = XY + YX$. From Eq. (D5) it follows, see Appendix C for details, that the commutators

$$Q := B_0 B_1 B_0^\dagger B_1^\dagger, \quad Q' := B_2 B_0 B_2^\dagger B_0^\dagger, \quad Q'' := B_1 B_2 B_1^\dagger B_2^\dagger$$

commute with each B_i ($i = 0, 1, 2$) and hence are scalars $\lambda \mathbb{1}$ with $\lambda^3 = 1$. Fixing the orientation $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ selects $\lambda = \omega$ (the opposite gives ω^2), yielding the ω -commutation relations

$$B_0 B_1 = \omega B_1 B_0, \quad B_1 B_2 = \omega B_2 B_1, \quad B_2 B_0 = \omega B_0 B_2. \quad (\text{D6})$$

These generate the HW algebra in dimension 3. The same relations for A_0, A_1, A_2 follow either by symmetry or by using the swapped SOPO, $6\mathbb{1} - \mathcal{B}_3 = D_0^\dagger D_0 + D_1^\dagger D_1 + D_2^\dagger D_2$ with $D_j := B_j - \frac{1}{\sqrt{3}} \sum_{k=0,1,2} e^{-i\phi_{j,k}} A_k^\dagger$. $\square$

Appendix E: Isomorphism

The ideal identification also follows from the finite Stone–von Neumann theorem (see main text); here we provide an explicit isometry via a SWAP-type circuit and propagate errors to obtain the $O(\sqrt{\varepsilon})$ bounds with stated constants.

We explicitly construct the local isometry $V_A \otimes V_B$ where each V_A, V_B is described by the circuit in Fig. 2. To prove Theorem 1, we spell out the action of circuit layer-by-layer. We start with two auxiliary qubits and the state $|\psi\rangle$, and apply Fourier transform on both auxiliary [1]. Then, apply $A_0^\dagger A_1$ and $B_0^\dagger B_1$ operators [2], and again the Fourier transform on both sides [3]. Lastly, apply A_0^{-1} and B_0^{-1} operators [4]. Namely, it is

$$\begin{aligned} & |\psi\rangle \otimes |0, 0\rangle \\ \xRightarrow{[1]} & \frac{1}{d} \sum_{j,k \in \mathbb{Z}_d} |j, k\rangle \otimes |\psi\rangle \end{aligned} \quad (\text{E1})$$

$$\xRightarrow{[2]} \frac{1}{d} \sum_{j,k \in \mathbb{Z}_d} |j, k\rangle \otimes (\omega^{-2^{-1}j} A_0^\dagger A_j) (\omega^{2^{-1}k} B_0^\dagger B_k) |\psi\rangle \quad (\text{E2})$$

$$\xRightarrow{[3]} \frac{1}{d^2} \sum_{n,m \in \mathbb{Z}_d} |n, m\rangle \otimes \left(\sum_{j \in \mathbb{Z}_d} \omega^{-jn} (\omega^{-2^{-1}j} A_0^\dagger A_j) \right) \left(\sum_{k \in \mathbb{Z}_d} \omega^{-km} (\omega^{-2^{-1}k} B_0^\dagger B_k) \right) |\psi\rangle \quad (\text{E3})$$

$$\xRightarrow{[4]} \frac{1}{d^2} \sum_{n,m \in \mathbb{Z}_d} |n, m\rangle \otimes \left((A_0^n)^\dagger \sum_{j \in \mathbb{Z}_d} \omega^{-jn} (\omega^{-2^{-1}j} A_0^\dagger A_j) \right) \left((B_0^m)^\dagger \sum_{k \in \mathbb{Z}_d} \omega^{-km} (\omega^{-2^{-1}k} B_0^\dagger B_k) \right) |\psi\rangle \quad (\text{E4})$$

We evaluate Eq. (E4), first by permuting the A_0 and B_0 with the A_1 and B_1 operators, then apply the relations from Eq. (D1). For convenience, we denote $\bar{B}_m := (B_0^m)^\dagger \sum_{k \in \mathbb{Z}_d} \omega^{-km} (\omega^{-2^{-1}k} B_0^\dagger B_k)$,

$$(V_A \otimes V_B) (|\psi\rangle) = \frac{1}{d^2} \sum_{n,m \in \mathbb{Z}_d} |n, m\rangle \otimes \left((A_0^n)^\dagger \sum_{j \in \mathbb{Z}_d} \omega^{-jn} (\omega^{-2^{-1}j} A_0^\dagger A_j) \right) \bar{B}_m |\psi\rangle \quad (\text{E5})$$

$$= \frac{1}{d^2} \sum_{n,m,j \in \mathbb{Z}_d} |n, m\rangle \otimes \omega^{-2^{-1}jn} A_{-n-1,j}^\dagger \bar{B}_m |\psi\rangle \quad (\text{E6})$$

$$= \frac{1}{d^3} \sum_{n,m,j,l,s \in \mathbb{Z}_d} |n, m\rangle \otimes \bar{B}_m \omega^{-2^{-1}jn} \omega^{(j-l)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}} B_{n-1,l}^n |\psi\rangle \quad (\text{E7})$$

$$= \frac{1}{d^2} \sum_{n,m,l,s \in \mathbb{Z}_d} |n, m\rangle \otimes \bar{B}_m \delta_{s,2^{-1}n} \omega^{-ls + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}} B_{n-1,l}^n |\psi\rangle \quad (\text{E8})$$

$$= \frac{1}{d^2} \sum_{n,m,l \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, m\rangle \otimes \bar{B}_m \omega^{-2^{-1}nl} B_{n-1,l}^n |\psi\rangle \quad (\text{E9})$$

$$\left(\begin{smallmatrix} r=n-m \\ m=n+r \end{smallmatrix} \right) = \frac{1}{d^2} \sum_{n,r,k,l \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n+r\rangle \otimes \omega^{-kr} B_0^{-r} \omega^{-2^{-1}(k+l)n} B_{-n-1,k}^{-n} B_{n-1,l}^n |\psi\rangle \quad (\text{E10})$$

$$= \frac{1}{d^2} \sum_{n,r,k,l \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n+r\rangle \otimes \omega^{-kr} B_0^{-r} \omega^{-2^{-1}(l+k)n} B_0^\dagger B_{k+l} |\psi\rangle \quad (\text{E11})$$

$$\left(\begin{smallmatrix} t=k+l \\ l=t-k \end{smallmatrix} \right) = \frac{1}{d^2} \sum_{n,r,k,t \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n+r\rangle \otimes \omega^{-kr} B_0^{-r} \omega^{-2^{-1}t} B_0^\dagger B_t |\psi\rangle \quad (\text{E12})$$

$$= \frac{1}{d} \sum_{n,r,t \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n+r\rangle \otimes \delta_{r,0} B_0^{-r} \omega^{-2^{-1}t} B_0^\dagger B_t |\psi\rangle \quad (\text{E13})$$

$$= \frac{1}{d} \sum_{n,t \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n\rangle \otimes \omega^{-2^{-1}t} B_0^\dagger B_t |\psi\rangle. \quad (\text{E14})$$

We identify $|aux\rangle = \frac{1}{\sqrt{d}} \sum_{t \in \mathbb{Z}_d} \omega^{-2^{-1}t} B_0^\dagger B_t |\psi\rangle$ and $(U_\nu \otimes \mathbb{1}) |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{n \in \mathbb{Z}_d} \omega^{\nu_n} |n, n\rangle$ with the Bell state $|\Phi\rangle$ and $U_\nu = \sum_{k \in \mathbb{Z}_d} \omega^{\nu_k} |k\rangle \langle k|$ and a global phase ω^{ν_0} . From Eq. (E5) to Eq. (E6), use $A_{-n,0} A_{-1,0} A_{1,j} = \omega^{2^{-1}(n+1)j} A_{-n,j}$. From Eq. (E6) to Eq. (E7), use Eq. (D1). From Eq. (E7) to Eq. (E8), sum over j . From Eq. (E8) to Eq. (E9), sum over s . From Eq. (E9) to Eq. (E10), unpack $\bar{B}_m$ and substitute $r = n - m$ for m . From Eq. (E10) to Eq. (E11), use

$\omega^{-2^{-1}(k+l)n} B_{-n-1-k}^{-n} B_{n-1-l}^n = \omega^{-2^{-1}(l+k)} B_0^\dagger B_{k+l}$. From Eq. (E11) to Eq. (E12), substitute $t = k + l$ for l . From Eq. (E12) to Eq. (E13), sum over k . From Eq. (E13) to Eq. (E14), sum over m .

Then, we show that $(V_A \otimes V_B) (A_u^v |\psi\rangle) = (T_{(v,vu)} \otimes \mathbb{1}) |\psi\rangle \otimes |aux\rangle$:

$$(V_A \otimes V_B) (A_u^v |\psi\rangle) = \frac{1}{d^2} \sum_{n,m \in \mathbb{Z}_d} |n, m\rangle \otimes \left((A_0^n)^\dagger \sum_{j \in \mathbb{Z}_d} \omega^{-jn} (\omega^{-2^{-1}j} A_0^\dagger A_j) \right) \bar{B}_m (A_u^v |\psi\rangle) \quad (\text{E15})$$

$$= \frac{1}{d^2} \sum_{n,m,j \in \mathbb{Z}_d} |n, m\rangle \otimes \omega^{-2^{-1}j(n-v)+2^{-1}nu} A_{(-n+1)^{-1}(j+uv)}^{-n+v} \bar{B}_m |\psi\rangle \quad (\text{E16})$$

$$\left(\begin{smallmatrix} o=n-v \\ n=o+v \end{smallmatrix} \right) = \frac{1}{d^2} \sum_{o,m,j \in \mathbb{Z}_d} |o+v, m\rangle \otimes \omega^{-2^{-1}jo+2^{-1}(o+v)uv} A_{-o^{-1}(j+uv)}^{-o} \bar{B}_m |\psi\rangle \quad (\text{E17})$$

$$\left(\begin{smallmatrix} j'=j+uv \\ j=j'-uv \end{smallmatrix} \right) = \frac{1}{d^2} \sum_{o,m,j' \in \mathbb{Z}_d} |o+v, m\rangle \otimes \omega^{-2^{-1}(j'-uv)o+2^{-1}(o+v)uv} A_{-o^{-1}j'}^{-o} \bar{B}_m |\psi\rangle \quad (\text{E18})$$

$$= \frac{1}{d} \sum_{o,t \in \mathbb{Z}_d} \omega^{\nu_o - \nu_0} \omega^{2^{-1}uv^2} X^v Z^{uv} |o, o\rangle \otimes \omega^{-2^{-1}t} B_0^\dagger B_t |\psi\rangle \quad (\text{E19})$$

$$= \frac{1}{d} \sum_{o,t \in \mathbb{Z}_d} \omega^{\nu_o - \nu_0} (T_{(v,vu)} \otimes \mathbb{1}) |o, o\rangle \otimes \omega^{-2^{-1}t} B_0^\dagger B_t |\psi\rangle \quad (\text{E20})$$

$$= (T_{(v,vu)} \otimes \mathbb{1}) |\psi\rangle \otimes |aux\rangle. \quad (\text{E21})$$

From Eq. (E15) to Eq. (E16), we use that $\omega^{-jn-2^{-1}j} (A_0^n)^\dagger A_0^\dagger A_j = \omega^{-2^{-1}j(n-v)+2^{-1}nu} A_{(-n+1)^{-1}(j+uv)}^{-n+v}$ from Eq. (6). From Eq. (E16) to Eq. (E17), we substitute $o = n - v$ for n . From Eq. (E17) to Eq. (E18), we substitute $j' = j + uv$ for j . From Eq. (E18) to Eq. (E19), apply the steps from Eqs. (E6) - (E14). From Eq. (E19) to Eq. (E20), we identify the operators $T_{(v,vu)}$. From Eq. (E20) to Eq. (E21), we identify the states $|\psi\rangle$ and $|aux\rangle$. Due to the symmetry of the state and the operators $A_j B_j$, we can show in the exact same fashion that $(V_A \otimes V_B) (B_u^v |\psi\rangle) = (\mathbb{1} \otimes T_{(v,vu)}) |\psi\rangle \otimes |aux\rangle$.

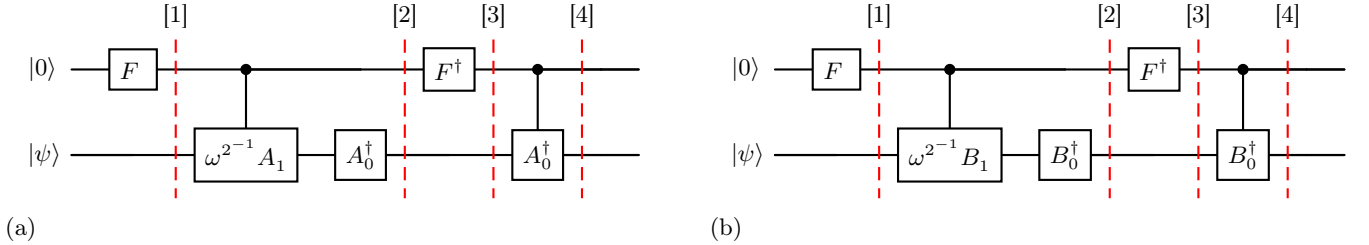


FIG. 2: Two circuits describing the action of the isometry V_A in 2a and V_B in 2b, both on the states $|\psi\rangle$ and $|0\rangle$, with the Fourier operation F , a generalized Hadamard matrix, $F = \frac{1}{\sqrt{d}} \sum_{i,j \in \mathbb{Z}_d} \omega^{jk} |j\rangle \langle k|$, and operators A_0, A_1, B_0, B_1 . The slices [1] - [4] correspond to Eqs. (E1) - (E4).

Appendix F: Robust self-testing relations for $d > 3$

Under the assumptions of Lemma 1, $\langle \psi | d(d-1)\mathbb{1} - \mathcal{B}_d | \psi \rangle \leq \epsilon$ for an $\epsilon > 0$. The operator has a sum-of-positive-operators (SOPO) decomposition: $d(d-1)\mathbb{1} - \mathcal{B}_d = \sum_{n=1}^{(d-1)/2} \sum_{j \in \mathbb{Z}_d} C_{n,j}^\dagger C_{n,j}$ with $C_{n,j}$ from Eq. (B2). It follows that $\langle \psi | C_{n,j}^\dagger C_{n,j} | \psi \rangle \leq \epsilon$, and consequently,

$$\sqrt{\langle \psi | C_{n,j}^\dagger C_{n,j} | \psi \rangle} = \|C_{n,j} |\psi\rangle\|_1 \leq \sqrt{\epsilon}, \quad (\text{F1})$$

for all $j \in \mathbb{Z}_d$, $n \in \mathbb{Z}_d^*$, and the trace norm $\|\rho\|_1 := \text{Tr} \sqrt{\rho^\dagger \rho}$. From here onward, we follow the steps of the proof for the ideal case and bound the terms one after the other.

Recall that $C_{n,j} = A_j^n - P_{n,j}^\dagger$ and $P_{n,j} = \sum_{k \in \mathbb{Z}_d} g(j, k, n) B_k^n$ with $g(j, k, n) = \frac{1}{d} \sum_{s \in \mathbb{Z}_d} \omega^{n(j+k)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}}$.

Applying $(A_j^n)^\dagger (A_j^{n'})^\dagger = (A_j^{n+n'})^\dagger$,

$$\begin{aligned}
& \| [P_{n,j} P_{n',j} - P_{n+n',j}] |\psi\rangle \|_1 \\
&= \left\| \left[P_{n,j} P_{n',j} - P_{n+n',j} + \left((A_j^{n+n'})^\dagger - (A_j^n)^\dagger (A_j^{n'})^\dagger \right) + \left((A_j^{n'})^\dagger P_{n,j} - (A_j^n)^\dagger P_{n',j} \right) \right] |\psi\rangle \right\|_1 \\
&\leq \|P_{n,j}\| \left\| \left[P_{n',j} - (A_j^{n'})^\dagger \right] |\psi\rangle \right\|_1 + \left\| (A_j^{n'})^\dagger \right\| \left\| \left[(A_j^n)^\dagger - P_{n,j} \right] |\psi\rangle \right\|_1 + \left\| \left[P_{n+n',j} - (A_j^{n+n'})^\dagger \right] |\psi\rangle \right\|_1 \\
&\leq \sqrt{\epsilon} (\|P_{n,j}\| + 2) \\
&\leq \sqrt{\epsilon} (\gamma(\nu) + 2),
\end{aligned} \tag{F2}$$

where we introduce $\gamma(\nu) \leq \|P_{n,j}\|$ for the following reason: In the worst case, $\gamma(\nu) = d$ due to

$$\|P_{n,j}\| = \frac{1}{d} \left\| \sum_{k,s \in \mathbb{Z}_d} \omega^{\iota(j,k,s,\nu)} B_k^n \right\| \leq \frac{1}{d} \sum_{k,s \in \mathbb{Z}_d} \|B_k^n\| \leq d, \tag{F3}$$

for all $j \in \mathbb{Z}_d$. But, we find that, when ν is a polynomial of degree three, $\deg(\nu) = 3$, it is $\|P_{n,j}\| \leq \sqrt{d}$ for all $j \in \mathbb{Z}_d$. In this case, $\gamma(\nu) = \sqrt{d}$. For every $j \in \mathbb{Z}_d$, Equation (F2), together with the identity (C1), is then

$$\left\| \sum_{k,l \in \mathbb{Z}_d} g(j,k,n) g(j,l,n') \left(B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle \right\|_1 \leq \sqrt{\epsilon} (\gamma(\nu) + 2). \tag{F4}$$

Now, we define the vector $R = (R_{k,l})_{(k,l) \in \mathbb{Z}_d^2}$ with

$$R_{k,l} := B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'},$$

and the matrix $G_n = (g(j,k,n))_{j,k \in \mathbb{Z}_d}$. The matrix G_n is unitary because $\sum_{j \in \mathbb{Z}_d} g(j,k,n) g(j,k',n)^* = \delta_{k,k'}$ and $\sum_{k \in \mathbb{Z}_d} g(j,k,n) g(j',k,n)^* = \delta_{j,j'}$. There exists a basis $\{T_{(k,l)}\}_{(k,l) \in \mathbb{Z}_d^2}$ such that

$$R_{(k,l)} = \sum_{p,q \in \mathbb{Z}_d} c_{(k,l)}^{(p,q)} T_{(p,q)}, \tag{F5}$$

for coefficients $c_{(k,l)}^{(p,q)}$. Furthermore, we write

$$b_j^{(p,q)} = \sum_{k,l \in \mathbb{Z}_d} g(j,k,n) g(j,l,n') c_{(k,l)}^{(p,q)}. \tag{F6}$$

Then, we convert the expression to the Hilbert-Schmidt norm

$$\sqrt{\sum_{p,q \in \mathbb{Z}_d} |b_j^{(p,q)}|^2} = \left\| \sum_{k,l \in \mathbb{Z}_d} b_j^{(p,q)} \right\|_2 = \left\| \sum_{p,q \in \mathbb{Z}_d} b_j^{(p,q)} T_{(p,q)} |\psi\rangle \right\|_2 \leq \sqrt{d} \left\| \sum_{p,q \in \mathbb{Z}_d} b_j^{(p,q)} T_{(p,q)} |\psi\rangle \right\|_1 \leq \sqrt{d} \sqrt{\epsilon} (\gamma(\nu) + 2). \tag{F7}$$

Then, we can bound the norm of the vector of coefficients $c_{(k,l)}^{(p,q)}$. First,

$$\sqrt{\sum_{k,l \in \mathbb{Z}_d} |c_{(k,l)}^{(p,q)}|^2} = \left\| \sum_{k,l \in \mathbb{Z}_d} c_{(k,l)}^{(p,q)} \right\|_2 = \left\| \sum_{k,l \in \mathbb{Z}_d} (G_n \otimes G_{n'})^{-1} b_{(k,l)}^{(p,q)} \right\|_2 = \left\| \sum_{k,l \in \mathbb{Z}_d} b_j^{(p,q)} \right\|_2 \leq \sqrt{d} \sqrt{\epsilon} (\gamma(\nu) + 2), \tag{F8}$$

and then,

$$|c_{(k,l)}^{(p,q)}| \leq \sqrt{d} \sqrt{\epsilon} (\gamma(\nu) + 2) \tag{F9}$$

Then, we bound

$$\left\| \left(B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle \right\|_1 = \|R_{(k,l)}|\psi\rangle\|_1 = \left\| \sum_{p,q \in \mathbb{Z}_d} c_{(k,l)}^{(p,q)} T_{(p,q)} |\psi\rangle \right\|_1 \quad (\text{F10})$$

$$\leq \sum_{p,q \in \mathbb{Z}_d} |c_{(k,l)}^{(p,q)}| \|T_{(p,q)} |\psi\rangle\|_1 \quad (\text{F11})$$

$$\leq \sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2), \quad (\text{F12})$$

Due to the symmetry of A and B , it holds that

$$\left\| \left(A_k^n A_l^{n'} - \omega^{2^{-1}nn'(k-l)} A_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle \right\|_1 \leq \sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2), \quad (\text{F13})$$

$$\left\| \left(B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle \right\|_1 \leq \sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2). \quad (\text{F14})$$

For the robust ω -commutation relations, we get

$$\left\| \left(B_k^n B_l^{n'} - \omega^{nn'(k-l)} B_l^{n'} B_k^n \right) |\psi\rangle \right\|_1 \quad (\text{F15})$$

$$= \left\| \left(B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} + \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} - \omega^{nn'(k-l)} B_l^{n'} B_k^n \right) |\psi\rangle \right\|_1 \quad (\text{F16})$$

$$\leq \left\| \left(B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle \right\|_1 + \left\| \left(\omega^{2^{-1}nn'(k-l)} B_k^n B_l^{n'} - \omega^{nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'} \right) |\psi\rangle \right\|_1 \quad (\text{F17})$$

$$\leq 2\sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2). \quad (\text{F18})$$

Due to the symmetry of A and B , it holds that

$$\left\| \left(A_k^n A_l^{n'} - \omega^{nn'(k-l)} A_l^{n'} A_k^n \right) |\psi\rangle \right\|_1 \leq 2\sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2), \quad (\text{F19})$$

$$\left\| \left(B_k^n B_l^{n'} - \omega^{nn'(k-l)} B_l^{n'} B_k^n \right) |\psi\rangle \right\|_1 \leq 2\sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2). \quad (\text{F20})$$

Appendix G: Robust self-testing relations for $d = 3$

Under the assumptions of Lemma 1, $\langle \psi | 6\mathbb{1} - \mathcal{B}_3 | \psi \rangle \leq \epsilon$ for an $\epsilon > 0$. The operator has a sum-of-positive-operators (SOPO) decomposition: $6\mathbb{1} - \mathcal{B}_3 = C_0^\dagger C_0 + C_1 C_1^\dagger + C_2 C_2^\dagger$ with C_j from Eq. (A16) for $j = 0, 1, 2$. It follows that $\langle \psi | C_j^\dagger C_j | \psi \rangle \leq \epsilon$, and consequently,

$$\sqrt{\langle \psi | C_j^\dagger C_j | \psi \rangle} = \|C_j |\psi\rangle\|_1 \leq \sqrt{\epsilon}, \quad (\text{G1})$$

for $j = 0, 1, 2$, and the trace norm $\|\rho\|_1 := \text{Tr} \sqrt{\rho^\dagger \rho}$. From here onward, we follow the steps of the proof for the ideal case and bound the terms one after the other.

1. Bound $\|[\{B_0^2, B_1^2\} + B_2]|\psi\rangle\|_1$, $\|[\{B_0, B_1\} + B_2^2]|\psi\rangle\|_1$, and index permutations thereof

The first step is to bound $\|[\{B_0^2, B_1^2\} + B_2]|\psi\rangle\|_1$ and index permutations thereof. To this aim, $A_0^2 = A_0^\dagger$ leads to

$$\left\| \left[\frac{e^{-2i\phi_1}}{3} (B_0^2 + \omega B_1^2 + B_2^2)^2 - \frac{e^{i\phi_1}}{\sqrt{3}} (B_0 + \omega^2 B_1 + B_2) \right] |\psi\rangle \right\|_1 \quad (\text{G2})$$

$$\leq \left\| \left[\frac{e^{-2i\phi_1}}{3} (B_0^2 + \omega B_1^2 + B_2^2)^2 - A_0^\dagger \right] |\psi\rangle \right\|_1 + \left\| \left[A_0^\dagger - \frac{e^{i\phi_1}}{\sqrt{3}} (B_0 + \omega^2 B_1 + B_2) \right] |\psi\rangle \right\|_1 \quad (\text{G3})$$

$$\leq \left\| \left[\frac{e^{2i\phi_1}}{3} (B_0^2 + \omega B_1^2 + B_2^2)^2 - A_0^2 \right] |\psi\rangle \right\|_1 + \sqrt{\epsilon} \quad (\text{G4})$$

$$\leq \left\| \left[\frac{e^{-i\phi_1}}{\sqrt{3}} (B_0^2 + \omega B_1^2 + B_2^2) - A_0 \right] |\psi\rangle \right\|_1 + \left\| \left[\frac{e^{-i\phi_1}}{\sqrt{3}} (B_0^2 + \omega B_1^2 + B_2^2) + A_0 \right] |\psi\rangle \right\|_1 + \sqrt{\epsilon} \quad (\text{G5})$$

$$\leq \sqrt{\epsilon} \left\| \left[\frac{e^{-i\phi_1}}{\sqrt{3}} (B_0^2 + \omega B_1^2 + B_2^2) + A_0 \right] |\psi\rangle \right\|_1 + \sqrt{\epsilon} \quad (\text{G6})$$

$$\leq \frac{\sqrt{\epsilon}}{\sqrt{3}} \left(\|B_0^2\| + \|B_1^2\| + \|B_2^2\| \right) + \sqrt{\epsilon} \|A_0\| + \sqrt{\epsilon} \quad (\text{G7})$$

$$\leq \sqrt{\epsilon} (\sqrt{3} + 2). \quad (\text{G8})$$

From Eq. (G2) to Eq. (G3), add $A_0^\dagger - A_0^\dagger$ and use the triangle inequality. From Eq. (G3) to Eq. (G4), use Eq. (G1). From Eq. (G4) to Eq. (G5), use the inequality $\|D^2 - E^2\| \leq \|D - E\| \|D + E\|$. From Eq. (G5) to Eq. (G6), use Eq. (G1). From Eq. (G6) to Eq. (G7), use the triangle inequality and $\|[D]|\psi\rangle\|_1 \leq \|D\|$. From Eq. (G7) to Eq. (G8), all operators are unitary such that their norm is $\|A_j\| = \|B_j\| = 1$ for $j = 0, 1, 2$.

One can show the same for $A_1^2 = A_1^\dagger$ and $A_2^2 = A_2^\dagger$ where the reasoning is equivalent up to permutation of indices such that

$$\left\| \left[\frac{e^{-2i\phi_1}}{3} (B_0^2 + \omega B_1^2 + B_2^2)^2 - \frac{e^{i\phi_1}}{\sqrt{3}} (B_0 + \omega^2 B_1 + B_2) \right] |\psi\rangle \right\|_1 \leq \sqrt{\epsilon} (\sqrt{3} + 2) \quad (\text{G9})$$

$$\left\| \left[\frac{e^{-2i\phi_1}}{3} (\omega B_0^2 + B_1^2 + B_2^2)^2 - \frac{e^{i\phi_1}}{\sqrt{3}} (\omega^2 B_0 + B_1 + B_2) \right] |\psi\rangle \right\|_1 \leq \sqrt{\epsilon} (\sqrt{3} + 2) \quad (\text{G10})$$

$$\left\| \left[\frac{e^{-2i\phi_1}}{3} (B_0^2 + B_1^2 + \omega B_2^2)^2 - \frac{e^{i\phi_1}}{\sqrt{3}} (B_0 + B_1 + \omega^2 B_2) \right] |\psi\rangle \right\|_1 \leq \sqrt{\epsilon} (\sqrt{3} + 2) \quad (\text{G11})$$

We evaluate the left-hand side of Eqs. (G9)-(G11) as for the ideal case using $1 - \sqrt{3}e^{3i\phi_1} = \omega$, to arrive at $\|F_j|\psi\rangle\|_1 \leq 3\sqrt{\epsilon}(\sqrt{3} + 2)$ for $j \in \{0, 1, 2\}$ with

$$F_0 := \omega\{B_0^2, B_1^2\} + \{B_0^2, B_2^2\} + \omega\{B_1^2, B_2^2\} + \omega B_0 + B_1 + \omega B_2, \quad (\text{G12})$$

$$F_1 := \omega\{B_0^2, B_1^2\} + \omega\{B_0^2, B_2^2\} + \{B_1^2, B_2^2\} + B_0 + \omega B_1 + \omega B_2, \quad (\text{G13})$$

$$F_2 := \{B_0^2, B_1^2\} + \omega\{B_0^2, B_2^2\} + \omega\{B_1^2, B_2^2\} + \omega B_0 + \omega B_1 + B_2. \quad (\text{G14})$$

With these,

$$\|[\{B_0^2, B_1^2\} + B_2]|\psi\rangle\|_1 = \frac{1}{3} \|F_0 + F_1 + \omega F_2\|_1 \leq \frac{1}{3} (\|F_0\|_1 + \|F_1\|_1 + \|F_2\|_1) \leq 3\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G15})$$

$$\|[\{B_0^2, B_2^2\} + B_1]|\psi\rangle\|_1 = \frac{1}{3} \|\omega F_0 + F_1 + F_2\|_1 \leq \frac{1}{3} (\|F_0\|_1 + \|F_1\|_1 + \|F_2\|_1) \leq 3\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G16})$$

$$\|[\{B_1^2, B_2^2\} + B_0]|\psi\rangle\|_1 = \frac{1}{3} \|F_0 + \omega F_1 + F_2\|_1 \leq \frac{1}{3} (\|F_0\|_1 + \|F_1\|_1 + \|F_2\|_1) \leq 3\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G17})$$

The same bounds hold for the complex conjugate of the expressions, which can be achieved by starting from $A_j = A_j^\dagger A_j^\dagger$ for all $j \in \{0, 1, 2\}$.

$$\|[\{B_0, B_1\} + B_2^2]|\psi\rangle\|_1 \leq 3\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G18})$$

$$\|[\{B_0, B_2\} + B_1^2]|\psi\rangle\|_1 \leq 3\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G19})$$

$$\|[\{B_1, B_2\} + B_0^2]|\psi\rangle\|_1 \leq 3\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G20})$$

All other steps follow in the exact same manner.

For later use, we derive the bounds

$$\|[B_0 B_1 B_2 - B_1 B_2 B_0] |\psi\rangle\|_1 \leq \|[B_0 B_1 B_2 + B_1 B_0 B_2 - B_1 B_0 B_2 - B_1 B_2 B_0] |\psi\rangle\|_1 \quad (\text{G21})$$

$$\begin{aligned} &= \|\{B_0, B_1\} B_2 - B_1 \{B_0, B_2\} |\psi\rangle\|_1 \\ &= \|\{B_0, B_1\} B_2 - B_1 \{B_0, B_2\} + (B_2^3 - B_1^3) |\psi\rangle\|_1 \\ &\leq 6\sqrt{\epsilon}(\sqrt{3} + 2), \end{aligned} \quad (\text{G22})$$

$$\|[B_0 B_1 B_2 - B_2 B_0 B_1] |\psi\rangle\|_1 \leq 6\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G23})$$

$$\|[B_1 B_2 B_0 - B_2 B_0 B_1] |\psi\rangle\|_1 \leq 6\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G24})$$

$$\|[B_0 B_2 B_1 - B_2 B_1 B_0] |\psi\rangle\|_1 \leq 6\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G25})$$

$$\|[B_0 B_2 B_1 - B_1 B_0 B_2] |\psi\rangle\|_1 \leq 6\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G26})$$

$$\|[B_2 B_1 B_0 - B_1 B_0 B_2] |\psi\rangle\|_1 \leq 6\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G27})$$

2. Commutation elements

We consider the three commutation elements

$$Q = B_0 B_1 B_0^2 B_1^2, \quad (\text{G28})$$

$$Q' = B_2 B_0 B_2^2 B_0^2, \quad (\text{G29})$$

$$Q'' = B_1 B_2 B_1^2 B_2^2, \quad (\text{G30})$$

It is useful to write the commutation elements Q, Q', Q'' as

$$Q = B_0 B_1 (\{B_0^2, B_1^2\} + B_2) - \mathbb{1} - B_0 B_1 B_2, \quad (\text{G31})$$

$$Q' = B_2 B_0 (\{B_2^2, B_0^2\} + B_1) - \mathbb{1} - B_2 B_0 B_1, \quad (\text{G32})$$

$$Q'' = B_1 B_2 (\{B_1^2, B_2^2\} + B_0) - \mathbb{1} - B_1 B_2 B_0. \quad (\text{G33})$$

Moreover, note that

$$\|[Q + Q' + \mathbb{1}] |\psi\rangle\|_1 = \|\{B_0, B_1\} \{B_0^2, B_1^2\} - \mathbb{1} |\psi\rangle\|_1 \quad (\text{G34})$$

$$= \|\{B_0, B_1\} \{B_0^2, B_1^2\} + \{B_0, B_1\} B_2 - \{B_0, B_1\} B_2 - \mathbb{1} |\psi\rangle\|_1 \quad (\text{G35})$$

$$\leq 6\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G36})$$

In the following, we show that Q is ‘almost’ central: First, bound

$$\|[B_0, Q] |\psi\rangle\|_1 = \|[B_0, B_0 B_1 (\{B_0^2, B_1^2\} + B_2) - B_0 B_1 B_2 - \mathbb{1}] |\psi\rangle\|_1 \quad (\text{G37})$$

$$\begin{aligned} &= \|[B_0^2 B_1 (\{B_0^2, B_1^2\} + B_2) - B_0^2 B_1 B_2 \\ &\quad - B_0 B_1 (\{B_0^2, B_1^2\} + B_2) B_0 + B_0 B_1 B_2 B_0] |\psi\rangle\|_1 \end{aligned} \quad (\text{G38})$$

$$\leq 6\sqrt{\epsilon}(\sqrt{3} + 2) + \|[B_0^2 B_1 B_2 - B_0 B_1 B_2 B_0] |\psi\rangle\|_1 \quad (\text{G39})$$

$$= 12\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G40})$$

$$\|[B_1, Q] |\psi\rangle\|_1 \leq 12\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G41})$$

$$\|[B_2, Q] |\psi\rangle\|_1 \leq 12\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G42})$$

In Eq. (G38), we inflated the term, then used (G15) in Eq. (G16), and in Eq. (G17) the final bound comes from Eq. (G22). Write $S := Q - \frac{1}{3} \text{Tr}(Q) \mathbb{1}$, then $2\|S\| \geq \|[U, S]\| = \|[U, Q]\|$ for $\|U\| = 1$. As a result, $\|S\| = \frac{1}{2} \sup_{\|U\|=1} \|[U, Q]\|$, and

$$\|[Q - \lambda \mathbb{1}] |\psi\rangle\|_1 \leq \|Q - \lambda \mathbb{1}\| = \frac{1}{2} \sup_{\|U\|=1} \|[Q, U]\| \leq 6\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G43})$$

Since Q is unitary, thus normal, we can apply the spectral mapping theorem

$$\|Q - \lambda \mathbb{1}\| = \max_{\mu \in \sigma(Q)} |\mu - \lambda| \leq 6\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G44})$$

3. Bounding $\|Q - \omega^k \mathbf{1}\|$

From Eq. (G36),

$$|\lambda + \lambda^* + 1| = \left\| [(\lambda \mathbf{1} - Q) + (\lambda^* \mathbf{1} - Q^\dagger) + (\mathbf{1} + Q + Q^\dagger)] |\psi\rangle \right\|_1 \quad (\text{G45})$$

$$\leq 18\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G46})$$

Write $\lambda = e^{i\theta}$, then $\lambda + \lambda^* + 1 = 2\cos\theta + 1$. Assuming that $\epsilon' := 18\sqrt{\epsilon}(\sqrt{3} + 2) < 1/2 < |\omega - \omega^2|$, confines θ to a ϵ' -neighborhood of $\theta = 2\pi/3$ or $\theta = 4\pi/3$, such that $|\lambda - \omega^k| \leq \epsilon'$ for $k = 1, 2$.

More formally, we show, first, that the eigenvalues of Q are ‘almost’ ω^k for $k = 1$ or $k = 2$, starting with

$$\left\| [Q^3 - \mathbf{1}] |\psi\rangle \right\|_1 = \left\| [Q^2 B_0 B_1 B_0^2 B_1^2 - Q B_0 Q B_1 B_0^2 B_1^2 + Q B_0 Q B_1 B_0^2 B_1^2 - \mathbf{1}] |\psi\rangle \right\|_1 \quad (\text{G47})$$

$$\leq 12\sqrt{\epsilon}(\sqrt{3} + 2) + \left\| [Q B_0 Q B_1 B_0^2 B_1^2 - \mathbf{1}] |\psi\rangle \right\|_1 \quad (\text{G48})$$

$$= 12\sqrt{\epsilon}(\sqrt{3} + 2) + \left\| [Q B_0^2 B_1 B_0 B_1^2 - \mathbf{1}] |\psi\rangle \right\|_1 \quad (\text{G49})$$

$$\leq 12\sqrt{\epsilon}(\sqrt{3} + 2) + \left\| [Q B_0^2 B_1 B_0 B_1^2 - B_0 Q B_0 B_1 B_0 B_1^2 + B_0 Q B_0 B_1 B_0 B_1^2 - \mathbf{1}] |\psi\rangle \right\|_1 \quad (\text{G50})$$

$$\leq 24\sqrt{\epsilon}(\sqrt{3} + 2) + \left\| [B_0 Q B_0 B_1 B_0 B_1^2 - B_0^2 Q B_1 B_0 B_1^2 + B_0^2 Q B_1 B_0 B_1^2 - \mathbf{1}] |\psi\rangle \right\|_1 \quad (\text{G51})$$

$$\leq 36\sqrt{\epsilon}(\sqrt{3} + 2) + \left\| [B_0^2 Q B_1 B_0 B_1^2 - \mathbf{1}] |\psi\rangle \right\|_1 \quad (\text{G52})$$

$$= 36\sqrt{\epsilon}(\sqrt{3} + 2) + \left\| [B_0^2 B_0 B_1 B_0^2 B_1^2 B_1 B_0 B_1^2 - \mathbf{1}] |\psi\rangle \right\|_1 \quad (\text{G53})$$

$$= 36\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G54})$$

As a result, the eigenvalues of Q are close to the third roots of unity $\{1, \omega, \omega^2\}$. In particular, the spectral mapping theorem of normals (applies since the B_j are unitary) gives

$$\|Q^3 - \mathbf{1}\| = \max_{\lambda \in \sigma(Q)} |\lambda^3 - 1| \leq 36\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G55})$$

In the following, we define $\epsilon' = 36\sqrt{\epsilon}(\sqrt{3} + 2)$. From Eq. (G44), every eigenvalue $\lambda \in \sigma(Q)$ lies in the union of the three ϵ' -disks $D_{\epsilon'}(\omega^k)$ for $k = 0, 1, 2$. By assuming $\epsilon' < 1/2$, we can make the three disk disjoint since $|\omega^k - \omega^j| = \sqrt{3}$ for $k \neq j$. Then, it is

$$|\lambda - \omega^k| \leq \epsilon' < \frac{1}{2}, \quad (\text{G56})$$

for all λ , which implies that all eigenvalues lie in the same disk, i.e. $k = k^*$ is fixed. Finally,

$$\|Q - \omega^{k^*} \mathbf{1}\| = \max_{\lambda \in \sigma(Q)} |\lambda - \omega^{k^*}| \leq \epsilon' \quad (\text{G57})$$

Lastly, we show that $k^* \neq 0$ by contradiction. So assume $\|Q - \mathbf{1}\| \leq \epsilon'$, then $\|[B_0, B_1]\| = \|Q - \mathbf{1}\| \leq \epsilon'$. For the anticommutator, we use the inequality, given that U and V are unitary, therefore UV is unitary and $\|UV\| = 1$,

$$\|\{U, V\}\| = \|UV + 2VU - VU\| \geq |||[U, V]| - 2|. \quad (\text{G58})$$

Then,

$$\|\{B_0, B_1\}\| \geq 2 - \|[B_0, B_1]\|, \quad (\text{G59})$$

and, furthermore,

$$\|\{B_0, B_1\} + B_2^2\| \geq |||\{B_0, B_1\}\| - \|B_2^2\|| = \|\{B_0, B_1\}\| - 1 \geq 1 - \epsilon'. \quad (\text{G60})$$

This, however, contradicts $\|\{B_0, B_1\} + B_2^2\| \leq 3\sqrt{\epsilon}(\sqrt{3} + 2)$ from Eq. (G15) for a small enough ϵ' .

4. Commutation elements Q' and Q''

All inequalities and arguments also hold for Q' and Q'' and cyclic permutations of B_0, B_1, B_2 . As a final step, we need to show that k^* is the same for $Q, Q',$ and Q'' . First note that Q, Q', Q'' are mutually close:

$$\| [Q - Q'] |\psi\rangle \|_1 = \| [B_0 B_1 B_0^2 B_1^2 - B_2 B_0 B_2^2 B_0^2] |\psi\rangle \|_1 \quad (\text{G61})$$

$$= \| [B_0 B_1 (\{B_0^2, B_1^2\} + B_2) - B_0 B_1 B_2 - \mathbb{1} - B_2 B_0 (\{B_2^2, B_0^2\} + B_1) + B_2 B_0 B_1 + \mathbb{1}] |\psi\rangle \|_1 \quad (\text{G62})$$

$$\leq 9\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G63})$$

$$\| [Q - Q''] |\psi\rangle \|_1 \leq 9\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G64})$$

$$\| [Q' - Q''] |\psi\rangle \|_1 \leq 9\sqrt{\epsilon}(\sqrt{3} + 2). \quad (\text{G65})$$

From Eq. (G61) to Eq. (G62), we inflate the expression. From Eq. (G62) to Eq. (G63), we use the triangle inequality and the bounds from Eq. (G15), Eq. (G16), and Eq. (G22). As a result, the eigenvalues of the commutation elements, fixed by k^* , are mutually ‘almost’ the same:

$$|\omega^{k^*} - \omega^{k'^*}| = \| [\omega^{k^*} \mathbb{1} - \omega^{k'^*} \mathbb{1} + Q - Q + Q' - Q'] |\psi\rangle \|_1 \quad (\text{G66})$$

$$\leq \| [\omega^{k^*} \mathbb{1} - Q] |\psi\rangle \|_1 + \| [Q' - \omega^{k'^*} \mathbb{1}] |\psi\rangle \|_1 + \| [Q - Q'] |\psi\rangle \|_1 \quad (\text{G67})$$

$$\leq 2\epsilon' + 9\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G68})$$

$$|\omega^{k^*} - \omega^{k''*}| \leq 2\epsilon' + 9\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G69})$$

$$|\omega^{k'^*} - \omega^{k''*}| \leq 2\epsilon' + 9\sqrt{\epsilon}(\sqrt{3} + 2) \quad (\text{G70})$$

Since $2\epsilon' + 9\sqrt{\epsilon}(\sqrt{3} + 2) = (2 + \frac{1}{4})\epsilon' < 1 + \frac{1}{8} < \sqrt{3}$, which is the distance between any ω^j and ω^k for $j \neq k$, we conclude that $k^* = k'^* = k''* \in \{1, 2\}$.

5. Twisted commutation relations

For a small enough ϵ , i.e. $36\sqrt{\epsilon}(\sqrt{3} + 2) < 1/2$, it follows from Eq. (G44) and its following considerations that

$$\| [B_0 B_1 - \omega^{k^*} B_1 B_0] |\psi\rangle \|_1 \leq 9\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G71})$$

$$\| [B_2 B_0 - \omega^{k^*} B_0 B_2] |\psi\rangle \|_1 \leq 9\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G72})$$

$$\| [B_1 B_2 - \omega^{k^*} B_2 B_1] |\psi\rangle \|_1 \leq 9\sqrt{\epsilon}(\sqrt{3} + 2), \quad (\text{G73})$$

for a $k^* \in \{1, 2\}$. The choice of k^* remains, as for the ideal case $\epsilon = 0$, a degree of freedom. Here, we choose $k^* = 1$.

Appendix H: Robust isometry

To bound the isometry, we require two bounds: First, the bounds on the twisted commutation relations from Eq. (F13) and Eq. (F14), for $d > 3$,

$$\| (A_k^n A_l^{n'} - \omega^{2^{-1}nn'(k-l)} A_{(n+n')^{-1}(nk+n'l)}^{n+n'}) |\psi\rangle \|_1 \leq \sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2) =: \sqrt{\epsilon}\mu_d, \quad (\text{H1})$$

$$\| (B_k^n B_l^{n'} - \omega^{2^{-1}nn'(k-l)} B_{(n+n')^{-1}(nk+n'l)}^{n+n'}) |\psi\rangle \|_1 \leq \sqrt{d}\sqrt{\epsilon}(\gamma(\nu) + 2) =: \sqrt{\epsilon}\mu_d, \quad (\text{H2})$$

and, for $d = 3$, those from Eqs. (G71) - (G73):

$$\| [B_i B_j - \omega^{j-i} B_j B_i] |\psi\rangle \|_1 \leq 9\sqrt{\epsilon}(\sqrt{3} + 2) =: \sqrt{\epsilon}\mu_3. \quad (\text{H3})$$

Second, the bound on the relations between the A_j and B_j for all $j \in \mathbb{Z}_d$: For $d > 3$, Eq. (F1) and Eq. (B2) give

$$\left\| \left[A_j^n - \sum_{k \in \mathbb{Z}_d} g(j, k, n)^* (B_k^n)^\dagger \right] |\psi\rangle \right\|_1 \leq \sqrt{\epsilon}. \quad (\text{H4})$$

For $d = 3$, Eq. (G1) and Eq. (A16) give

$$\left\| \left[A_0 - \frac{1}{\sqrt{3}} (e^{-i\phi_1} B_0^2 + e^{-i\phi_2} B_1^2 + e^{-i\phi_1} B_2^2) \right] |\psi\rangle \right\|_1 \leq \sqrt{\epsilon}, \quad (\text{H5})$$

$$\left\| \left[A_1 - \frac{1}{\sqrt{3}} (e^{-i\phi_2} B_0^2 + e^{-i\phi_1} B_1^2 + e^{-i\phi_1} B_2^2) \right] |\psi\rangle \right\|_1 \leq \sqrt{\epsilon}, \quad (\text{H6})$$

$$\left\| \left[A_2 - \frac{1}{\sqrt{3}} (e^{-i\phi_1} B_0^2 + e^{-i\phi_1} B_1^2 + e^{-i\phi_2} B_2^2) \right] |\psi\rangle \right\|_1 \leq \sqrt{\epsilon}. \quad (\text{H7})$$

Then, we go through Eqs. (E5) - (E14) and bound them using the triangle inequality:

$$\|(V_A \otimes V_B)|\psi\rangle - |\psi\rangle \otimes |aux\rangle\|_1 \quad (\text{H8})$$

$$= \left\| \frac{1}{d^2} \sum_{n,m \in \mathbb{Z}_d} |n, m\rangle \otimes \left((A_0^n)^\dagger \sum_{j \in \mathbb{Z}_d} \omega^{-jn} (\omega^{-2^{-1}j} A_0^\dagger A_j) \right) \bar{B}_m |\psi\rangle - |\psi\rangle \otimes |aux\rangle \right\|_1 \quad (\text{H9})$$

$$\leq \sqrt{\epsilon} d(d-1) \mu_d + \left\| \frac{1}{d^2} \sum_{n,m,j \in \mathbb{Z}_d} |n, m\rangle \otimes \omega^{-2^{-1}jn} A_{-n-1}^{-n} \bar{B}_m |\psi\rangle - |\psi\rangle \otimes |aux\rangle \right\|_1 \quad (\text{H10})$$

$$\leq \sqrt{\epsilon} d(d-1)(\mu_d + 1) + \left\| \frac{1}{d^3} \sum_{n,m,j,l,s \in \mathbb{Z}_d} |n, m\rangle \otimes \bar{B}_m \omega^{-2^{-1}jn} \omega^{(j-l)s + \nu_{s+2^{-1}n} - \nu_{s-2^{-1}n}} B_{n-1}^n |\psi\rangle - |\psi\rangle \otimes |aux\rangle \right\|_1 \quad (\text{H11})$$

$$\leq \sqrt{\epsilon} d(d-1)(2\mu_d + 1) + \left\| \frac{1}{d^2} \sum_{n,r,k,l \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n+r\rangle \otimes \omega^{-kr} B_0^{-r} \omega^{-2^{-1}(k+l)n} B_{n-1}^{-n} B_{n-1}^n |\psi\rangle - |\psi\rangle \otimes |aux\rangle \right\|_1 \quad (\text{H12})$$

$$\leq \sqrt{\epsilon} d(d-1)(4\mu_d + 1) + \left\| \frac{1}{d^2} \sum_{n,r,k,l \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n+r\rangle \otimes \omega^{-kr} B_0^{-r} \omega^{-2^{-1}(l+k)} B_0^\dagger B_{k+l} |\psi\rangle - |\psi\rangle \otimes |aux\rangle \right\|_1 \quad (\text{H13})$$

$$= \sqrt{\epsilon} d(d-1)(4\mu_d + 1) + \left\| \frac{1}{d} \sum_{n,t \in \mathbb{Z}_d} \omega^{\nu_n - \nu_0} |n, n\rangle \otimes \omega^{-2^{-1}t} B_0^\dagger B_t |\psi\rangle - |\psi\rangle \otimes |aux\rangle \right\|_1 \quad (\text{H14})$$

$$= \sqrt{\epsilon} d(d-1)(4\mu_d + 1). \quad (\text{H15})$$

For the bound on $\|(V_A \otimes V_B)A_u^v |\psi\rangle - (T_{(v,vu)} \otimes \mathbf{1}) |\psi\rangle \otimes |aux\rangle\|_1$, $\|(V_A \otimes V_B)B_u^v |\psi\rangle - (\mathbf{1} \otimes T_{(v,vu)}) |\psi\rangle \otimes |aux\rangle\|_1$, we apply the same inequalities as above but, since we apply the ω -relation (F13) and (F14) twice, it is

$$\|(V_A \otimes V_B)A_u^v |\psi\rangle - (T_{(v,vu)} \otimes \mathbf{1}) |\psi\rangle \otimes |aux\rangle\|_1 \leq \sqrt{\epsilon} d(d-1)(\mu_d(4 + 1/d) + 1), \quad (\text{H16})$$

$$\|(V_A \otimes V_B)B_u^v |\psi\rangle - (\mathbf{1} \otimes T_{(v,vu)}) |\psi\rangle \otimes |aux\rangle\|_1 \leq \sqrt{\epsilon} d(d-1)(\mu_d(4 + 1/d) + 1). \quad (\text{H17})$$