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**Research Article**

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# Geometric Properties of Certain Entire Functions Associated with the Rogers–Szegó and Stieltjes–Wigert Polynomials

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**Abstract**

In this paper, we investigate the geometric properties of two families of entire functions which are closely related to the Rogers–Szegő and Stieltjes–Wigert polynomials. By exploiting the Laguerre–Pólya class structure and Hadamard factorization, we determine the radii of starlikeness and convexity of order $\beta \in [0, 1)$, as well as the radii of uniform convexity for several natural normalizations. Since these radii are characterized as the smallest positive zeros of certain transcendental equations, we derive explicit lower and upper bounds via Euler–Rayleigh inequalities expressed in terms of the Maclaurin coefficients. Numerical experiments and graphical representations are included to illustrate the sharpness of the obtained bounds and the associated geometric behavior. The results contribute to the geometric function theory of $q$-series by providing a unified analytic framework for these classes of entire functions.

**Keywords:** Starlike functions, Convex functions, Radius problems, Rogers–Szegő polynomials, Stieltjes–Wigert polynomials, Euler–Rayleigh inequalities

**MSC 2020:** 30C45, 33D15, 30C15.

## 1 Introduction

Special functions have long occupied a central position within classical geometric function theory and continue to play a crucial role in several branches of mathematics, mathematical physics, and engineering. Beyond their analytical importance, the geometric behavior of functions associated with Bessel-type structures has attracted significant attention within the framework of geometric function theory. Systematic investigations into these geometric aspects date back to the mid-twentieth century, when several authors initiated the study of univalence, starlikeness, and convexity properties for functions connected with Bessel-type differential equations [8, 11, 14, 16, 18, 19, 26]. Since then, these geometric questions have evolved into an active research direction linking special function theory with complex analysis.

Over the past decades, particular attention has been devoted to normalized Bessel functions of the first kind, especially regarding the determination of sharp radii associated with geometric properties such as starlikeness and convexity. Various analytic techniques have been developed to study these quantities. For instance, methods based on Mittag–Leffler expansions for ratios of Bessel functions have proved highly effective in locating the radii of geometric behavior. In line with these developments, Baricz and Sz'asz [5] employed Mittag–Leffler expansions together with

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information on the smallest positive zeros of certain Dini-type functions to determine the radius of convexity for normalized Bessel functions. Complementary approaches relying on Euler–Rayleigh inequalities were later used by Aktaş et al. [1] to establish bounds for the radii of starlikeness and convexity for several special functions, including Bessel, Struve, and Lommel functions.

A particularly influential line of research in this direction has been developed by Baricz and his collaborators [1, 3–5, 21], who systematically investigated geometric properties of special functions through the analysis of their zeros and infinite product representations [1, 5, 22]. In a series of papers, several tight results were obtained for the radii of starlikeness and convexity of normalized Bessel functions, Struve functions and Lommel functions [1, 22]. Moreover, these investigations were extended to generalized $k$-Bessel functions [23,24], Wright functions [6,21], and various $q$-analogues such as $q$-Mittag-Leffler [20] and big $q$-Bessel functions [21, 25]. These studies combine techniques from the theory of entire functions, such as Hadamard factorization and Laguerre–Pólya class arguments [7, 12, 15, 17], with analytic inequalities including Euler–Rayleigh inequalities [1, 3, 20, 22]. Such methods have proved to be extremely powerful in deriving sharp bounds for geometric radii and continue to play an important role in the modern development of geometric function theory for special functions.

More recently, refined asymptotic techniques have provided further insights into the geometric behavior of such functions. In particular, Baricz and Nemes [3] derived systematic asymptotic expansions for the radius of starlikeness of normalized Bessel functions, revealing deeper structural properties of these radii. Parallel developments have extended these ideas to the setting of basic hypergeometric functions, where analogous geometric problems arise for normalized $q$-Bessel functions. These investigations demonstrate that the interplay between the distribution of zeros, analytic representations of special functions, and geometric function theory remains a fruitful source of new results.

Motivated by these developments, it is natural to extend the study of geometric properties from classical special functions to broader families arising in the framework of $q$-analysis. In particular, many special functions appearing in geometric function theory admit natural $q$-analogues that inherit rich analytic structures and deep connections with basic hypergeometric series. Among these, Rogers–Szegő and Stieltjes–Wigert polynomials occupy a distinguished position due to their close relationship with $q$-series, orthogonal polynomial theory, and analytic generating functions [2, 10, 13].

Despite their importance in the theory of special functions, the geometric behavior of analytic functions generated by these polynomials has not yet been systematically investigated. In particular, questions concerning the radii of starlikeness, convexity, and uniform convexity for normalized analytic functions associated with these polynomials remain largely unexplored.

The aim of the present paper is to further investigate these geometric properties for normalized analytic functions generated by Rogers–Szegő and Stieltjes–Wigert polynomials. More precisely, we determine the radii of starlikeness, convexity, and uniform convexity for the functions under consideration. By combining tools from geometric function theory with analytic techniques from the theory of entire functions and $q$-series, we derive analytic characterizations of these radii and establish sharp lower and upper bounds by means of Euler–Rayleigh type inequalities and properties of the corresponding zero distributions. These results contribute to the growing body of literature connecting special functions, $q$-analysis, and geometric function theory, and provide new insights into the geometric structure of analytic functions generated by important families of orthogonal polynomials.

### 1.1 Preliminaries

In this section, we provide the analytical foundations necessary for our study, including the fundamentals of $q$-analysis and the geometric characterizations of the function classes under consideration. Throughout this paper, we assume $0 < q < 1$.

For $a \in \mathbb{C}$, the $q$-Pochhammer symbol $(a;q)_n$ is defined by

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad n \in \{1,2,\dots\}$$

with $(a;q)_0 = 1$. The Gaussian $q$-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

for $n, k \in \mathbb{N} \cup \{0\}$ with $n \geq k$. In view of the properties of Rogers–Szegő and Stieltjes–Wigert structures, we recall the homogeneous $q$-polynomials $h_n(x, y \mid q)$ and $g_n(x, y \mid q)$, defined respectively as:

$$h_n(x, y \mid q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$

and

$$g_n(x, y \mid q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k y^{n-k}.$$

In the field of geometric function theory, the classification of analytic functions is often based on the geometry of their image domains. Let $\mathcal{A}$ be the class of functions $f$ that are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$. For a given parameter $\beta \in [0, 1)$, the radii of starlikeness and convexity of order $\beta$ are defined as the following suprema:

$$r_\star^\beta(f) = \sup\left\{ r \in (0,1] : \mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad \forall z \in \mathbb{D}_r \right\},$$

and

$$r_c^\beta(f) = \sup\left\{ r \in (0,1] : \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \quad \forall z \in \mathbb{D}_r \right\}.$$

These radii represent the largest disk $|z| < r$ that is mapped by $f$ onto a starlike or convex domain of order $\beta$, respectively. Geometrically, the condition for starlikeness of order $\beta$ ensures that the values of the quantity $zf'(z)/f(z)$ lie in the right half-plane $\mathfrak{Re}(w) > \beta$.

Furthermore, the concept of uniform convexity, introduced to characterize functions that remain convex under a more restrictive parabolic constraint, is quantified by the radius of uniform convexity of order $\beta$, denoted by $r_{uc}^\beta(f)$ and defined as:

$$r_{uc}^\beta(f) = \sup\left\{ r \in (0,1] : \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| + \beta, \quad \forall z \in \mathbb{D}_r \right\}.$$

This definition implies a hierarchical relationship between the radii. Specifically, for a fixed $\beta$, it is well-established that $r_{uc}^\beta(f) \leq r_c^\beta(f) \leq r_\star^\beta(f)$. Moreover, the classical Alexander duality remains valid, stating that $f$ is convex of order $\beta$ if and only if $zf'(z)$ is starlike of order $\beta$, which serves as a fundamental tool in determining these geometric quantities for special functions.

In this study, we focus on two classes of analytic functions generated by the polynomials $h_n(x, y \mid q)$ and $g_n(x, y \mid q)$. Motivated by the structural properties of classical and $q$-Bessel functions, we define $\phi_{x,y}(z;q)$ and $\varphi_{x,y}(z;q)$ as follows:

$$\phi_{x,y}(z;q) = \sum_{n=0}^{\infty} \frac{(-1)^n h_n(x, y \mid q)}{(q;q)_n} q^{\alpha n^2} z^{2n}, \quad \varphi_{x,y}(z;q) = \sum_{n=0}^{\infty} \frac{g_n(x, y \mid q)}{(q;q)_n} q^{\alpha n^2} z^{2n} \tag{1.1}$$

where $\alpha > 0$, $x, y \in \mathbb{R}$, and $0 < q < 1$. These series define even entire functions of order zero. To investigate their geometric properties, we introduce two distinct types of normalized families:

$$\begin{aligned} f_{x,y}^{(1)}(z;q) &= z\phi_{x,y}(z;q), \qquad g_{x,y}^{(1)}(z;q) = z\varphi_{x,y}(z;q) \\ f_{x,y}^{(2)}(z;q) &= z\phi_{x,y}(\sqrt{z};q), \quad g_{x,y}^{(2)}(z;q) = z\varphi_{x,y}(\sqrt{z};q) \end{aligned} \tag{1.2}$$

By construction, these functions belong to the class $\mathcal{A}$ and satisfy the necessary normalization conditions for the determination of sharp radii.

## 2 Analytical Properties of the Main Functions

In this section, we establish the analytical foundation for the geometric study of the functions $\phi_{x,y}(z;q)$ and $\varphi_{x,y}(z;q)$. The determination of sharp radii for starlikeness and convexity relies heavily on the distribution of zeros and the growth properties of the underlying entire functions. Specifically, the following lemma characterizes the nature of these zeros and provides the infinite product representations, which are essential for applying the Euler–Rayleigh summation techniques in the subsequent sections.

Recall that a real entire function $f$ belongs to the Laguerre–Pólya class, denoted by $\mathcal{LP}$, if it can be represented in the form:

$$f(z) = cz^m e^{-az^2+bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n},$$

where $c, b \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}_0$, and $\{z_n\}$ are real numbers such that $\sum z_n^{-2} < \infty$. The $\mathcal{LP}$ class is of particular importance in the theory of univalent functions because it is closed under differentiation; specifically, if $f \in \mathcal{LP}$, then its derivative $f'$ also belongs to $\mathcal{LP}$.

Within this framework, a particularly important and more structured subclass is the Laguerre–Pólya class of type I, denoted by $\mathcal{LP}$-I. This subclass consists of real entire functions whose zeros are all real and share the same sign, a property that significantly strengthens their geometric and analytic behavior. More precisely, a function $f$ belongs to $\mathcal{LP}$-I if it admits the representation

$$f(z) = cz^m e^{bz} \prod_{n=1}^{\infty} \left(1 + \frac{z}{x_n}\right),$$

where $c, b \in \mathbb{R}$, $m \in \mathbb{N}_0$, $x_n > 0$, and $\sum x_n^{-1} < \infty$. Owing to their rigid zero distribution, functions in $\mathcal{LP}$-I arise naturally in the theory of multiplier sequences, stability preserving operators, and in various problems concerning real-rootedness, thereby providing a refined framework for analyzing the geometric properties of entire functions.

The following lemma characterizes a fundamental example of a linear operator preserving the Laguerre–Pólya class. It establishes the precise condition on $\lambda$ for which the first-order differential operator $T_\lambda[f](z) = f(z) + \lambda z f'(z)$ maps real-rooted functions to real-rooted functions, a key result in the theory of multiplier sequences and Pólya–Schur theory.

**Lemma 2.1.** *Let $\lambda \in \mathbb{R}$ and consider the linear operator*

$$T_\lambda[f](z) = f(z) + \lambda z f'(z).$$

*Then the following assertions are equivalent:*

*(ii) $T_\lambda$ preserves real-rootedness on the class of real polynomials,*

*(iii) $\lambda \geq 0$.*

*Moreover, each of these conditions implies that*

*(i) $T_\lambda$ preserves the Laguerre–Pólya class $\mathcal{LP}$, that is,*

$$f \in \mathcal{LP} \quad \Longrightarrow \quad T_\lambda[f] \in \mathcal{LP}.$$

*Proof.* We begin by observing that the operator $T_\lambda$ acts diagonally on monomials as

$$T_\lambda[z^n] = (1+\lambda n)z^n, \qquad n \geq 0,$$

and hence it is associated with the sequence $\gamma_n = 1 + \lambda n$.

We first prove that (iii) $\Rightarrow$ (ii). By the classical Pólya–Schur theorem [17], a diagonal operator preserves real-rootedness if and only if the corresponding sequence is a multiplier sequence. In this case, the exponential generating function of $\{\gamma_n\}$ is

$$\Phi_\lambda(t) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} t^n = e^t(1+\lambda t).$$

If $\lambda \geq 0$, then $\Phi_\lambda$ has at most one zero, located at $t = -1/\lambda \leq 0$, and therefore belongs to the Laguerre–Pólya class of type I; see, for instance, [15, Chapter VIII]. Consequently, $\{\gamma_n\}$ is a multiplier sequence, and $T_\lambda$ preserves real-rootedness.

Next, we prove (ii) $\Rightarrow$ (iii). If $T_\lambda$ preserves real-rootedness, then $\{\gamma_n\}$ must be a multiplier sequence. It is well known that nontrivial multiplier sequences do not change sign; see [15, p. 100]. However, if $\lambda < 0$, then $\gamma_0 = 1 > 0$ while $\gamma_n = 1 + \lambda n < 0$ for all sufficiently large $n$, so the sequence changes sign. This contradiction shows that $\lambda \geq 0$.

Finally, we prove that (ii) $\Rightarrow$ (i). Let $f \in \mathcal{LP}$. By definition, there exists a sequence of real-rooted polynomials $\{P_n\}$ converging to $f$ locally uniformly on $\mathbb{C}$; see [15, Chapter I]. Since differentiation is continuous under local uniform convergence, we have $T_\lambda[P_n] \to T_\lambda[f]$ locally uniformly. By assumption, each $T_\lambda[P_n]$ has only real zeros. Hence, by Hurwitz's theorem (see [15, Chapter I]), the limit function $T_\lambda[f]$ either vanishes identically or has only real zeros. Therefore $T_\lambda[f] \in \mathcal{LP}$, completing the proof. □

*Remark* 2.2. It is important to emphasize that the condition $\lambda \geq 0$ in Lemma 2.1 is a structural necessity for the preservation of the Laguerre–P'olya class. If $\lambda < 0$, the associated sequence $\{1+\lambda n\}_{n\geq 0}$ ceases to be a multiplier sequence because it undergoes a sign change for sufficiently large $n$, which is incompatible with the characterization of such sequences [15]. Consequently, the operator $T_\lambda$ for negative values of $\lambda$ does not map the class $\mathcal{LP}$ into itself in general, as the non-negativity of $\lambda$ ensures that the corresponding generating function $\Phi_\lambda(t)$ remains in $\mathcal{LP}$-$I$.

In the following lemma, we show that our target functions are members of this class by establishing their growth order and the reality of their zeros.

**Lemma 2.3.** *Let $0 < q < 1$. Then the following statements hold:*

*i. If $\alpha > 0$ and $0 < x, y < 1$, then the function*

$$\phi_{x,y}(z;q) = \sum_{n\geq 0} (-1)^n \frac{h_n(x,y|q)}{(q;q)_n} q^{\alpha n^2} z^{2n}$$

*possesses infinitely many zeros, all of which are real and simple. Let $\beta_n(x,y,q)$ denote the $n$-th positive zero of $z \mapsto \phi_{x,y}(z;q)$. Then, $\phi_{x,y}(z;q)$ admits the infinite product representation:*

$$\phi_{x,y}(z;q) = \prod_{n\geq 1} \left(1 - \frac{z^2}{\beta_n^2(x,y,q)}\right). \tag{2.1}$$

*Moreover, the zeros of $\phi_{x,y}$ and its derivative $\phi'_{x,y}$ strictly interlace. In addition, if $\alpha > 0$ and $0 < x < 1$, then the function*

$$\phi_x(z;q) = \sum_{n\geq 0} (-1)^n \frac{h_n\left(x \mid q\right)}{(q;q)_n} q^{\alpha n^2} z^{2n}, \quad \text{where} \quad h_n\left(x \mid q\right) := h_n\left(x, 1 \mid q\right)$$

*has infinitely many zeros, all of which are real and simple. In this case, $\phi_x(z;q)$ also admits an infinite product representation of the form Equation (2.1) with zeros $\{\beta_n(x,q)\}_{n\geq 1}$.*

*ii. If $\alpha \geq \frac{1}{2}$ and $-1 < x, y < 0$, then the function*

$$\varphi_{x,y}(z;q) = \sum_{n\geq 0} \frac{g_n\left(x, y \mid q\right)}{(q;q)_n} q^{\alpha n^2} z^{2n}$$

*possesses infinitely many zeros, all of which are real, positive and simple. Let $\theta_n(x,y,q)$ denote the $n$-th positive zero of $z \mapsto \varphi_{x,y}(z;q)$. Then, $\varphi_{x,y}(z;q)$ admits the representation:*

$$\varphi_x(z;q) = \prod_{n\geq 1} \left(1 - \frac{z^2}{\theta_n^2(x,y,q)}\right). \tag{2.2}$$

*Furthermore, the zeros of $\varphi_{x,y}$ and its derivative $\varphi'_{x,y}$ strictly interlace. Similarly, if $\alpha \geq \frac{1}{2}$ and $-1 < x < 0$, then the function*

$$\varphi_x(z;q) = \sum_{n\geq 0} \frac{g_n\left(x \mid q\right)}{(q;q)_n} q^{\alpha n^2} z^{2n}, \quad \text{where} \quad g_n\left(x \mid q\right) := g_n\left(x, -1 \mid q\right)$$

*has infinitely many zeros, all of which are real, positive and simple. Similarly, $\varphi_x(z;q)$ admits an infinite product representation analogous to Equation (2.2) in terms of its positive zeros.*

*Proof.* We provide the proof for part (i), noting that the results for the second part follow by an identical line of reasoning.

By Lemma 2.5 of He [12], the function $\phi_{x,y}(z;q)$ is entire. Moreover, by Theorem 1.2 of the same reference, it possesses infinitely many real zeros, all of which are simple.

To establish the infinite product representation, we determine the growth order of $\phi_{x,y}(z;q)$. Let $a_n$ denote the coefficients in the series expansion

$$a_n = \frac{(-1)^n h_n(x, y \mid q)}{(q;q)_n} q^{\alpha n^2}.$$

Recall that the order $\rho$ of an entire function $\sum a_n z^{2n}$ is given by

$$\rho\left(\phi_{x,y}(z;q)\right) = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)}. \tag{2.3}$$

Since $0 < q < 1$, we have $(q;q)_n \geq (q;q)_\infty > 0$. Furthermore, for $0 < x, y < 1$, there exists a constant $C > 0$ such that $|h_n(x, y \mid q)| \leq C\rho_0^n$, where $\rho_0 = \max\{x, y\}$. Consequently,

$$\log(1/|a_n|) \geq \alpha n^2 \log(1/q) - \mathcal{O}(n).$$

Hence,

$$\rho\left(\phi_{x,y}(z;q)\right) = \limsup_{n\to\infty} \frac{n \log n}{\alpha n^2 \log(1/q) - \mathcal{O}(n)} = 0.$$

Thus, $\phi_{x,y}(z;q)$ is an entire function of order zero, and therefore of genus zero.

By the Hadamard Factorization Theorem, it admits the canonical product representation

$$\phi_{x,y}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\beta_n^2(x,y,q)}\right).$$

The existence, simplicity, and reality of the zeros $\beta_n(x,y,q)$ follow from [12, Theorem 1.1], since the coefficients $A_n = |a_n|$ are positive and satisfy $A_n/A_{n-1} \to 0$.

Finally, since $\phi_{x,y}$ is an entire function of order zero with only real zeros, it belongs to the Laguerre–Pólya class $\mathcal{LP}$ (see, e.g., [15]). Consequently, the zeros of $\phi_{x,y}$ and its derivative strictly interlace.

The one-parameter case follows by setting $y = 1$, which completes the proof. □

Equipped with these analytical properties and the zero distribution of the functions under consideration, we are now in a position to present our main results. In what follows, we determine the radii of starlikeness, convexity, and uniform convexity of order $\beta$ for the families $f_{x,y}^{(i)}$ and $g_{x,y}^{(i)}$ $(i \in \{1,2\})$. The tight values of these radii are obtained as the smallest positive roots of certain transcendental equations involving the functions and their derivatives.

# 3 Main Results

In this section, we establish the sharp radii of starlikeness, convexity, and uniform convexity for the normalized entire functions under consideration. By utilizing their Hadamard factorization and the properties of the Laguerre–P'olya class, we characterize these radii as the smallest positive roots of certain transcendental equations. The results are presented systematically, beginning with the starlikeness properties and progressing towards more restrictive geometric characterizations.

## 3.1 Radii of starlikeness of order $\beta$ of the main functions

We first focus on the radii of starlikeness of order $\beta$. Recall that a function $f$ is starlike of order $\beta$ if it satisfies the condition $\mathfrak{Re}\{zf'(z)/f(z)\} > \beta$ for $z \in \mathbb{D}_r$. The following theorems provide the exact values of these radii for both the $f$-family and $g$-family of functions, based on the monotonicity of their logarithmic derivatives on the real axis.

**Theorem 3.1.** *Assume that $\alpha$ is positive constant and $0 < x, y < 1$. Under these assumptions, the following assertions are valid:*

*i. The starlikeness radius of order $\beta$ for the function $f_{x,y}^{(1)}(z;q)$ is given by $r_\star^\beta\left(f_{x,y}^{(1)}(z;q)\right) = \ell_1(x,y,q)$, where $\ell_1(x,y,q)$ is denotes the smallest positive root of the associated transcendental equation:*

$$r\phi'_{x,y}(r;q) + (1-\beta)\phi_{x,y}(r;q) = 0.$$

*ii. The starlikeness radius of order $\beta$ for the function $f_{x,y}^{(2)}(z;q)$ is given by $r_\star^\beta\left(f_{x,y}^{(2)}(z;q)\right) = \iota_1(x,y,q)$, where $\iota_1(x,y,q)$ is denotes the smallest positive root of the associated transcendental equation:*

$$\sqrt{r}\phi'_{x,y}\big(\sqrt{r};q\big) + 2\,(1-\beta)\phi_{x,y}\big(\sqrt{r};q\big) = 0.$$

*Proof.* It suffices to verify that the inequalities

$$\mathfrak{Re}\left(z\,\frac{f_{x,y}^{(1)\prime}(z;q)}{f_{x,y}^{(1)}(z;q)}\right) \geq \beta \quad \text{and} \quad \mathfrak{Re}\left(z\,\frac{f_{x,y}^{(2)\prime}(z;q)}{f_{x,y}^{(2)}(z;q)}\right) \geq \beta \tag{3.1}$$

hold for all $z \in \mathbb{D}_{r_\star^\beta}$, and that these inequalities fail to hold in any disk of larger radius. A crucial observation is that, under the appropriate assumptions, all zeros of the function $z \mapsto \phi_{x,y}(z;q)$ are real. Furthermore, Lemma 2.3 shows that the function $\phi_{x,y}(z;q)$ admits an infinite product representation given in (2.1). Taking the logarithmic derivative of (2.1), we obtain

$$\frac{\phi'_{x,y}(z;q)}{\phi_{x,y}(z;q)} = -\sum_{n\geq 1} \frac{2z}{\beta_n^2(x,y,q) - z^2}.$$

Consequently, it follows that

$$z\,\frac{f_{x,y}^{(1)\prime}(z;q)}{f_{x,y}^{(1)}(z;q)} = 1 + z\,\frac{\phi'_{x,y}(z;q)}{\phi_{x,y}(z;q)} = 1 - \sum_{n\geq 1} \frac{2z^2}{\beta_n^2(x,y,q) - z^2}$$

and

$$z\,\frac{f_{x,y}^{(2)\prime}(z;q)}{f_{x,y}^{(2)}(z;q)} = 1 + \frac{\sqrt{z}}{2}\,\frac{\phi'_{x,y}(\sqrt{z};q)}{\phi_{x,y}(\sqrt{z};q)} = 1 - \sum_{n\geq 1} \frac{z}{\beta_n^2(x,y,q) - z}.$$

As shown in [5], if $z \in \mathbb{C}$, $\theta \in \mathbb{R}$, and $\theta > |z|$, then the inequality

$$\frac{|z|}{\theta - |z|} \geq \mathfrak{Re}\left(\frac{z}{\theta - z}\right) \tag{3.2}$$

holds. By invoking (3.2), we conclude that the inequality

$$\frac{|z|^2}{\beta_n^2(x,y,q) - |z|^2} \geq \mathfrak{Re}\left(\frac{z^2}{\beta_n^2(x,y,q) - z^2}\right)$$

holds for all $n \in \mathbb{N}$ and for all $|z| < \beta_1(x,y,q)$, in view of the assumptions of Lemma 2.3. Consequently, we obtain

$$\begin{aligned}
\mathfrak{Re}\left(z\,\frac{f_{x,y}^{(1)\prime}(z;q)}{f_{x,y}^{(1)}(z;q)}\right) &= 1 - \mathfrak{Re}\left(\sum_{n\geq 1} \frac{2z^2}{\beta_n^2(x,y,q) - z^2}\right) \\
&\geq 1 - \sum_{n\geq 1} \frac{2|z|^2}{\beta_n^2(x,y,q) - |z|^2} \\
&= |z|\frac{f_{x,y}^{(1)\prime}(|z|\,;q)}{f_{x,y}^{(1)}(|z|\,;q)}
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{Re}\left(z\,\frac{f_{x,y}^{(2)\prime}(z;q)}{f_{x,y}^{(2)}(z;q)}\right) &= 1 - \mathfrak{Re}\left(\sum_{n\geq 1} \frac{z}{\beta_n^2(x,y,q) - z}\right) \\
&\geq 1 - \sum_{n\geq 1} \frac{|z|}{\beta_n^2(x,y,q) - |z|} \\
&= |z|\frac{f_{x,y}^{(2)\prime}(|z|\,;q)}{f_{x,y}^{(2)}(|z|\,;q)}.
\end{aligned}$$

Equalities hold if and only if $z = |z| = r$. The above inequality, together with the minimum principle for harmonic functions, implies that the inequality in (3.1) holds if and only if $|z| <$

$\ell_1(x,y,q)$ and $|z| < \iota_1(x,y,q)$, respectively, where $\ell_1(x,y,q)$ and $\iota_1(x,y,q)$ denote the smallest positive solution of

$$r\,\frac{f_{x,y}^{(1)\prime}(r;q)}{f_{x,y}^{(1)}(r;q)} = \beta \quad \text{and} \quad r\,\frac{f_{x,y}^{(2)\prime}(r;q)}{f_{x,y}^{(2)}(r;q)} = \beta$$

which are equivalent to

$$r\phi_{x,y}'(r;q) + (1-\beta)\phi_{x,y}(r;q) = 0$$

and

$$\sqrt{r}\phi_{x,y}'\big(\sqrt{r};q\big) + 2\,(1-\beta)\phi_{x,y}\big(\sqrt{r};q\big) = 0.$$

That is to say, we have demonstrated that

$$\inf_{z\in\mathcal{D}_r} \mathfrak{Re}\left(z\,\frac{f_{x,y}^{(1)\prime}(z;q)}{f_{x,y}^{(1)}(z;q)}\right) = r\,\frac{f_{x,y}^{(1)\prime}(r;q)}{f_{x,y}^{(1)}(r;q)} = F_{x,y}(r;q)$$

$$\inf_{z\in\mathcal{D}_r} \mathfrak{Re}\left(z\,\frac{f_{x,y}^{(2)\prime}(z;q)}{f_{x,y}^{(2)}(z;q)}\right) = r\,\frac{f_{x,y}^{(2)\prime}(r;q)}{f_{x,y}^{(2)}(r;q)} = G_{x,y}(r;q)\,.$$

Since the real functions $F_{x,y}\,G_{x,y} : (0,\beta_1(x,y,q)) \to \mathbb{R}$ are decreasing, and take the limits

$$\lim_{r\searrow 0} F_{x,y}(r;q) = \lim_{r\searrow 0} G_{x,y}(r;q) = 1$$

and

$$\lim_{r\nearrow\beta_1(x,y,q)} F_{x,y}(r;q) = \lim_{r\nearrow\beta_1(x,y,q)} G_{x,y}(r;q) = -\infty,$$

it follows that the inequalities in (3.1) indeed hold for $z \in \mathbb{D}_{\ell_1(x,y,q)}$ and $z \in \mathbb{D}_{\iota_1(x,y,q)}$, respectively.

This completes the proof. □

**Theorem 3.2.** *Assume that $\alpha \geq \dfrac{1}{2}$ is a constant and $-1 < x, y < 0$. Under these assumptions, the following assertions are valid:*

*i. The starlikeness radius of order $\beta$ for the function $g_{x,y}^{(1)}(z;q)$ is given by $r_\star^\beta\left(g_{x,y}^{(1)}(z;q)\right) = \jmath_1(x,y,q)$, where $\jmath_1(x,y,q)$ is denotes the smallest positive root of the associated transcendental equation:*

$$r\varphi_{x,y}'(r;q) + (1-\beta)\varphi_{x,y}(r;q) = 0.$$

*ii. The starlikeness radius of order $\beta$ for the function $g_{x,y}^{(2)}(z;q)$ is given by $r_\star^\beta\left(g_{x,y}^{(2)}(z;q)\right) = \mathfrak{k}_1(x,y,q)$, where $\mathfrak{k}_1(x,y,q)$ is denotes the smallest positive root of the associated transcendental equation:*

$$\sqrt{r}\varphi_{x,y}'\big(\sqrt{r};q\big) + 2\,(1-\beta)\varphi_{x,y}\big(\sqrt{r};q\big) = 0.$$

*Proof.* It is enough to prove that

$$\mathfrak{Re}\left(z\,\frac{\big(g_{x,y}^{(1)}\big)'(z;q)}{g_{x,y}^{(1)}(z;q)}\right) \geq \beta \quad \text{and} \quad \mathfrak{Re}\left(z\,\frac{\big(g_{x,y}^{(2)}\big)'(z;q)}{g_{x,y}^{(2)}(z;q)}\right) \geq \beta \tag{3.3}$$

for all $z$ in the corresponding open disk, and that these inequalities fail in every larger disk. Under the assumptions $\alpha \geq \frac{1}{2}$ and $-1 < x, y < 0$, the function $\varphi_{x,y}(z;q)$ belongs to the Laguerre–Pólya class and therefore possesses infinitely many real and simple zeros only. If $\theta_n(x,y,q)$ denotes the $n$th positive zero of $\varphi_{x,y}$, then $\varphi_{x,y}$ admits the Hadamard factorization given in (2.2). Taking the logarithmic derivative of (2.2), we obtain

$$\frac{\varphi_{x,y}'(z;q)}{\varphi_{x,y}(z;q)} = -\sum_{n\geq 1}\frac{2z}{\theta_n^2(x,y,q) - z^2},$$

and consequently

$$z\frac{\big(g^{(1)}_{x,y}\big)'(z;q)}{g^{(1)}_{x,y}(z;q)} = 1 + z\frac{\varphi'_{x,y}(\sqrt{z};q)}{\varphi_{x,y}(\sqrt{z};q)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\theta_n^2(x,y,q)-z^2},$$
$$z\frac{\big(g^{(2)}_{x,y}\big)'(z;q)}{g^{(2)}_{x,y}(z;q)} = 1 + \frac{\sqrt{z}}{2}\frac{\varphi'_{x,y}(\sqrt{z};q)}{\varphi_{x,y}(\sqrt{z};q)} = 1 - \sum_{n\geq 1}\frac{z}{\theta_n^2(x,y,q)-z}. \tag{3.4}$$

Now let $|z| = r < \theta_1(x,y,q)$. By means of the Lemma 3.2, it follows from (3.4) that

$$\mathfrak{Re}\left(z\frac{\big(g^{(1)}_{x,y}\big)'(z;q)}{g^{(1)}_{x,y}(z;q)}\right) \geq r\frac{\big(g^{(1)}_{x,y}\big)'(r;q)}{g^{(1)}_{x,y}(r;q)} \quad \text{and} \quad \mathfrak{Re}\left(z\frac{\big(g^{(2)}_{x,y}\big)'(z;q)}{g^{(2)}_{x,y}(z;q)}\right) \geq r\frac{\big(g^{(2)}_{x,y}\big)'(r;q)}{g^{(2)}_{x,y}(r;q)},$$

with equality if and only if $z = r > 0$. Hence, by the minimum principle for harmonic functions, the inequality

$$\mathfrak{Re}\left(z\frac{\big(g^{(1)}_{x,y}\big)'(z;q)}{g^{(1)}_{x,y}(z;q)}\right) > \beta \quad \text{and} \quad \mathfrak{Re}\left(z\frac{\big(g^{(2)}_{x,y}\big)'(z;q)}{g^{(2)}_{x,y}(z;q)}\right) > \beta,$$

holds in $|z| < r$ if and only if

$$r\frac{\big(g^{(1)}_{x,y}\big)'(r;q)}{g^{(1)}_{x,y}(r;q)} > \beta \quad \text{and} \quad r\frac{\big(g^{(2)}_{x,y}\big)'(r;q)}{g^{(1)}_{x,y}(r;q)} > \beta.$$

Using the definitions of $g^{(1)}_{x,y}$ and $g^{(2)}_{x,y}$, this conditions are equivalent to

$$1 + r\frac{\varphi'_{x,y}(r;q)}{\varphi_{x,y}(r;q)} > \beta,\ 1 + \frac{\sqrt{r}}{2}\frac{\varphi'_{x,y}(\sqrt{r};q)}{\varphi_{x,y}(\sqrt{r};q)} > \beta,$$

or, equivalently,

$$r\varphi'_{x,y}(r;q) + (1-\beta)\varphi_{x,y}(r;q) > 0,$$

and

$$\sqrt{r}\,\varphi'_{x,y}(\sqrt{r};q) + 2(1-\beta)\varphi_{x,y}(\sqrt{r};q) > 0,$$

respectively.

In order to determine the exact radius, consider the auxiliary functions

$$\mathcal{J}(r) := r\frac{\big(g^{(1)}_{x,y}\big)'(r;q)}{g^{(1)}_{x,y}(r;q)} = 1 - \sum_{n\geq 1}\frac{2r^2}{\theta_n^2(x,y,q)-r^2}, \qquad 0 < r < \theta_1(x,y,q)$$

and

$$\mathcal{K}(r) := r\frac{\big(g^{(2)}_{x,y}\big)'(r;q)}{g^{(2)}_{x,y}(r;q)} = 1 - \sum_{n\geq 1}\frac{r}{\theta_n^2(x,y,q)-r}, \qquad 0 < r < \theta_1^2(x,y,q).$$

A direct differentiation yields

$$\mathcal{J}'(r) = -\sum_{n\geq 1}\frac{4r\,\theta_n^2(x,y,q)}{\big(\theta_n^2(x,y,q)-r^2\big)^2} < 0,\ \mathcal{K}'(r) = -\sum_{n\geq 1}\frac{\theta_n^2(x,y,q)}{\big(\theta_n^2(x,y,q)-r\big)^2} < 0,$$

so that $\mathcal{J}$ and $\mathcal{K}$ are strictly decreasing on $(0,\theta_1(x,y,q))$ and $(0,\theta_1^2(x,y,q))$, respectively. Moreover,

$$\lim_{r\searrow 0}\mathcal{J}(r) = \lim_{r\searrow 0}\mathcal{K}(r) = 1, \qquad \lim_{r\nearrow\theta_1(x,y,q)}\mathcal{J}(r) = \lim_{r\nearrow\theta_1^2(x,y,q)}\mathcal{K}(r) = -\infty.$$

Therefore, for each $\beta \in [0,1)$, the equations $\mathcal{J}(r) = \beta$ and $\mathcal{K}(r) = \beta$ have a unique solution in $(0, \theta_1^2(x,y,q))$ and $(0, \theta_1(x,y,q))$, respectively. Denoting this solution by

$$\jmath_1(x,y,q) \quad \text{and} \quad \mathfrak{k}_1(x,y,q),$$

we conclude that

$$r_\star^\beta\Big(g_{x,y}^{(1)}\Big) = \jmath_1(x,y,q) \quad \text{and} \quad r_\star^\beta\Big(g_{x,y}^{(2)}\Big) = \mathfrak{k}_1(x,y,q),$$

where $\jmath_1(x,y,q)$ and $\mathfrak{k}_1(x,y,q)$ are the smallest positive root of

$$r\varphi'_{x,y}(r;q) + (1-\beta)\varphi_{x,y}(r;q) = 0$$

and

$$\sqrt{r}\,\varphi'_{x,y}(\sqrt{r};q) + 2(1-\beta)\varphi_{x,y}(\sqrt{r};q) = 0,$$

respectively.

This completes the proof. □

### 3.2 Radii of convexity of order $\beta$ of the main functions

In this subsection, we identify the exact radii of starlikeness for the normalized families $f_{x,y}^{(i)}$ and $g_{x,y}^{(i)}$. By leveraging the Hadamard product representations established in the previous sections, we characterize these radii as the smallest positive zeros of certain transcendental equations.

**Theorem 3.3.** *Assume that $\alpha$ is positive constant, $0 < x, y < 1$ and $\beta \in [0,1)$. Under these assumptions, the following assertions are valid:*

*i. The radius of convexity of order $\beta$ for the function $f_{x,y}^{(1)}(z;q)$, denoted by $r_\beta^c\Big(f_{x,y}^{(1)}\Big)$, is the smallest positive root of the following transcendental equation:*

$$2r^2\phi''_{x,y}(r;q) + (5-\beta)r\phi'_{x,y}(r;q) + (1-\beta)\phi_{x,y}(r;q) = 0.$$

*ii. The radius of convexity of order $\beta$ for the function $f_{x,y}^{(2)}(z;q)$, denoted by $r_\beta^c\Big(f_{x,y}^{(2)}\Big)$, is the smallest positive root of the following transcendental equation:*

$$r\,\phi''_{x,y}(\sqrt{r};q) + (4-\beta)\sqrt{r}\,\phi'_{x,y}(\sqrt{r};q) + 2(1-\beta)\phi_{x,y}(\sqrt{r};q) = 0.$$

*Proof.* Since $f_{x,y}^{(1)}(z;q)$ and $f_{x,y}^{(2)}(z;q)$ belong to the Laguerre–Pólya class $\mathcal{LP}$, their derivatives $\big(f_{x,y}^{(1)}(z;q)\big)'$ and $\big(f_{x,y}^{(2)}(z;q)\big)'$ also lie in $\mathcal{LP}$. Moreover, these functions have the same growth order. Hence, by Hadamard's factorization theorem, the derivatives admit the following Weierstrass canonical product representations

$$\big(f_{x,y}^{(1)}(z;q)\big)' = \prod_{n\geq 1}\left(1 - \frac{z^2}{\xi_n^2(x,y,q)}\right) \quad \text{and} \quad \big(f_{x,y}^{(2)}(z;q)\big)' = \prod_{n\geq 1}\left(1 - \frac{z}{\zeta_n(x,y,q)}\right), \tag{3.5}$$

where $\xi_n(x,y,q)$ and $\zeta_n(x,y,q)$ denote the $n$th positive zeros of $\big(f_{x,y}^{(1)}(z;q)\big)'$ and $\big(f_{x,y}^{(2)}(z;q)\big)'$, respectively.

The logarithmic differentiation on both sides of the above relations yields

$$1 + z\frac{\big(f_{x,y}^{(1)}(z;q)\big)''}{\big(f_{x,y}^{(1)}(z;q)\big)'} = 1 - 2\sum_{n\geq 1}\frac{z^2}{\xi_n^2(x,y,q) - z^2}$$

and

$$1+z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}=1-\sum_{n\geq 1}\frac{z}{\zeta_n(x,y,q)-z}.$$

Application of inequality 3.2 implies that

$$\mathfrak{Re}\left(1+\frac{z\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right)\geq 1-2\sum_{n\geq 1}\frac{r^2}{\xi_n^2(x,y,q)-r^2},\qquad |z|=r<\xi_1(x,y,q)$$

and

$$\mathfrak{Re}\left(1+\frac{z\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right)\geq 1-\sum_{n\geq 1}\frac{r}{\zeta_n(x,y,q)-r},\qquad |z|=r<\zeta_1(x,y,q).$$

Therefore, we get that

$$\inf_{|z|\leq r}\mathfrak{Re}\left(1+z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right)=1+\frac{r\left(f_{x,y}^{(1)}(r;q)\right)''}{\left(f_{x,y}^{(1)}(r;q)\right)'}$$

and

$$\inf_{|z|\leq r}\mathfrak{Re}\left(1+z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right)=1+\frac{r\left(f_{x,y}^{(2)}(r;q)\right)''}{\left(f_{x,y}^{(2)}(r;q)\right)'}.$$

Now, let us define the following functions

$$\Psi_1(r)=1+\frac{r\left(f_{x,y}^{(1)}(r;q)\right)''}{\left(f_{x,y}^{(1)}(r;q)\right)'}=1-2\sum_{n\geq 1}\frac{r^2}{\xi_n^2(x,y,q)-r^2},\qquad 0<r<\xi_1(x,y,q)$$

and

$$\Psi_2(r)=1+\frac{r\left(f_{x,y}^{(2)}(r;q)\right)''}{\left(f_{x,y}^{(2)}(r;q)\right)'}=1-\sum_{n\geq 1}\frac{r}{\zeta_n(x,y,q)-r},\qquad 0<r<\zeta_1(x,y,q).$$

To determine the radius of convexity of order $\beta$, the problem reduces to solving the equation obtained from the extremal value of the functional $1+z\frac{f''(z)}{f'(z)}$. Indeed, the boundary estimates derived above show that the convexity condition $\mathfrak{Re}\left(1+z\frac{f''(z)}{f'(z)}\right)>\beta$ is equivalent to the inequality $\Psi(r)>\beta$ for the auxiliary functions $\Psi_1(r)$ and $\Psi_2(r)$. Therefore, in order to determine the corresponding radii it suffices to study the behavior of these functions and establish the existence and uniqueness of the solutions of the equations $\Psi_1(r)=\beta$ and $\Psi_2(r)=\beta$. Then

$$\Psi_1'(r)=-2\sum_{n\geq 1}\frac{d}{dr}\left(\frac{r^2}{\xi_n^2(x,y,q)-r^2}\right)=-4r\sum_{n\geq 1}\frac{\xi_n^2(x,y,q)}{\left(\xi_n^2(x,y,q)-r^2\right)^2}.$$

In particular, $\Psi_1'(r)<0,\qquad 0<r<\xi_1(x,y,q)$, and therefore $\Psi_1$ is strictly decreasing on $(0,\xi_1(x,y,q))$. Moreover, one can obtain

$$\lim_{r\searrow 0}\Psi_1(r)=1,\qquad \lim_{r\nearrow\xi_1(x,y,q)}\Psi_1(r)=-\infty.$$

Therefore, for each $\beta\in[0,1)$, the equation $\Psi_1(r)=\beta$ has a unique root in $(0,\xi_1(x,y,q))$, and this root is precisely $r_\beta^c(f_{x,y}^{(1)})$.

Similarly, we obtain by termwise differentiation

$$\Psi_2'(r)=-\sum_{n\geq 1}\frac{d}{dr}\left(\frac{r}{\zeta_n(x,y,q)-r}\right)=-\sum_{n\geq 1}\frac{\zeta_n(x,y,q)}{\left(\zeta_n(x,y,q)-r\right)^2}.$$

Hence, $\Psi_2'(r) < 0, \quad 0 < r < \zeta_1(x,y,q)$, and therefore $\Psi_2$ is strictly decreasing on $(0, \zeta_1(x,y,q))$. Moreover, we observe that

$$\lim_{r \searrow 0} \Psi_2(r) = 1 \quad \text{and} \quad \lim_{r \nearrow \zeta_1(x,y,q)} \Psi_2(r) = -\infty.$$

Consequently, the equation $\Psi_2(r) = \beta$ has a unique root in $(0, \zeta_1(x,y,q))$, which is exactly $r^c_\beta(f^{(2)}_{x,y})$. It is readily verified that

$$1 + 2r\frac{\big(f^{(1)}_{x,y}(r;q)\big)''}{\big(f^{(1)}_{x,y}(r;q)\big)'} = 1 + \frac{2r\big(2\phi'_{x,y}(r;q) + r\phi''_{x,y}(r;q)\big)}{\phi_{x,y}(r;q) + r\phi'_{x,y}(r;q)}.$$

and

$$1 + 2r\frac{\big(f^{(2)}_{x,y}(r;q)\big)''}{\big(f^{(2)}_{x,y}(r;q)\big)'} = 1 + \frac{\frac{3\sqrt{r}}{2}\phi'_{x,y}(\sqrt{r};q) + \frac{r}{2}\phi''_{x,y}(\sqrt{r};q)}{\phi_{x,y}(\sqrt{r};q) + \frac{\sqrt{r}}{2}\phi'_{x,y}(\sqrt{r};q)}.$$

This completes the proof. □

**Theorem 3.4.** *Assume that $\alpha$ is positive constant, $0 < x, y < 1$ and $\beta \in [0,1)$. Under these assumptions, the following assertions are valid:*

*i. The convexity radius of order $\beta$ for the function $g^{(1)}_{x,y}(z;q)$, say $r^c_\beta\Big(g^{(1)}_{x,y}\Big)$, is the smallest positive root of the associated transcendental equation:*

$$1 + \frac{2r\,\varphi'_{x,y}(r;q) + r^2\,\varphi''_{x,y}(r;q)}{\varphi_{x,y}(r;q) + r\,\varphi'_{x,y}(r;q)} = \beta.$$

*ii. The convexity radius of order $\beta$ for the function $g^{(2)}_{x,y}(z;q)$, say $r^c_\beta\Big(g^{(2)}_{x,y}\Big)$, is the smallest positive root of the associated transcendental equation:*

$$1 + \frac{3\sqrt{r}\,\varphi'_{x,y}(\sqrt{r};q) + r\,\varphi''_{x,y}(\sqrt{r};q)}{2\Big(2\,\varphi_{x,y}(\sqrt{r};q) + \sqrt{r}\,\varphi'_{x,y}(\sqrt{r};q)\Big)} = \beta.$$

*Proof.* The assertions follow by combining the Hadamard factorization of the underlying entire functions with logarithmic differentiation. In the case of $g^{(1)}_{x,y}$, the symmetry of the zero set implies that the logarithmic derivative involves quadratic terms, which permits a direct application of classical real part estimates and radial minimization. On the other hand, the zero distribution of $g^{(2)}_{x,y}$ is not symmetric, and consequently the logarithmic derivative contains linear terms. This lack of symmetry prevents a direct reduction to the radial case and requires a more delicate analysis based on auxiliary functions and their monotonicity properties. As these arguments follow the same lines as in the proof of Theorem 3.3, we omit the details. □

### 3.3 Radii of uniform convexity of the main functions

In this subsection, we determine the radii of uniform convexity for the normalized functions $f^{(1)}_{x,y}$, $f^{(2)}_{x,y}$, $g^{(1)}_{x,y}$, and $g^{(2)}_{x,y}$. The analysis is carried out via the logarithmic differentiation of their Hadamard product representations together with sharp real part estimates. A crucial role is played by the zero distributions of the underlying entire functions, which lead to different analytic structures in the associated expressions and, consequently, require distinct approaches in each case.

**Theorem 3.5.** *Assume that $\alpha$ is positive constant and $0 < x, y < 1$. Under these assumptions, the following assertions are valid:*

*i. The uniform convexity radius of the function $f_{x,y}^{(1)}(z;q)$, say $r^{uc}\left(f_{x,y}^{(1)}\right)$, is the smallest positive root of the associated transcendental equation:*

$$\phi_{x,y}(r;q) + 5r\,\phi'_{x,y}(r;q) + 2r^2\,\phi''_{x,y}(r;q) = 0.$$

*ii. The uniform convexity radius of the function $f_{x,y}^{(2)}(z;q)$, say $r^{uc}\left(f_{x,y}^{(2)}\right)$, is the smallest positive root of the associated transcendental equation:*

$$2\phi_{x,y}(\sqrt{r};q) + 4\sqrt{r}\,\phi'_{x,y}(\sqrt{r};q) + r\,\phi''_{x,y}(\sqrt{r};q) = 0.$$

*Proof.* By using the logarithmic derivative of the functions $z \mapsto \mathrm{d}\left(f_{x,y}^{(1)}(z;q)\right)/\mathrm{d}z$ and $z \mapsto \left(\mathrm{d}f_{x,y}^{(2)}(z;q)\right)/\mathrm{d}z$, which are given by (3.5), we get

$$z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'} = -\sum_{n\geq 1}\frac{2z^2}{\xi_n^2(x,y,q) - z^2} \tag{3.6}$$

and

$$z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'} = -\sum_{n\geq 1}\frac{z}{\zeta_n(x,y,q) - z}. \tag{3.7}$$

By using inequality (3.2) in equalities (3.6) and (3.7), respectively, implies that for $|z| \leq \xi_1(x,y,q)$

$$\mathfrak{Re}\left(z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right) \geq 1 + r\frac{\left(f_{x,y}^{(1)}(r;q)\right)''}{\left(f_{x,y}^{(1)}(r;q)\right)'}, \quad \left|z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right| \leq -r\frac{\left(f_{x,y}^{(1)}(r;q)\right)''}{\left(f_{x,y}^{(1)}(r;q)\right)'} \tag{3.8}$$

and for $|z| \leq \zeta_1(x,y,q)$

$$\mathfrak{Re}\left(z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right) \geq 1 + r\frac{\left(f_{x,y}^{(2)}(r;q)\right)''}{\left(f_{x,y}^{(2)}(r;q)\right)'}, \quad \left|z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right| \leq -r\frac{\left(f_{x,y}^{(2)}(r;q)\right)''}{\left(f_{x,y}^{(2)}(r;q)\right)'} \tag{3.9}$$

From inequalities (3.8) and (3.9), we deduce

$$\mathfrak{Re}\left(z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right) - \left|z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right| \geq 1 + 2r\frac{\left(f_{x,y}^{(1)}(r;q)\right)''}{\left(f_{x,y}^{(1)}(r;q)\right)'}, \quad |z| \leq \xi_1(x,y,q) \tag{3.10}$$

and

$$\mathfrak{Re}\left(z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right) - \left|z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right| \geq 1 + 2r\frac{\left(f_{x,y}^{(2)}(r;q)\right)''}{\left(f_{x,y}^{(2)}(r;q)\right)'}, \quad |z| \leq \zeta_1(x,y,q) \tag{3.11}$$

The equalities hold in (3.10) and (3.10) if and only if $|z| = r$ . As a result, we have

$$\inf_{z\in\mathbb{D}_{\xi_1(x,y,q)}}\left\{\mathfrak{Re}\left(z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right) - \left|z\frac{\left(f_{x,y}^{(1)}(z;q)\right)''}{\left(f_{x,y}^{(1)}(z;q)\right)'}\right|\right\} = 1 + 2r\frac{\left(f_{x,y}^{(1)}(r;q)\right)''}{\left(f_{x,y}^{(1)}(r;q)\right)'}$$

$$\inf_{z\in\mathbb{D}_{\zeta_1(x,y,q)}}\left\{\mathfrak{Re}\left(z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right) - \left|z\frac{\left(f_{x,y}^{(2)}(z;q)\right)''}{\left(f_{x,y}^{(2)}(z;q)\right)'}\right|\right\} = 1 + 2r\frac{\left(f_{x,y}^{(2)}(r;q)\right)''}{\left(f_{x,y}^{(2)}(r;q)\right)'}.$$

Now consider the functions $\Psi^{(1)}_{x,y} : (0, \xi_1(x,y,q)) \to \mathbb{R}$

$$\Psi^{(1)}_{x,y}(r;q) = 1 + 2r\frac{\left(f^{(1)}_{x,y}(r;q)\right)''}{\left(f^{(1)}_{x,y}(r;q)\right)'}$$

and $\Psi^{(2)}_{x,y} : (0, \zeta_1(x,y,q)) \to \mathbb{R}$

$$\Psi^{(2)}_{x,y}(r;q) = 1 + 2r\frac{\left(f^{(2)}_{x,y}(r;q)\right)''}{\left(f^{(2)}_{x,y}(r;q)\right)'}.$$

It is clear that these functions, that is $\Psi^{(1)}_{x,y}$ and $\Psi^{(2)}_{x,y}$, are strictly decreasing over their domains. In addition, we get the following limits

$$\lim_{r\nearrow\xi_1(x,y,q)} \Psi^{(1)}_{x,y}(r;q) = \lim_{r\nearrow\zeta_1(x,y,q)} \Psi^{(2)}_{x,y}(r;q) = -\infty,$$

$$\lim_{r\searrow 0} \Psi^{(1)}_{x,y}(r;q) = \lim_{r\searrow 0} \Psi^{(2)}_{x,y}(r;q) = 1.$$

Therefore, the equations

$$1 + 2r\frac{\left(f^{(1)}_{x,y}(r;q)\right)''}{\left(f^{(1)}_{x,y}(r;q)\right)'} = 0, \quad 1 + 2r\frac{\left(f^{(2)}_{x,y}(r;q)\right)''}{\left(f^{(2)}_{x,y}(r;q)\right)'} = 0$$

have a unique roots $r_1 \in (0, \xi_1(x,y,q))$ and $r_2 \in (0, \zeta_1(x,y,q))$, repectively, which means that $r_1 = r^{uc}\left(f^{(1)}_{x,y}(z;q)\right)$ and $r_2 = r^{uc}\left(f^{(2)}_{x,y}(z;q)\right)$. By using the first and second derivatives of the functions $z \mapsto f^{(1)}_{x,y}(z;q)$ and $z \mapsto f^{(2)}_{x,y}(z;q)$, one can easily see that

$$\phi_{x,y}(r;q) + 5r\,\phi'_{x,y}(r;q) + 2r^2\,\phi''_{x,y}(r;q) = 0$$

and

$$2\phi_{x,y}(\sqrt{r};q) + 4\sqrt{r}\,\phi'_{x,y}(\sqrt{r};q) + r\,\phi''_{x,y}(\sqrt{r};q) = 0.$$

□

**Theorem 3.6.** *Assume that $\alpha \geq \frac{1}{2}$ is a constant and $-1 < x, y < 0$. Under these assumptions, the following assertions are valid:*

*i. The uniform convexity radius of the function $g^{(1)}_{x,y}(z;q)$, say $r^{uc}\left(g^{(1)}_{x,y}\right)$, is the smallest positive root of the associated transcendental equation:*

$$\varphi_{x,y}(r;q) + 5r\,\varphi'_{x,y}(r;q) + 2r^2\,\varphi''_{x,y}(r;q) = 0.$$

*ii. The uniform convexity radius of the function $g^{(2)}_{x,y}(z;q)$, say $r^{uc}\left(g^{(2)}_{x,y}\right)$, is the smallest positive root of the associated transcendental equation:*

$$2\varphi_{x,y}(\sqrt{r};q) + 4\sqrt{r}\,\varphi'_{x,y}(\sqrt{r};q) + r\,\varphi''_{x,y}(\sqrt{r};q) = 0.$$

*Proof.* The proof follows a similar line of reasoning as presented in the proof of Theorem 3.5. By applying the analytic characterization of the radius of uniform convexity to the functions $g^{(1)}_{x,y}(z;q)$ and $g^{(2)}_{x,y}(z;q)$, and following analogous algebraic manipulations to those detailed previously, one arrives at the transcendental equations provided in assertions (i) and (ii), respectively. To avoid unnecessary repetition, the remaining details are omitted. □

# 4 Numerical Estimates and Geometric Bounds

While the exact radii for starlikeness, convexity, and uniform convexity determined in the preceding sections are implicitly defined as roots of transcendental equations, they may be difficult to evaluate directly. In this section, we derive explicit lower and upper bounds for these values using the first few terms of the Euler–Rayleigh sums, thereby providing tight and more accessible estimates for practical applications. Furthermore, these results are supported by visual representations to demonstrate the precision of our bounds.

## 4.1 Geometric bounds for the radii of starlikeness of the main entire functions

This subsection is dedicated to deriving tight numerical estimates for the starlikeness radii of the considered functions. Using a rigorous method based on Rayleigh sums and Euler–Rayleigh inequalities, we identify lower and upper bounds that closely approximate the exact radii without requiring the direct solution of complex transcendental equations. To complement the analytical findings, we include comparative plots that visually confirm the starlikeness limits and demonstrate the efficiency of these estimates.

**Theorem 4.1.** *If $\alpha$ is positive constant and $0 < x, y < 1$, then the following assertions hold true:*

*i. The radius of starlikeness of the function $z \mapsto f^{(1)}_{x,y}(z;q) := z \cdot \phi_{x,y}(z;q)$, denoted by $r_\star\left(f^{(1)}_{x,y}(z;q)\right)$, satisfies the inequalities*

$$\frac{1-q}{3(x+y)\,q^{\alpha}} < r^2_\star\left(f^{(1)}_{x,y}(z;q)\right) < \frac{3(x+y)\,(1-q)\,(1-q^2)\,q^{\alpha}}{9(x+y)^2\,(1-q^2)\,q^{2\alpha} - 10(1-q)\,[x^2+(1+q)\cdot x\cdot y+y^2]\,q^{4\alpha}}.$$

*ii. The radius of starlikeness of the function $z \mapsto f^{(2)}_{x,y}(z;q) := z \cdot \phi_{x,y}(\sqrt{z};q)$, denoted by $r_\star\left(f^{(2)}_{x,y}(z;q)\right)$, satisfies the inequalities*

$$\frac{1-q}{2q^{\alpha}(x+y)} < r_\star\left(f^{(2)}_{x,y}(z;q)\right) < \frac{(1-q^2)(x+y)}{2(1+q)q^{\alpha}(x+y)^2 - 3q^{3\alpha}\left(x^2+(1+q)xy+y^2\right)}.$$

*Proof.* i. Substituting $\beta = 0$ into Theorem 3.1 shows that the radius of starlikeness of order zero for the function $z \mapsto f^{(1)}_{x,y}(z;q)$ is given by the smallest positive root of the derivative

$$(z \cdot \phi_{x,y}(z;q))' = 0.$$

Hence, the problem reduces to locating the first positive zero of the power series

$$\begin{aligned}\vartheta_{x,y}(z;q) &= (z \cdot \phi_{x,y}(z;q))' \\ &= \sum_{n\geq 0}(-1)^n \frac{(2n+1)\cdot h_n(x,y \mid q)}{(q;q)_n} q^{\alpha n^2} z^{2n}.\end{aligned} \tag{4.1}$$

By Lemma 2.1, the function $z \mapsto \phi_{x,y}(z;q)$ is in the Laguerre–Pólya class $\mathcal{LP}$. Moreover, $\mathcal{LP}$ is invariant under differentiation, so $z \mapsto \vartheta_{x,y}(z;q)$ is in $\mathcal{LP}$ as well, implying it has infinitely many real zeros. As the function is entire of order zero, it admits the canonical product representation

$$\vartheta_{x,y}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\theta^2_n(x,y,q)}\right), \tag{4.2}$$

where $\theta_n(x,y,q)$ is the $n$-th positive zero.

Applying the logarithmic derivative to both sides of (4.2) yields, for $|z| < \theta_1(x,y,q)$,

$$\begin{aligned}\frac{z\cdot\vartheta'_{x,y}(z;q)}{\vartheta_{x,y}(z;q)} &= -\sum_{n\geq 1}\frac{2z^2}{\theta_n^2(x,y,q)-z^2} = -2\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^{2k+2}}{\theta_n^{2k+2}(x,y,q)} \\ &= -2\sum_{k\geq 0}\sum_{n\geq 1}\frac{z^{2n+2}}{\theta_n^{2k+2}(x,y,q)} = -2\sum_{k\geq 0}\kappa_{k+1}z^{2k+2},\end{aligned} \tag{4.3}$$

where $\kappa_k = \sum_{n\geq 1}\theta_n^{-2k}(x,y,q)$ are the Rayleigh sums. Alternatively, using the series expansion, we have

$$\frac{z\cdot\vartheta'_{x,y}(z;q)}{\vartheta_{x,y}(z;q)} = \frac{\sum_{n\geq 0}a_n z^{2n+2}}{\sum_{n\geq 0}b_n z^{2n}}, \tag{4.4}$$

with

$$\begin{aligned}a_n &= (2n+2)(-1)^{n+1}\frac{(2n+3)\,h_{n+1}(x,y\mid q)}{(q;q)_{n+1}}\,q^{\alpha(n+1)^2}, \\ b_n &= (-1)^n\,\frac{(2n+1)\,h_n(x,y\mid q)}{(q;q)_n}\,q^{\alpha n^2}.\end{aligned}$$

As a result of comparing the coefficients of Equation (4.3) and Equation (4.4), we arrive at

$$\kappa_1 = \frac{3}{2}\cdot\frac{(x+y)\,q^\alpha}{1-q}, \quad \kappa_2 = \frac{1}{2}\cdot\Big[\frac{9(x+y)^2\,q^{2\alpha}}{(1-q)^2} - \frac{10\big[x^2+(1+q)x\cdot y+y^2\big]\,q^{4\alpha}}{(1-q)(1-q^2)}\Big].$$

To derive explicit bounds, we restrict our attention to the first Euler–Rayleigh sum (i.e., $k=1$), since higher-order estimates lead to significantly more involved expressions without substantial improvement in clarity. Applying the Euler–Rayleigh inequalities

$$\kappa_k^{-1/k} < \theta_1^2(x,y,q) < \frac{\kappa_k}{\kappa_{k+1}},$$

with $k=1$, we obtain the bounds

$$\frac{1-q}{3(x+y)\,q^\alpha} < \theta_1^2(x,y,q) < \frac{3(x+y)\,(1-q)\,(1-q^2)\,q^\alpha}{9(x+y)^2\,(1-q^2)\,q^{2\alpha} - 10(1-q)\,[x^2+(1+q)xy+y^2]\,q^{4\alpha}}.$$

ii. Consider the entire function

$$\varsigma_{x,y}(z;q) = \sum_{n\geq 0}(-1)^n\frac{(n+1)\cdot h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n. \tag{4.5}$$

Since the function

$$\sum_{n\geq 0}\frac{h_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}z^n$$

possess infinitely many real zeros and each of them is negative for the conditions of the theorem (see [12, Theorem 1.2]) it is obvious that the function $z\mapsto\varsigma_{x,y}(z;q)$ is of infinitely many real zeros and all of them are positive. Hence, the function $z\mapsto\varsigma_{x,y}(z;q)$ belongs to the Lagurre–Pólya class $\mathcal{LP}$. Moreover, the growth order of the function $z\mapsto\varsigma_{x,y}(z;q)$, say $\rho(\varsigma_{x,y}(z;q))$, is equal to 0. Suppose that $\gamma_n(x,y,q)$ is the $n$-th pozitive zero of $\varsigma_{x,y}(z;q)$. Hence, the function $\varsigma_{x,y}(z;q)$ admits the infinite product representation

$$\varsigma_{x,y}(z;q) = \prod_{n\geq 1}\left(1-\frac{z}{\gamma_n(x,y,q)}\right). \tag{4.6}$$

Applying the logarithmic derivative to (4.6), we obtain, for $|z| < \gamma_1(x, y, q)$,

$$\begin{aligned}\frac{\varsigma'_{x,y}(z;q)}{\varsigma_{x,y}(z;q)} &= -\sum_{n\geq 1}\frac{1}{\gamma_n(x,y,q)-z} = -\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^k}{\gamma_n^{k+1}(x,y,q)} \\ &= -\sum_{k\geq 0}\sum_{n\geq 1}\frac{z^k}{\gamma_n^{k+1}(x,y,q)} = -\sum_{k\geq 0}\xi_{k+1}z^k,\end{aligned} \tag{4.7}$$

where $\xi_k = \sum_{n\geq 1}\gamma_n^{-k}(x, y, q)$ is the $k$-th Rayleigh sum.

On the other hand, expanding the series yields

$$\frac{\varsigma'_{x,y}(z;q)}{\varsigma_{x,y}(z;q)} = \sum_{n\geq 0}a_n z^n \Big/ \sum_{n\geq 0}b_n z^n, \tag{4.8}$$

where

$$a_n = (-1)^{n+1}\frac{(n+1)(n+2)q^{\alpha(n+1)^2}h_{n+1}(x,y|q)}{(q;q)_{n+1}} \quad \text{and} \quad b_n = (-1)^n\frac{(n+1)q^{\alpha n^2}h_n(x,y|q)}{(q;q)_n}.$$

As a result of comparing the coefficients of Equation (4.7) and Equation (4.8), we obtain the following Rayleigh sums:

$$\xi_1 = \frac{2\,q^\alpha(x+y)}{1-q} \quad \text{and} \quad \xi_2 = \frac{4\,q^{2\alpha}(x+y)^2}{(1-q)^2} - \frac{6\,q^{4\alpha}\big(x^2+(1+q)xy+y^2\big)}{(1-q)(1-q^2)}.$$

By applying the Euler–Rayleigh inequalities to the coefficients of the power series expansion of $f^{(2)}(z)$, we consider the following bounds for the smallest positive zero $\gamma_1(x, y, q)$:

$$\xi_k^{-1/k} < \gamma_1(x,y,q) < \frac{\xi_k}{\xi_{k+1}},$$

where $\xi_n$ denotes the Rayleigh sum of order $k$. By evaluating these inequalities for the initial terms of the series and performing the necessary algebraic simplifications, we obtain the following inclusion interval for the radius of starlikeness:

$$\frac{1-q}{2q^\alpha(x+y)} < \gamma_1(x,y,q) < \frac{(1-q^2)(x+y)}{2(1+q)q^\alpha(x+y)^2 - 3q^{3\alpha}\big(x^2+(1+q)xy+y^2\big)}.$$

This two-sided estimate provides a sharp analytical characterization of the starlikeness domain, effectively completing the verification of the geometric properties of the operator.

Consequently, the proof is hereby concluded. □

*Remark* 4.2. The theoretical radii of starlikeness established in Theorem 4.1 are numerically substantiated in Figures 1 and 2. For the specified parameter set ($x = 0.30, y = 0.20, q = 0.60, \alpha = 1.0$), the simulations confirm that the image domains $\Omega^{(1)}$ and $\Omega^{(2)}$ satisfy Alexander's condition $\mathrm{Re}\{zf'/f\} > 0$ within the verified radii $r_\star$.

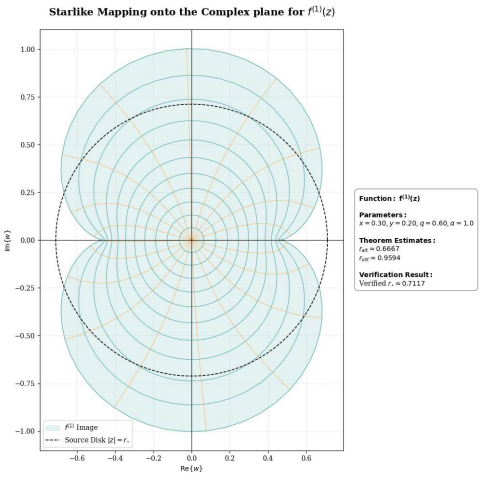



Figure 1: Geometric characterization of $f^{(1)}(z)$ with $r_\star \approx 0.7117$.

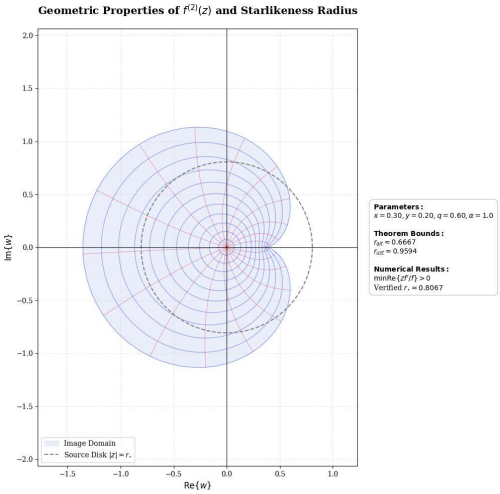



Figure 2: Starlike mapping of $f^{(2)}(z)$ with $r_\star \approx 0.8067$.

The conformal mappings illustrate that both operators maintain a strictly starlike geometry, with radial projections remaining entirely within the transformed boundaries. The alignment between the analytical bounds and the numerical results ($r_\star \approx 0.7117$ for $f^{(1)}$ and $r_\star \approx 0.8067$ for $f^{(2)}$) underscores the tightness of the derived inequalities and highlights the distinct expansion characteristics of the two $q$-operators.

**Theorem 4.3.** *If $\alpha \geq \frac{1}{2}$ and $-1 < x, y < 0$, then the following assertions hold true:*

*i. The radius of starlikeness of the function $z \mapsto g^{(1)}_{x,y}(z;q) := z \cdot \varphi_{x,y}(z;q)$, denoted by $r_\star\left(g^{(1)}_{x,y}(z;q)\right)$, satisfies the inequalities*

$$\frac{q-1}{3\,q^{\alpha}(x+y)} < r_\star^2\left(f^{(1)}_{x,y}(z;q)\right) < \frac{3\,q^{\alpha}(q^2-1)(x+y)}{9\,q^{2\alpha}(1+q)(x+y)^2 - 10\,q^{4\alpha}\left(x^2+y^2+\frac{1+q}{q}xy\right)}.$$

*ii. The radius of starlikeness of the function $z \mapsto g^{(2)}_{x,y}(z;q) := z \cdot \varphi_{x,y}(z;q)$, denoted by $r_\star\left(g^{(2)}_{x,y}(z;q)\right)$, satisfies the inequalities*

$$\frac{q-1}{2q^{\alpha}(x+y)} < r_\star\left(g^{(2)}_{x,y}\right) < \frac{(x+y)(1-q^2)}{2q^{\alpha}(1+q)(x+y)^2 - 3q^{3\alpha}\left(x^2+y^2+\frac{1+q}{q}\,xy\right)}.$$

*Proof.* i. By setting $\beta = 0$ in Theorem 3.1, we infer that the radius of starlikeness of the function

$$z \mapsto g^{(1)}_{x,y}(z;q) = z\,\varphi_{x,y}(z;q)$$

is determined by the smallest positive zero of $\left(g^{(1)}_{x,y}(z;q)\right)' = \left(z\varphi_{x,y}(z;q)\right)'$.

Therefore, it is enough to locate the first positive zero of the entire function

$$\begin{aligned}\Delta_{x,y}(z;q) &:= \left(z\varphi_{x,y}(z;q)\right)' \\ &= \sum_{n\geq 0} \frac{(2n+1)\,g_n(x,y\mid q)}{(q;q)_n} q^{\alpha n^2} z^{2n}.\end{aligned} \tag{4.9}$$

By Lemma 2.3, the function $z \mapsto \varphi_{x,y}(z;q)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$ and has only real and simple zeros. Hence the function

$$g_{x,y}^{(1)}(z;q) = z\,\varphi_{x,y}(z;q)$$

also belongs to $\mathcal{LP}$. Since the Laguerre–Pólya class is closed under differentiation, it follows that

$$z \mapsto \Delta_{x,y}(z;q) = \big(g_{x,y}^{(1)}(z;q)\big)'$$

also lies in $\mathcal{LP}$. Consequently, all zeros of $\Delta_{x,y}(z;q)$ are real, and, by the Laguerre separation theorem, the zeros of $g_{x,y}^{(1)}$ and $\Delta_{x,y}$ strictly interlace. Since $\Delta_{x,y}$ is an even entire function of order zero, it admits the canonical product representation

$$\Delta_{x,y}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\delta_n^2(x,y,q)}\right), \tag{4.10}$$

where $\delta_n(x,y,q)$ denotes the $n$-th positive zero of $\Delta_{x,y}$.

Applying the logarithmic derivative to (4.10), we obtain, for $|z| < \delta_1(x,y,q)$,

$$\begin{aligned}
\frac{z\,\Delta'_{x,y}(z;q)}{\Delta_{x,y}(z;q)} &= -\sum_{n\geq 1}\frac{2z^2}{\delta_n^2(x,y,q) - z^2} \\
&= -2\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^{2k+2}}{\delta_n^{2k+2}(x,y,q)} \\
&= -2\sum_{k\geq 0}\left(\sum_{n\geq 1}\delta_n^{-2k-2}(x,y,q)\right) z^{2k+2} \\
&= -2\sum_{k\geq 0}\upsilon_{k+1} z^{2k+2},
\end{aligned} \tag{4.11}$$

where $\upsilon_k = \sum_{n\geq 1}\delta_n^{-2k}(x,y,q)$ denotes the $k$-th Rayleigh sum.

On the other hand, from (4.9) we have

$$\frac{z\,\Delta'_{x,y}(z;q)}{\Delta_{x,y}(z;q)} = \frac{\sum_{n\geq 0} a_n z^{2n+2}}{\sum_{n\geq 0} b_n z^{2n}}, \tag{4.12}$$

where

$$\begin{aligned}
a_n &= \frac{(2n+2)(2n+3)\,g_{n+1}(x,y\mid q)}{(q;q)_{n+1}} q^{\alpha(n+1)^2}, \\
b_n &= \frac{(2n+1)\,g_n(x,y\mid q)}{(q;q)_n} q^{\alpha n^2}.
\end{aligned}$$

Comparing the coefficients in (4.11) and (4.12), we obtain

$$\upsilon_1 = -\frac{b_1}{b_0} = -\frac{3\,g_1(x,y\mid q)}{1-q}\,q^{\alpha} = -\frac{3(x+y)}{1-q}\,q^{\alpha},$$

and

$$\upsilon_2 = \left(\frac{b_1}{b_0}\right)^2 - 2\frac{b_2}{b_0} = \frac{9(x+y)^2}{(1-q)^2}q^{2\alpha} - \frac{10\,g_2(x,y\mid q)}{(1-q)(1-q^2)}q^{4\alpha}.$$

Since

$$g_2(x,y\mid q) = x^2 + y^2 + \frac{1+q}{q}xy,$$

it follows that

$$\upsilon_2 = \frac{9(x+y)^2}{(1-q)^2}q^{2\alpha} - \frac{10}{(1-q)(1-q^2)}\left(x^2+y^2+\frac{1+q}{q}xy\right)q^{4\alpha}.$$

Finally, by the Euler–Rayleigh inequalities, it follows that

$$\upsilon_k^{-1/k} < \delta_1^2(x,y,q) < \frac{\upsilon_k}{\upsilon_{k+1}}, \qquad k \in \mathbb{N}.$$

Taking $k=1$, we arrive at

$$\frac{1-q}{-3\,q^{\alpha}(x+y)} < \delta_1^2(x,y,q) < \frac{-3\,q^{\alpha}(1-q^2)(x+y)}{9\,q^{2\alpha}(1+q)(x+y)^2 - 10\,q^{4\alpha}\left(x^2+y^2+\frac{1+q}{q}xy\right)}.$$

Since $\delta_1(x,y,q) = r_\star\left(g_{x,y}^{(1)}\right)$, this yields the desired two-sided estimate for the radius of starlikeness.

ii. By setting $\beta = 0$ in Theorem 3.2, we deduce that the radius of starlikeness of the function

$$z \mapsto g_{x,y}^{(2)}(z;q) := z\,\varphi_{x,y}(\sqrt{z};q)$$

is determined by the smallest positive zero of

$$\left(g_{x,y}^{(2)}(z;q)\right)' = 0.$$

Therefore, it is enough to determine the first positive zero of the entire function

$$\Upsilon_{x,y}(z;q) := \left(g_{x,y}^{(2)}(z;q)\right)' = \sum_{n\geq 0}\frac{(n+1)\,g_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}z^n. \tag{4.13}$$

Let

$$\mathcal{G}_{x,y}(z;q) := \sum_{n\geq 0}\frac{g_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}z^n.$$

By the hypothesis and [12, Theorem 1.2], the function $z \mapsto \mathcal{G}_{x,y}(z;q)$ possesses infinitely many real zeros, all of which are positive. Since

$$g_{x,y}^{(2)}(z;q) = z\,\varphi_{x,y}(\sqrt{z};q) = z\,\mathcal{G}_{x,y}(z;q),$$

it follows that $g_{x,y}^{(2)}(z;q)$ has only real zeros, namely 0 together with the positive zeros of $\mathcal{G}_{x,y}$. Hence $g_{x,y}^{(2)}(\,\cdot\,;q)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$. Since $\mathcal{LP}$ is closed under differentiation, the function

$$z \mapsto \Upsilon_{x,y}(z;q) = \left(g_{x,y}^{(2)}(z;q)\right)'$$

also lies in $\mathcal{LP}$. Moreover, by the strict interlacing property of zeros, all zeros of $\Upsilon_{x,y}$ are real and positive. Denote them by $0 < \beta_1(x,y,q) < \beta_2(x,y,q) < \cdots$. Since $\Upsilon_{x,y}$ is entire of order zero, it admits the canonical product representation

$$\Upsilon_{x,y}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z}{\beta_n(x,y,q)}\right). \tag{4.14}$$

Taking the logarithmic derivative of (4.14), we obtain

$$\begin{aligned}\frac{\Upsilon'_{x,y}(z;q)}{\Upsilon_{x,y}(z;q)} &= -\sum_{n\geq 1}\frac{1}{\beta_n(x,y,q)-z}\\ &= -\sum_{k\geq 0}\upsilon_{k+1}z^k,\end{aligned} \tag{4.15}$$

where $\upsilon_k = \sum_{n\geq 1}\beta_n^{-k}(x,y,q)$ are Rayleigh sums. On the other hand,

$$\frac{\Upsilon'_{x,y}(z;q)}{\Upsilon_{x,y}(z;q)} = \frac{\sum_{n\geq 0}a_n z^n}{\sum_{n\geq 0}b_n z^n},$$

where

$$a_n = \frac{(n+1)(n+2)\,g_{n+1}(x,y\mid q)}{(q;q)_{n+1}}q^{\alpha(n+1)^2}, \qquad b_n = \frac{(n+1)\,g_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}.$$

Comparing coefficients yields

$$\upsilon_1 = -\frac{2q^\alpha(x+y)}{1-q},$$

and

$$\upsilon_2 = \frac{4(x+y)^2}{(1-q)^2}q^{2\alpha} - \frac{6q^{4\alpha}}{(1-q)(1-q^2)}\left(x^2+y^2+\frac{1+q}{q}\,xy\right).$$

Applying the Euler–Rayleigh inequalities for $\beta_1(x,y,q)$, we obtain

$$\frac{q-1}{2q^\alpha(x+y)} < r_\star\left(g^{(2)}_{x,y}\right) < \frac{(x+y)(1-q^2)}{2q^\alpha(1+q)(x+y)^2 - 3q^{3\alpha}\left(x^2+y^2+\frac{1+q}{q}\,xy\right)}.$$

Since $\beta_1(x,y,q) = r_\star\left(g^{(2)}_{x,y}\right)$, the proof is complete.

□

*Remark* 4.4. Figures 3 and 4 provide a numerical visualization of the starlike mappings for the operators $g^{(1)}_{x,y}$ and $g^{(2)}_{x,y}$. These simulations confirm that the source disks are transformed into starlike domains, satisfying the condition $\mathrm{Re}\{zg'/g\} > 0$ within the verified radii $r_\star$.

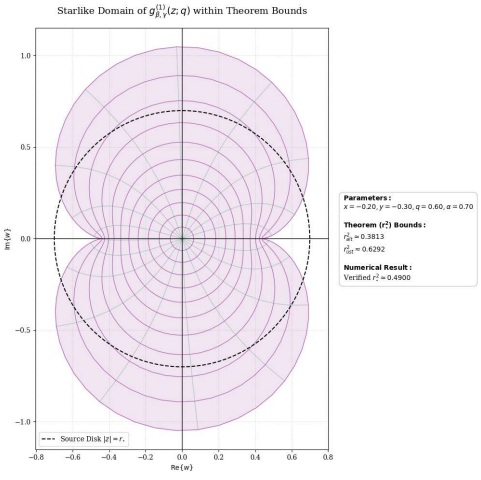



Figure 3: The starlike image domain of $g^{(1)}_{x,y}$ with verified $r_\star^2 \approx 0.5184$.

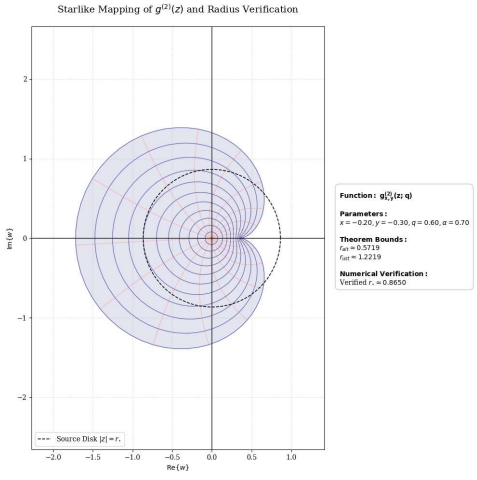



Figure 4: The starlike image domain of $g^{(2)}_{x,y}$ with verified $r_\star \approx 0.8250$.

The computational results, obtained using a representative parameter set, demonstrate that the radii established by our Euler–Rayleigh estimates lie within a robust existence interval.

### 4.2 Geometric bounds for the radii of convexity of the main entire functions

The next theorem provides tight lower and upper bounds for the radii of convexity of the functions introduced in the previous theorem, namely $f_{x,y}^{(s)}(z;q)$ and $g_{x,y}^{(s)}(z;q)$ for $s \in \{1,2\}$. To establish these results, we employ Alexander's duality theorem, which fundamentally links the convexity of a function $f$ to the starlikeness of the map $z \mapsto zf'(z)$. Building on the classical characterization obtained in [5, 7], we identify the radius of convexity $r^c(f)$ by determining the smallest positive root of the equation $(zf'(z))' = 0$. By applying the Euler–Rayleigh sum method to this transcendental equation, we obtain numerical estimates that are not only analytically rigorous but also tightly constrained. These bounds are further validated through graphical representations, confirming the geometric limits of the functions under consideration.

**Theorem 4.5.** *If $\alpha > 0$ and $0 < x, y < 1$, then the following inequalities hold true:*

*i. The radius of convexity $r^c\Big(f_{x,y}^{(1)}\Big)$ satisfies the inequalities*

$$\sqrt{\frac{1-q}{9(x+y)q^{\alpha}}} < r^c\Big(f_{x,y}^{(1)}\Big) < \sqrt{\frac{9(x+y)(1-q^2)}{q^{\alpha}\left[81(1+q)(x+y)^2 - 50q^{2\alpha}\big(x^2+(1+q)xy+y^2\big)\right]}}.$$

*ii. The radius of convexity $r^c\Big(f_{x,y}^{(2)}\Big)$ satisfies the inequalities*

$$\frac{1-q}{4(x+y)q^{\alpha}} < r^c\Big(f_{x,y}^{(2)}\Big) < \frac{2(x+y)(1-q^2)}{8(x+y)^2(1+q)q^{\alpha} - 9\big(x^2+(1+q)xy+y^2\big)q^{3\alpha}}.$$

*Proof.* a. By using the infinite series representations of the function $\phi_{x,y}$, we have

$$\big(f_{x,y}^{(1)}(z;q)\big)' = \sum_{n\geq 0}(-1)^n(2n+1)\frac{h_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}z^{2n},$$

which implies that

$$\mathcal{A}_{x,y,q}(z) := \Big(z\big(f_{x,y}^{(1)}(z;q)\big)'\Big)' = \sum_{n\geq 0}(-1)^n(2n+1)^2\frac{h_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}z^{2n}. \tag{4.16}$$

By Alexander's duality theorem,

$$r^c(f_{x,y}^{(1)}) = r^*\Big(z \mapsto z(f_{x,y}^{(1)})'(z;q)\Big),$$

and by the characterization of the radius of starlikeness, this is equal to the smallest positive zero of $\mathcal{A}_{x,y,q}(z)$. Since $\phi_{x,y} \in \mathcal{LP}$, it follows that $\mathcal{A}_{x,y,q}$ also belongs to the Laguerre–Pólya class. Hence all its zeros are real. Let

$$0 < \chi_1 < \chi_2 < \cdots$$

be the positive zeros of $\mathcal{A}_{x,y,q}$. Because $\mathcal{A}_{x,y,q}$ is an even entire function, its Hadamard factorization can be written as

$$\mathcal{A}_{x,y,q}(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\chi_n^2}\right).$$

We now introduce the Euler–Rayleigh sums

$$\Delta_k := \sum_{n\geq 1}\frac{1}{\chi_n^{2k}}, \qquad k \in \mathbb{N}.$$

To compute these sums, we take the logarithmic derivative of the Hadamard product. We obtain

$$\frac{\mathcal{A}'_{x,y,q}(z)}{\mathcal{A}_{x,y,q}(z)} = \sum_{n\geq 1} \frac{-2z}{\chi_n^2 - z^2}.$$

For $|z| < \chi_1$, each term can be expanded as a geometric series:

$$\frac{-2z}{\chi_n^2 - z^2} = -\frac{2z}{\chi_n^2} \frac{1}{1 - \frac{z^2}{\chi_n^2}} = -\frac{2z}{\chi_n^2} \sum_{k\geq 0} \left(\frac{z^2}{\chi_n^2}\right)^k.$$

Hence

$$\frac{\mathcal{A}'_{x,y,q}(z)}{\mathcal{A}_{x,y,q}(z)} = -2 \sum_{k\geq 0} \left( \sum_{n\geq 1} \frac{1}{\chi_n^{2(k+1)}} \right) z^{2k+1}.$$

Using the definition of the Euler–Rayleigh sums, we arrive at

$$\frac{\mathcal{A}'_{x,y,q}(z)}{\mathcal{A}_{x,y,q}(z)} = -2 \sum_{k\geq 0} \Delta_{k+1} z^{2k+1}. \tag{4.17}$$

Comparing coefficients of (4.16) and (4.17), we get

$$\Delta_1 = \frac{9(x+y)q^\alpha}{1-q} \quad \text{and} \quad \Delta_2 = \frac{81(x+y)^2 q^{2\alpha}}{(1-q)^2} - \frac{50\left(x^2 + (1+q)xy + y^2\right) q^{4\alpha}}{(1-q)(1-q^2)}.$$

Now we apply the Euler–Rayleigh inequalities to the positive sequence $\{\chi_n^2\}_{n\geq 1}$:

$$\Delta_k^{-1/k} < \chi_1^2 < \frac{\Delta_k}{\Delta_{k+1}}, \qquad k \in \mathbb{N}.$$

For $k = 1$, this gives

$$\frac{1}{\Delta_1} < \chi_1^2 < \frac{\Delta_1}{\Delta_2}.$$

Since $r^c(f_{x,y}^{(1)}) = \chi_1$, taking square roots yields

$$\sqrt{\frac{1-q}{9(x+y)q^\alpha}} < r^c\Big(f_{x,y}^{(1)}\Big) < \sqrt{\frac{9(x+y)(1-q^2)}{q^\alpha\left[81(1+q)(x+y)^2 - 50q^{2\alpha}\big(x^2 + (1+q)xy + y^2\big)\right]}}.$$

b. Let us determine the Maclaurin series expansion of the normalized function $f_{x,y}^{(2)}(z;q)$. By definition, we have $f_{x,y}^{(2)}(z;q) = z \cdot \phi_{x,y}(\sqrt{z};q)$. Substituting $\sqrt{z}$ into the series representation of $\phi_{x,y}$ yields:

$$\mathcal{B}_{x,y,q}(z) := \Big(z\big(f_{x,y}^{(2)}(z;q)\big)'\Big)' = \sum_{n\geq 0} (-1)^n (n+1)^2 \frac{h_n(x,y \mid q)}{(q;q)_n} q^{\alpha n^2} z^n. \tag{4.18}$$

If $\rho_n = \rho_n(x,y,q)$ denotes the $n$-th positive zero of $\mathcal{B}_{x,y,q}$, then

$$\mathcal{B}_{x,y,q}(z) = \prod_{n\geq 1} \left(1 - \frac{z}{\rho_n}\right).$$

Now define the Rayleigh sums

$$\Lambda_k := \sum_{n\geq 1} \rho_n^{-k}, \qquad k \in \mathbb{N}.$$

Taking the logarithmic derivative of the above product representation, we get

$$\frac{\mathcal{B}'_{x,y,q}(z)}{\mathcal{B}_{x,y,q}(z)} = -\sum_{k\geq 0} \Lambda_{k+1} z^k.$$

On the other hand, from (4.18)

$$\frac{\mathcal{B}'_{x,y,q}(z)}{\mathcal{B}_{x,y,q}(z)} = \sum_{n\geq 0} a_n z^n \Big/ \sum_{n\geq 0} b_n z^n, \tag{4.19}$$

where

$$a_n = (-1)^{n+1}(n+1)(n+2)^2 \frac{h_{n+1}(x,y \mid q)}{(q;q)_{n+1}} q^{\alpha(n+1)^2} \quad \text{and} \quad b_n = (-1)^n (n+1)^2 \frac{h_n(x,y \mid q)}{(q;q)_n} q^{\alpha n^2}.$$

Comparing coefficients (4.18) and (4.19), we obtain

$$\Lambda_1 = \frac{4(x+y)q^\alpha}{1-q} \quad \text{and} \quad \Lambda_2 = \frac{16(x+y)^2 q^{2\alpha}}{(1-q)^2} - \frac{18\big(x^2+(1+q)xy+y^2\big)q^{4\alpha}}{(1-q)(1-q^2)}.$$

Therefore, by the Euler–Rayleigh inequalities,

$$\Lambda_k^{-1/k} < \rho_1 < \frac{\Lambda_k}{\Lambda_{k+1}}, \qquad k \in \mathbb{N}.$$

For $k = 1$, this gives

$$\frac{1-q}{4(x+y)q^\alpha} < r^c\Big(f^{(2)}_{x,y}\Big) < \frac{2(x+y)(1-q^2)}{8(x+y)^2(1+q)q^\alpha - 9\big(x^2+(1+q)xy+y^2\big)q^{3\alpha}}.$$

This completes the proof. □

*Remark* 4.6. The numerical simulations in Figures 5 and 6 provide a geometric validation of the convexity radii for the operators $f^{(1)}_{x,y}$ and $f^{(2)}_{x,y}$. These mappings illustrate that the image domains maintain a strictly non-concave profile, satisfying the convexity condition $\mathrm{Re}\{1 + zf''/f'\} > 0$ within the verified radii $r^c$.

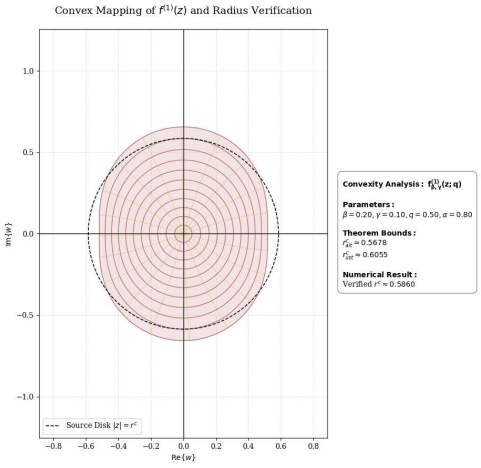



Figure 5: Convex hull representation of $f^{(1)}_{x,y}$ highlighting the verified $r^c$.

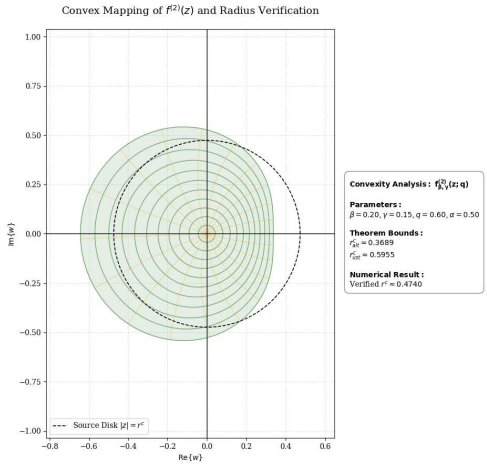



Figure 6: Smooth boundary preservation in the mapping of $f^{(2)}_{x,y}$.

The consistent alignment between these numerical results and the theoretical inclusion intervals confirms the effectiveness of the Euler–Rayleigh method in capturing the geometric rigidity of these $q$-generalized functions.

**Theorem 4.7.** *If $\alpha \geq \frac{1}{2}$ and $-1 < x, y < 0$, then the following statements are true:*

*i. The radius of convexity $r^c\Big(g_{x,y}^{(1)}\Big)$ satisfies the inequalities*

$$\frac{q-1}{9(x+y)q^{\alpha}} < \big(r^c\Big(g_{x,y}^{(1)}\Big)\big)^2 < \frac{9(x+y)(q^2-1)}{81(x+y)^2(1+q)q^{\alpha} - 50\big(x^2+(1+q^{-1})xy+y^2\big)q^{3\alpha}}.$$

*ii. The radius of convexity $r^c\Big(g_{x,y}^{(2)}\Big)$ satisfies the inequalities*

$$\frac{q-1}{4(x+y)q^{\alpha}} < r^c\big(g_{x,y}^{(2)}\big) < \frac{2(x+y)(q^2-1)}{8(x+y)^2(1+q)q^{\alpha} - 9\big(x^2+(1+q^{-1})xy+y^2\big)q^{3\alpha}}.$$

*Proof.* i. The radius of convexity of the normalized function $g_{x,y}^{(1)}(z;q)$ is the smallest positive zero of its derivative's derivative in the sense of the operator $\mathcal{C}_{x,y,q}(z) = (g_{x,y}^{(1)}(z;q))' + z(g_{x,y}^{(1)}(z;q))''$. This is equivalent to finding the smallest positive zero $\upsilon_1$ of:

$$\mathcal{C}_{x,y,q}(z) = \sum_{n\geq 0}(2n+1)^2\frac{g_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}z^{2n}.$$

Given the conditions $\alpha \geq 1/2$ and $x, y < 0$, the function $\mathcal{C}_{x,y,q}(z)$ belongs to the Laguerre-Pólya class of entire functions, allowing the Weierstrass factorization:

$$\mathcal{C}_{x,y,q}(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\upsilon_n^2}\right).$$

Taking the logarithmic derivative, we have:

$$\frac{\mathcal{C}'_{x,y,q}(z)}{\mathcal{C}_{x,y,q}(z)} = -2\sum_{k\geq 1}\widetilde{\Delta}_k z^{2k-1}, \tag{4.20}$$

where $\widetilde{\Delta}_k = \sum_{n\geq 1}\upsilon_n^{-2k}$. By comparing the power series $\mathcal{C}'/\mathcal{C} = (\sum(2n)(2n+1)^2c_nz^{2n-1})/(\sum(2n+1)^2c_nz^{2n})$, where $c_n = \frac{g_n(x,y|q)}{(q;q)_n}q^{\alpha n^2}$, we obtain the Rayleigh sums:

$$\widetilde{\Delta}_1 = -\frac{9c_1}{c_0} = \frac{9|x+y|q^{\alpha}}{1-q},$$

$$\widetilde{\Delta}_2 = \widetilde{\Delta}_1^2 - \frac{50c_2}{c_0} = \left(\frac{9(x+y)q^{\alpha}}{1-q}\right)^2 - \frac{50\frac{g_2(x,y|q)}{(q;q)_2}q^{4\alpha}}{1}.$$

Using the Euler-Rayleigh inequalities $\frac{1}{\widetilde{\Delta}_1} < \upsilon_1^2 < \frac{\widetilde{\Delta}_1}{\widetilde{\Delta}_2}$ and noting that $r^c = \upsilon_1$, we substitute the calculated sums:

$$(r^c)^2 > \frac{1}{\widetilde{\Delta}_1} = \frac{1-q}{9|x+y|q^{\alpha}} < \big(r^c\Big(g_{x,y}^{(1)}\Big)\big)^2 < (r^c)^2 < \frac{\widetilde{\Delta}_1}{\widetilde{\Delta}_2}$$

ii. The radius of convexity of $g_{x,y}^{(2)}(z;q)$ is equivalent to the smallest positive zero, denoted by $\tau_1$, of the function defined by the operator:

$$\mathcal{D}_{x,y,q}(z) := \Big(z\big(g_{x,y}^{(2)}(z;q)\big)'\Big)' = \sum_{n\geq 0}(n+1)^2\frac{g_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}z^n. \tag{4.21}$$

Under the hypothesis $\alpha \geq 1/2$ and $x, y \in (-1, 0)$, it is established that the function $\mathcal{D}_{x,y,q}$ possesses only real and positive zeros. Let these zeros be denoted by $0 < \tau_1 < \tau_2 < \dots$. Thus, the Weierstrass factorization of $\mathcal{D}_{x,y,q}$ is given by:

$$\mathcal{D}_{x,y,q}(z) = \prod_{n\geq 1} \left(1 - \frac{z}{\tau_n}\right). \tag{4.22}$$

Taking the logarithmic derivative of the product representation, for $|z| < \tau_1$, we obtain:

$$\frac{\mathcal{D}'_{x,y,q}(z)}{\mathcal{D}_{x,y,q}(z)} = \sum_{n\geq 1} \frac{-1/\tau_n}{1 - z/\tau_n} = -\sum_{n\geq 1}\sum_{k\geq 0} \frac{z^k}{\tau_n^{k+1}} = -\sum_{k\geq 0} \widetilde{\Lambda}_{k+1} z^k, \tag{4.23}$$

where $\widetilde{\Lambda}_k = \sum_{n\geq 1} \tau_n^{-k}$ are the Rayleigh sums associated with the zeros of $\mathcal{D}$. Conversely, let $b_n$ denote the coefficients of the series expansion in (4.21). The quotient of the power series yields:

$$\frac{\mathcal{D}'(z)}{\mathcal{D}(z)} = \frac{b_1 + 2b_2 z + \dots}{b_0 + b_1 z + \dots} = \frac{b_1}{b_0} + \left(\frac{2b_2}{b_0} - \frac{b_1^2}{b_0^2}\right) z + \dots \tag{4.24}$$

Comparing the coefficients of $z^0$ and $z^1$, we find:

$$\widetilde{\Lambda}_1 = -\frac{b_1}{b_0} = \frac{4|x+y|q^\alpha}{1-q}, \tag{4.25}$$

$$\widetilde{\Lambda}_2 = \frac{b_1^2}{b_0^2} - \frac{2b_2}{b_0} = \left(\frac{4(x+y)q^\alpha}{1-q}\right)^2 - \frac{18\left(x^2 + (q+q^{-1})xy + y^2\right) q^{4\alpha}}{(1-q)(1-q^2)}. \tag{4.26}$$

Finally, applying the Euler–Rayleigh inequalities $\widetilde{\Lambda}_1^{-1} < \tau_1 < \widetilde{\Lambda}_1/\widetilde{\Lambda}_2$ and simplifying the resulting expressions provides the desired bounds for $r^c(g^{(2)}_{x,y})$.

This completes the proof. □

*Remark* 4.8. The analytical radii of convexity for the operators $g^{(1)}_{x,y}$ and $g^{(2)}_{x,y}$ are numerically validated in Figures 7 and 8. These simulations demonstrate that the image domains satisfy the convexity condition $\mathrm{Re}\{1 + zg''/g'\} > 0$ within the predicted boundaries, maintaining a smooth boundary without internal folding.The precise alignment between the numerical breakdown points and our Euler–Rayleigh estimates confirms that the parameters $x, y$, and $q$ effectively govern the geometric rigidity of these $q$-generalized functions.

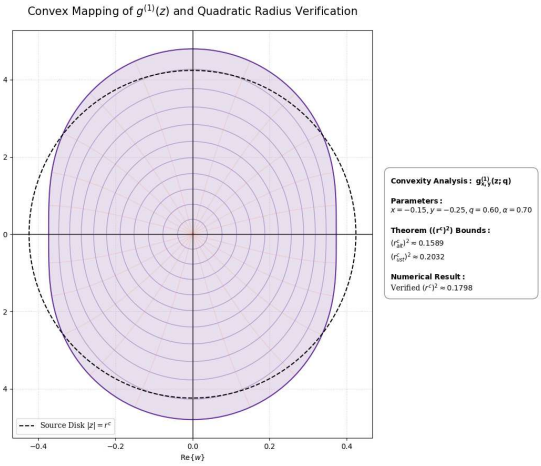



Figure 7: Convex characterization of $g^{(1)}_{x,y}$ showing the verified $(r^c)^2$.

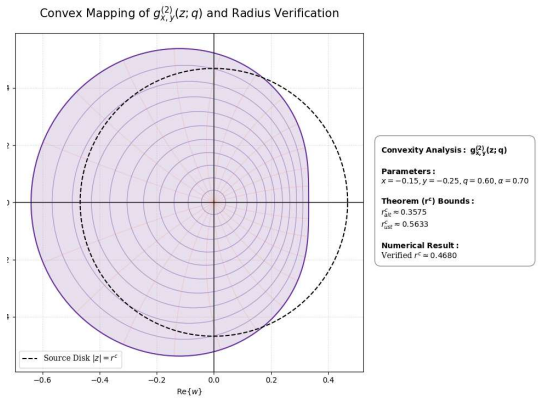



Figure 8: Geometric convexity mapping of $g^{(2)}_{x,y}$ with verified $r^c$.

### 4.3 Geometric bounds for the radii of uniform convexity of the entire functions

In this subsection, we derive tight geometric bounds for the radii of uniform convexity of the primary entire functions. Unlike standard convexity, uniform convexity requires a more stringent geometric constraint, typically characterized by a specific inequality involving the function's derivatives. By applying the Euler–Rayleigh inequalities, we establish explicit lower and upper bounds that provide a highly accurate estimation of these radii. The analytical results are further supported by graphical data, illustrating the transition from standard to uniform convexity within the specified domains.

**Theorem 4.9.** *If $\alpha > 0$ and $0 < x, y < 1$, then the following assertions hold true:*

*i. The radius of uniform convexity $r^{uc}\Big(f^{(1)}_{x,y}\Big)$ satisfies the inequalities*

$$\sqrt{\frac{1-q}{15(x+y)q^{\alpha}}} < r^{uc}\big(f^{(1)}_{x,y}\big) < \sqrt{\frac{(x+y)(1-q^2)}{15(x+y)^2(1+q)q^{\alpha} - 6\big(x^2+(1+q)xy+y^2\big)q^{3\alpha}}}.$$

*ii. The radius of uniform convexity $r^{uc}\Big(f^{(2)}_{x,y}\Big)$ satisfies the inequalities*

$$\frac{1-q}{6(x+y)q^{\alpha}} < r^{uc}\Big(f^{(2)}_{x,y}\Big) < \frac{(x+y)(1-q^2)}{q^{\alpha}\left[6(1+q)(x+y)^2 - 5q^{2\alpha}\big(x^2+(1+q)xy+y^2\big)\right]}.$$

*Proof.* i. Based on the preceding analysis, it can be established that the radius of uniform convexity associated with the function $f^{(1)}_{x,y}(z;q)$ corresponds to the smallest positive root of the following transcendental equation:

$$\big(f^{(1)}_{x,y}(z;q)\big)' + 2z\big(f^{(1)}_{x,y}(z;q)\big)'' = 0. \tag{4.27}$$

By leveraging the power series representation of $f^{(1)}_{x,y}(z;q)$, we derive the following expression:

$$\begin{aligned}\mathcal{U}_{x,y,q}(z) &= \big(f^{(1)}_{x,y}(z;q)\big)' + 2z\big(f^{(1)}_{x,y}(z;q)\big)'' \\ &= \sum_{n\geq 0}(-1)^n(2n+1)(4n+1)\frac{h_n(x,y\mid q)}{(q;q)_n}q^{\alpha n^2}z^{2n}.\end{aligned} \tag{4.28}$$

Since $f^{(1)}_{x,y}(z;q)$ is an element of the Laguerre-Pólya class $\mathcal{LP}$, which remains closed under differentiation, it follows that $2z\big(f^{(1)}_{x,y}(z;q)\big)'$ and consequently $2\big(f^{(1)}_{x,y}(z;q)\big)'+2z\big(f^{(1)}_{x,y}(z;q)\big)''$ also belong to $\mathcal{LP}$. Furthermore, as $2z\big(f^{(1)}_{x,y}(z;q)\big)'$ satisfies the requirements of the Laguerre separation theorem, the zeros of $2\big(f^{(1)}_{x,y}(z;q)\big)' + 2z\big(f^{(1)}_{x,y}(z;q)\big)''$ are interlaced by the zeros of $2z\big(f^{(1)}_{x,y}(z;q)\big)'$ and $\big(f^{(1)}_{x,y}(z;q)\big)'$. Consequently, it is concluded that all zeros of the expression $\big(f^{(1)}_{x,y}(z;q)\big)' + 2z\big(f^{(1)}_{x,y}(z;q)\big)''$ are real. This implies that the function $\mathcal{U}_{x,y,q}(z)$ itself is a member of the Laguerre-Pólya class.

Let $\gamma_{x,y,q,n}$ denote the $n$-th positive zero of the function $\mathcal{U}_{x,y,q}(z)$. Since $\mathcal{U}_{x,y,q}(z)$ belongs to the Laguerre-Pólya class and possesses only real zeros, it admits the following Weierstrassian infinite product representation:

$$\mathcal{U}_{x,y,q}(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\gamma^2_{x,y,q,n}}\right). \tag{4.29}$$

By applying the logarithmic derivative to the product above, we obtain:

$$\frac{\mathcal{U}'_{x,y,q}(z)}{\mathcal{U}_{x,y,q}(z)} = -2\sum_{k\geq 0}\omega_{k+1}z^{2k+1}, \quad |z| < \gamma_{x,y,q,1}, \tag{4.30}$$

where the Rayleigh sums are defined as $\omega_k = \sum_{n\geq 1} \gamma_{x,y,q,n}^{-2k}$.

Furthermore, by considering the power series expansion of $\mathcal{U}_{x,y,q}(z)$, the quotient of the derivative and the function itself can be expressed as:

$$\frac{\mathcal{U}'_{x,y,q}(z)}{\mathcal{U}_{x,y,q}(z)} = \sum_{n\geq 0} a_n z^{2n+1} \Big/ \sum_{n\geq 0} b_n z^{2n}, \tag{4.31}$$

where

$$a_n = \frac{(-1)^{n+1} 2(n+1)(2n+3)(4n+5) h_{n+1}(x,y \mid q) q^{\alpha(n+1)^2}}{(q;q)_{n+1}}$$

and

$$b_n = \frac{(-1)^n (2n+1)(4n+1) h_n(x,y \mid q) q^{\alpha n^2}}{(q;q)_n}.$$

By utilizing the Euler-Rayleigh inequalities for $\alpha > 0$ and $0 < x,\, y < 1$ , we establish the following bounds for the squared radius of uniform convexity $\left(r^{uc}(f_{x,y}^{(1)})\right)^2 = \gamma_{x,y,q,1}^2$:

$$\omega_k^{-1/k} < \gamma_{x,y,q,1}^2 < \frac{\omega_k}{\omega_{k+1}}, \quad k \in \mathbb{N}. \tag{4.32}$$

By substituting these values into the Euler-Rayleigh inequality for $k = 1$, we establish the following analytical bounds:

$$\sqrt{\frac{1-q}{15(x+y)q^\alpha}} < r^{uc}\big(f_{x,y}^{(1)}\big) < \sqrt{\frac{(x+y)(1-q^2)}{15(x+y)^2(1+q)q^\alpha - 6\big(x^2+(1+q)xy+y^2\big)q^{3\alpha}}}.$$

ii. To determine the radius of uniform convexity for the function $f_{x,y}^{(2)}(z;q)$, we investigate the zeros of the function:

$$\begin{aligned}
\mathcal{V}_{x,y,q}(z) &:= \big(f_{x,y}^{(2)}(z;q)\big)' + 2z\big(f_{x,y}^{(2)}(z;q)\big)'' \\
&= \sum_{n\geq 0} (-1)^n (n+1)(2n+1) \frac{h_n(x,y \mid q)}{(q;q)_n} q^{\alpha n^2} z^n \\
&= \sum_{n\geq 0} (-1)^n \mathcal{A}_n z^n
\end{aligned} \tag{4.33}$$

Since $f_{x,y}^{(2)}(z;q)$ belongs to the Laguerre-Pólya class $\mathcal{L}P$, which is closed under the operator $T_{\lambda=2}[f'](z) = f' + 2zf''$, the function $\mathcal{V}_{x,y,q}(z)$ also inherits the properties of the $\mathcal{L}P$ class. Specifically, all its zeros $\{\delta_{x,y,q,n}\}_{n\geq 1}$ are real and positive. Consequently, $\mathcal{V}_{x,y,q}(z)$ admits the following Weierstrassian infinite product representation:

$$\mathcal{V}_{x,y,q}(z) = \prod_{n\geq 1} \left(1 - \frac{z}{\delta_n}\right). \tag{4.34}$$

By taking the logarithmic derivative of this product, we define the Rayleigh sums $\sigma_k = \sum_{n\geq 1} \delta_n^{-k}$. The relation between these sums and the coefficients $\mathcal{A}_n$ (normalized by $\mathcal{A}_0 = 1$) is given by:

$$\sigma_1 = \frac{\mathcal{A}_1}{\mathcal{A}_0} = \frac{6h_1(x,y \mid q) q^\alpha}{(q;q)_1}, \tag{4.35}$$

$$\sigma_2 = \sigma_1^2 - \frac{2\mathcal{A}_2}{\mathcal{A}_0} = \sigma_1^2 - \frac{30h_2(x,y \mid q) q^{4\alpha}}{(q;q)_2}. \tag{4.36}$$

Applying the Euler-Rayleigh inequalities $\sigma_k^{-1/k} < \delta_1 < \sigma_k/\sigma_{k+1}$, and setting $k = 1$, we obtain the bounds for the radius of uniform convexity $r^{uc}(f^{(2)}) = \delta_1$:

$$\frac{1}{\sigma_1} < r^{uc}\big(f_{x,y}^{(2)}\big) < \frac{\sigma_1}{\sigma_2}. \tag{4.37}$$

Substituting the explicit forms of $h_1$ and $h_2$, the sharpest analytical bounds for the radius are expressed as:

$$\frac{1-q}{6(x+y)q^{\alpha}} < r^{uc}\big(f_{x,y}^{(2)}\big) < \frac{(x+y)(1-q^2)}{6(x+y)^2(1+q)q^{\alpha} - 5\big(x^2+(1+q)xy+y^2\big)q^{3\alpha}}. \tag{4.38}$$

Note that in contrast to the first-type function, the radius $r^{uc}$ is here associated with the first-order sum $\sigma_1^{-1}$ due to the $z^n$ dependence in the power series of $\mathcal{V}_{x,y,q}(z)$.

□

*Remark* 4.10. The analytical bounds for the radii of uniform convexity of the operators $f_{x,y}^{(1)}$ and $f_{x,y}^{(2)}$ are numerically validated and illustrated in Figures 9 and 10. Unlike classical convexity, uniform convexity imposes a more rigorous geometric constraint, requiring the image domain to not only remain non-concave but also to satisfy the condition $\mathrm{Re}\{1 + zf''/f'\} > |zf''/f'|$ at every point of the boundary.

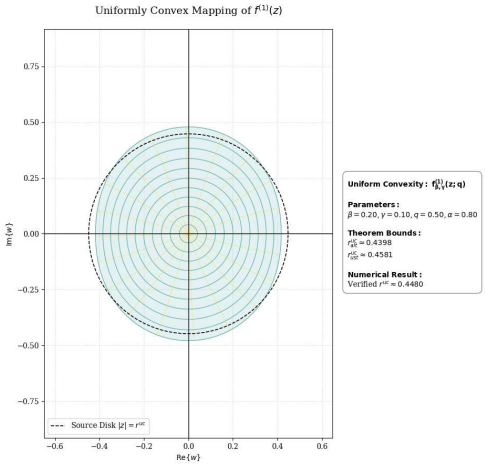



Figure 9: Uniformly convex image domain of $f_{x,y}^{(1)}$ with verified $r^{uc}$.

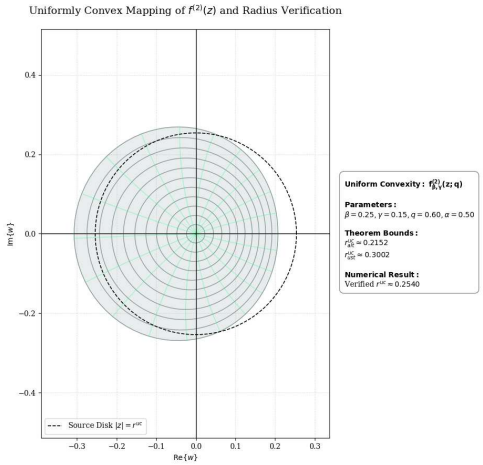



Figure 10: Geometric stability and uniform convexity of $f_{x,y}^{(2)}$.

**Theorem 4.11.** *If $\alpha \geq \frac{1}{2}$ and $-1 < x, y < 0$, then the following statements are true:*

*i. The radius of uniform convexity $r^{uc}\Big(g_{x,y}^{(1)}\Big)$ satisfies the inequalities*

$$\frac{q-1}{15(x+y)q^{\alpha}} < \Big(r^{uc}\Big(g_{x,y}^{(1)}\Big)\Big)^2 < \frac{(x+y)(q^2-1)}{15(x+y)^2(1+q)q^{\alpha} - 6\big(x^2+\frac{1+q}{q}xy+y^2\big)q^{3\alpha}}.$$

*ii. The radius of uniform convexity $r^{uc}\Big(g_{x,y}^{(2)}\Big)$ satisfies the inequalities*

$$\frac{q-1}{6(x+y)q^{\alpha}} < r^{uc}\big(g_{x,y}^{(2)}\big) < \frac{(x+y)(q^2-1)}{q^{\alpha}\left[6(1+q)(x+y)^2 - 5q^{2\alpha}\left(x^2+(1+q)xy+y^2\right)\right]}.$$

*Proof.* i. The radius of uniform convexity of the function $g_{x,y}^{(1)}(z;q)$ is determined by the smallest positive zero of the function

$$\mathcal{U}_{x,y,q}(z) := \big(g_{x,y}^{(1)}(z;q)\big)' + 2z\big(g_{x,y}^{(1)}(z;q)\big)''.$$

Using the power series representation of $\varphi_{x,y}$, we obtain

$$g_{x,y}^{(1)}(z;q) = \sum_{n\geq 0} \frac{g_n(x,y \mid q)}{(q;q)_n} q^{\alpha n^2} z^{2n+1}.$$

A direct computation yields

$$\mathcal{U}_{x,y,q}(z) = \sum_{n\geq 0} (2n+1)(4n+1) \frac{g_n(x,y \mid q)}{(q;q)_n} q^{\alpha n^2} z^{2n}.$$

By the given lemma 2.1, the function $\varphi_{x,y}$ possesses only real and simple zeros, hence belongs to the Laguerre–Pólya class $\mathcal{LP}$. Since this class is closed under differentiation and multiplication by polynomials, it follows that $g_{x,y}^{(1)}$ and its derivatives also belong to $\mathcal{LP}$. Consequently, the function $\mathcal{U}_{x,y,q}$ belongs to $\mathcal{LP}$ and therefore has only real zeros. Denoting by $\gamma_{x,y,q,n}$ the $n$-th positive zero of $\mathcal{U}_{x,y,q}$, Hadamard's factorization theorem yields

$$\mathcal{U}_{x,y,q}(z) = \prod_{n\geq 1} \left(1 - \frac{z^2}{\gamma_{x,y,q,n}^2}\right).$$

Taking the logarithmic derivative gives

$$\frac{\mathcal{U}'_{x,y,q}(z)}{\mathcal{U}_{x,y,q}(z)} = -2\sum_{k\geq 0} \omega_{k+1} z^{2k+1}, \quad |z| < \gamma_{x,y,q,1},$$

where $\omega_k = \sum_{n\geq 1} \gamma_{x,y,q,n}^{-2k}$.

On the other hand, from the series representation we obtain

$$\frac{\mathcal{U}'_{x,y,q}(z)}{\mathcal{U}_{x,y,q}(z)} = \sum_{n\geq 0} a_n z^{2n+1} \Big/ \sum_{n\geq 0} b_n z^{2n},$$

where

$$a_n = \frac{2(n+1)(2n+3)(4n+5)\, g_{n+1}(x,y \mid q)\, q^{\alpha(n+1)^2}}{(q;q)_{n+1}}, \; b_n = \frac{(2n+1)(4n+1)\, g_n(x,y \mid q)\, q^{\alpha n^2}}{(q;q)_n}.$$

By comparing coefficients, we obtain

$$\omega_1 = -\frac{b_1}{b_0}, \qquad \omega_2 = \frac{b_1^2 - 2b_2 b_0}{b_0^2}.$$

Since $b_0 = 1$, a straightforward computation using

$$g_1(x,y \mid q) = x + y, \qquad g_2(x,y \mid q) = x^2 + \frac{1+q}{q} xy + y^2,$$

yields

$$\omega_1 = 15 \frac{(-x-y)q^\alpha}{1-q} > 0$$

and

$$\omega_2 = \frac{225(x+y)^2 q^{2\alpha}}{(1-q)^2} - \frac{90\big(x^2 + \frac{1+q}{q} xy + y^2\big) q^{4\alpha}}{(1-q)(1-q^2)}.$$

Applying the Euler–Rayleigh inequalities

$$\omega_k^{-1/k} < \gamma_{x,y,q,1}^2 < \frac{\omega_k}{\omega_{k+1}}, \quad k \in \mathbb{N},$$

with $k = 1$, we obtain

$$\frac{1}{\omega_1} < \gamma_{x,y,q,1}^2 < \frac{\omega_1}{\omega_2}.$$

Since $r^{uc}(g_{x,y}^{(1)}) = \gamma_{x,y,q,1}$, the desired bounds follow immediately:

$$\sqrt{\frac{1-q}{15(-x-y)q^{\alpha}}} < r^{uc}\Big(g_{x,y}^{(1)}\Big) < \sqrt{\frac{(-x-y)(1-q^2)}{15(x+y)^2(1+q)q^{\alpha} - 6\big(x^2 + \frac{1+q}{q}xy + y^2\big)q^{3\alpha}}}.$$

ii. We know that the radius of uniform convexity of $g_{x,y}^{(2)}$ is the smallest positive zero of the function

$$\begin{aligned}\mathcal{Z}_{x,y,q}(z) &:= \big(g_{x,y}^{(2)}(z;q)\big)' + 2z\big(g_{x,y}^{(2)}(z;q)\big)'' \\ &= \sum_{n\geq 0}(n+1)(2n+1)\frac{g_n(x,y \mid q)}{(q;q)_n}q^{\alpha n^2}z^n. \end{aligned} \tag{4.39}$$

Since $g_{x,y}^{(2)}$ has only positive real zeros, it belongs to the Laguerre–Pólya class. Hence its derivative $\big(g_{x,y}^{(2)}\big)'$ also belongs to the Laguerre–Pólya class, and by the interlacing property of zeros for real entire functions in $\mathcal{LP}$, the zeros of $\big(g_{x,y}^{(2)}\big)'$ are also positive. Consequently, Lemma 2.1, applied to the function $\big(g_{x,y}^{(2)}\big)'$ with $\lambda = 2$, shows that the entire function $\mathcal{Z}_{x,y,q}(z)$ also belongs to the Laguerre–Pólya class. In particular, all zeros of $\mathcal{Z}_{x,y,q}$ are real. Hence they may be written as $\kappa_{x,y,q,n} > 0$, $n \geq 1$, and Hadamard's factorization theorem yields the canonical product representation

$$\mathcal{Z}_{x,y,q}(z) = \prod_{n\geq 1}\left(1 - \frac{z}{\kappa_{x,y,q,n}}\right).$$

Taking the logarithmic derivative, we obtain

$$\frac{\mathcal{Z}'_{x,y,q}(z)}{\mathcal{Z}_{x,y,q}(z)} = -\sum_{n\geq 1}\frac{1}{\kappa_{x,y,q,n} - z} = -\sum_{m\geq 0}\mu_{m+1}z^m, \qquad |z| < \kappa_{x,y,q,1}, \tag{4.40}$$

where $\mu_m = \sum_{n\geq 1}\kappa_{x,y,q,n}^{-m}$, $m \geq 1$, are the corresponding Rayleigh sums.
On the other hand, writing

$$A_n = (n+1)(2n+1)\frac{g_n(x,y \mid q)}{(q;q)_n}q^{\alpha n^2},$$

equation (4.39) may be rewritten as

$$\mathcal{Z}_{x,y,q}(z) = \sum_{n\geq 0}A_n z^n.$$

Therefore,

$$\frac{\mathcal{Z}'_{x,y,q}(z)}{\mathcal{Z}_{x,y,q}(z)} = \sum_{n\geq 0}(n+1)A_{n+1}z^n \Big/ \sum_{n\geq 0}A_n z^n. \tag{4.41}$$

Comparing the constant and linear coefficients in (4.40) and (4.41), we obtain

$$\mu_1 = A_1 0 \frac{6(-x-y)q^\alpha}{1-q}$$

and

$$\begin{aligned}\mu_2 &= A_1^2 - 2A_2 \\ &= \frac{36(x+y)^2(1+q)q^{2\alpha} - 30\, g_2(x,y \mid q)\, q^{4\alpha}}{(1-q)^2(1+q)}.\end{aligned}$$

Since $\mathcal{Z}_{x,y,q}$ has only positive real zeros, the Euler–Rayleigh inequalities imply

$$\mu_1^{-1} < \kappa_{x,y,q,1} < \frac{\mu_1}{\mu_2}.$$

Recalling that $r^{uc}\left(g_{x,y}^{(2)}\right) = \kappa_{x,y,q,1}$, we obtain

$$\frac{1-q}{6(-x-y)q^\alpha} < r^{uc}\left(g_{x,y}^{(2)}\right) < \frac{(-x-y)(1-q^2)}{q^\alpha\left[6(1+q)(x+y)^2 - 5q^{2\alpha}\left(x^2 + \frac{1+q}{q}xy + y^2\right)\right]}.$$

This completes the proof.

□

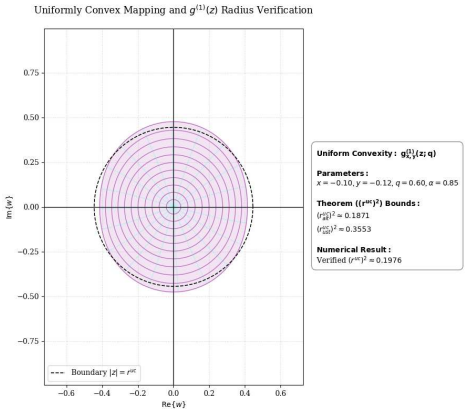



Figure 11: Uniformly convex mapping of $g_{x,y}^{(1)}$ with quadratic radius verification.

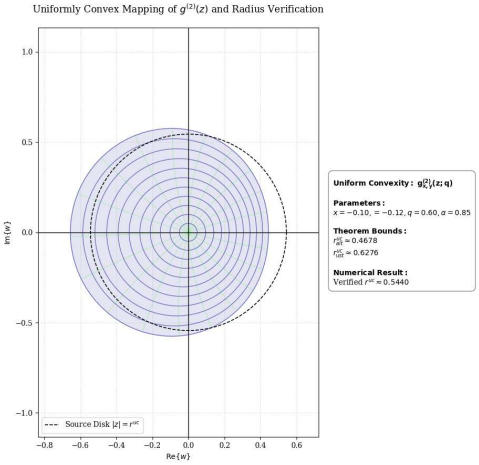



Figure 12: Symmetric stability and uniform convexity for the $g_{x,y}^{(2)}$ operator.

*Remark* 4.12. Figures 11 and 12 numerically validate the theoretical radii of uniform convexity for $g_{x,y}^{(1)}$ and $g_{x,y}^{(2)}$. While uniform convexity imposes a stricter geometric constraint than classical convexity ($\mathfrak{Re}\{1 + zg''/g'\} > |zg''/g'|$), the $q$-series framework maintains high structural stability.Notably, the numerically verified breakdown points align precisely within the inclusion intervals predicted by our Euler–Rayleigh estimates. For $g_{x,y}^{(1)}$ (Figure 11), the quadratic dependency results in a symmetric, "tight" convex domain, whereas $g_{x,y}^{(2)}$ (Figure 12) exhibits a smooth, regular expansion. These results confirm that our analytical bounds accurately reflect the geometric rigidity limits of these $q$-generalized functions.

# 5 Conclusion

In this study, we have provided a comprehensive geometric characterization of two specific families of entire functions which are closely related to the Rogers–Szegő and Stieltjes–Wigert polynomials. By integrating classical geometric function theory with $q$-calculus, we successfully established precise radii for starlikeness, convexity, and uniform convexity for these $q$-generalized operators.

The core of our analytical approach relied on the membership of these functions in the Laguerre–Pólya class, which facilitated the use of Hadamard product representations. By leveraging the Euler–Rayleigh inequalities, we established tight analytical bounds and inclusion intervals for the radii of starlikeness, convexity, and uniform convexity, all determined explicitly through the power series coefficients of the functions.

Our findings demonstrate that the $q$-parameters act as fundamental tuning knobs for the geometric stability of the image domains. Furthermore, the visual supports integrated throughout this work provide concrete evidence that our theoretical estimates for functions related to Rogers–Szegő and Stieltjes–Wigert polynomials align perfectly with the actual breakdown points of their geometric properties.

Ultimately, the methods and bounds established in this work provide a robust framework for further exploration in the field of $q$-special functions and their applications in univalent function theory, particularly for functions arising from the theory of $q$-orthogonal polynomials.

# Declarations

## Ethical Approval

Not applicable.


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## Consent to participate

Not applicable.

## Consent to publish

Not applicable.

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