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On the Meaning of Urban Scaling

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Urban scaling laws describe how an urban quantity Y varies with city population P, typically as $Y \sim P^{\beta}$. These relations are usually obtained from cross-sectional comparisons across cities at a given time (transversal scaling), but their link to the temporal evolution of individual cities (longitudinal scaling) remains unclear. Here we derive explicit expressions for the transversal exponent from the longitudinal dynamics of cities. We show that the measured exponent does not directly reflect individual city dynamics, but instead arises from a snapshot of a heterogeneous ensemble of cities with distinct growth trajectories. As a result, transversal scaling combines intrinsic dynamics with statistical effects due to the distribution of city sizes and correlations between population and city-specific parameters. Consequently, cross-sectional scaling laws cannot, in general, be used to infer the dynamics of individual cities. In particular, apparent sub- or superlinear scaling can emerge even when all cities follow linear longitudinal dynamics, as we demonstrate for the area–population relation. Strikingly, the behavior associated with the transversal exponent is in general not observed in the trajectory of any individual city, underscoring its collective, rather than dynamical, nature. More broadly, the transversal exponent has a clear dynamical meaning only under restrictive conditions—when cities behave as scaled versions of one another and path dependence is weak. Outside of these limits, it is not a law of urban growth, but a statistical artefact of heterogeneity.

I. INTRODUCTION

Urban systems are paradigmatic complex systems in which many interacting agents generate robust macroscopic regularities. Among the most studied of these is the existence of *urban scaling laws*, whereby urban—socio-economic, environmental, or infrastructural—quantities Y scale with population P as a power law [1–19]

$$Y \sim P^{\beta}. \tag{1}$$

Empirically, socioeconomic quantities such as GDP, innovation, or wages typically exhibit superlinear scaling ($\beta > 1$), whereas infrastructural quantities tend to be sublinear ($\beta < 1$). These regularities have motivated extensive theoretical efforts to explain the origin of the exponent β in terms of interaction networks, spatial constraints, and collective efficiencies in large cities [20–25].

Precisely, most empirical estimates, however, are obtained from *transversal* (cross-sectional) comparisons across cities. At a given time t, one considers a set of cities and measures

$$Y_i(t) \approx Y_0(t) P_i(t)^{\beta_T}, \tag{2}$$

where β_T is the transversal exponent, Y_0 a prefactor that can depend on time, $Y_i(t)$ is the value of the quantity Y for city i at time t, and $P_i(t)$ is the population of city i at time t. Implicit in this approach is the assumption that such a relation reflects, at least approximately, the dynamics of individual cities, and can be used to discuss the evolution of cities [5].

At the city level, however, the most general description is a *longitudinal* relation of the form

$$Y_i(t) = F_i\big(P_i(t), t\big), \tag{3}$$

where the function F_i may differ across cities and evolve in time. In some cases, individual trajectories may be approximated by a power law,

$$Y_i(t) = a_i P_i(t)^{\beta_i}, \tag{4}$$

which defines a city-specific longitudinal exponent β_i. In general, however, there is no reason for the functions F_i (or the exponents β_i when they exist) to be identical across cities.

This distinction is crucial. While transversal scaling is largely driven by data availability, and is often interpreted as a fundamental property of urban systems, it does not directly follow from the longitudinal dynamics. Several studies have shown that measured exponents can depend on methodological choices such as the definition of city boundaries or aggregation procedures [21, 26], and have questioned whether deviations from unity necessarily imply nonlinear dynamics [27–29]. More fundamentally, a key conceptual question remains unresolved: *what does the transversal exponent represent at the level of individual cities?*

Early evidence of this issue was provided in the context of congestion-induced traffic delays [30], where aggregating data across cities and years yields an apparent

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nonlinear scaling, while individual trajectories display strong path dependence and do not follow a common law. Similar conclusions emerge from micro-level analyses of wage data [31], where superlinear transversal scaling coexists with heterogeneous longitudinal behaviors across cities. More generally, historical path dependence and technological shifts further complicate the interpretation of cross-sectional relations [32, 33].

These observations highlight a fundamental point: a transversal scaling law does not imply that individual cities follow identical trajectories. Conversely, even if each city obeys a well-defined longitudinal relation between Y and P, the exponent measured across cities need not coincide with these city-level exponents. Instead, as we will show the transversal exponent reflects a statistical aggregate shaped by heterogeneous dynamics, historical trajectories, inter-city correlations and the distribution of city sizes.

In this paper, we clarify the relation between transversal and longitudinal scaling. We show that transversal power laws can emerge from a broad class of heterogeneous longitudinal dynamics, but that the resulting exponent generally does not correspond to a single city-level behavior. Rather, it encodes an effective property of the ensemble, controlled not only by individual growth relations $Y_i(P_i)$ but also by correlations between city size and dynamical parameters. These results imply that transversal scaling relations cannot, in general, be interpreted as direct signatures of the dynamics of individual cities, and should therefore be used with caution when inferring mechanisms of urban growth.

II. RESULTS

A. Earlier approaches: Temporal versus transversal scaling

The relation between cross-sectional and temporal scaling has been examined in several recent works [31–36]. These studies consistently show that exponents inferred from the temporal evolution of individual cities can differ substantially from cross-sectional exponents obtained from comparisons across cities [31, 33]. At the same time, they generally argue that the transversal exponent remains the most relevant quantity for characterizing urban scaling.

In particular, in [34], the authors establish a mapping between temporal and cross-sectional scaling forms. While mathematically correct, this construction raises several conceptual and interpretational issues. The starting point of temporal scaling discussed in this paper is to assume that the individual city behavior is described by

$$Y_i(t) = a_i P_i(t)^{\beta_i} e^{\chi_i(t)}, \tag{5}$$

with a time-independent, city-specific prefactor a_i (the residual χ_i for the temporal fit vanishes for each city under time averaging). This assumption is not innocuous: it effectively absorbs all temporal variations unrelated to population growth into the longitudinal exponent β_i. As a consequence, the exponent β_i is not a pure measure of scale effects, but a composite quantity mixing scale dependence, temporal trends, and fluctuations. This point is made explicit in [34], where β_i is shown to depend on the temporal evolution of the prefactor, on variations of the cross-sectional exponent $\beta_T(t)$, and on residual terms. More precisely, using this assumption (Eq. (5)), it can be shown that the temporal scaling is fundamentally ill-conditioned when population growth is small. In particular, in the simplified case of constant growth rates (a similar relation is also derived in [36]),

$$\beta_i \simeq \beta_T + \frac{\eta}{\gamma_i}, \tag{6}$$

so that β_i diverges as $\gamma_i \to 0$ (here, η and γ_i denote the assumed constant growth rates of the prefactor and population of city i, respectively). This divergence is not a property of the underlying urban dynamics, but an artefact connected to the specific form chosen for the temporal scaling (Eq. (5)). As emphasized in [34], this leads to unstable and even arbitrarily large exponents for slowly growing or shrinking cities, a phenomenon absent in the examples that we will discuss below.

Although the authors present the mapping between temporal and cross-sectional exponents as an equivalence, this equivalence is purely formal. The conditions under which transversal and longitudinal exponents coincide are extremely restrictive: absence of intensive growth, time-independent cross-sectional exponents, and negligible residuals. These conditions are rarely satisfied in empirical urban systems, where intensive growth and heterogeneity are ubiquitous. In practice, the temporal exponent therefore does not measure the same quantity as the transversal exponent, despite the formal mapping.

As noted above, this discussion relies crucially on the assumption that the evolution of $Y_i(t)$ involves a city-specific prefactor a_i that is independent of time. Empirical evidence suggests that this assumption does not always hold. In particular, as we will discuss below, many cities follow well-defined trajectories in the area (A) versus population (P) plane of the form [37]

$$A_i(t) \simeq a_i(t) P_i(t)\,. \tag{7}$$

In some cases, a_i is approximately constant in time, while in others it exhibits a piecewise behavior, taking values a_1 for $t < t^*$ and a_2 for $t > t^*$, where t^* is city-dependent. Such piecewise linear relations are naturally associated with different phases of urban development. Under these conditions, the divergence mechanism discussed in [34] does not arise. The temporal evolution remains effectively linear, with exponent $\beta_i = 1$, while the dynamics are captured by a time-dependent prefactor.

More fundamentally, the framework proposed in [34] takes the cross-sectional scaling relation as the primary object and infers temporal behavior from it. In this

view, the transversal scaling law is treated as fundamental, while temporal exponents are interpreted as derived quantities. Here, we propose the opposite perspective: the dynamics of individual cities are taken as fundamental, and transversal scaling is viewed as an emergent statistical pattern arising from an ensemble of heterogeneous trajectories. From this standpoint, the observed cross-sectional exponent does not encode a universal growth law, but reflects a combination of intrinsic city dynamics, inter-city heterogeneity, and correlations within the urban system.

In this context, the key issue is not the instability of temporal scaling, but the diversity of city dynamics. Different cities follow distinct relations $Y_i(P)$ shaped by their history, spatial constraints, and institutional context. The exponent measured from cross-sectional data is therefore an aggregate statistical quantity rather than a dynamical invariant. Understanding how transversal scaling emerges from this heterogeneity is essential for a correct interpretation of urban scaling laws.

Moreover, in [36], the authors analyze both transversal and longitudinal scaling across more than 5000 Brazilian cities and observe clear discrepancies between the corresponding exponents. They argue, however, that these differences can be reconciled by removing so-called "external factors," namely the temporal variation of the cross-sectional prefactor $Y_0(t)$ (see Eq. (2)). Once this correction is applied, the longitudinal exponents appear, in some cases, to cluster around the transversal one. As noted by the authors, however, there is no general agreement between the distribution of the decomposed exponents and the value of the transversal exponent; this correspondence appears to hold for GDP, but not for the water network length [36]. To address this issue, they propose an alternative method, similar to that of [34], and obtain a relation between longitudinal and transversal exponents akin to Eq. (6).

In [38] Hong et al. investigate employment trajectories across cities and over time, and argue for the existence of a universal pathway of urban economic development. According to their findings, cities transition from economies dominated by manual labor to ones centered on the tertiary sector, with an estimated threshold of approximately $1.2 \cdot 10^6$ inhabitants in the United States. They introduce the concept of urban recapitulation, whereby observing a single city over time yields patterns equivalent to comparing different cities of different sizes at a fixed point in time. To support this claim, they define a recapitulation score

$$S = 1 - \frac{|\hat{\beta} - \beta|}{\beta}, \tag{8}$$

where $\hat{\beta}$ is a longitudinal exponent obtained after removing the nation-wide growth factor, i.e., the transversal scaling prefactor $Y_0(t)$ (β is the transversal exponent). By construction, S is large when transversal and longitudinal exponents are close. They report an average value of about 0.7, which they interpret as strong evidence of agreement between the two. We argue that this interpretation is biased. The apparent agreement between exponents only emerges after normalizing by a system-wide quantity, namely $Y_0(t)$, and is therefore not intrinsic to the dynamics of individual cities. More broadly, we show in the following that the transversal scaling hypothesis leads to a distorted view of urban growth: transversal exponents do not possess a clear longitudinal dynamical meaning and may, in fact, be misleading when used to infer or predict the evolution of individual cities.

B. From individual cities to the transversal exponent

In this part, we derive the relation between the transversal exponent and the longitudinal behavior of cities. We define $x_i = \ln P_i(t)$ and $y_i = \ln Y_i(t)$, and the transversal exponent is usually estimated by ordinary least squares (OLS) in the log–log plane and is given by the following expression (see Supp. Mat.)

$$\beta_T = \frac{\mathrm{Cov}(x, y)}{\mathrm{Var}(x)}. \tag{9}$$

where $\mathrm{Cov}(x, y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x\rangle)(y_i - \langle y\rangle)$ (the brackets $\langle\cdot\rangle$ denote the average over the N cities at a given time t). This identity (Eq. 9) holds independently of the functional form of the longitudinal dynamics.

Although alternative estimators can be used—such as maximum likelihood estimation, robust regressions, or Bayesian approaches—OLS remains the most commonly used method. It provides a reliable estimate of the power-law exponent and, importantly, allows one to express explicitly the different contributions (in particular, correlation terms) to the transversal exponent. These contributions are typically present in other methods as well, albeit in different combinations and not always in an analytically transparent form. In the Supplementary Material, we show that alternative fitting methods yield similar values of the transversal exponent. The discrepancies discussed here are therefore not specific to OLS; rather, OLS has the advantage of making these deviations explicit through a decomposition in terms of correlation terms.

1. *Individual power-law case*

We start with the simplest case (although probably not the most common) where individual cities behave as power laws with various exponents

$$Y_i(t) = a_i P_i(t)^{\beta_i} \tag{10}$$

where the prefactor a_i could depend on time or population, and the exponent β_i varies from a city to another. In this case, we obtain

$$y_i(t) = \alpha_i + \beta_i x_i(t) \tag{11}$$

where $\alpha_i = \ln a_i$. Using the estimate (eq. (9)) of β_T, we find

$$\beta_T(t) = \frac{\mathrm{Cov}(x,\alpha)}{\mathrm{Var}(x)} + \frac{\mathrm{Cov}(x,\beta x)}{\mathrm{Var}(x)}. \tag{12}$$

The first term in Eq. 12 corresponds to the OLS slope obtained when regressing the prefactor α against x. The second term is more subtle. In the special case where all cities share the same exponent, $\beta_i = \beta$, it reduces simply to β. In that situation, the difference between the longitudinal exponent β and the transversal exponent β_T is entirely captured by the first covariance term. This shows that β_T does not purely reflect intrinsic city dynamics, but is also influenced by correlations between prefactors and population. In the absence of such correlations, one recovers $\beta_T = \beta$. When the exponents β_i are heterogeneous, however, the situation becomes more intricate.

Altogether, the transversal exponent emerges as a combination of the average longitudinal exponent and additional statistical contributions that depend on the joint distribution of β and x. As a result, β_T is generally a nontrivial aggregate quantity, whose interpretation is not straightforward.

2. General case

To connect in the general case the cross-sectional scaling exponent to the longitudinal evolution of individual cities, we estimate for each city i and year t a local longitudinal relation in logarithmic variables,

$$y_i(t) \approx \alpha_i(t) + \beta_i(t) x_i(t), \tag{13}$$

Here $\beta_i(t)$ is the local longitudinal elasticity of city i, estimated from a rolling regression of $\ln Y_i$ against $\ln P_i$ along the temporal trajectory of that city, while $\alpha_i(t)$ is the corresponding local intercept. More precisely, for each city and each year t, we consider a short time window centered on t (typically five years for the data studied below) and perform a linear regression of $\ln Y_i$ on $\ln P_i$ using the observations within this window. The resulting slope provides a local estimate of the derivative

$$\beta_i(t) = \frac{d \ln Y_i}{d \ln P_i} \tag{14}$$

which defines the instantaneous longitudinal elasticity $\beta_i(t)$, while the intercept defines $\alpha_i(t)$. This procedure provides a local linear approximation of the city trajectory in the $(\ln P, \ln Y)$ plane, allowing us to capture gradual changes in scaling behavior over time while reducing the noise that would arise from estimating the derivative from only two consecutive observations.

At a fixed time t, the transversal exponent is defined as the ordinary least-squares slope of the cross-sectional relation between $y_i(t)$ and $x_i(t)$,

$$\begin{aligned}\beta_T(t) &= \frac{\mathrm{Cov}(x,y)}{\mathrm{Var}(x)} \\ &= \frac{\mathrm{Cov}(x,\alpha)}{\mathrm{Var}(x)} + \frac{\mathrm{Cov}(x,\beta x)}{\mathrm{Var}(x)}\end{aligned} \tag{15}$$

Writing $\beta = \beta - \langle\beta\rangle + \langle\beta\rangle$, we get

$$\beta_T = \langle\beta\rangle + \frac{\mathrm{Cov}(x,\alpha)}{\mathrm{Var}(x)} + \frac{\mathrm{Cov}(x,(\beta - \langle\beta\rangle)x)}{\mathrm{Var}(x)} \tag{16}$$

where $x = x(t)$, $y = y(t)$, $\alpha = \alpha(t)$, and $\beta = \beta(t)$ are all estimated at time t.

This relation shows that the transversal exponent can be predicted directly from the longitudinal behavior of individual cities. We observe in this relation that additional correlations (compared to pure power law cases) are now induced by the time variations of β_i and a_i. Here also, Equation (16) makes explicit that the transversal exponent is not, in general, equal to the average longitudinal exponent. Besides the mean local elasticity $\langle\beta(t)\rangle = 1/N \sum_i \beta_i(t)$, two correction terms appear: a covariance term involving the intercepts α_i, and a term measuring the correlation between city size and local longitudinal elasticity.

C. Empirical illustrations

We now illustrate and empirically assess the theoretical results derived above through two main examples. First, we consider the "fundamental allometry" between urbanized area and population [23], which provides a benchmark case where individual city dynamics are simple (essentially piecewise linear). Second, we analyze wages, for which the longitudinal behavior is more complex. We also briefly revisit in the Supplementary Material the case of traffic delay due to congestion, previously analyzed in [30].

The goal of these empirical analyses is not to test the existence of scaling laws per se, but to validate the theoretical decomposition of the transversal exponent. In particular, we test whether the measured transversal can be quantitatively reconstructed from the longitudinal dynamics of individual cities together with the covariance terms predicted by the theory (e.g. Eq. (16)).

Empirically, we proceed as follows. For each city, we estimate $\beta_i(t)$ and $\alpha_i(t)$ from rolling temporal regressions. Using these quantities, we compute the predicted transversal exponent from Eq. (16) by evaluating numerically the terms $\mathrm{Cov}(x,\alpha)$ and $\mathrm{Cov}(x,\beta x)$. We then compare this prediction to the transversal exponent measured directly from cross-sectional data using a power law fit.

This comparison allows us to assess the relative importance of the different correlations that enter the expression for β_T. Its relevance is (i) to demonstrate that the

behavior observed at the cross-sectional level does not correspond to the dynamics of any individual city, and (ii) to quantify the respective contributions of the various correlations shaping the transversal exponent.

1. *Area-population relation*

In this part we discuss the important case of the area–population relation. Understanding how the built area A increases with population P is a crucial component for describing the spatial dynamics of cities. This specific urban scaling was discussed in many papers [16, 39–44]. Early results reported $\beta_T \approx 3/4$ [39], and $\beta_T \approx 2/3$ [40]. In [41], Batty and Ferguson collected the results of various papers and reported that mostly $\beta_T < 1$ for this area-population relation (see Supp. Mat.). In a recent study [44], it was reported that for half of the cities studies (18 out of 38 countries) the exponent is closer to 5/6 while for the other the scaling is indistinguishable from linearity.

In contrast with these studies, longitudinal measurements [37] were performed for the period 1800–2000 using the Angel *et al.* dataset [45], and for the period 1985–2015 using the dataset [46, 47]. In all the cases studied, the longitudinal behavior of cities can be described by a linear relation of the form (for large enough population)

$$A_i(t) \approx a_i(t) P_i(t)\,, \tag{17}$$

where $a_i(t)$ is the inverse density. Typical cases include strictly linear trajectories with $a_i = \text{const.}$, as well as piecewise linear behaviors in which a_i takes two values ($a_i = a_1$ for $t < t^*$ and $a_i = a_2$ for $t > t^*$, usually with $a_2 < a_1$, and where t^* depends on the city). This observation indicates that urban expansion typically proceeds at approximately constant density. Typical linear relations are illustrated (a) yearly over 30 years in Fig. 1a) and (b) sporadically over approximately 200 years in Fig. 1b). Density-breaking situations, for which the relation between area and population is piecewise linear, are discussed in [37]. These results therefore support the interpretation that, to a good approximation, population growth and spatial expansion remain linearly coupled over time.

Although in both datasets (1800–2000 and 1985–2015) the longitudinal behavior of individual cities is well described by a linear or piecewise linear relation, the transversal exponent obtained from cross-sectional fits is different from unity. This is expected from the general discussion, since the prefactor a_i is not constant but varies across cities and in time. This example illustrates that nonlinear behavior observed at the transversal level does not necessarily reflect the underlying dynamics of individual cities. In particular, the transversal exponent mixes together the heterogeneous trajectories of all cities and is therefore sensitive to correlations between the prefactor a_i, the population P_i, and their temporal evolution.

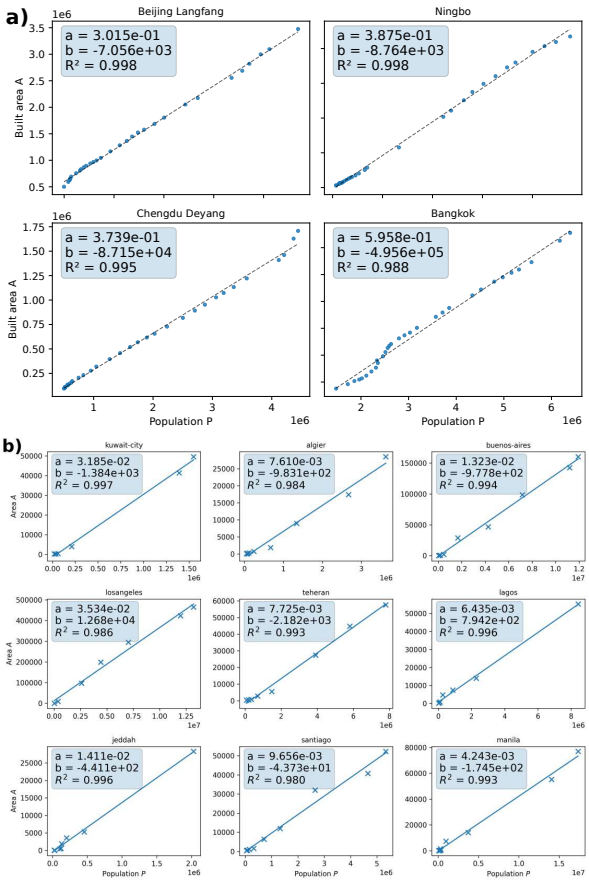


FIG. 1. **Examples of longitudinal linear dependency between area (A) and population (P).** a) For four cities of the WSFEvo dataset [46] studied in [37] and b) for nine cities in the [45] dataset. On each panel, the points represent the data (annual for the WSFEvo dataset, sporadic for the [45] dataset). For each city, the inverse density (a), the intercept (b) and the goodness-of-fit (R^2), estimated by ordinary least-square regression, are reported for each city. The line on each panel represent the best linear fit $A = aP + b$.

An even more puzzling feature is that the sign of the deviation from linearity changes between the two datasets (Fig. 2). For the long historical period (1800–2000), the transversal exponent is larger than one ($\beta_T \simeq 1.12 \pm 0.06 > 1$), corresponding to an apparent superlinear behavior and suggesting that larger cities occupy disproportionately more area, i.e. that density decreases with population. In contrast, for the more recent period (1985–2015), the transversal analysis yields $\beta_T \simeq 0.5 - 0.7 < 1$, which would lead to the opposite interpretation that density increases with population. The temporal evolution and associated error bars are shown in Fig. 2a), while a snapshot of the transversal scaling is displayed in Fig. 2b). In this case, we observe a decrease of the transversal exponent over time.

These opposite conclusions arise even though the longitudinal dynamics of individual cities remain essentially linear in both cases. The difference therefore originates

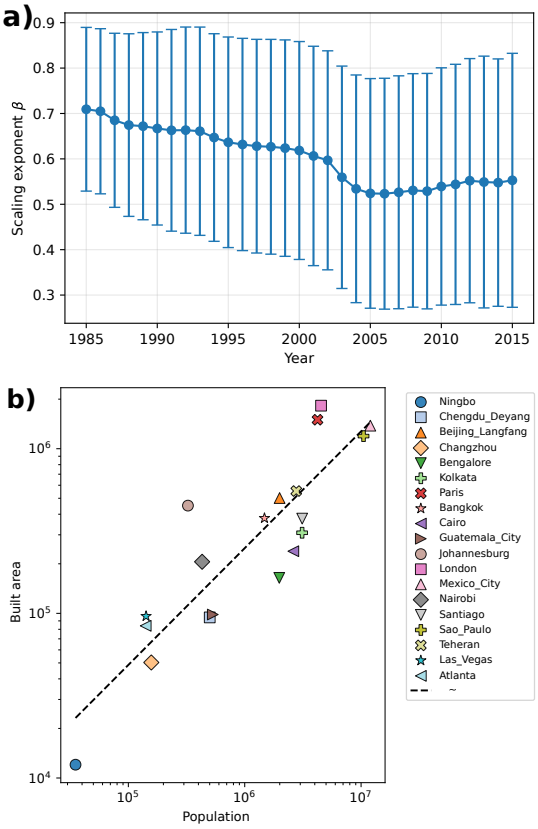


FIG. 2. **Transversal scaling: area–population.** a) Temporal evolution of the exponent β_T for the 19 cities studied in [37], estimated using OLS in log–log space. Error bars indicate 95% confidence intervals. b) Cross-sectional relation in 1985. The dashed line shows the best fit, with $\beta_T = 0.71$ ($R^2 = 0.78$). The population spans more than 2.5 orders of magnitude. Alternative estimators (Theil–Sen, weighted least squares) yield similar results (see Supplementary Material).

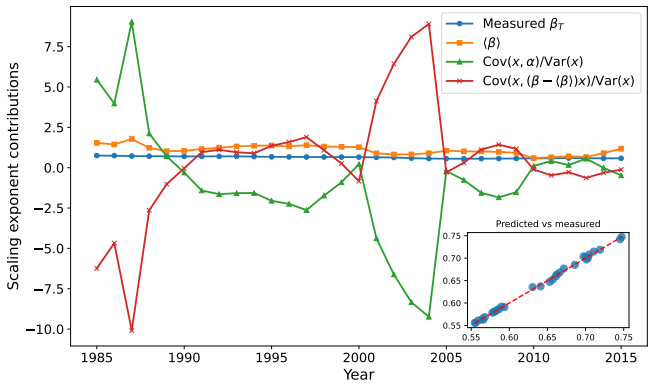


FIG. 3. **Statistical contributions to the transversal exponent.** On the data discussed in Fig. 2, the local longitudinal exponent $\beta_i(t)$ is measured for each city. Its average $\langle\beta\rangle$ is shown in the orange curve. The two other contributions (red and green curve), introduced in Eq. 16, are of opposite sign and the deviation between β_T and $\langle\beta\rangle$ emerges from their non-null compensation. Inset: comparison between the measured exponent (done by directly fitting the data) and the predicted exponent given by Eq. 16.

from the correlations that enter the expression of the transversal exponent. As shown in the theoretical derivation (Eq. (16)), the transversal elasticity can be written as the average of the local longitudinal elasticities plus correction terms involving correlations between the prefactor a_i, the population P_i, and their temporal evolution. In particular, covariance terms such as $\mathrm{Cov}(\alpha, x)$ can shift the effective exponent away from unity even when each city individually follows a linear relation. Figure. 3 illustrates the decomposition of the transversal exponent β_T in various contributions given by Eq. 16. The deviation between the local longitudinal exponent (blue curve) and the transversal exponent (orange curve) arises from the incomplete compensation of two contributions with opposite signs (red and green curves). The inset highlights the excellent agreement between the measured exponent and the prediction obtained by summing these contributions.

The transversal exponent thus reflects an aggregate statistical effect over the ensemble of cities rather than a genuine dynamical law governing the growth of individual cities, and its interpretation requires taking into account the heterogeneity of city trajectories and the correlations present in the dataset.

To illustrate this mechanism and clarify the origin of super- and sublinear transversal behavior, we consider the empirically observed area–population relation (for details of the calculation, see Supplementary Material).

$$A_i(t) = a_i(t)\, P_i(t), \tag{18}$$

which is linear in population, with a time-dependent prefactor $a_i(t)$. We assume that the random variable $x = \ln P$ is distributed according to some law $P(x)$. Motivated by empirical observations, we assume that the prefactor a_i depends on x_i through a threshold mechanism:

$$a_i(t) = \begin{cases} a_1 & \text{if } x_i < x_i^*, \\ a_2 & \text{if } x_i \geq x_i^*, \end{cases} \tag{19}$$

where the thresholds x_i^* are random variables distributed according to $\rho(x^*)$. In the simple case where all longitudinal exponents are identical ($\beta_i = 1$), one can show that the transversal exponent takes the form

$$\beta_T = 1 + C\,(\ln a_2 - \ln a_1), \tag{20}$$

where $C > 0$ depends on the distributions of x and x^*.

This result immediately implies that $\beta_T > 1$ if $a_2 > a_1$ (superlinear scaling), and $\beta_T < 1$ if $a_2 < a_1$ (sublinear scaling). This simple mechanism accounts for the empirical observations: both superlinear exponents (as in Angel *et al.* [45]) and sublinear exponents (as in the 19 cities studied in [37]) can arise from the same underlying linear longitudinal dynamics. The apparent nonlinearity at the

transversal level is therefore not intrinsic, but emerges from heterogeneity in the prefactor a_i across cities. In particular, superlinearity reflects a positive correlation between the prefactor and city size. While the sign of the deviation (super- or sublinear) can thus be meaningfully interpreted, the precise value of the exponent carries no clear dynamical meaning.

We also propose a simulation (see Supp. Mat.) of this simple scenario in which cities undergo a change in slope at either a fixed or a random threshold. The underlying dynamics is identical for all cities; however, in the simulation the initial populations are drawn at random (and in the absence of a threshold, this would lead to $\beta_T = 1$). These simulations confirm that β_T recovers the true elasticity $\beta_i = 1$ only when all cities follow identical dynamics and share the same initial conditions. Any source of heterogeneity—whether in the initial populations, in the transition thresholds, or more generally in the dynamical parameters—can induce cross-sectional correlations between α_i and x_i, causing β_T to depart from the (true) longitudinal exponent.

2. Wages

A prototypical example of urban scaling concerns the total wages produced in a city. It was reported that wages grow superlinearly with population size [8], a result often interpreted as a consequence of enhanced social and economic interactions in larger cities. In particular, interaction-based theories predict a scaling exponent $\beta = 7/6 \simeq 1.17$ [20].

This prediction can be tested empirically: using data for 363 Metropolitan Statistical Areas (MSAs) in the United States in 2006, one obtains a measured transversal exponent $\beta_T = 1.13 \pm 0.02$, in reasonable agreement with the theoretical expectation. Here, we go beyond this single-year analysis and use the full dataset from [20], which provides yearly observations of wages and population over the period 1969–2015.

The temporal evolution of wages as a function of population is shown in Fig. 4(a). Each trajectory corresponds to the longitudinal evolution of a given city. While the transversal exponent β_T, obtained from cross-sectional fits at each year, remains remarkably stable over time (Fig. 4(b)) – similarly to what is observed for congestion delays – the individual trajectories display strong heterogeneity, both in slope and amplitude (see the distribution of individual exponents β_i in the SM).

To better understand this discrepancy, we decompose the transversal exponent according to our theoretical expression (Eq. (16)), as shown in Fig. 4(c). This decomposition reveals that the average longitudinal exponent is of order $\langle\beta\rangle \simeq 4 - 5$, which is much larger than both the transversal exponent and the theoretical prediction 7/6. This large difference reflects the fact that individual city trajectories are significantly steeper than what is suggested by the cross-sectional analysis.

This contrast is clearly visible in Fig. 4(a): individual cities exhibit rapid increases of wages with population, whereas the cross-sectional scaling remains comparatively shallow. The observed stability of β_T over time therefore results from a balance between heterogeneous longitudinal dynamics and statistical correlations across cities, rather than from a universal mechanism governing all urban systems.

The agreement between the measured transversal exponent and the value predicted by Eq. (16) is shown in Fig. 4(d). While the overall trend is well reproduced, some deviations remain, which can be attributed to the large dispersion of individual longitudinal exponents around $\langle\beta\rangle$.

III. DISCUSSION

Our results clarify the interpretation of transversal (cross-sectional) scaling exponents. In general, the transversal exponent cannot be viewed as a direct estimator of a universal longitudinal exponent. Rather, it is an aggregate quantity that combines several effects, including the distribution of city sizes, the heterogeneity of city-specific growth functions, local elasticities at current population levels, and the correlations between these quantities. As such, transversal scaling reflects not only the intrinsic dynamics of individual cities, but also the statistical structure of the urban system. In particular, no individual city follows the behavior characterized by the transversal exponent, indicating that it is a statistical construct rather than a descriptor of any real trajectory.

From this perspective, the fundamental relations are the longitudinal dynamics, which describe the temporal evolution of individual cities and admit a clear dynamical interpretation. By contrast, transversal exponents mix heterogeneous behaviors across cities and can be strongly affected by correlations and cross-sectional variability. This aggregation can generate apparent scaling laws that do not correspond to any underlying city-level mechanism. In this sense, the transversal exponent effectively describes a "fictitious city" whose behavior is not present in reality.

Importantly, transversal exponents do have a clear meaning in systems where larger entities are effectively scaled-up versions of smaller ones, and where path dependence is weak. In such cases, comparing different system sizes is equivalent to probing a single underlying structure across scales. A classical example is Kleiber's law [48–51] in biology, which relates metabolic rate to body mass as $B \sim M^{3/4}$. This relation is understood as a consequence of the geometry and efficiency of hierarchical distribution networks (such as the respiratory or circulatory systems), which are broadly comparable across organisms of the same family. In this context, cross-sectional scaling reflects a genuine structural principle.

Urban systems, however, differ fundamentally from such biological examples. Cities are shaped by history,

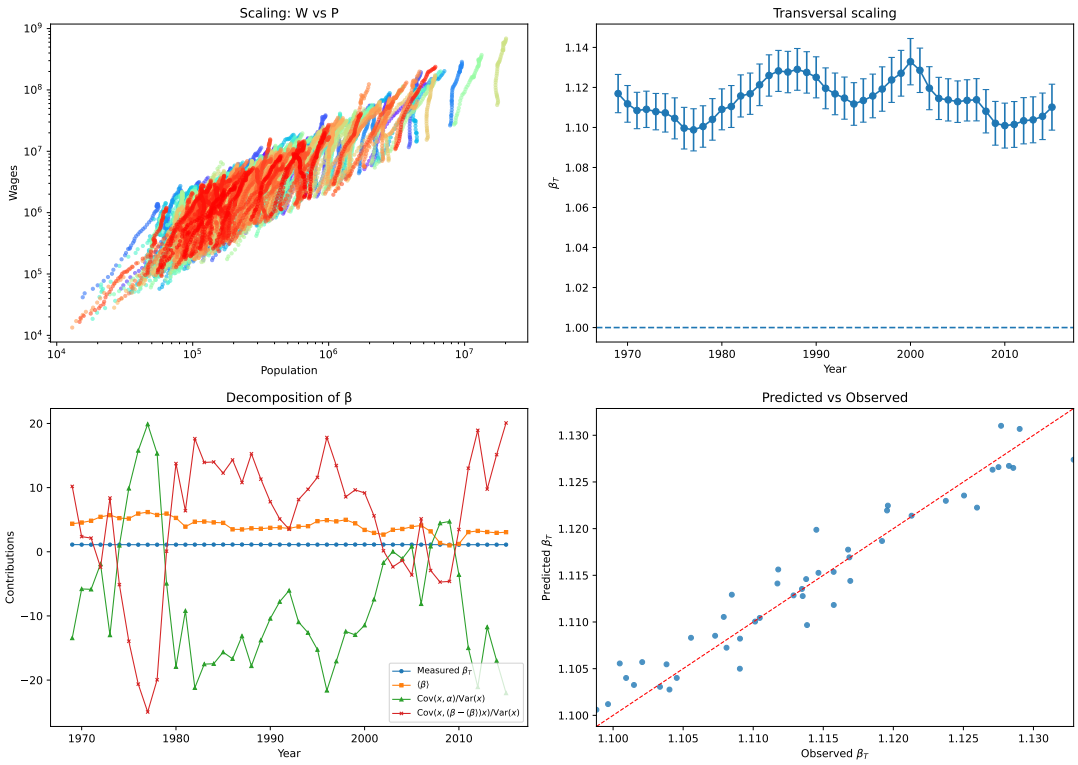


FIG. 4. **Wages: transversal against longitudinal scaling.** a) Evolution of wages as a function of city size. Each city is represented by contiguous colored points. There are 363 MSAs, with data for each year between 1969 and 2015. b) Measure of the transversal exponent (using OLS in log-log coordinates) over years. Error bars represent the 95% confidence interval. Data: from [20]. c) Decomposition of the contributions to the transversal exponent, discussed in Sec. II B 2. It is important to note the large deviation between the average longitudinal exponent and the transversal exponent. d) Comparison between predicted and measured transversal exponents. Red line: identity.

geography, technological change, and institutional factors, all of which generate strong path dependence and a wide diversity of individual trajectories. As a result, cities cannot, in general, be regarded as scaled replicas of one another. The heterogeneity in their growth functions and in their dynamical responses implies that transversal exponents aggregate fundamentally different behaviors.

A clean transversal power law can therefore emerge only under restrictive conditions, namely when longitudinal dynamics are sufficiently homogeneous and when correlations between size and dynamical parameters remain weak. In most empirical situations, these conditions are not satisfied. Transversal exponents should thus be interpreted with caution: rather than revealing universal laws of urban growth, they typically reflect effective statistical properties of a heterogeneous ensemble.

[1] T. Oke, "City size and the urban heat island," *Atmospheric Environment (1967)*, vol. 7, no. 8, pp. 769–779, 1973. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0004698173901406

[2] L. Sveikauskas, "The productivity of cities," *The Quarterly Journal of Economics*, vol. 89, no. 3, pp. 393–413, 1975. [Online]. Available: http://www.jstor.org/stable/1885259

[3] E. L. Glaeser and B. Sacerdote, "Why is there more crime in cities?" *Journal of Political Economy*, vol. 107, no. S6, pp. S225–S258, 1999. [Online]. Available: http://www.jstor.org/stable/10.1086/250109

[4] D. Pumain, "Scaling laws and urban systems," Santa Fe Institute, Working Paper 2004-02-002, 2004.

[5] D. Pumain, F. Paulus, C. Vacchiani-Marcuzzo, and J. Lobo, "An evolutionary theory for interpreting urban scaling laws," *Cybergeo: European Journal of Geography*, vol. 343, 2006.

[6] C. Kuhnert, D. Helbing, and G. B. West, "Scaling laws in urban supply networks," *Physica A*, vol. 363, no. 1, pp. 96–103, 2006.

[7] L. M. A. Bettencourt, J. Lobo, and D. Strumsky, "Invention in the city: increasing returns to patenting as a scaling function of metropolitan size," *Research Policy*, vol. 36, pp. 107–120, 2007.

[8] L. M. Bettencourt, J. Lobo, D. Helbing, C. Kühnert, and G. B. West, "Growth, innovation, scaling, and the pace of life in cities," *Proceedings of the national academy of*

sciences, vol. 104, no. 17, pp. 7301–7306, 2007.

[9] H. Samaniego and M. E. Moses, "Cities as organisms: Allometric scaling of urban road networks," *Journal of Transport and Land Use*, vol. 1, no. 1, pp. 21–39, 2008.

[10] S. Arbesman, J. M. Kleinberg, and S. H. Strogatz, "Superlinear scaling for innovation in cities," *Phys. Rev. E*, vol. 79, p. 016115, Jan 2009. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevE.79.016115

[11] M. Fragkias, J. Lobo, D. Strumsky, and K. C. Seto, "Does size matter? scaling of co2 emissions and us urban areas," *Plos one*, vol. 8, no. 6, p. e64727, 2013.

[12] R. Louf and M. Barthelemy, "Scaling: Lost in the smog," *Environment and Planning B: Planning and Design*, vol. 41, no. 5, p. 767–769, Oct. 2014. [Online]. Available: http://dx.doi.org/10.1068/b4105c

[13] M. Schläpfer, J. Lee, and L. M. A. Bettencourt, "Urban skylines: building heights and shapes as measures of city size," 2015. [Online]. Available: https://arxiv.org/abs/1512.00946

[14] E. L. Glaeser and M. E. Kahn, "The greenness of cities: Carbon dioxide emissions and urban development," *Journal of urban economics*, vol. 67, no. 3, pp. 404–418, 2010.

[15] E. A. Oliveira, J. S. Andrade, and H. A. Makse, "Large cities are less green," *Scientific Reports*, vol. 4, p. 4235, 2014.

[16] D. Rybski, D. E. Reusser, A.-L. Winz, C. Fichtner, T. Sterzel, and J. P. Kropp, "Cities as nuclei of sustainability?" *Environment and Planning B: Urban Analytics and City Science*, vol. 44, no. 3, pp. 425–440, 2017.

[17] C. Cabrera-Arnau, R. Prieto Curiel, and S. R. Bishop, "Uncovering the behaviour of road accidents in urban areas," *Royal Society Open Science*, vol. 7, no. 4, p. 191739, 04 2020. [Online]. Available: https://doi.org/10.1098/rsos.191739

[18] A. J. Stier, K. E. Schertz, N. W. Rim, C. Cardenas-Iniguez, B. B. Lahey, L. M. A. Bettencourt, and M. G. Berman, "Evidence and theory for lower rates of depression in larger us urban areas," *Proceedings of the National Academy of Sciences*, vol. 118, no. 31, p. e2022472118, 2021. [Online]. Available: https://www.pnas.org/doi/abs/10.1073/pnas.2022472118

[19] S. Puttock, U. Barros, D. Pinheiro, and M. Oliveira, "Larger cities, more commuters, more crime? the role of inter-city commuting in the scaling of urban crime," *Crime Science*, vol. 14, no. 1, p. 19, 2025.

[20] L. M. A. Bettencourt, "The origins of scaling in cities," *Science*, vol. 340, pp. 1438–1441, 2013.

[21] R. Louf and M. Barthelemy, "How congestion shapes cities: From mobility patterns to scaling," *Scientific Reports*, vol. 4, p. 5561, 2014.

[22] V. Verbavatz and M. Barthelemy, "Critical factors for mitigating car traffic in cities," *PLoS ONE*, vol. 14, p. e0219559, 2019.

[23] F. L. Ribeiro and D. Rybski, "Mathematical models to explain the origin of urban scaling laws," *Physics Reports*, vol. 1012, pp. 1–39, 2023.

[24] A. Gomez-Lievano, O. Patterson-Lomba, and R. Hausmann, "Explaining the prevalence, scaling and variance of urban phenomena," *Nature Human Behaviour*, vol. 1, no. 1, Dec. 2016. [Online]. Available: http://dx.doi.org/10.1038/s41562-016-0012

[25] V. C. Yang, A. V. Papachristos, and D. M. Abrams, "Modeling the origin of urban-output scaling laws," *Physical Review E*, vol. 100, no. 3, Sep. 2019. [Online]. Available: http://dx.doi.org/10.1103/PhysRevE.100.032306

[26] E. Arcaute, E. Hatna, P. Ferguson, H. Youn, A. Johansson, and M. Batty, "Constructing cities, deconstructing scaling laws," *Journal of the Royal Society Interface*, vol. 12, p. 20140745, 2015.

[27] C. R. Shalizi, "Scaling and hierarchy in urban economies," *arXiv*, 2011.

[28] J. C. Leitao, J. M. Miotto, M. Gerlach, and E. G. Altmann, "Is this scaling nonlinear?" *Royal Society Open Science*, vol. 3, p. 150649, 2016.

[29] M. Barthelemy, "Tomography of scaling," *Journal of the Royal Society Interface*, vol. 16, no. 160, 2019.

[30] J. Depersin and M. Barthelemy, "From global scaling to the dynamics of individual cities," *Proceedings of the National Academy of Sciences*, vol. 115, no. 10, pp. 2317–2322, 2018.

[31] M. Keuschnigg, "Scaling trajectories of cities," *Proceedings of the National Academy of Sciences*, vol. 116, no. 28, pp. 13 759–13 761, 2019.

[32] D. S. Connor, T. Kemeny, and M. Storper, "Frontier workers and the seedbeds of inequality and prosperity," *Journal of Economic Geography*, vol. 24, no. 3, pp. 393–414, 2024.

[33] D. S. Connor, S. Xie, J. Jang, A. E. Frazier, P. Kedron, G. Jain, Y. Yu, and T. Kemeny, "Big cities fuel inequality within and across generations," *PNAS nexus*, vol. 4, no. 2, p. pgae587, 2025.

[34] L. Bettencourt, V. C. Yang, J. Lobo, C. P. Kempes, D. Rybski, and M. J. Hamilton, "The interpretation of urban scaling analysis in time," *Journal of the Royal Society Interface*, vol. 17, no. 163, 2020.

[35] J. Lobo, L. Bettencourt, and S. G. Ortman, "Urban scaling theory: Answers to frequent questions," *Environment and Planning B: Urban Analytics and City Science*, vol. 52, no. 7, pp. 1701–1716, 2025.

[36] F. L. Ribeiro, J. Meirelles, V. M. Netto, C. R. Neto, and A. Baronchelli, "On the relation between transversal and longitudinal scaling in cities," *PloS one*, vol. 15, no. 5, p. e0233003, 2020.

[37] U. Marquis, O. Artime, R. Gallotti, and M. Barthelemy, "Universal roughness and the dynamics of urban expansion," *Physical Review Letters*, vol. 135, no. 18, p. 187403, 2025.

[38] I. Hong, M. R. Frank, I. Rahwan, W.-S. Jung, and H. Youn, "The universal pathway to innovative urban economies," *Science Advances*, vol. 6, no. 34, p. eaba4934, 2020. [Online]. Available: https://www.science.org/doi/abs/10.1126/sciadv.aba4934

[39] J. Q. Stewart, "Suggested principles of "social physics"," *Science*, vol. 106, no. 2748, pp. 179–180, 1947.

[40] S. Nordbeck, "Urban allometric growth," *Geografiska Annaler. Series B, Human Geography*, vol. 53, no. 1, pp. 54–67, 1971.

[41] M. Batty and P. Ferguson, "Defining city size," *Environment and Planning B: Planning and Design*, vol. 38, no. 5, pp. 753–756, 2011.

[42] S. G. Ortman, A. H. F. Cabaniss, J. O. Sturm, and L. M. A. Bettencourt, "The pre-history of urban scaling," *PLoS ONE*, vol. 9, no. 2, p. e87902, 2014.

[43] M. J. Hamilton, B. T. Milne, R. S. Walker, and J. H. Brown, "Nonlinear scaling of space use in human hunter–gatherers," *Proceedings of the National Academy of Sciences*, vol. 104, no. 111, pp. 4765–4769, 2007.

[44] J. R. Burger, J. G. Okie, I. Hatton, V. P. Weinberger, M. Shrestha, K. J. Liedtke, T. Be, A. R. Cruz, X. Feng, C. Hinojo-Hinojo, A. S. M. G. Kibria, K. C. Ernst, and B. J. Enquist, "Global city densities: re-examining urban scaling theory," *arXiv preprint arXiv:2210.08067*, 2022.

[45] S. Angel, J. Parent, D. L. Civco, and A. Blei, *Atlas of Urban Expansion – 2012 Edition, Volume 1: Areas and Densities*. New York; Nairobi; Cambridge, MA: New York University; UN-Habitat; Lincoln Institute of Land Policy, 2012.

[46] DLR - German Aerospace Center, "World settlement footprint evolution (wsf evolution)," https://download.geoservice.dlr.de/WSF_EVO/, n.d., accessed: 2026.

[47] World Population Review and Macrotrends, "Historical population (1950–2025) for urban areas," https://worldpopulationreview.com/ and https://www.macrotrends.net/global-metrics/cities, n.d., accessed: 2026.

[48] M. Kleiber, "Body size and metabolism," *Hilgardia*, vol. 6, no. 11, 1932.

[49] ——, "Body size and metabolic rate," *Physiological Reviews*, vol. 27, no. 4, pp. 511–541, 1947.

[50] B. J. Enquist, J. H. Brown, and G. B. West, "Allometric scaling of plant energetics and population density," *Nature*, vol. 395, no. 6698, pp. 163–165, Sep. 1998.

[51] G. B. West, J. H. Brown, and B. J. Enquist, "A general model for the origin of allometric scaling laws in biology," *Science*, vol. 276, no. 5309, pp. 122–126, Apr. 1997.

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