Supplementary Information

Spectral Moment Embedding: A canonical generalisation of
Characteristic Response Analysis for covariance-based signal classification

Supplementary Note: Full proofs of Theorems 1–5

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Notation and setup

Throughout, $\boldsymbol{C} \in \mathcal{S}_+(N)$ denotes the regularised sample covariance, $\boldsymbol{C} = \sum_{i=1}^{N} \lambda_i \boldsymbol{P}_i$ with $\boldsymbol{P}_i = \boldsymbol{v}_i \boldsymbol{v}_i^\top$ the rank-1 spectral projectors, v_{ij} the j-th component of $\boldsymbol{v}_i$, and $\mathcal{G}_N = \{-1, +1\}^N$ the sign-flip group acting on eigenvectors.

Theorem 1: Factorisation through spectral projectors

Theorem 1 (Factorisation)**.** *Any polynomial $\varphi : \mathcal{S}_+(N) \to \mathbb{R}$ that is well-defined on $\boldsymbol{C}$ (independent of eigendecomposition choice) is a polynomial in the entries of $\{\boldsymbol{C}^k\}_{k \geq 1}$.*

Proof. Well-definedness imposes two invariance conditions.

Step 1 ($\mathcal{G}_N$-invariance). The group $\mathcal{G}_N$ acts by independent sign-flips: $\boldsymbol{v}_i \mapsto s_i \boldsymbol{v}_i$ leaves $\boldsymbol{P}_i = (s_i \boldsymbol{v}_i)(s_i \boldsymbol{v}_i)^\top = \boldsymbol{P}_i$ and $\boldsymbol{C}$ unchanged. A monomial $\prod_{i,j} v_{ij}^{a_{ij}}$ transforms under $\boldsymbol{s} = (s_1, \ldots, s_N)$ as $\prod_i s_i^{\sum_j a_{ij}} \cdot \prod_{i,j} v_{ij}^{a_{ij}}$.

For invariance: $s_i^{\sum_j a_{ij}} = 1$ for all $s_i \in \{-1, +1\}$ requires $\sum_j a_{ij}$ to be even for each i. Any monomial with even row-degree in every row i decomposes as products of quadratic factors $v_{ij} v_{ij'} = P_i^{jj'}$ within each row. Hence the $\mathcal{G}_N$-invariant polynomial ring in $\{v_{ij}\}$ is generated by $\{P_i^{jj'} : 1 \leq i, j, j' \leq N\}$, the entries of $\boldsymbol{P}_i$. Therefore φ is a polynomial in $\{(\lambda_i, \boldsymbol{P}_i)\}_{i=1}^{N}$.

Step 2 (Permutation symmetry). The labelling of eigenvectors is arbitrary: any permutation $\sigma \in S_N$ sends $(\lambda_i, \boldsymbol{P}_i) \mapsto (\lambda_{\sigma(i)}, \boldsymbol{P}_{\sigma(i)})$ while leaving $\boldsymbol{C}$ unchanged. Hence φ must be symmetric in the atom pairs $\{(\lambda_i, \boldsymbol{P}_i)\}$.

Step 3 (Expression in moments by induction). The matrix moments satisfy $(\boldsymbol{C}^k)_{jj'} = \sum_i \lambda_i^k P_i^{jj'}$, so they are the multivariate power sums of the atoms. We show by induction on total degree d that every symmetric polynomial in $\{(\lambda_i, \boldsymbol{P}_i)\}$ is a polynomial in $\{(\boldsymbol{C}^k)_{jj'} : k \geq 1\}$.

Base $d = 0$: Constants are trivially in the algebra.

Inductive step: Consider a symmetric monomial $M = \frac{1}{|\mathrm{Orb}|} \sum_\sigma \prod_i \lambda_{\sigma(i)}^{a_i} \prod_{j,j'} (P_{\sigma(i)}^{jj'})^{b_i^{jj'}}$. Expanding the product $\prod_{r=1}^{R} (\boldsymbol{C}^{k_r})_{j_r j'_r} = \prod_r (\sum_i \lambda_i^{k_r} P_i^{j_r j'_r})$ yields a sum over $(i_1, \ldots, i_R) \in$

$[N]^R$. The *injective* part (all i_r distinct) gives exactly M (up to a combinatorial coefficient). The *collision* part (at least one repeated index) involves fewer than R distinct atoms and has strictly lower total degree in atom-count. By the inductive hypothesis, collision parts are already in $\mathbb{R}[(\boldsymbol{C}^k)_{jj'}]$. Rearranging therefore expresses M in the moment algebra. $\square$

Theorem 2: Generic injectivity

Theorem 2 (Injectivity)**.** *The map* $\Phi : \boldsymbol{C} \mapsto [\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{C}^2, \dots, \boldsymbol{C}^{N-1}]$ *is injective on the Zariski-open dense set* $\mathcal{U} = \{\boldsymbol{C} \in \mathcal{S}_+(N) : \lambda_i \neq \lambda_j$ *for all* $i \neq j\}$.

Proof. We show $\Phi(\boldsymbol{C})$ uniquely determines $\{\lambda_i\}$ and $\{\boldsymbol{P}_i\}$.

Step 1 (Eigenvalues via Newton's identities). The power sums $p_k = \mathrm{tr}(\boldsymbol{C}^k) = \sum_i \lambda_i^k$ for $k = 1, \dots, N$ determine the elementary symmetric polynomials e_ℓ via $e_\ell = \frac{1}{\ell}\sum_{r=1}^{\ell}(-1)^{r-1}e_{\ell-r}p_r$, with $e_0 = 1$. These are the coefficients of the characteristic polynomial $\chi(\lambda) = \sum_{\ell=0}^{N}(-1)^\ell e_\ell \lambda^{N-\ell}$, whose roots are $\{\lambda_i\}$. For $\boldsymbol{C} \in \mathcal{U}$, all roots are distinct, so $\{\lambda_i\}$ is uniquely determined. The set $\mathcal{S}_+(N) \setminus \mathcal{U}$ is the zero locus of $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)^2$, a non-zero polynomial; hence $\mathcal{U}$ is Zariski-open and dense.

Step 2 (Projectors via Vandermonde inversion). Fix any pair (j, j'). The N equations $(\boldsymbol{C}^k)_{jj'} = \sum_i \lambda_i^k P_i^{jj'}$ for $k = 0, \dots, N-1$ form the linear system $\boldsymbol{W}\boldsymbol{p}^{(jj')} = \boldsymbol{c}^{(jj')}$, where $W_{ki} = \lambda_i^k$ is an $N \times N$ Vandermonde matrix, $\boldsymbol{p}^{(jj')} = (P_1^{jj'}, \dots, P_N^{jj'})^\top$, and $\boldsymbol{c}^{(jj')} = (\delta_{jj'}, (\boldsymbol{C})_{jj'}, (\boldsymbol{C}^2)_{jj'}, \dots, (\boldsymbol{C}^{N-1})_{jj'})^\top$ uses $(\boldsymbol{C}^0)_{jj'} = \delta_{jj'}$.

Since $\det \boldsymbol{W} = \prod_{1 \le i < i' \le N}(\lambda_{i'} - \lambda_i) \neq 0$ for distinct eigenvalues, the system has the unique solution $\boldsymbol{p}^{(jj')} = \boldsymbol{W}^{-1}\boldsymbol{c}^{(jj')}$.

Step 3 (Reconstruction). Given $\{\lambda_i\}$ and $\{\boldsymbol{P}_i\}$, we can recover $\boldsymbol{C} = \sum_i \lambda_i \boldsymbol{P}_i$ exactly. $\square$

Theorem 3: POCR as a distorted upper bound

Theorem 3 (POCR bound and non-injectivity)**.** *With* $\mathrm{POCR}_j = \sum_i \lambda_i |v_{ij}|$ *and* $\mathrm{SME}_j^{(1)} = (\boldsymbol{C})_{jj} = \sum_i \lambda_i v_{ij}^2$*:* $\mathrm{POCR}_j(\boldsymbol{C})^2 \le \mathrm{tr}(\boldsymbol{C}) \cdot \mathrm{SME}_j^{(1)}(\boldsymbol{C})$*, with equality iff all* $|v_{ij}|$ *are equal. POCR is non-injective: there exist distinct* $\boldsymbol{C}, \boldsymbol{C}' \in \mathcal{S}_+(N)$ *with* $\mathrm{POCR}(\boldsymbol{C}) = \mathrm{POCR}(\boldsymbol{C}')$.

Proof. **Part 1 (Cauchy-Schwarz).** Let $\boldsymbol{\alpha} = (\sqrt{\lambda_i})$, $\boldsymbol{\beta} = (\sqrt{\lambda_i}|v_{ij}|)$. Then $\mathrm{POCR}_j^2 = \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle^2 \le \|\boldsymbol{\alpha}\|^2 \|\boldsymbol{\beta}\|^2 = (\sum_i \lambda_i)(\sum_i \lambda_i v_{ij}^2) = \mathrm{tr}(\boldsymbol{C}) \cdot (\boldsymbol{C})_{jj}$. Equality holds iff $\boldsymbol{\beta} = c\boldsymbol{\alpha}$, i.e., all $|v_{ij}| = c$.

Part 2 (Explicit non-injective pair). Let $N = 2$, $\lambda_1 > \lambda_2 > 0$, $c, s > 0$, $c^2 + s^2 = 1$, $c \neq s$. Define $\boldsymbol{v}_1 = (c, s)^\top$, $\boldsymbol{v}_2 = (-s, c)^\top$ and $\boldsymbol{v}_1' = (c, -s)^\top$, $\boldsymbol{v}_2' = (s, c)^\top$. Both pairs are orthonormal (verify: $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$, $\boldsymbol{v}_1' \cdot \boldsymbol{v}_2' = cs + (-s)c = 0$). Since $|v_{ij}| = |v_{ij}'|$ for all i, j, POCR vectors coincide.

Direct computation yields:

$$\boldsymbol{C} = \begin{pmatrix} \lambda_1 c^2 + \lambda_2 s^2 & (\lambda_1 - \lambda_2)cs \\ (\lambda_1 - \lambda_2)cs & \lambda_1 s^2 + \lambda_2 c^2 \end{pmatrix}, \quad \boldsymbol{C}' = \begin{pmatrix} \lambda_1 c^2 + \lambda_2 s^2 & -(\lambda_1 - \lambda_2)cs \\ -(\lambda_1 - \lambda_2)cs & \lambda_1 s^2 + \lambda_2 c^2 \end{pmatrix}.$$

Since $(\lambda_1 - \lambda_2)cs > 0$ we have $\boldsymbol{C} \neq \boldsymbol{C}'$ (opposite off-diagonal signs). Note $\boldsymbol{v}_2' = (s, c) \neq \pm(-s, c) = \pm\boldsymbol{v}_2$ for $c \neq s$, confirming these are genuinely distinct matrices. Meanwhile $\mathrm{vech}(\boldsymbol{C}) \neq \mathrm{vech}(\boldsymbol{C}')$ since $C_{12} \neq C_{12}'$, so $\mathrm{SME}_{\mathrm{vech},1}$ separates them. $\square$

Theorem 4: Lipschitz stability

Theorem 4 (Lipschitz continuity)**.** *For bounded* $\mathcal{B} \subset \mathcal{S}_+(N)$*,* $M = \sup_{\mathcal{B}} \|\boldsymbol{C}\|_{\mathrm{op}}$*, and projection* P *with norm* $\|P\|$*:* $\|\mathrm{SME}_{P,p}(\boldsymbol{C}) - \mathrm{SME}_{P,p}(\boldsymbol{C}')\| \leq \|P\| \left(\sum_{k=1}^{p} kM^{k-1}\right) \|\boldsymbol{C} - \boldsymbol{C}'\|_{\mathrm{op}}$*.*

Proof. **Step 1 (Telescoping identity).** For any square matrices $\boldsymbol{A}, \boldsymbol{B}$, $\boldsymbol{A}^k - \boldsymbol{B}^k = \sum_{r=0}^{k-1} \boldsymbol{A}^r(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{B}^{k-1-r}$. This is proved by induction: the base case $k = 1$ is $\boldsymbol{A}^0(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{B}^0 = \boldsymbol{A} - \boldsymbol{B}$; the inductive step uses $\boldsymbol{A}^k - \boldsymbol{B}^k = \boldsymbol{A}(\boldsymbol{A}^{k-1} - \boldsymbol{B}^{k-1}) + (\boldsymbol{A} - \boldsymbol{B})\boldsymbol{B}^{k-1}$ and substitutes the inductive hypothesis for the first term.

Step 2 (Norm bound). Applying sub-multiplicativity to the telescoping sum with $\boldsymbol{A} = \boldsymbol{C}$, $\boldsymbol{B} = \boldsymbol{C}'$:

$$\|\boldsymbol{C}^k - \boldsymbol{C}'^k\| \leq \sum_{r=0}^{k-1} M^r \|\boldsymbol{C} - \boldsymbol{C}'\| M^{k-1-r} = kM^{k-1}\|\boldsymbol{C} - \boldsymbol{C}'\|.$$

Step 3 (Final combination).

$$\|\mathrm{SME}_{P,p}(\boldsymbol{C}) - \mathrm{SME}_{P,p}(\boldsymbol{C}')\| \leq \sum_{k=1}^{p} \|P\| \|\boldsymbol{C}^k - \boldsymbol{C}'^k\| \leq \|P\| \left(\sum_{k=1}^{p} kM^{k-1}\right) \|\boldsymbol{C} - \boldsymbol{C}'\|.$$

Since $M, p < \infty$, the Lipschitz constant is finite. □

Theorem 5: Approximation density

Theorem 5 (Stone-Weierstrass density)**.** *The algebra* $\mathcal{A}$ *of polynomials in* $\{(\boldsymbol{C}^k)_{jj'}\}_{k\geq 1}$ *is uniformly dense in* $C(K)$ *for any compact* $K \subset \mathcal{S}_+(N)$*.*

Proof. We verify the three hypotheses of the Stone-Weierstrass theorem.

(i) Constants. The zero polynomial plus a constant belongs to $\mathcal{A}$.

(ii) Closed under multiplication. Products of polynomials in $\{(\boldsymbol{C}^k)_{jj'}\}$ are again such polynomials.

(iii) Separates points. For any distinct $\boldsymbol{C} \neq \boldsymbol{C}' \in K$, there exist j, j' with $C_{jj'} \neq C'_{jj'}$. The function $f : \boldsymbol{C} \mapsto (\boldsymbol{C}^1)_{jj'} = C_{jj'}$ belongs to $\mathcal{A}$ and satisfies $f(\boldsymbol{C}) \neq f(\boldsymbol{C}')$. This holds for all $\boldsymbol{C}, \boldsymbol{C}' \in K$ without any genericity condition.

By Stone-Weierstrass, $\mathcal{A}$ is uniformly dense in $C(K)$.

Interpretation: Combined with Theorem 1 (every $\mathcal{G}_N$-invariant polynomial is in $\mathcal{A}$), this means any continuous, well-defined function of the covariance spectrum and eigenvectors can be uniformly approximated by SME features. SME is therefore a universal covariance representation. □