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A Novel approach for the solvability of Monotone Bilevel framework for split variational inequality problems within Hilbert setting optimization

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A Novel approach for the solvability of Monotone Bilevel framework for split variational inequality problems within Hilbert setting optimization

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Abstract

This paper presents a novel algorithm modified inertial Tseng extragradient approach with a self-adaptive step size for the approximation of the solutions for the celebrated bilevel split variational inequality problem in the framework of Hilbert setting. Using the approach of projections onto a constructible half-space, we establish the required strong convergence result for the proposed iterative technique. Our method does not require prior techniques of the Lipschitz type constant for the cost operators. Finally, we present several numerical experiments that show how much efficient and applicable the proposed techniques are in contrast with the well-known existing results.

Keywords: Bilevel variational inequality problem, Split bilvel variational inequality problem, iterative techniques

MSC Classification: 47H06 , 47H09 , 47J05 , 47J25

1 Introduction

Considering a real Hilbert space denoted by H endowed norm $\|\cdot\|$ is induced from its inner product $\langle\cdot,\cdot\rangle$. Moreover, we also consider a $\emptyset \neq C \subset H$ to be a closed, and convex set. Let $A, F : H \to H$ be nonlinear operators. It is well known that the bilevel variational inequality problem (BVIP) can be formulated in brief as:

$$\text{Seek } x \text{ in } VI(C,A) \text{ s.t. } \langle Fx, y - x\rangle \geq 0 \qquad \text{for all } y \text{ in } VI(C,A). \tag{1}$$

where $VI(C,A)$ symbolizes the solution set corresponding to the associated classical Variational Inequality Problem (VIP). We recall that the VIP is designed by:

$$\text{Determine } x \in C \text{ s.t. } \langle Ax, x - y\rangle \leq 0 \qquad \text{for all } y \text{ in } C. \tag{2}$$

We notice from (1) that the bilevel variational inequality problem can be seen as a generalization to the classical VIP (2).
The notion of the variational inequality problem (VIP) was independently formulated by Stampacchia [1] and Fichera [2, 3]. It was developped to tackle model challenges to various problems in mechanics particularly those related to the types of boundary conditions known as Signorini. Many mathematical and physical Scineces problems, particularly those arising in financial mathematics and economics, can be formulated as VIPs. Owing to the broad range of application of the VIPs, numerous iterative techniques have been developed to solve (2). In particular, Goldstein [4] presented the given technique (iterative scheme):

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda A x_n), \forall n \in \mathbb{N}, \end{cases} \tag{3}$$

where λ is chosen in the range $0 < \lambda < \frac{2\alpha}{L^2}$, A stands for the α-strongly monotone operator which is Lipschitz continuous with constant L, while P_C stands for the projection map on C and defined on H. The author established that the iterative method (3) converges to the solutions set of the variational inequality given in relation (2). Nevertheless, He and Liao [5] observed that the iterative scheme (3) may not converge to the solution set of (2) whenever the operator A is simply Lipschitz-continuous. For further detailed subsequent development in this direction, see the references therein. Moreover, we must emphasis that to compute the appropriate value of λ may pause very difficult challenges or even at time completely impossible in practice which causes the drawbacks for the mentioned techniques. In order to tacke these issues, Korpelevich [6] suggested the EM:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \text{ for all } n \in \mathbb{N}, \end{cases} \tag{4}$$

where λ_n are the parameters chosen so that $0 < \lambda_n < \frac{1}{L}$, A is monotone and L-Lipschitz continuous, and P_C denotes the metric projection of H onto C. This method was developed under weaker assumption on the cost operator; however, the computation of the step size λ_n remains a challenge. In addition to the aforementioned drawbacks, there are further limitations of the iterative scheme (4), most notably the requirement of two projections on C per iteration, which can be computationally expensive when C has no simple structure.

Since the introduction of the extragradient method, several authors have proposed and analyzed various modified versions under the conditions of monotonicity and pseudomonotonicity of A; see, for instance, He et al. [7], Apostol et al. [8], He et al. [5], Ceng et al. [9, 10], Nadezhkina and Takahashi [11], and the references therein. Motivated by the need to overcome the limitations of the extragradient method, Censor et al. [12] presented and extensively investigated the subgradient extragradient method, which is defined by:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ T_n = \{w \in H : \langle x_n - \lambda_n A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda_n A y_n), \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\lambda_n \in \left(0, \frac{1}{L}\right)$, A is monotone and Lipschitz continuous, and P_C, the metric mapping projecting H on C. It was established that the iterative method (5) converges to a solution of VIP in (2). However, the computation of the step size λ_n chosen above remains a major challenge.

The extension of VIP in (2) was introduced and investigated in the work [13, 14]. They proposed the split variational inequality problem (SVIP):

$$\text{Determine } x^* \in C \quad \text{s.t. } \langle Ax^*, x - x^* \rangle \geq 0 \quad \text{for all } x \in C \tag{6}$$

and

$$y^* = Tx^* \in Q \text{ for which } \langle By^*, y - y^* \rangle \geq 0 \quad \text{holds for all } y \in Q, \tag{7}$$

where H_1 and H_2 are both real Hilbert spaces, $\emptyset \neq C \subset H_1$ and $H_2 \supset Q \neq \emptyset$ are closed and convex subsets of, A and B are both self-mappings on H_1 and H_2, resp. and the map $T : H_1 \to H_2$ is a bounded linear operator. As we can see from the literature, the SVIP has been applied to many fields such as phase retrieval, medical image reconstruction, signal processing and radiation therapy treatment planning [15–18] and the references therein. We now show that the SVIP formulated by (1.1) - (1.2) can be viewed as a combination of the classical VIP (1.3) and the well-known Split Feasibility Problem (SFP) introduced and studied by Censor and Elfving in [18]: Solve

for $x^* \in C$ which fulfills

$$Tx^* = y^* \in Q. \tag{8}$$

In the recent work [19], Anh et al. developed a subgradient extragradient method for solving the BSVIP, where the operator A^- is strongly monotone, while the operators A_i are only pseudomonotone. The BSVIP is formulated as"

Determine $x^* \in \Gamma$ for which the inequality $\langle Fx^*, x - x^* \rangle \geq 0$ is fulfilled $\forall\, x \in \Gamma$, (9)

where Γ is the solution of the SVIP (6)-(7), $F : H_1 \to H_1$ is κ-strongly monotone and L_3- Lipschitz continuous on H_1. We denote the solution of problem (9) by Ω.

Arguing as in Anh et al. [19], we introduce the iterative method used in Algorithm 1 studied by the authors of [19].

Algorithm 1 Initialization:

Step 1. Let $x_0 \in H_1$, $0 < \mu < \frac{2\beta}{L^2}$, the sequences $\{\alpha_n\} \subset (0, 1)$ $\{\eta_n\}$, $\{\delta_n\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ such that $\{\delta_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|+1})$, $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, \frac{1}{L_1})$, and $\{\mu_n\} \subset [e, f]$ for some $e, f \in (0, \frac{1}{L_2})$ and $\mu \in (0, \frac{2\beta}{L^2})$.

Step 2. Compute $u_n = A(x_n)$

$$v_n = P_Q(u_n - \mu_n F_2 u_n) \tag{10}$$

$$w_n = P_{Q_n}(u_n - \mu_n F_2 v_n), \tag{11}$$

where $Q_n = \{\omega_2 \in H_2 : \langle u_n - \mu_n F_2 u_n - v_n, \omega_2 - v_n \rangle \leq 0\}$.
Compute

$$y_n = x_n + \delta_n A^*(w_n - u_n). \tag{12}$$

$$t_n = P_C(y_n - \lambda_n F_1 y_n), \tag{13}$$

$$z_n = P_{C_n}(y_n - \lambda_n F_1 t_n), \tag{14}$$

where $C_n = \{\omega_1 \in H_1 : \langle y_n - \lambda_n F_1 y_n - t_n, \omega_1 - t_n \rangle \leq 0\}$.

Step 3. Compute

$$x_{n+1} = \eta_n x_n + (1 - \eta_n) z_n - \alpha_n \mu F z_n, \tag{15}$$

where $F : H_1 \to H_1$ is κ-strongly monotone and L_3-Lipschitz continuous on H_1, $F_1 : H_1 \to H_1$ is pseudomonotone on C, L_1-Lipschitz continuous on H_1, and $F_2 : H_2 \to H_2$ is pseudomonotone on Q, L_2-Lipschitz continuous on H_1. The authors showed that the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the unique solution of the BSVIP (9), provided that the solution set of the SVIP (6)-(7) is nonempty. We observe the following drawbacks in the iterative processes; The above iterative method in Algorithm 1 requires four projections per each iteration which will definitely affect the efficiency. Also, the proposed algorithm requires the

operator A to be bounded and with Lipschitz constant.

Recently, based on these drawbacks, Huy et al. [20], proposed a modified Tseng's extra-gradient method for solving the BSVIP (9). They investigated the iterative method in:

Algorithm 2 Initialization:

Step 1. Let $x_0 \in H_1, \mu_0 > 0, \lambda_0 > 0, \mu_n \in (0,1), \lambda_n \in (0,1), \{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$.

Step 2. Compute $u_n = A(x_n)$

$$v_n = P_Q(u_n - \mu_n F_2 u_n) \tag{16}$$

$$w_n = v_n - \mu_n(F_2 v_n - F_2 u_n), \tag{17}$$

where

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - v_n\|}{\|F_2 u_n - F_2 v_n\|}, \mu_n\right\}, & \text{if } F_2 u_n \neq F_2 v_n \\ \mu_n, & \text{otherwise.} \end{cases} \tag{18}$$

Compute

$$y_n = x_n + \delta_n A^*(w_n - u_n), \tag{19}$$

where the step size

$$\delta_n = \begin{cases} \min\left\{\frac{\|w_n - u_n\|}{2\|A^*(w_n - u_n)\|}, \mu_n\right\}, & \text{if } A^*(w_n - u_n) \neq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

$$z_n = P_C(y_n - \lambda_n F_1 y_n), \tag{21}$$

$$t_n = z_n - \lambda_n(F_1 z_n - F_1 y_n), \tag{22}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\lambda\|y_n - z_n\|}{\|F_1 y_n - F_1 z_n\|}, \lambda_n\right\}, & \text{if } F_1 y_n \neq F_1 z_n \\ \lambda_n, & \text{otherwise,} \end{cases} \tag{23}$$

Step 3. Compute

$$x_{n+1} = t_n - \alpha_n F t_n. \tag{24}$$

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where $F : H_1 \to H_1$ is κ-strongly monotone and L_3-Lipschitz continuous on H_1, $F_i : H_i \to H_i$, is $F_1 : H_1 \to H_1$ is pseudomonotone on C, and Q, L_i- Lipschitz continuous on H_i, $i = 1, 2$, resp. The authors ([20]) showed that the sequence $\{x_n\}$ generated by Algorithm 2 has a unique limit, which is the only solution of the BSVIP (9), whenever the solution set of the SVIP (6)-(7) is nonempty. We observe the following drawbacks in the iterative processes; The above iterative method Algorithm 2 requires four projections per each iteration, which definitely has an impact on the

efficiency. Also, the proposed algorithm requires the operator A to be bounded and with Lipschitz constant.

Recently, the basis of these drawbacks, Huy et al. [20], proposed a modified Tseng's extragradient method for solving the BSVIP (9). They investigated:

Algorithm 3. Initialization:

Step 1. Let $x_0 \in H_1, \mu_0 > 0, \lambda_0 > 0, \mu_n \in (0, 1), \lambda_n \in (0, 1), \{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Step 2. Compute $u_n = A(x_n)$

$$v_n = P_Q(u_n - \mu_n F_2 u_n) \tag{25}$$

$$w_n = v_n - \mu_n(F_2 v_n - F_2 u_n), \tag{26}$$

where

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - v_n\|}{\|F_2 u_n - F_2 v_n\|}, \mu_n\right\}, & \text{if } F_2 u_n \neq F_2 v_n \\ \mu_n, & \text{otherwise.} \end{cases} \tag{27}$$

Compute

$$y_n = x_n + \delta_n A^*(w_n - u_n), \tag{28}$$

where the step size

$$\delta_n = \begin{cases} \min\left\{\frac{\|w_n - u_n\|}{2\|A^*(w_n - u_n)\|}, \mu_n\right\}, & \text{if } A^*(w_n - u_n) \neq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

$$z_n = P_C(y_n - \lambda_n F_1 y_n), \tag{30}$$

$$t_n = z_n - \lambda_n(F_1 z_n - F_1 y_n), \tag{31}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\lambda\|y_n - z_n\|}{\|F_1 y_n - F_1 z_n\|}, \lambda_n\right\}, & \text{if } F_1 y_n \neq F_1 z_n \\ \lambda_n, & \text{otherwise,} \end{cases} \tag{32}$$

Step 3. Compute

$$x_{n+1} = t_n - \alpha_n F t_n. \tag{33}$$

where $F : H_1 \to H_1$ is κ-strongly monotone and L_3-Lipschitz continuous on H_1, $F_1 : H_1 \to H_1$ is pseudomonotone on C, L_1-Lipschitz continuous on H_1, and $F_2 : H_2 \to H_2$ is pseudomonotone on Q, L_2-Lipschitz continuous on H_1. They established that the sequence $\{x_n\}$ generated by Algorithm 2 is strongly convergent and its limit is the unique solution of the BSVIP (9), whenever, the solution set of the SVIP (6)-(7) is nonempty.

Remark 1 In the above iterative method Algorithm 2, one needs to calculate two projections per iteration, onto each feasible subsets of the space H_1 and H_2 respectively. However, whenever the feasible sets lack simple structures, then the algorithm becomes really challenging to implement.

This study is motivated by the findings in [19, 20] and the growing body of ongoing research work in this field. The main purpose of this work is to introduce and study a new form of Algorithms 1, 2 and 2 for solving problem (9) in infinite-dimensional real Hilbert spaces. This iterative algorithm resolves the drawbacks in the above iterative methods. The method has the following important features:

1. Our method only assumes two projections onto a half-space per iteration, which we evaluate without any hurdle. Hence, our method is less expensive when compared to several iterative methods (see, [19, 20] and the references therein) for solving BSVIP (9).
2. Our proposed iterative method does not depend on the knowledge of the bounded linear operator $||T||$ unlike the following iterative methods in which knowledge of the bounded linear operator is relevant for their implementation (see [13, 19, 21, 22]).
3. In the iterative scheme we introduced the modified inertial extrapolation steps. The modified inertial extrapolation step makes the convergence rate of the iterative method to be better. The inertial steps brought a large gain in the rate of convergence of the algorithm compared with other similar works in the literature, (for instance, see [19, 20] and the references therein).

This paper is organized in several folds, Section 2 helps us to recall very crucial and relevant basic concepts and results which we deem super important for this investigation. In Section 3, the proof of our proposed method is given. In Section 4, the strong convergence of our given scheme is proved. In Section 5, some numerical results are given to show the performance of the new scheme in infinite dimensional Hilbert spaces. Finally, in Section 6, conclusion of the paper is given.

2 preliminaries

This background section helps us to recall several well-known concepts that maybe needed throughout the paper. Given a real Hilbert space H, the fixed points set of a nonlinear map $T : H \to H$ will be denoted by $F(T)$. Note that $F(T) := \{x \in H : Tx = x\}$. It is customary to use the symbols "$\rightharpoonup$" and "$\to$", respectively to represent the weak and the strong convergence. For any $x, y \in H$ we find $\alpha \in R$ particularly $0 \leq \alpha \leq 1$ such that

$$||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2. \tag{34}$$

$$||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2. \tag{35}$$

$$\|x-y\|^2 \leq \|x\|^2 + 2\langle y, x-y\rangle. \tag{36}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{37}$$

Definition 1 Let $T : H \to H$ be an operator. Then T is called

(a) L-Lipschitz continuous whenever

$$\|Tx - Ty\| \leq L\|x-y\|,$$

for all $x, y \in H$; and some $L > 0$.

(b) nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|,$$

for all $x, y \in H$;

(c) quasinonexpansive, if

$$\|Tx - y\| \leq \|x-y\|,$$

for all $x \in H$ and $y \in F(T)$;

(d) monotone if

$$\langle Tx - Ty, x-y\rangle \geq 0, \quad \forall x, y \in H;$$

(e) pseudomonotone if

$$\langle Tx, y-x\rangle \geq 0 \Rightarrow \langle Ty, y-x\rangle \geq 0, \quad \forall x, y \in H;$$

(f) α-strongly monotone if there exists $\alpha > 0$, such that

$$\langle Tx - Ty, x-y\rangle \geq \alpha\|x-y\|^2, \quad \forall\, x, y \in H;$$

Let C be a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\| \quad \forall y \in C.$$

The operator P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies the following properties.

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{38}$$

for all $x, y \in H$. Furthermore, P_C is characterized by the property

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{39}$$

for all $x \in H$ and $y \in C$.

Lemma 4 [23] Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \; n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty} \{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim_{k\to\infty} a_n = 0$.

Definition 2 A function $k : H \to R$ is said to be Gateaux differentiable at $x \in H$, if there exist an element denoted by $k'(x) \in H$ such that

$$\lim_{h\to 0} \frac{k(x + hy) - k(x)}{h} = \langle y, k'(x) \rangle$$

for all $y \in H$, $h \in [0, 1]$, where $k'(x)$ is called the Gateaux differential of k at x. We also recall that if for each $x \in H$, k is Gateaux differentiable at x, then k is Gateaux differentiable on H.

Definition 3 Let H be a real Hilbert space. A function $k : H \to R \cup \infty$ is said to be weakly lower semicontinuous at $x \in H$ if

$$\lim_{n\to\infty} k(x_n) \geq k(x)$$

hold for any arbitrary sequence $\{x_n\}$ in H satisfying the fact that $\{x_n\}$ converges weakly to x.

Definition 4 A convex set $h : H \to R$ is said to be subdifferentiable at a point $x \in H$ if the set

$$\partial h(x) = \{p \in H : h(y) \geq h(p) + \langle p, y - x \rangle \quad \forall y \in H\} \tag{40}$$

is nonemepty. Each element in $\partial h(x)$ is called a subgradient of h at x. We note that if h is subdifferentiable at each $x \in H$ then h is subdifferentiable on H. It is also known that if h is Gateaux differentiable at x, then h is subdifferential at x and $\partial h(x) = \{h'(x)\}$.

Lemma 5 Let $VI(C, A)$ be the solution set for the VIP (2) such that $VI(C, A) \neq \emptyset$ and C is define as $C := \{x \in H : h(x) \leq 0\}$, where $h : H \to R$ is a continuously differentiable convex function. Let $p \in C$, then $p \in VI(C, A)$ if and only if

1. $A(p) = 0$ or
2. $p \in \partial C$ and there exist $\tau > 0$ such that $A(p) = \tau h'(p)$, where ∂C denotes the boundary of C.

3 Proposed Algorithm

In this subsection we propose an algorithm to solve the quasimonotone variational inequality problem and the fixed point problem.

Assumption 6 Condition A. Suppose

1. The feasible sets C and Q are nonempty, closed and convex subsets of the real Hilbert spaces H_1 and H_2 respectively. They are defined as

$$C = \{x \in H_1 : k(x) \leq 0\}$$
$$Q = \{x \in H_2 : h(x) \leq 0\},$$

where $k : H_1 \to R$ and $h : H_2 \to R$ satisfies the following conditions:
   (a) k and h are continuously differentiable convex functions such that $k'(.)$ and $h'(.)$ are L_4-Lipschitz continuous and L_5-Lipschitz continuous, respectively;
   (b) k and h are weakly lower semicontinuous on H_1 and H_2 respectively.
2. $A : H_2 \to H_2$ and $B : H_1 \to H_1$ are monotone and Lipschitz continuous with Lipschitz constant L_2 and L_1 respectively.
3. $F : H_1 \to H_1$ is κ-strongly monotone and L_3-Lipschitz continuous on H_1.
4. $T : H_1 \to H_2$ is a bounded linear operator.
5. The solution set $\Omega \neq \emptyset$.

Condition B.

1. $\alpha_n \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
2. $\lambda_0 > 0, \nu_0 > 0, \mu, \delta, \tau, \tau_1 \in (0, \frac{1}{2})$ such that $1 - \tau^2 - \mu^2 > 0$ and $1 - \tau^2 - \delta_1^2 > 0$

We give here the main results of the iterative method obtained in this paper in the algorithm below.

Algorithm 7 Initialization Step:

Step 1: Select $x_0, x_1 \in H_1$, considering the iterates x_{n-1} and x_n for each $n \in N$, select ϑ_n s.t. $0 \leq \vartheta_n \leq \bar{\vartheta}_n$, where

$$\bar{\vartheta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\delta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_{n-1} \neq x_n \\ \frac{n-1}{n+\delta-1}, & \text{otherwise,} \end{cases} \tag{41}$$

with $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = o(\alpha_n)$.

Step 2. Set

$$w_n = x_n + \vartheta_n(x_n - x_{n-1}).$$

Evaluate

$$y_n = P_{Q_n}(Tw_n - \lambda_n ATw_n), \tag{42}$$

$$z_n = y_n - \lambda_n(Ay_n - ATw_n) \tag{43}$$

where $Q_n = \{x \in H_2 : h(Tw_n) + \langle h'(Tw_n), x - Tw_n \rangle \leq 0\}$ and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|Tw_n - y_n\|}{\|ATw_n - Ay_n\| + \|h'(Tw_n) - h'(y_n)\|}, \lambda_n\right\}, & \text{if } \|ATw_n - Ay_n\| + \|h'(Tw_n) - h'(y_n)\| \neq 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{44}$$

Step 3. Evaluate

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{45}$$

the parameters γ_n are selected so that for sufficiently small $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right)$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.

Step 4. Evaluate

$$u_n = P_{C_n}(v_n - v_n Bv_n), \tag{46}$$

$$t_n = u_n - v_n(Bu_n - Bv_n), \tag{47}$$

$C_n = \{y \in H_1 : v_n + \langle k'(v_n), y - v_n \rangle \leq 0\}$ and

$$v_{n+1} = \begin{cases} \min\left\{\frac{\delta\|v_n - u_n\|}{\|Bv_n - Bu_n\| + \|k'(v_n) - k'(u_n)\|}, v_n\right\}, & \text{if } \|Bv_n - Bu_n\| + \|k'(v_n) - k'(u_n)\| \neq 0, \\ v_n, & Bv_n = Bu_n \text{ and } k'(v_n) = k'(u_n). \end{cases} \tag{48}$$

Step 5. Compute

$$x_{n+1} = t_n + \alpha_n F t_n \tag{49}$$

Remark 2 It is easy to see from Algorithm 7 that $\lambda_{n+1} \leq \lambda_n$ for all $n \in \mathbb{N}$. Since A is monotone and L_2-Lipschitz continuous and h is Lipschitz continuous, we have

$$\|ATw_n - Ay_n\| + \|h'(Tw_n) - h'(y_n)\| \neq 0,$$

thus we have

$$\frac{\mu\|Tw_n - y_n\|}{\|ATw_n - Ay_n\| + \|h'(Tw_n) - h'(y_n)\|} \geq \frac{\mu\|Tw_n - y_n\|}{(L_2 + L_5)\|Tw_n - y_n\|} = \frac{\mu}{L_2 + L_5} \tag{50}$$

as such, we have

$$\lambda_{n+1} = \min\left\{\frac{\mu\|Tw_n - y_n\|}{\|ATw_n - Ay_n\| + \|h'(Tw_n) - h'(y_n)\|}, \lambda_n\right\} \geq \min\left\{\frac{\mu}{L_2 + L_5}, \lambda_n\right\}, \tag{51}$$

using induction techniques, we assert that $\{\lambda_{n+1}\}$ is bounded by $\min\left\{\frac{\mu}{L_2+L_5}, \lambda_1\right\}$. Using the fact that $\{\lambda_n\}$ is monotone nonincreasing, we have that

$$\lim_{n\to\infty} \lambda_n \geq \min\left\{\frac{\mu}{L_2 + L_5}, \lambda_0\right\} > 0.$$

4 Convergence Analysis

Lemma 8 The step-sizes γ_n, in Algorithm 7 are well defined.

Proof To prove that γ_n is well defined, we obviously follow a similar approach as in the lines for Lemma 3.6 of [11], thus we omit it. □

Lemma 9 Any sequence $\{x_n\}$ generated by Algorithm 7 is bounded whenever Assumption 6 holds.

Proof Let $p \in \Omega$, it then follows that $Tp \in VI(A, Q) \subset Q$. Since $\lim_{n\to\infty} \frac{\vartheta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\vartheta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$, for all $n \in \mathbb{N}$. Then using Algorithm 2, we have

$$\begin{aligned}\|w_n - p\| &= \|x_n + \vartheta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \vartheta_n\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \alpha_n \frac{\vartheta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \alpha_n N_1. \end{aligned} \tag{52}$$

Also, using Algorithm 7, we have

$$\begin{aligned}\|z_n - Tp\|^2 &= \|y_n - \lambda_n(Ay_n - ATw_n) - Tp\|^2 \\ &= \|y_n - Tp\|^2 + \lambda_n^2\|Ay_n - ATw_n\|^2 - 2\lambda_n\langle y_n - Tp, Ay_n - ATw_n\rangle. \end{aligned} \tag{53}$$

Suppose $Tp \in \Omega$, we have $Tp \in Q \subset Q_n$. Since $y_n = P_{Q_n}(Tw_n - \lambda_n ATw_n)$, and $Tp \in Q_n$, using the characteristics of the metric projection, we obtain

$$\langle Tw_n - \lambda_n ATw_n - y_n, y_n - Tp\rangle \geq 0,$$

which implies that

$$2\langle Tw_n - y_n, y_n - Tp\rangle - 2\lambda_n\langle ATw_n - Ay_n, y_n - Tp\rangle \geq 2\lambda_n\langle Ay_n, y_n - Tp\rangle. \tag{54}$$

Using (35), we have

$$2\langle Tw_n - y_n, y_n - Tp\rangle = \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - \|y_n - Tp\|^2, \tag{55}$$

substituting this into (54), we have

$$\|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - \|y_n - Tp\|^2 - 2\lambda_n\langle ATw_n - Ay_n, y_n - Tp\rangle \geq 2\lambda_n\langle Ay_n, y_n - Tp\rangle. \tag{56}$$

Using the fact that A is monotone, we have

$$\begin{aligned}\|y_n - Tp\|^2 &\leq \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - 2\lambda_n\langle ATw_n - Ay_n, y_n - Tp\rangle - 2\lambda_n\langle Ay_n, y_n - Tp\rangle \\ &= \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - 2\lambda_n\langle ATw_n - Ay_n, y_n - Tp\rangle - 2\lambda_n(\langle Ay_n - ATp, y_n - Tp\rangle \\ &\quad + \langle ATp, y_n - Tp\rangle) \\ &\leq \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - 2\lambda_n\langle ATw_n - Ay_n, y_n - Tp\rangle - 2\lambda_n\langle ATp, y_n - Tp\rangle. \end{aligned} \tag{57}$$

Substituting (57) into (53), we have

$$\begin{aligned}\|z_n - Tp\|^2 &\leq \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 - 2\lambda_n\langle ATw_n - Ay_n, y_n - Tp\rangle - 2\lambda_n\langle ATp, y_n - Tp\rangle \\ &\quad + \lambda_n^2\|Ay_n - ATw_n\|^2 - 2\lambda_n\langle y_n - Tp, Ay_n - ATw_n\rangle \\ &= \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 + 2\lambda_n\langle ATp, Tp - y_n\rangle + \lambda_n^2\|Ay_n - ATw_n\|^2. \end{aligned} \tag{58}$$

In what follows, we consider two cases in respect to Lemma 9

From the step size (44), we obtain that

$$\|ATw_n - Ay_n\| = \frac{\mu}{\lambda_{n+1}}\|Tw_n - y_n\| - \|h'(Tw_n) - h'(y_n)\|$$

$$\leq \frac{\mu}{\lambda_{n+1}}\|Tw_n - y_n\|,$$

which implies that

$$\|ATw_n - Ay_n\|^2 \leq \frac{\mu^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2$$

Case I: Suppose that $ATp = 0$, and using the stepsize (44), we have (58) becomes

$$\begin{aligned}\|z_n - Tp\|^2 &\leq \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 + \lambda_n^2\|Ay_n - ATw_n\|^2\\ &\leq \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 + \frac{\mu^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2\\ &\leq \|Tw_n - Tp\|^2 - (1 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|Tw_n - y_n\|^2\\ &\leq \|Tw_n - Tp\|^2 - (1 - \tau^2 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|Tw_n - y_n\|^2, \end{aligned} \tag{59}$$

where $\tau > 0$, since $\lim_{n\to\infty}\lambda_{n+1} = \lim_{n\to\infty}\lambda_n > 0$ and item 2. of **Condition B**, we $\lim_{n\to\infty}(1 - \tau^2 - \frac{\mu^2\lambda_n}{\lambda_{n+1}^2}) = 1 - \tau^2 - \mu > 0$. Thus, we have

$$\|z_n - Tp\| \leq \|Tw_n - Tp\|. \tag{60}$$

Case II: Suppose that $ATp \neq 0$. Using Lemma 5, we have $Tp \in \partial Q$ and there exists $\tau > 0$ such that $ATp = -\tau h'(Tp)$. Clearly, $h(Tp) = 0$ since $Tp \in \partial Q$. Then, using (40), we obtain

$$\begin{aligned} h(Tp) + \langle h'(Tp), y_n - Tp\rangle &\leq h(y_n)\\ \Rightarrow -\frac{1}{\tau}\langle ATp, Tp - y_n\rangle &\leq h(y_n)\\ \Rightarrow \langle ATp, Tp - y_n\rangle &\leq \tau h(y_n). \end{aligned} \tag{61}$$

Using the fact that $y_n \in Q_n$, we have

$$h(Tw_n) + \langle h'(Tw_n), y_n - Tw_n\rangle \leq 0. \tag{62}$$

Using (40), we have

$$h(y_n) + \langle h'(y_n), Tw_n - y_n\rangle \leq h(Tw_n). \tag{63}$$

Combining (62) and (63), we have

$$h(y_n) \leq \langle h'(y_n) - h'(Tw_n), y_n - Tw_n\rangle. \tag{64}$$

Thus, (61) becomes

$$\langle ATp, Tp - y_n\rangle \leq \tau\langle h'(y_n) - h'(Tw_n), y_n - Tw_n\rangle. \tag{65}$$

Using the stepsize (44), we have

$$\|ATw_n - Ay_n\| + \|h'(Tw_n) - h'(y_n)\| = \frac{\mu}{\lambda_{n+1}}\|Tw_n - y_n\|,$$

observe that

$$\|ATw_n - Ay_n\|^2 + \|h'(Tw_n) - h'(y_n)\|^2 \leq (\|ATw_n - Ay_n\| + \|h'(Tw_n) - h'(y_n)\|)^2 = \frac{\mu^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2, \tag{66}$$

Using (65) and the stepsize, we have (58) becomes

$$
\begin{aligned}
\|z_n - Tp\|^2 &\le \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 + 2\tau\lambda_n\langle h'(y_n) - h'(Tw_n), y_n - Tw_n\rangle + \lambda_n^2\|Ay_n - ATw_n\|^2 \\
&\le \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 + \lambda_n^2\, \|h'(y_n) - h'(Tw_n)\|^2 + \tau^2\|y_n - Tw_n\|^2 + \lambda_n^2\|Ay_n - ATw_n\|^2 \\
&= \|Tw_n - Tp\|^2 - (1-\tau^2)\|Tw_n - y_n\|^2 + \lambda_n^2(\|h'(y_n) - h'(Tw_n)\|^2 + \|Ay_n - ATw_n\|^2) \\
&\le \|Tw_n - Tp\|^2 - (1-\tau^2)\|Tw_n - y_n\|^2 + \lambda_n^2(\|h'(y_n) - h'(Tw_n)\| + \|Ay_n - ATw_n\|)^2 \\
&= \|Tw_n - Tp\|^2 - (1-\tau^2)\|Tw_n - y_n\|^2 + \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2 \\
&= \|Tw_n - Tp\|^2 - (1-\tau^2 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2})\|Tw_n - y_n\|^2. \qquad (67)
\end{aligned}
$$

Using the fact that $\lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}\lambda_{n+1}$ and the condition that $1-\tau^2 - \frac{\mu^2\lambda_n^2}{\lambda_{n+1}^2} > 0$,

we have

$$\|z_n - Tp\|^2 \le \|Tw_n - Tp\|^2. \qquad (68)$$

It then follows that

$$\|z_n - Tp\| \le \|Tw_n - Tp\|. \qquad (69)$$

Furthermore, using Algorithm 2, step-size (γ_n) and (56), we have

$$
\begin{aligned}
\|v_n - p\|^2 &= \|w_n + \gamma_n T^*(z_n - Tw_n) - p\| \\
&= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + 2\gamma_n\langle w_n - p, T^*(z_n - Tw_n)\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + 2\gamma_n\langle Tw_n - Tp, z_n - Tw_n\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + \gamma_n\|z_n - Tp\|^2 - \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|z_n - Tw_n\|^2 \\
&\le \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|z_n - Tw_n\|^2 \\
&\le \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 - \gamma_n(\gamma_n + \epsilon)\|T^*(z_n - Tw_n)\|^2 \\
&= \|w_n - p\|^2 - \gamma_n\epsilon\|T^*(z_n - Tw_n)\|^2 \\
&\le \|w_n - p\|^2, \qquad (70)
\end{aligned}
$$

which implies that

$$\|v_n - p\| \le \|w_n - p\|. \qquad (71)$$

Using similar approach as in (53), we obtain

$$\|t_n - p\|^2 \le \|v_n - p\|^2 - (1 - \tau_1^2 - \frac{\delta^2 v_n^2}{v_{n+1}^2})\|u_n - v_n\|^2 \qquad (72)$$

also, we have

$$\|t_n - p\|^2 \le \|v_n - p\|^2 - (1 - \tau_1^2 - \frac{\delta^2 v^2}{v_{n+1}^2})\|u_n - v_n\|^2 \qquad (73)$$

which implies that

$$\|t_n - p\| \le \|v_n - p\|. \qquad (74)$$

Using the fact that $\eta \in (0, \frac{2\kappa}{L_3^2})$ and F is κ- strongly monotone and L_3-Lipschitz continuous, we have

$$\|t_n - p - \eta(Ft_n - Fp)\|^2 = \|t_n - p\|^2 - 2\eta\langle t_n - p, Ft_n - Fp\rangle + \eta^2\|Ft_n - Fp\|^2$$

$$\leq \|t_n - p\|^2 - 2\eta\kappa\|t_n - p\|^2 + \eta^2 L_3^2 \|t_n - p\|^2$$
$$= (1 - \eta(2\kappa - \eta L_3^2))\|t_n - p\|^2. \tag{75}$$

It follows that

$$\|t_n - p - \eta(Ft_n - Fp)\| \leq \sqrt{(1 - \eta(2\kappa - \eta L_3^2))}\|t_n - p\|. \tag{76}$$

In addition, using (76), we have

$$\begin{aligned}\|t_n - \alpha_n F t_n - (p - \alpha_n F p)\| &= \|(1 - \frac{\alpha_n}{\eta})(t_n - p) + \frac{\alpha_n}{\eta}(t_n - p - \eta(Ft_n - Fp))\| \\ &\leq (1 - \frac{\alpha_n}{\eta})\|t_n - p\| + \frac{\alpha_n}{\eta}\|t_n - p - \eta(Ft_n - Fp))\| \\ &\leq (1 - \frac{\alpha_n}{\eta})\|t_n - p\| + \frac{\alpha_n}{\eta}\sqrt{(1 - \eta(2\kappa - \eta L_3^2))}\|t_n - p\| \\ &= (1 - \frac{\alpha_n}{\eta}(1 - \sqrt{(1 - \eta(2\kappa - \eta L_3^2))}))\|t_n - p\| \\ &\leq (1 - \frac{\alpha_n \rho}{\eta})\|t_n - p\|, \end{aligned} \tag{77}$$

where $\rho = 1 - \sqrt{(1 - \eta(2\kappa - \eta L_3^2))} \in (0, 1)$.

$$\begin{aligned}\|x_{n+1} - p\| &= \|t_n - \alpha_n F t_n - p\| \\ &= \|t_n - \alpha_n F t_n - p + \alpha_n F p - \alpha_n F p\| \\ &\leq \|t_n - \alpha_n F t_n - (p - \alpha_n F p)\| + \alpha_n \|F p\| \\ &\leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)\|t_n - p\| + \alpha_n \|F p\| \\ &\leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)\|x_n - p\| + \alpha_n N_1 + \alpha_n \|F p\| \\ &= \left(1 - \frac{\alpha_n \rho}{\eta}\right)\|x_n - p\| + \frac{\alpha_n \rho}{\eta}\frac{\eta N_1 + \eta \|F p\|}{\rho}. \end{aligned} \tag{78}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\gamma N_1 + \gamma \|F p\|}{\rho}\}. \tag{79}$$

Thus, $\{x_n\}$ is bounded. □

Lemma 10 Given the sequence $\{x_n\}$ as in Algorithm 2 and assume that Assumption 6 holds. Moreover, if for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to $x^* \in H_1$ and $\lim_{k\to\infty} \|w_{n_k} - v_{n_k}\| = 0 = \lim_{k\to\infty} \|t_{n_k} - v_{n_k}\|$, then, $x^* \in \Gamma$.

Proof Let $p \in \Gamma$. It is easy to see from (70) that

$$\begin{aligned}\|v_{n_k} - p\|^2 &\leq \|w_{n_k} - p\|^2 - \gamma_{n_k}\epsilon\|T^*(z_{n_k} - T w_{n_k})\|^2 \\ &\leq \|w_{n_k} - p\|^2 - \epsilon^2\|T^*(z_{n_k} - T w_{n_k})\|, \end{aligned} \tag{80}$$

which implies that

$$\epsilon^2\|T^*(z_{n_k} - T w_{n_k})\|^2$$

$$\leq \|w_{n_k} - p\|^2 - \|v_{n_k} - p\|^2$$
$$\leq (\|w_{n_k} - v_{n_k}\| + \|v_{n_k} - p\|)^2 - \|v_{n_k} - p\|^2$$
$$\leq \|w_{n_k} - v_{n_k}\|^2 + 2\|w_{n_k} - v_{n_k}\|\|v_{n_k} - p\| + \|v_{n_k} - p\|^2 - \|v_{n_k} - p\|^2$$
$$= \|w_{n_k} - v_{n_k}\|^2 + 2\|w_{n_k} - v_{n_k}\|\|v_{n_k} - p\| \tag{81}$$

using our hypothesis, we have

$$\lim_{k\to\infty} \|T^*(z_{n_k} - Tw_{n_k})\| = 0. \tag{82}$$

Moreover, we have

$$\|v_{n_k} - p\|^2 \leq \|w_{n_k} - p\|^2 + \gamma_n^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 - \gamma_n\|z_{n_k} - Tw_{n_k}\|^2, \tag{83}$$

which implies that

$$\gamma_{n_k}\|z_{n_k} - Tw_{n_k}\|^2 \leq \|w_{n_k} - p\|^2 - \|v_{n_k} - p\|^2 + \gamma_{n_k}^2\|T^*(z_{n_k} - Tw_{n_k})\|^2$$
$$\leq \|w_{n_k} - v_{n_k}\|^2 + 2\|w_{n_k} - v_{n_k}\|\|v_{n_k} - p\| + \gamma_{n_k}^2\|T^*(z_{n_k} - Tw_{n_k})\|^2, \tag{84}$$

using the fact that $0 < \epsilon < \gamma_{n_k}$, we have

$$\lim_{k\to\infty} \|z_{n_k} - Tw_{n_k}\| = 0. \tag{85}$$

From (59) or (67), we have

$$\|z_{n_k} - Tp\|^2 \leq \|Tw_{n_k} - Tp\|^2 - (1 - \tau^2 - \frac{\mu^2\lambda_{n_k}^2}{\lambda_{n_k+1}^2})\|Tw_{n_k} - y_{n_k}\|^2. \tag{86}$$

Now, observe that

$$\|z_{n_k} - Tp\|^2 = \|z_{n_k} - Tw_{n_k} + Tw_{n_k} - Tp\|^2$$
$$= \|Tw_{n_k} - Tp - (Tw_{n_k} - z_{n_k})\|^2$$
$$= \|Tw_{n_k} - Tp\|^2 - 2\langle Tw_{n_k} - Tp, Tw_{n_k} - z_n\rangle + \|Tw_{n_k} - z_{n_k}\|^2$$
$$\geq \|Tw_{n_k} - Tp\|^2 - 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| + \|Tw_n - z_n\|^2 \tag{87}$$

which implies that

$$-\|z_{n_k} - Tp\|^2 \leq -\|Tw_{n_k} - Tp\|^2 + 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_n - z_n\|^2 \tag{88}$$

Adding (86) and (88), we have

$$\|z_{n_k} - Tp\|^2 - \|z_{n_k} - Tp\|^2$$
$$\leq \|Tw_{n_k} - Tp\|^2 - (1 - \tau^2 - \frac{\mu^2\lambda_{n_k}^2}{\lambda_{n_k+1}^2})\|Tw_{n_k} - y_{n_k}\|^2 - \|Tw_{n_k} - Tp\|^2 + 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} -$$

$$\Rightarrow 0 \leq -(1 - \tau^2 - \frac{\mu^2\lambda_{n_k}^2}{\lambda_{n_k+1}^2})\|Tw_{n_k} - y_{n_k}\|^2 + 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} - z_{n_k}\|^2$$

$$\Rightarrow (1 - \tau^2 - \frac{\mu^2\lambda_{n_k}^2}{\lambda_{n_k+1}^2})\|Tw_{n_k} - y_{n_k}\|^2 \leq 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} - z_{n_k}\|^2, \tag{89}$$

using (85), $\lim_{n\to\infty}\lambda_{n+1} = \lim_{n\to\infty}\lambda_n > 0$ and Condition B (2) , we have

$$\lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0. \tag{90}$$

Observe that

$$\|w_{n_k} - x_{n_k}\| = \vartheta_{n_k}\|x_{n_k} - x_{n_k-1}\| = \alpha_{n_k} \cdot \frac{\vartheta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{91}$$

Since, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point x^*. We assert that $\{Tw_{n_k}\}$ converges to the limit, Tx^* since T is bounded. Hence, using the fact that $\lim_{n\to\infty}\|Tw_{n_k} - y_{n_k}\| = 0$, we have that the subsequence $\{y_{n_k}\}$ is therefore converging to the same limit, Tx^*. Also, since $y_{n_k} \in Q_{n_k}$, using the definition of Q_n, we obtain

$$h(Tw_{n_k}) + \langle h'(Tw_{n_k}), y_{n_k} - Tw_{n_k}\rangle \leq 0, \tag{92}$$

and using the Cauchy Schwarz inequality, we obtain

$$h(Tw_{n^k}) \leq \|h'(Tw_{n^k})\|\,\|y_{n^k} - Tw_{n^k}\|. \tag{93}$$

Furthermore, since $h'(.)$ is Lipschitz continuous, therefore, it is bounded on any subset of H_2. Thus, we find a constant $N > 0$ satisfying

$$\|h'(Tw_{n^k})\| \leq N$$

for all $j \in \mathbb{N}$. Clearly, we have (93) becomes

$$h(Tw_{n_k}) \leq N\|y_{n_k} - Tw_{n_k}\|, \tag{94}$$

and using (90), we have

$$\lim_{k\to\infty} h(Tw_{n_k}) = 0. \tag{95}$$

In addition, using the fact that h is weakly lower semicontinuous, we obtain $h(Tx^*) \leq \liminf_{k\to\infty} h(Tw_{n_k}) \leq 0$. This implies that $Tx^* \in Q$. Since $y_{n_k} = P_Q(Tw_{n_k} - \lambda_{n_k} ATw_{n_k})$, then from the characteristic of the metric projection, we have

$$\langle Tw_{n_k} - \lambda_{n_k} ATw_{n_k} + y_{n_k}, x - y_{n_k}\rangle \geq 0 \quad \forall \quad x \in Q \subset Q_n, \tag{96}$$

and using the fact that A is monotone, we have

$$\begin{aligned}
0 &\leq \langle y_{n_k} - Tw_{n_k}, x - y_{n_k}\rangle + \lambda_{n_k}\langle ATw_{n_k}, x - y_{n_k}\rangle \\
&= \langle y_{n_k} - Tw_{n_k}, x - y_{n_k}\rangle + \lambda_{n_k}\langle ATw_{n_k}, x - Tw_{n_k}\rangle + \lambda_{n_k}\langle ATw_{n_k}, Tw_{n_k} - y_{n_k}\rangle \\
&= \langle y_{n_k} - Tw_{n_k}, x - y_{n_k}\rangle + \lambda_{n_k}\langle Ax, x - Tw_{n_k}\rangle + \lambda_{n_k}\langle ATw_{n_k} - Ax, x - Tw_{n_k}\rangle + \lambda_{n_k}\langle ATw_{n_k}, Tw_{n_k} - y_{n_k}\rangle \\
&\leq \langle y_{n_k} - Tw_{n_k}, x - y_{n_k}\rangle + \lambda_{n_k}\langle Ax, x - Tw_{n_k}\rangle + \lambda_{n_k}\langle ATw_{n_k}, Tw_{n_k} - y_{n_k}\rangle,
\end{aligned}$$

using (90) and the fact that $\lim_{k\to\infty}\lambda_{n_k} > 0$, we have

$$\langle Ax, x - Tx^*\rangle \geq 0$$

Hence, $Tx^* \in VI(A, Q)$.

Using a similar approach, we have $x^* \in VI(B, C)$. Hence, we conclude that $x^* \in \Gamma$.

□

Theorem 11 *Let Assumption 6 holds and given the sequence $\{x_n\}$ as in Algorithm 2, then, $\{x_n\}$ converges strongly to $p \in \Omega$.*

Proof Let $p \in \Omega$. Using Algorithm 2, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \vartheta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\vartheta_n\langle x_n - p, x_n - x_{n-1}\rangle + \vartheta_n^2\|x_n - x_{n-1}\|^2
\end{aligned}$$

$$\leq \|x_n - p\|^2 + 2\vartheta_n \|x_n - x_{n-1}\| \|x_n - p\| + \vartheta_n^2 \|x_n - x_{n-1}\|^2$$
$$\leq \|x_n - p\|^2 + \vartheta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \vartheta_n \|x_n - x_{n-1}\|]$$
$$= \|x_n - p\|^2 + \vartheta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n \frac{\vartheta_n}{\alpha_n} \|x_n - x_{n-1}\|]$$
$$\leq \|x_n - p\|^2 + \vartheta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n N_1]$$
$$\leq \|x_n - p\|^2 + \vartheta_n \|x_n - x_{n-1}\| N_2, \tag{97}$$

In addition, using Algorithm 2, (72), (75), (77) and (97), we have

$$\|x_{n+1} - p\|^2 = \|t_n - \alpha_n F t_n - (p - \alpha_n F p) - \alpha_n F p\|^2$$
$$\leq \|t_n - \alpha_n F t_n - (p - \alpha_n F p)\|^2 + 2\alpha_n \langle Fp, p - x_{n+1}\rangle$$
$$\leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 \|t_n - p\|^2 + 2\alpha_n \langle Fp, p - x_{n+1}\rangle$$
$$\leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 \|t_n - p\|^2 - 2\alpha_n \langle Fp, x_{n+1} - p\rangle$$
$$\leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 \|w_n - p\|^2 + 2\alpha_n \langle Fp, p - x_{n+1}\rangle$$
$$\leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 [\|x_n - p\|^2 + \vartheta_n \|x_n - x_{n-1}\| N_2] + 2\alpha_n \langle Fp, p - x_{n+1}\rangle$$
$$\leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 \|x_n - p\|^2 + \vartheta_n \|x_n - x_{n-1}\| N_2 + 2\alpha_n \langle Fp, p - x_{n+1}\rangle$$
$$= \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 \|x_n - p\|^2 + \frac{\alpha_n \rho}{\eta} \frac{\vartheta_n \eta}{\alpha_n \rho} \|x_n - x_{n-1}\| N_2 + 2\frac{\eta}{\rho} \langle Fp, p - x_{n+1}\rangle$$
$$= \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 \|x_n - p\|^2 + \frac{\alpha_n \rho}{\eta} \Psi_n, \tag{98}$$

where $\Psi_n = \frac{\vartheta_n \eta}{\alpha_n \rho} \|x_n - x_{n-1}\| N_2 + 2\frac{\eta}{\rho} \langle Fp, p - x_{n+1}\rangle$. According to Lemma 4, to deduce our proof, we only need to show that $\limsup_{k\to\infty} \Psi_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty} \{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \tag{99}$$

For this purpose, we assume that for any subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (99) holds. To establish that $\limsup_{k\to\infty} \Psi_n \leq 0$, we suppose that for any subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (99) holds. Then,

$$\liminf_{k\to\infty} \{\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\}$$
$$= \liminf_{k\to\infty} \{(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|)(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|)\} \geq 0. \tag{100}$$

It is easy to see from (98) and (72), that

$$\|x_{n_k+1} - p\|^2 \leq \left(1 - \frac{\alpha_n \rho}{\eta}\right)^2 \|t_n - p\|^2 - 2\alpha_n \langle Fp, x_{n+1} - p\rangle$$
$$\leq \|t_n - p\|^2 - 2\alpha_n \langle Fp, x_{n+1} - p\rangle$$
$$\leq \|v_n - p\|^2 - (1 - \tau_1^2 - \frac{\delta^2 v_n^2}{v_{n+1}^2}) \|u_n - v_n\|^2 - 2\alpha_n \langle Fp, x_{n+1} - p\rangle$$

$$\leq \|w_n - p\|^2 - (1 - \tau_1^2 - \frac{\delta^2 v_n^2}{v_{n+1}^2})\|u_n - v_n\|^2 - 2\alpha_n\langle Fp, x_{n+1} - p\rangle$$

$$\leq \|x_n - p\|^2 + \vartheta_n\|x_n - x_{n-1}\|N_2 - (1 - \tau_1^2 - \frac{\delta^2 v_n^2}{v_{n+1}^2})\|u_n - v_n\|^2 - 2\alpha_n\langle Fp, x_{n+1} - p\rangle, \tag{101}$$

which implies that

$$\begin{aligned}
&\limsup_{k\to\infty} \left(1 - \tau_1^2 - \frac{\delta^2 v_{n_k}^2}{v_{n_k+1}^2}\right)\|u_{n_k} - v_{n_k}\|^2 \\
&\leq \limsup_{k\to\infty} \|x_{n_k} - p\|^2 + \alpha_{n_k}\frac{\vartheta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\|N_2 \\
&- 2\alpha_{n_k}\langle Fp, x_{n_k+1} - p\rangle - \|x_{n_k+1} - p\|^2 \\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2] \leq 0.
\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty} \|v_{n_k} - u_{n_k}\| = 0. \tag{102}$$

Using Algorithm 2 and (102)

$$\lim_{k\to\infty} \|t_{n_k} - u_{n_k}\| = 0 \tag{103}$$

Furthermore, by the triangular inequality and (102), we have

$$\lim_{k\to\infty} \|t_{n_k} - v_{n_k}\| \leq \lim_{k\to\infty} \|t_{n_k} - u_{n_k}\| + \lim_{k\to\infty} \|u_{n_k} - v_{n_k}\| = 0. \tag{104}$$

$$\begin{aligned}
\|x_{n_k+1} - p\|^2 &\leq \left(1 - \frac{\alpha_n\rho}{\eta}\right)\|t_n - p\|^2 - 2\alpha_n\langle Fp, x_{n+1} - p\rangle \\
&\leq \|t_n - p\|^2 - 2\alpha_n\langle Fp, x_{n+1} - p\rangle \\
&\leq \|v_n - p\|^2 - 2\alpha_n\langle Fp, x_{n+1} - p\rangle \\
&\leq \|w_n - p\|^2 - \gamma_n\epsilon\|T^*(z_n - Tw_n)\|^2 - 2\alpha_n\langle Fp, x_{n+1} - p\rangle \\
&\leq \|x_n - p\|^2 + \vartheta_n\|x_n - x_{n-1}\|N_2 - \gamma_n\epsilon\|T^*(z_n - Tw_n)\|^2 - 2\alpha_n\langle Fp, x_{n+1} - p\rangle,
\end{aligned} \tag{105}$$

which implies

$$\begin{aligned}
&\limsup_{k\to\infty} \gamma_{n_k}\epsilon^2\|T^*(z_{n_k} - Tw_{n_k})\|^2 \\
&\leq \limsup_{k\to\infty} \|x_{n_k} - p\|^2 + \alpha^{n_k}\frac{\vartheta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\|N_2 \\
&+ -2\alpha_{n_k}\langle Fp, x_{n_k+1} - p\rangle - \|x_{n_k+1} - p\|^2 \\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2] \leq 0.
\end{aligned}$$

$$\lim_{k\to\infty} \|T^*(z_{n_k} - Tw_{n_k})\| = 0. \tag{106}$$

In the proof of Lemma 8 in [11] (establishing that γ_n is well defined), the authors obtained that

$$\|Tw_n - z_n\|^2 \leq 2\|T^*(z_{n_k} - Tw_{n_k})\|\|w_n - z_n\|, \tag{107}$$

see Equation 3.14 of [11]. Using (106) and with the above inequality, we have

$$\lim_{k\to\infty} \|z_{n_k} - Tw_{n_k}\| = 0. \tag{108}$$

From Algorithm 2 and (106), we have

$$\lim_{k\to\infty} \|v_{n_k} - w_{n_k}\| = \gamma_{n_k}\|T^*(z_{n_k} - Tw_{n_k})\| \to 0 \quad k \to \infty. \tag{109}$$

In addition, we have

$$\begin{aligned}\|z_{n_k} - Tp\|^2 &= \|Tw_{n_k} - Tp - Tw_{n_k} + z_{n_k}\|^2 \\ &= \|Tw_{n_k} - Tp\|^2 - 2\langle T(w_{n_k} - p), Tw_{n_k} - z_{n_k}\rangle + \|Tw_{n_k} - z_{n_k}\|^2 \\ &\geq \|Tw_{n_k} - Tp\|^2 - 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| + \|Tw_n - z_n\|^2, \end{aligned} \tag{110}$$

which implies that

$$-\|z_{n_k} - Tp\|^2 \leq -\|Tw_{n_k} - Tp\|^2 + 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} - z_{n_k}\|^2. \tag{111}$$

Adding Adding (111) and (86), we have

$$(1 - \tau^2 - \frac{\mu^2\lambda_{n_k}^2}{\lambda_{n_k+1}^2})\|Tw_{n_k} - y_{n_k}\|^2 \leq 2\|T(w_{n_k} - p)\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} - z_{n_k}\|^2, \tag{112}$$

using (85), we have

$$\lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0. \tag{113}$$

Using Algorithm 2 and (113), we have

$$\lim_{k\to\infty} \|z_{n_k} - y_{n_k}\| = 0. \tag{114}$$

It is easy to see that, as $k \to \infty$, we have

$$\|w_{n_k} - x_{n_k}\| = \vartheta_{n_k}\|x_{n_k} - x_{n_k-1}\| = \alpha_{n_k} \cdot \frac{\vartheta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{115}$$

In addition, we have that

$$\|v_{n_k} - x_{n_k}\| \leq \|w_{n_k} - x_{n_k}\| + \gamma_n \|T^*(z_{n_k} - Tw_{n_k})\| \to 0 \quad \text{as} \quad k \to \infty. \tag{116}$$

$$\|w_{n_k} - v_{n_k}\| \leq \|w_{n_k} - x_{n_k}\| + \|x_{n_k} - v_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{117}$$

$$\|t_{n_k} - x_{n_k}\| \leq \|t_{n_k} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{118}$$

$$\|t_{n_k} - w_{n_k}\| \leq \|t_{n_k} - x_{n_k}\| + \|x_{n_k} - w_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{119}$$

$$\|u_{n_k} - x_{n_k}\| \leq \|u_{n_k} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{120}$$

From Algorithm 2, it is also easy to see that

$$\|x_{n_k+1} - t_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{121}$$

Thus, we have

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - t_{n_k}\| + \|t_{n_k} - x_{n_k}\| \to 0 \quad \text{as } k \to \infty. \tag{122}$$

Note that $\{x_{n_k}\}$ is bounded, then, one extracts a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which weakly converges to $x^* \in H$. In addition, using (118) and the boundedness of $\{t_{n_k}\}$, there exists a subsequence $\{t_{n_{k_j}}\}$ of $\{t_{n_k}\}$ such that $\{t_{n_{k_j}}\}$ converges weakly to $x^* \in H_1$. Hence, combining (104), (109) with the resulkts in Lemma 10, we deduce that $x^* \in \Omega$. Moreover, since $x_{n_{k_j}}$ weakly converges to x^*, we infer that

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k}\rangle = \lim_{j\to\infty} \langle Fp, p - x_{n_{k_j}}\rangle = \langle Fp, p - x^*\rangle. \tag{123}$$

Note that p is a unique solution of Ω, from which we infer that

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k}\rangle = \langle Fp, p - x^*\rangle \leq 0, \tag{124}$$

we have obtain from (124) and (122)

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k+1}\rangle \leq 0. \tag{125}$$

Combining the above assumptions with (125), we obtain that $\lim_{k\to\infty} \Psi_{n_k} = \frac{\vartheta_n \eta}{\alpha_n \rho} \|x_n - x_{n-1}\| N_2 + 2\frac{\eta}{\rho}\langle Fp, p - x_{n+1}\rangle \leq 0$. Thus, applying results in Lemma 4, we deduce that $\lim_{n\to\infty} \|x_n - p\| = 0$.

□

5 Numerical Example

The example Illustrating the Performance of the New Method In this subsection we give several examples which can see the applicability and the efficiency of our new iterative method. The comparisons with Algorithm 1 and Algorithm 2 will be also shown.

Example 1 We endow the space $H_1 := L_2([0, 1]) =: H_2$ with the below inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \quad \forall \quad x, y \in L_2([0, 1]) \quad \text{and} \quad \|x\|^2 := \int_0^1 |x(t)|^2 dt \quad \forall x, y \in L_2([0, 1]).$$

Let $F, A, B, T : L_2([0, 1]) \to L_2([0, 1])$ be defined by

$$Bx(t) = Ax(t) = \max\{0, x(t)\}, \quad t \in [0, 1], x \in L_2([0, 1]);$$

$$Fx(t) = 2x(t), \quad t \in [0, 1], x \in L_2([0, 1]),$$

$$Tx(s) = \int_0^1 K(s, t)x(t)dt \quad x \in L_2([0, 1]),$$

This is an immediate consequence of the definitions. Throughout this section, we will denote by K a continuous real valued function on

$$[0, 1] \times [0, 1]$$

. Then it is easy to verify that A is 1-Lipschitz continuous and monotone, that F is κ-strongly monotone and that T is a bounded linear operator with the adjoint $T^*x(s) =$

$\int_0^1 K(t,s)x(t)dt$ $x \in L_2([0,1])$. Let $C = Q = \{x \in L_2([0,1]) : \|\phi x\| \leq 1\}$ and suppose that $h, k : L_2([0,1]) \to \mathbb{R}$ are defined by

$$h(x) = k(x) = \frac{1}{3}(\|\phi x\|^2 - 1) \ \forall \ x \in L^2([0,1]),$$

clearly, k and h are differentiable such that $h'(x) = k'(x) = \frac{2}{3}\phi^2 x$ for all $x \in L^2([0,1])$, see [24]. In the case of Algorithm 1 and Algorithm 2. Let C be defined by $C = Q = \{x \in L_2([0,1]) : \langle a, x\rangle = b\}$ where $a \neq 0$ and $b = 2$. Thus, we have

$$P_C(x^-) = P_Q(x^-) = \max\left\{0, \frac{b - \langle a, x^-\rangle}{\|a\|^2}\right\} a + x^-.$$

We select $\alpha_n = \frac{2}{200n+5}$, $\lambda_1 = 0.4$, $\nu_1 = 0.54$, $\delta = 0.3$, $\mu = 0.4$ for all $n \in \mathbb{N}$. Let $\epsilon = \|x_n - x_{n-1}\| \leq 10^{-4}$ as the stopping criterion and select the starting points:

Case 1: $x_0(t) = 2t^3 + 2$, $x_1(t) = t + 1$;

Case 2: $x_0(t) = e^{2t} + t$, $x_1(t) = 3$;

Case 3: $x_0(t) = t^2 + e^{3t}$, $x_1(t) = t + \cos(t)$;

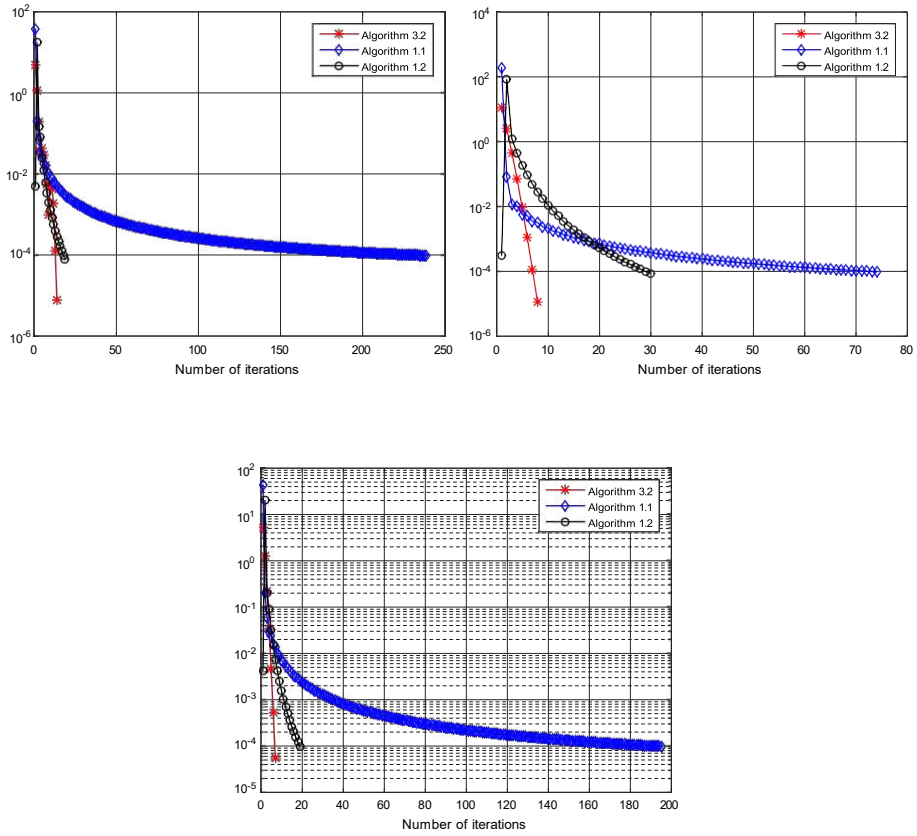


Fig. 1 Example 1, Top Left: Case 1; Top Right: Case 2; Case 3; Bottom Left.

In Figure 1, we observe that the numerical results for Example 1, comparing the proposed method (Algorithm 7, denoted as Algorithm 3.2 in the figures) with Algorithm 1 and Algorithm 2, are illustrated in Figure 1. The top left, top right, and bottom left plots represent the convergence behavior for Cases 1, 2, and 3, respectively. The error sequence $\{\mathrm{E}_n\}_{n\geq 1}$ for our proposed method is observed to decrease significantly faster than that of the existing algorithms. Specifically, in Case 1, our method reaches the stopping criterion of 10^4 in approximately 25 iterations, while Algorithm 1 and Algorithm 2 require more than 250 iterations to reach the same level of accuracy. This indicates that the inclusion of the modified inertial extrapolation step and the self-adaptive step size remarkably enhances the convergence speed and computational efficiency in the framework of infinite-dimensional Hilbert spaces.

Moreover, we noticed in Figure 1, the proposed method (Red Line/Stars) in Algorithm 7 actually dominates the remaining methods in Algorithms 1 and 2.

This can only make us observe and deduce that The proposed method is roughly 10 times faster in terms of iteration count to reach the 10^{-4} threshold.

Example 2 We consider the space $H_1 := \mathbb{R}^2 =: H_2$ with the following norm $\|x\| = (x_1^2 + x_2^2)^{\frac{1}{2}}$. Let $C = Q = \{x \in \mathbb{R}^2 : h(x) = k(x) = x_1^2 - x_2 \leq 0\}$. Let $T, A, B, F : \mathbb{R}^2 \to \mathbb{R}^2$ by $A(x) = (2k_1(x_1), 3x_1 + x_2)$, $B(x) = (3k_2(x_1), 2x_1 + x_2)$, and $Tx = (x_1, x_2)$, where

$$k_1(x) = k_2(x) = \begin{cases} e(x-1) + e & \text{if} \quad x > 1, \\ e^x & \text{if} \quad -1 \leq x \leq 1 \\ e^{-1}(x+1) + e^{-1} & \text{if} \quad x < -1, \end{cases} \tag{126}$$

and $F(x) = 5x$. As for the Algorithms 1 and 2, see [25]. We introduce the set $C = Q := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. Then, the $P_C = P_Q$ is given by

$$P_C = P_Q = \begin{cases} x & \text{if} \quad x \in C, \\ \frac{x}{\|x\|} & \text{if otherwise.} \end{cases} \tag{127}$$

We choose $\alpha_n = \frac{2}{200n+5}$, $\lambda_1 = 0.4$, $v_1 = 0.54$, $\delta = 0.3$, $\mu = 0.4$ for all $n \in \mathbb{N}$. Given $\epsilon = \|x_n - x_{n-1}\| \leq 10^{-4}$ as the stopping criterion and select starting points:

Case 1: $x_0 = [1, -2]'$ and $x_1 = [1, 2]'$;

Case 2: $x_0 = [-5, 0]'$ and $x_1 = [-1, 0]'$;

Case 3: $x_0 = [-1, 1]'$ and $x_1 = [-2, 2]'$;

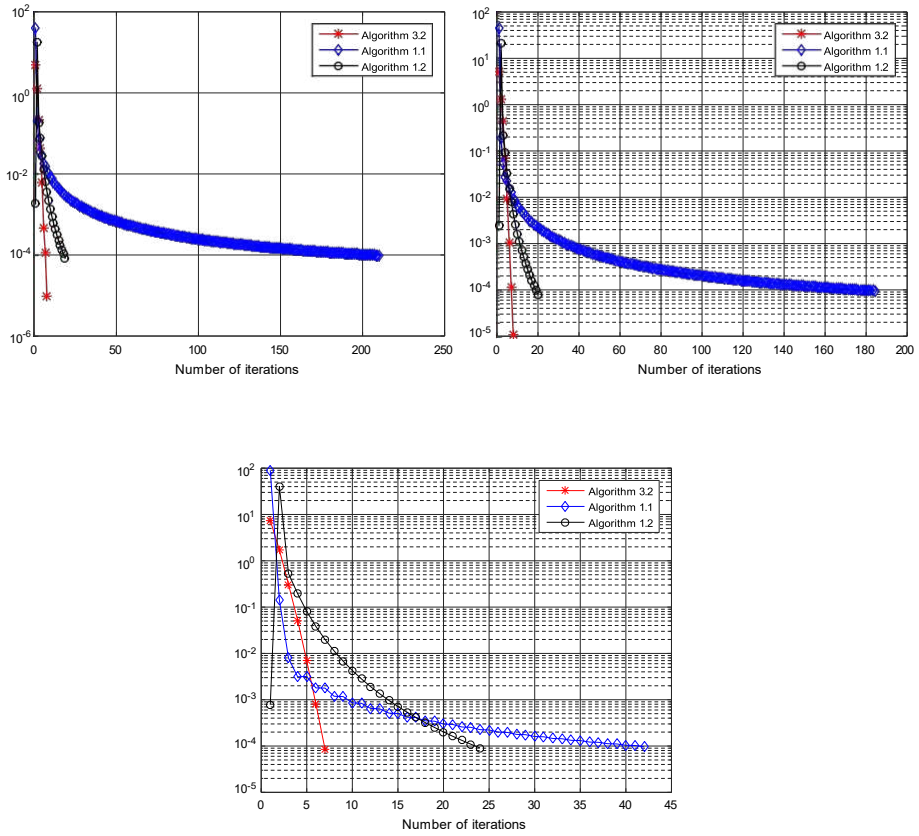


Fig. 2 Example 2, Top Left: Case 1; Top Right: Case 2; Case 3; Bottom Left.

As for the results in Figure 2, the performance of the algorithms for Example 2 in R^2 is depicted in Figure 2. Similar to the previous experiment, the top left, top right, and bottom left panels show the results for Cases 1, 2, and 3. As shown in Figure 2, the proposed iterative technique (Algorithm 3.2 in legend) exhibits a much steeper descent in the error function compared to the competing methods (Algorithm 1.1 and 1.2). For instance, in Case 3, our method achieves convergence in fewer than 10 iterations, while the other methods remain at a significantly higher error level after the same number of steps. These results confirm the applicability of our method and demonstrate that it is less expensive and more robust than existing results in the literature, regardless of the choice of starting points.

As for Figure 1, we also deduce that in Figure 2, we depict consistency across different starting points. In terms of the stability of the obtained method, we realised that Whether starting far from the solution (Case 2: $x_0 = [-5, 0)$ or even much

more closer than the other graphs. This only conludes that the red line maintains a consistent, and a very rapid convergence curve.

As a result, we duly observe that, the ”Inertial term” prevents the algorithm from slowing down near the solution set, which is in reality a very common problem with traditional extragradient methods as it is clearly also shown by the black and blue lines leveling off more slowly.

6 Conclusion

A new modified inertial Tseng extragradient method for solving monotone BSVIP (9) is introduced and studied in the framework of Hilbert spaces. By employing the techniques of projection onto a constructible half-space, we were able to establish the result of strong convergence for our iterative method. One of the advantages of this method is that it requires only two projection per iteration onto a constructible half-space, when compared to [19, 20] in which their methods requires four projections per iteration onto each, constructible half-spaces and feasible subsets of H_1 and H_2 and two projection per iteration onto each feasible subsets of H_1 and H_2, respectively. Clearly, our method is more applicable, efficient and less expensive in practice. However, the linear rate of convergence of the iterative technique introduced and studied in this paper was not investigated, neither was any application. Hence, we intend to look in this direction in the nearest future.

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Data Availability. The data in this article are derived from mathematical equations and computational simulations. There is no reference to primary, contextual or secondary data beyond that contained within the article.

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