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Fracture characteristics from well flowrate fluctuations: a statistical physics approach

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Fracture characteristics from well flowrate fluctuations: a statistical physics approach

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Author contributions

Kes Heffer contributed all of the study conception and design and material preparation, and wrote the draft of the manuscript. He also collected and analysed all the data that was used in companion publications that applied the theory of this manuscript.

Abstract

Previous studies have indicated that fluctuations in flowrates from or into wells over the detailed production history of a mature oilfield are approximate proxies for geomechanical strain changes. Inter-well temporal correlations in the flowrate fluctuations were found to be statistically significant over long distances between wells, raising the possibility that field structure can be analysed with a process analogous to tomography. This paper expounds a theory for the extraction of fracture characteristics from the inter-well correlations of flowrate "noise". I use the maximum information entropy principle of Jaynes, with a constraint of fixed rate of energy input to the reservoir, balanced with either dissipation by Darcy flow or fluctuation in rock strain energy, to make the case for the reservoir system being in a state of maximum entropy production rate or flux. Fluctuating departures from this state allow a simple relationship between the covariances of flowrate fluctuations across all pairs of wells and a matrix of parameters including non-local stiffness tensors, from which estimates of local stress interactions are derived. Best fit fracture densities and orientations are then extracted at each well using the theoretical orientational distribution of the tensile stress field around a given extensional fracture. The novel method is seen as complementary to more conventional fracture characterisation exercises, providing information at low cost from the all-important inter-well regions. Its several assumptions imply that it can only be verified by comparison of its predictions with multiple sets of independent field data: early results are encouraging.

Highlights

- Provides the detailed theory behind a new, low cost method of extracting the characteristics of subsurface reservoir fractures from the correlations over time of fluctuations in flowrates in production or injection wells in a mature field.
- The steps in the method minimise the chance of spurious correlations having an effect on the results.
- The method extracts stress interactions between nearest neighbour wells that are maximally consistent with the flowrate correlations between all pairs of wells.
- Shows that the usual poro-elastic storage and stress equilibrium equations are applicable when the reservoir rock-fluid system is in a state of minimum rate of energy input.
- Suggests a potential mechanism for high amplitude flowrate fluctuations
- Early results are encouraging. The strikes of interpreted fractures in 6 different reservoirs with water injection are correlated with the independently measured orientations of maximum horizontal principal stress axes. In reservoirs that have been depleted in pressure, there is no such correlation, and faults, or fault damage zones, appear to control the active fractures.

1. Introduction

1.1. Background

Characterisation of spatial and directional variations in fracture conductivities remains problematic despite many studies and applications of models of fractures in reservoirs that draw on seismic, induced microseismicity, outcrop, core, well-log, well test, interference test and analogue data. Production histories of individual wells have provided well-specific information in some cases, but

those do not characterise the all-important interwell regions. Geomechanical models, run with best estimates of in situ natural stresses (which may vary spatially) and coupled to the changes in fluid pressures and temperatures during the life of the field, can provide insight into fracture conductivity variations, but to be faithful, those models themselves beg the question by requiring detailed characterisation of natural fractures, faults and heterogeneous stress state (potentially in a near-critical system) as input. Inversion (history-matching) of such a model to extract estimates of fracture characteristics is as yet an enormous computing task.

Standard reservoir simulation typically incorporates geomechanics only through rock compressibility in local storage terms, assuming isotropic and constant properties. However, field observations reveal anisotropic geomechanical characteristics through three independent techniques:

a) Injected fluid pathways: Preferred flow directions of injected fluids strongly correlate with local Shmax orientation (Heffer & Dowokpor, 1990; Heffer & Lean, 1993). This was determined from tracer breakthroughs at production wells or fluid pressure responses to injection changes.

b) Shear-wave splitting: This phenomenon, where faster waves polarize sub-parallel to Shmax, characterizes nearly all crustal rocks and the upper mantle (Crampin, 1987; Crampin, 1994). This indicates that in-situ rocks contain stress-aligned, fluid-saturated microcracks. Stress-aligned grain-boundary cracks and possibly low aspect-ratio pores represent the most compliant reservoir elements.

c) Well flowrate correlations: Production and injection rates in oilfield production and injection wells show temporal fluctuations at a range of timescales, partly from operational interventions but primarily from inter-well communication through the subsurface. Heffer et al. (1995) found significantly higher flowrate correlations between injector-producer pairs when their subsurface locations align sub-parallel to the maximum horizontal stress axis (Shmax). Heffer (2012a & b) analysed flowrate fluctuations in six North Sea fields, finding that a parameter termed the ‘rate diffusivity tensor’ derived from flowrate correlations showed peaks in 'shearing' directions at approximately 30 degrees to Shmax; that is also the angle to fracture strikes at which their associated tensile stress field is a maximum.

In addition both flowrates and shear-wave splitting exhibit long-range spatial influence. Heffer et al. (1995), Main et al. (2007), and Heffer (2012a,b) documented long-range correlations in temporal flowrate fluctuations between well pairs—sometimes strongest between wells at opposite ends of a field, and associated with local stress fields and major faults (Heffer, 2007). Average correlations decreased slowly with distance, more slowly than normal Darcy diffusion would predict.

1.2. The concept of fracture criticality

Shear-wave splitting demonstrates extreme sensitivity to crustal stress changes with long-range effects. Examples include: a) Crampin (2003) observed time-delay changes in shear-waves before and after an Iceland volcanic eruption at 240 km distance across all reliable measurement stations, interpreted as stress relaxation from Mid-Atlantic Ridge adjustment to new stress regimes following magma injection; b) Decreased shear-wave splitting time-delays occurred alongside ground displacement, seismic velocity changes, and water well level changes approximately 70 km from minor Iceland earthquakes (Crampin, 2003; Crampin et al., 2003). Within the North Sea Valhall field shear-wave splitting properties changed over two months, attributed to evolving stress conditions (Teanby et al., 2004). Wuestefeld et al. (2011) found temporal shear-wave splitting changes during Texas tight gas hydraulic fracturing. Extensive fluid pressure changes can cause stress transfer; Jeanne et al. (2014) modelled the Geysers geothermal field, finding that "During injection, the reservoir expansion caused by the pressure increase led to mechanical stress

transfer through the reservoir…". In earthquake statistics, aftershock analysis (Li et al., 2025) indicates inelastic seismicity is sub-critical, though still near-critical.

Crampin (1994) introduced the concept of the fracture criticality of crustal rocks, stating that "proximity to this state of fracture criticality implies that most rocks are compliant, where small deformations or changes of conditions may have major effects on the behaviour of the rock mass".

Heffer & King (2006) argued that the effective stiffness modulus of a cracked body drops rapidly as the density of cracks approaches the critical point of percolation of intersecting cracks, leading to strong spatial correlation of strain. Crampin (1994) also considered fracture percolation to have a critical threshold at the point where fractures intersect and coalesce. Heffer (2014) extended the analysis of the percolation of cracks to include the elastic stress fields that surround each crack (as formulated by Pollard & Segall, 1987) which cause stresses and therefore strains to interfere as (micro-)crack densities increase towards a threshold that is lower than that required for crack intersections. Long-range strain correlations can therefore theoretically exist in a cracked medium with a purely elastic mechanism. Geomechanical changes through elastic behaviour, rather than inelastic shear slip, is consistent with significant induced seismicity being apparently atypical in conventional oilfields (albeit extended seismic monitoring is uncommon). In reality it is probable that some, locally variable, combination of elastic and inelastic processes does contribute to long-range correlation. Furthermore, theoretical crack densities in a range encompassing the theoretical percolation threshold for elastic stress fields are observed empirically, from analysis of field shear wave studies, to be commonplace throughout the lithosphere (Crampin, 1984; Heffer, 2014).

This concept extends beyond individual critically-stressed faults to envisage reservoir behaviour as a cooperative phenomenon which in general involve small-scale interactions cascading to larger-scale influences (Sornette, 2000). The mechanism is one of emergent collective behaviour, analogous for example to magnetization in a ferromagnet, which can arise spontaneously from local interactions among individual spins. A more recently described analogy is that of directional ordering in bird flocks (e.g. Bialek et al., 2012, 2014), in which a unique direction of flight is spontaneously chosen from the network of local pairwise interactions among neighbouring birds, with even fluctuations in velocities showing long-range correlation.

The stresses in a rock-fluid system are likely to be heterogeneous. Modelling by Mora & Place (2002) showed that the spatial correlation of stresses grows as a critical point is approached. At the laboratory scale, observations with the digital image correlation technique applied to samples undergoing deformation also demonstrate long-range correlation of strain (relative to the size of the sample) as failure is approached (Zhang, H. et al., 2012 and Zhang, J. et al., 2015 for 2D strain; Hall et al., 2009 for 3D strain).

Conventional reservoir simulation does not incorporate such complex physics. In addition, particular boundary conditions are normally assumed at each well, which are constraints in either a) production or injection rates; or b) bottom-hole or top-hole pressures. The variations over time in these constraints are averaged over the individual time-steps of the simulation, which can be long compared with the timescales of significant fluctuations in the field. In other words there is potential information in the fluctuations which is ignored by conventional reservoir simulation.

1.3. Extracting useful reservoir information in this context

Given the strong field evidence outlined above for the intimate and complex involvement of geomechanics in reservoir behaviour, the question arises: what methodology that incorporates geomechanics can be used to analyse reservoir history from the fluctuations in flowrates at wells?

In this paper I present a method to provide new characterisation of fractures and faults: fracture strikes and densities in particular. The method is applicable to mature oil or gas fields with a

reasonable history of production; some 70% of global oil production was projected to be from mature fields in 2018 by O'Brien et al. (2016). The method is also potentially applicable to geothermal fields, gas storage fields and Carbon Capture & Storage projects in mature depleted fields.

The fluctuations in the variables are partially driven by the perturbations to the reservoir-well system introduced either by deliberate actions of the operator at particular wells; or by unprompted changes to well flow characteristics, such as formation damage, asphaltene build-up etc.; or by areally extensive changes in fluid saturation, pressure or temperature that the operation is causing. The variations can also be caused by the freedom for individual well rates, either flowing into a common header pipe, or flowing from a common set of injection pumps, to vary amongst themselves, albeit keeping the total net rate to be more restricted in variability around a comparatively stationary mean. Each such perturbation can set up consequent changes in the hydro-mechanical system, which are spatially widespread in a geomechanically related pattern.

The fluctuations in the system will also tend to be high amplitude due to the strong sensitivity of a system in or near a critical state (e.g. Reichl, Ch.10). As Heffer et al. (1995) found empirically, the first few modes (or eigensolutions) of the associated flowrate fluctuations at wells, those with the largest amplitudes, will derive from the hydro-mechanical physics patterns that are stress-related and fault-related.

This paper discusses how the statistical mechanics governing such spontaneous fluctuations can be used to deduce geomechanical information about the system. Being spatially long-range in nature, the correlations allow an analysis analogous to tomography from which parameters of the reservoir structure can be extracted. Indeed the method has some similarities to hydraulic tomography as deployed in groundwater analysis (e.g. Hao et al., 2008), except that, unlike the latter, it does not require head levels (equivalent to fluid pressures) measurements to be available at wells.

2. Methods

2.1. Jaynes' MaxEnt approach

In a new approach to reservoir behaviour here, the energies associated with changes to both the fluid and the rock of the reservoir are considered within the framework of the well-established theory of maximum information entropy. Through the work of Jaynes (1957) we can use the Shannon entropy as a means of inferring the most probable position of a probabilistic system. It should be emphasised that the Shannon entropy is not necessarily the thermodynamic entropy, but has a deeper meaning with a more generic nature. The probability distribution that maximises the entropy subject to certain constraints can be used for statistical inference from incomplete information: the MaxEnt approach (Jaynes, 1957). MaxEnt is essentially a mathematical manipulation that translates assumptions of constraints into a most probable distribution. Niven (2009a,b) has extended the MaxEnt approach of Jaynes to flow-controlled systems. In these the mean flow rate through a system is a constraint on the distribution of instantaneous flow rates within many small discrete elements comprising the control volume ensemble; the MaxEnt method is used to describe the statistics of the maximum *flux* entropy, which represents the broadness of the probability distribution of instantaneous fluxes.

As illustrated by Niven, 2010, the most probable position of any multinomial system composed of N (tending to infinity) distinguishable entities allocated to number, s, of equiprobable, distinguishable categories can be obtained by maximising the entropy function.

Suppose a system which can exist in s equiprobable microstates m =1…s, and that there is a large number, N>>s, realisations of the system in an ensemble, distributed amongst the

microstates. If n_m is the number of systems in the m^{th} microstate, then the number of ways of obtaining a particular set of values $\{n_m\}$ is given by the multinomial distribution:

$$\Gamma[\{n_m\}] = \frac{N!}{n_1!n_2!\ldots n_s!} \qquad \ldots(1)$$

The Shannon entropy of the ensemble of microstates is defined as:

$$S = \ln(\Gamma) \approx -\sum_{m=1}^{s} \rho_m . \ln(\rho_m) \qquad \ldots(2)$$

apart from an additive constant, where $\rho_m = n_m/N$. The entropy is the expected value of the *surprisal* [= ln(1/ρ)] of a microstate which is 0 when probability ρ = 1 and increases to infinity as ρ tends to zero. The entropy is positive, is additive for independent sources of uncertainty and is a unique measure of the uncertainty represented by the probability distribution (Jaynes, 1957). The maximum entropy (MaxEnt) principle of Jaynes (1957) can be applied to infer the most probable distribution of microstates given a number of constraints. With the constraint of fixed average value of property, f; i.e. $\sum_{m=1}^{s} \rho_m f_m = \langle f \rangle$, and the natural constraint $\sum_{m=1}^{s} \rho_m = 1$, the most probable distribution of microstates is the Boltzmann distribution:

$$\rho_m^* = \exp(-\lambda_0 - \beta.f_m) = Z^{-1}. \exp(-\beta.f_m) \qquad (3)$$

where λ_0 and β are Lagrange multipliers ($\lambda_0 = \ln(Z)$ is known as the Massieu function), and the generalised partition function $Z = \sum_{m=1}^{s} \exp(-\beta.f_m)$. The maximum entropy is:

$$S^* = (\lambda_0 + \beta.\langle f \rangle) \qquad (4)$$

Although the situation is analogous to the canonical ensemble of systems in thermodynamics with the exception of a generalised fixed average property rather than fixed average energy, the MaxEnt approach does not in general require the physics to be that of a thermodynamical system; in particular the system need not be in equilibrium, and can be dynamic involving fluxes rather than static variables; so, as in Niven (2009a,b, 2010) I will consider flux entropies in this paper.

Note that I assume that all of the s categories are equiprobable a priori, and so dispense with the need for specifying prior probabilities, $q_m \equiv q_i$ in Niven (2009a).

2.2. Application of MaxEnt to a reservoir system

I now consider specifically flow through a geomechanically compliant porous medium in a reservoir system under production and injection, which has been conceptually discretised into a large number, N, of separate small regions of space; these are the "entities of the MaxEnt method. The small elements of the reservoir at any time can be allocated to a number of different, instantaneous microstates. Each microstate, labelled with a number, i, is characterised by particular *set* of values of variables, which I restrict to those potentially influential on crosswell interactions; those chosen variables are pressure, $p_i(\mathbf{r},t)$, strain tensor $\varepsilon_i(\mathbf{r},t)$ and their time derivatives $\dot{p}_i(\mathbf{r},t)$, $\dot{\varepsilon}_i(\mathbf{r},t)$, where **r** represents spatial location and t represents time. These microstates are the "categories" of the MaxEnt method.

The issue is now considered of what constraints are appropriate in applying MaxEnt to reservoir behaviour: I have here assumed just a single constraint, <f>. In order to be as general as possible, one cannot use constant average energy, as in the canonical ensemble of classical equilibrium thermodynamics: a reservoir under production is not in equilibrium. However, for a reservoir which is undergoing reasonably steady depletion, or perhaps steady net (production – injection) offtake, then the *rate* of total energy input or output from the reservoir varies reasonably slowly: this is the constraint that I have focussed upon. The theory will first be developed under the assumption that the average rate of energy input or output is fixed at a constant (a steady

state), but later adjusted to a less strict requirement: that the average rate is changing relatively slowly.

Each of the microstates will be associated with particular values of the *local* rate of energy input to specific small volumes of reservoir at a specific time, $\dot{E}(\mathbf{r},t)$. I assume that, within all the possible sets of variables $\mathbf{A} \equiv \{p(\mathbf{r},t), \varepsilon(\mathbf{r},t), \dot{p}(\mathbf{r},t), \dot{\varepsilon}(\mathbf{r},t)\}$, many combinations will have comparatively high

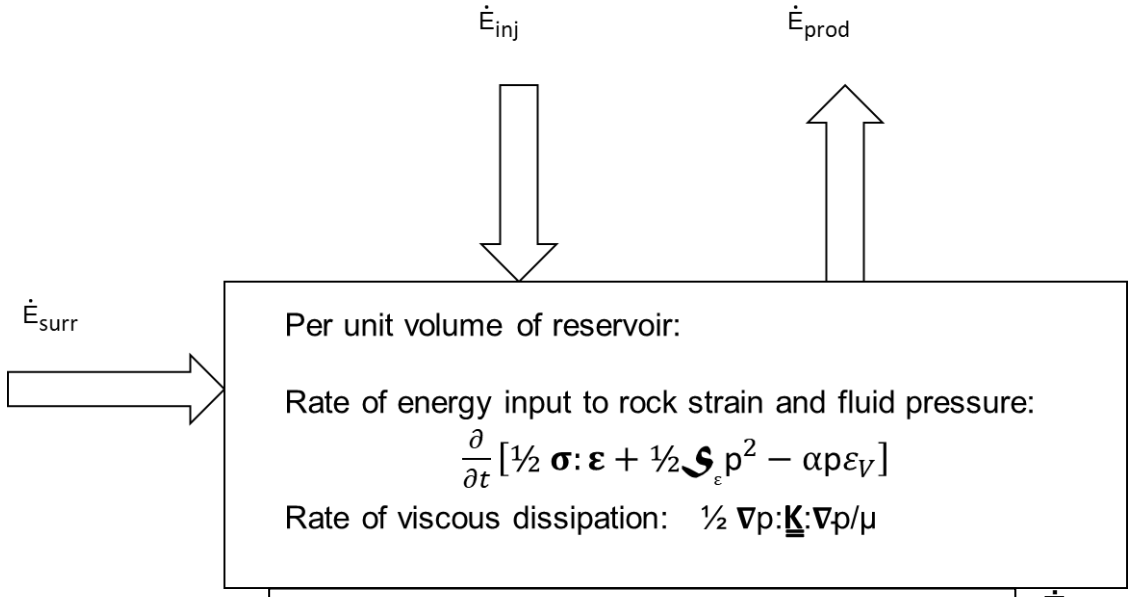


Figure 1: rates of energy change into and out of a reservoir

rates of energy input, $\dot{E}(\mathbf{r},t)$, so that their probabilities will be negligible according to the negative exponential distribution of equation (3), with local constraint value $f_m \equiv \{\dot{E}_m\}$, m ∈ {1, 2, ..., s}, in the discrete formulation. Hence only a more restricted set of "categories" is considered, so that s<<N, as required for the MaxEnt multinomial theory.

Figure 1 shows the energy flows into or out of a reservoir that are being considered. Energy obtained from the surroundings of the reservoir from (i) injection of fluids, (ii) aquifer influx or (iii) strain energy changes outside of the reservoir, must be balanced with (i) production of fluids from the reservoir, (ii) Darcy viscous fluid energy dissipation, or (iii) retained as rock strain and fluid pressure energy changes within the reservoir. Groenevelt (2003) discussed energy dissipation functions in terms of linear non-equilibrium thermodynamics, showing that Darcy's "law" provides the following equation for the dissipation rate of mechanical energy by viscous forces:

$\dot{E}_{Darcy} = ½\ \nabla p :\underline{\underline{K}}: \nabla p/\mu$. Energy changes due to other physical processes (e.g. heat, electric, capillary etc) are ignored in this analysis as being local in influence, inconsistent with my focus upon long-range interactions that are manifest as flowrate correlations between distant wells (an exception to this could be heat transfer, but I assume that this occurs relatively slowly with respect to both flowrate fluctuations and inter-well distances). Power balance requires that a net power input into the reservoir is realised as an increase in the rate of change of the elastic energy of the fluid and the rock of the reservoir plus the rate of dissipation of viscous fluid energy.

The time derivative of the energy input is thus:

$$\dot{E} = \dot{E}_{inj} + \dot{E}_{surr} - \dot{E}_{prod} = \int_V d^3\mathbf{r}.\ \{½\ \nabla p :\underline{\underline{K}}: \nabla p/\mu + \frac{\partial}{\partial t}[½\ \boldsymbol{\sigma}:\boldsymbol{\varepsilon} + ½:\mathcal{S}_{\varepsilon}.p^2 - \alpha.p.\varepsilon_V]\}$$

$$= \int_V d^3\mathbf{r}:\ \{½\ \nabla p :\underline{\underline{K}}: \nabla p/\mu + \frac{\partial}{\partial t}[½\ \boldsymbol{\varepsilon}:\underline{\underline{D}}:\boldsymbol{\varepsilon} + ½:\mathcal{S}_{\varepsilon}.p^2 - \alpha.p.\varepsilon_V]\}\ldots(5)$$

where the 3 power terms comprising $\dot{E}$ have subscripts denoting injection, surroundings and production respectively. The parameters (assumed constant in time) are: $\underline{\underline{K}}$, the permeability tensor; $\underline{\underline{D}}$, a stiffness tensor; μ, the fluid viscosity; α, Biot's constant, and $\mathcal{S}_\varepsilon$, the specific storage of the pore space at constant strain. The constraint for application of the MaxEnt method is chosen to be <f> = <$\dot{E}$>; I will show further justification for that choice later in the paper. The variables are pressure, p, components of the strain tensor (strain from a condition of zero stress), $\boldsymbol{\varepsilon}$, (ε_V is the total, or volumetric strain) and their derivatives with respect to space or time (the dependences of variables on position and time have been suppressed for conciseness). Note that I have omitted kinetic energies involving quadratic terms in time derivatives of variables, although they are implicit later in this paper.

Importantly for the further development of this theory, $\underline{\underline{D}}$ is not just a local stiffness tensor, but includes spatial interactions between fractures, even long-range interactions; later in this paper I relate components of $\underline{\underline{D}}$ to the tensile stresses produced at wells due to unit extensional strain at offset wells. It should also be noted that, because the strain can comprise both elastic and inelastic components, $\underline{\underline{D}}$**,** assumed constant with strain, represents an effective pseudo-elastic stiffness tensor. The energy represented by the final term on the right-hand side of equation (5), can be better appreciated when written as:

$$\tfrac{1}{2}\left[\,\boldsymbol{\varepsilon}:(\boldsymbol{\sigma}-\alpha p\boldsymbol{\delta})+\,(\mathcal{S}_\varepsilon p-\alpha\boldsymbol{\varepsilon}:\boldsymbol{\delta})p\right] \qquad (6)$$

where $\boldsymbol{\delta}$ is Kronecker's delta function written as a vector.

In this the two terms either side of the addition sign represent:

a) Energy change in the rock (strain x effective stress), plus:

b) Energy change in the fluid (volume of fluid added to storage x fluid pressure)

The parameters across the reservoir associated with viscous dissipation and strain energy increase are components of the permeability and stiffness tensors, which can either be treated as continuous functions or as taking discrete values across a grid of cells. The parameters themselves can be sensitive to the variables of pressure and strain, particularly permeability. The sensitivity implies that multiple locally stable flow patterns exist, potentially involving channelling; perturbations can trigger transitions between them, with each transition involving significant spatial reorganization of where and how energy dissipates. This sensitivity may be important to the applicability of Jaynes' MaxEnt: such variability will act to reduce the spatial coupling that pressure can exert, such that the number of degrees of freedom in the reservoir system is high, and that the system can self-organize to physically achieve the maximum entropy state (Niven, 2010). Nevertheless in the following sections, for the purposes of extracting information that will be useful to more conventional reservoir models, I will treat the parameters as constants over time, which implies that they are averages over their changes with the pressure or strain. $\dot{E}$ is a function of the set of variables, **A**, so $\dot{E}$(**A**).

2.3. Spontaneous changes over time in the MaxEnt state of the reservoir system

The reservoir system is unlikely to be in a *stable* steady-state of constant rate of energy input and maximum entropy. Changes are likely to be induced by both internal and external factors, for example due to: operator perturbations to production conditions; the introduction of new wells; changes over time to wells or reservoir conditions near wells; changes in saturations and/or pressures across the reservoir; the proximity of the system to a critical point. The reservoir system may often not have a constant value of the net rate of energy input.

As argued by Niven (2010), because MaxEnt can be applied to the system *and* its surroundings, any spontaneous change in the system (i.e. a change which makes the distribution of variables more probable) must be accompanied by an increase in the total entropy of the universe. That total increase is the sum of the increase in the flux entropy within the system and the flux entropy produced by the system and exported to its surroundings:

$$dS_{univ} = dS^* + dS_{prod} \geq 0 \qquad \ldots(9)$$

Spontaneous changes will occur to the system towards maximising the total entropy of the universe. As explained by Niven (2010), "the only means by which a system can produce and export more entropy is by reduction in the magnitude of one or more constraints", so that reductions in the constraints on the system represent increases in the flux entropy exported from the system:

$$dS_{prod} = -d[\beta\langle\dot{E}\rangle] \qquad \ldots(10)$$

In the context of a reservoir together with the storage equation and effective stress equilibrium , this describes the tendency of the system to evolve spontaneously towards a steady-state, when S* is a maximum and the net rate of energy input is a minimum. From equation (4), $dS^* = d\lambda_0 + d[\beta\langle\dot{E}\rangle]$, and so from equation (9) $dS_{univ} = d\lambda_0$. Niven (2010) illuminates how equation (9) is related to equilibrium thermodynamic concepts by introducing the potential function ϕ (negative Massieu function) which captures all possible changes in the system, due to changes in the entropy S*, or in the constraints. It is analogous to the Gibbs free energy concept of equilibrium thermodynamics.

$$d\phi = -dS_{univ} = -d\lambda_0 = -dS^* - dS_{prod} \leq 0 \qquad (11)$$

As discussed by Niven (2010, 2012) the maximum of total entropy of the universe can be achieved in one of three scenarios, depending on the signs of the terms:

I. $dS^* \geq 0$ and $dS_{prod} \geq 0$, giving max S* and max S_{prod}: pan-entropogenic

II. $dS^* \geq 0$ and $dS_{prod} \leq 0$, with $dS^* \geq |dS_{prod}| \geq 0$, giving max S* and min S_{prod}: endo-entropogenic

III. $dS^* \leq 0$ and $dS_{prod} \geq 0$, with $dS_{prod} \geq |dS^*| \geq 0$, giving min S^* and max S_{prod}: exo-entropogenic

These three possibilities throw light on the dichotomy of ostensibly contradictory extremal principles of non-equilibrium thermodynamics: maximum or minimum entropy production (max or min $S_{prod.}$; see also Dewar et al., 2014, for a review). Since $dS_{prod} = -d[\beta\langle\dot{E}\rangle]$ and assuming a constant analogue of "temperature", β, of the system, the average energy input rate $\langle\dot{E}\rangle$ progresses to be stationary, either a maximum (scenario II) or minimum (scenarios I & III) depending on which of the above 3 possibilities applies. Niven (2010) presented an argument, albeit not proof, (under his section 3(b) "Classification of spontaneous processes") that a pan-entropogenic process (scenario I. above) was more likely in the case of his flow system example, on the basis that an increase in the thermodynamic force conjugate to the mean local flux was *unlikely* to be associated with retention, or even reduction, of a restricted range of fluxes as would be implied by constant or reducing value of S*. A similar argument can be applied to reservoirs, noting that rates of change in strain or pressure can be positive or negative as with fluid fluxes: the distribution of fluxes is more likely to widen and flatten ($dS^* > 0$) at the same time as the mean flux and S_{prod} increase, such that $dS_{prod} = -d[\beta\langle\dot{E}\rangle] > 0$. By this argument the entropy production, S_{prod}, tends to a maximum, or, equivalently, $d[\beta\langle\dot{E}\rangle] < 0$ and, with a constant "temperature", β, $\langle\dot{E}\rangle$ tends to a minimum. This is consistent with the principle of minimum energy dissipation (e.g. Zupanovic et al., 2010). I will assume that this argument is valid. Main & Naylor (2008) have explained several observations of natural earthquakes in the brittle lithosphere by the system organizing to maximize entropy production in a near but strictly sub-critical state.

Niven (2010) also makes the profound point that systems in which flux entropy is being considered (in his case, flow systems) possess negative, as well as positive, flux levels. Therefore, as S*

increases, the system is more likely to access its states of reverse flux. "In other words, a high flux entropy is associated with *greater variability* in the fluxes…, which therefore implies *oscillatory* or *chaotic* processes". Niven (2010) makes the connection between "high S* and the defining features of many 'far from equilibrium' systems, such as turbulence, heat-induced convection cells, nonlinear diffusion phenomena and oscillatory chemical reactions". This connection might also apply to the changes in flowrates in mature reservoirs, and explain at least some of their high frequencies and amplitudes. As previously stated, the potential for strain, fracture conductivities, and therefore local average permeabilities, to change in response to stress changes implies a larger number of degrees of freedom and is another factor that boosts the possibility of oscillatory or even chaotic behaviour.

How the exported flux entropy is distributed across the surroundings has not been addressed in this work. It will likely be some combination of changes in the variables over the subsurface surrounding the reservoir, plus the wells and the surface equipment. It is possible that the exported flux entropy changes (dS_{prod}) due directly to spontaneous changes in the reservoir may only be part of the total entropy changes in the surroundings; there may be indirect external energy dissipation processes as a consequence of those due directly to dS_{prod} that increase the flux entropy changes in the surroundings further. This is immaterial to the further development of the methodology of this paper: it is that S_{prod} tends to a maximum and $<\dot{E}>$ to a minimum that is pertinent.

2.4. Fluctuation Theory - the relationship between covariances in variables and system parameters.

In a previous section I have given some justification that the system entropy, S*, and the entropy production, S_{prod}, both tend to maxima (the reservoir is a pan-entropogenic system), with the likelihood that the fluxes in the poroelastic reservoir will spontaneously fluctuate with high amplitude in addition to, but triggered by, any perturbations introduced by extraneous factors. Although operator-induced changes may have an arbitrary distribution, I rely on their correlation between wells over extended time being low compared with those occurring spontaneously. Spontaneous fluctuations are an intrinsic property of any finite system at finite “temperature”, arising from the probabilistic nature of statistical mechanics itself. This section describes the probabilities of such spontaneous departures from the current maximum of $S_{prod}(t)$, following Einstein's Fluctuation Theorem (Reichl, 1980). It leads to a relationship between the correlations in the fluctuating variables and the parameters of the energy fluxes. As explained by Reichl (1980) even though Einstein's Fluctuation Theorem is based on a Taylor series expansion of the entropy, which strictly breaks down where the fluctuations are very large, as near a critical point, it can still give valuable insight concerning the behaviour of fluctuations.

I will first expound the analysis with the assumption of constant power input, before, in the next section, extending it to changing power input with time.

With a spatial and temporal discretisation of equation (5), power input can be written. $\dot{E} = ½.\mathbf{A}.\underline{\underline{\mathbf{G}}}.\mathbf{A}$ where $\underline{\underline{\mathbf{G}}}$ is the matrix of parameters, combinations of permeability and stiffness tensors. $\dot{E}$ can be either positive or negative depending upon the balance of energy fluxes into and out of the reservoir.

For this analysis, the fluctuations of the variables, **A**, of the reservoir system from their values, **A***, at the maximum entropy state are expressed as deviations $\alpha = \mathbf{A} - \mathbf{A}^*$. I mainly follow the development by Reichl (1980) to derive the relationship between the moments of the deviations, α, and the parameters, $\underline{\underline{\mathbf{G}}}$.

In the Taylor series S_{prod} may be expanded with small fluctuations, α, about that maximum value to second order as (dropping the time dependency for clarity):

$$S_{prod}(\langle \dot{E} \rangle, \mathbf{A}) = S_{prod}(\langle \dot{E} \rangle, \mathbf{A^*}) - \frac{1}{2}\sum_{i=1}^{Nv}\sum_{j=1}^{Nv} g_{ij}\alpha_i\,\alpha_j + \ldots\ldots \qquad \ldots(12)$$

where: $g_{ij} \equiv -\left[\frac{\partial^2 S_{prod}}{\partial A_i \partial A_j}\right]_{A_i=A_i^*;\, A_j=A_j^*} = +\beta.\left[\frac{\partial^2 \dot{E}}{\partial A_i \partial A_j}\right]_{A_i=A_i^*;\, A_j=A_j^*} = +\beta.\left[\frac{\partial^2 \dot{E}}{\partial \alpha_i \partial \alpha_j}\right]_{\alpha_i=0;\, \alpha_j=0}$

given a constant "temperature" (Lagrangian multiplier), β. Due to the fact that $S_{prod}(\langle \dot{E} \rangle, \mathbf{A^*})$ is a maximum and the fluctuations must cause the entropy to decrease, the first order term in the expansion has been omitted because it must be identically zero, and g_{ij} must be positive definite. g_{ij} is also symmetric because the second derivatives of S_{prod} commute. Note that the averaging of $\dot{E}$ may be dropped because, with the rate of energy input given by equation (5), the second derivatives of its quadratic form are independent of the variables of the microstate.

Reichl's development then invokes Boltzmann's entropy, based on the number of microstates required to realise a macrostate. It is well established that Boltzmann's and Jaynes' approaches are entirely consistent, so, as in Reichl (1980), the probability distribution of fluctuations about the maximum entropy production state is given by:

$$\pi^*(\alpha) = C.\exp(-\beta.\alpha^T.\underline{\underline{G}}.\alpha) \qquad \ldots(13)$$

where $\underline{\underline{G}}$ represents the full tensor g_{ij} and the normalization factor, $C = \sqrt{\left[\left(\frac{\beta}{\pi}\right)^{Nv}|\underline{\underline{G}}|\right]}$

The subsequent development of Einstein's fluctuation theorem leads (Reichl, 1980) to the following**:**

a) the second moment or covariance in fluctuations of the variables $\langle \alpha\alpha^T \rangle = 1/(2\beta).\underline{\underline{G}}^{-1}$; or in terms of the i^{th} and j^{th} components of the variables $\langle \alpha_i.\alpha_j \rangle = 1/(2\beta).(\underline{\underline{G}}^{-1})_{ij}$. Here it is important to recognise the implicit existence of kinetic energies involving quadratic terms in time derivatives of variables in $\alpha^T.\underline{\underline{G}}.\alpha$, such that the matrix $\underline{\underline{G}}$ is non-singular. It is more expedient to take the inverse of the relationship. $\underline{\underline{G}} = 1/(2\beta).\langle \alpha\alpha^T \rangle^{-1}$, and the parameters associated with kinetic energies can be ignored by partitioning the matrix inverse (see Appendix 3).

b) The first moment $\langle \alpha_i \rangle = 0$, as required by the starting context of this theory.

I focus on the first two terms in the power input that most likely include cross-well influence and therefore significant to inter-well correlations. In that case the energy input rate is quadratic in the reduced variable set, $\mathbf{A} \equiv \{\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}\}$, or $\alpha = \mathbf{A} - \mathbf{A^*} \equiv \{\Delta\boldsymbol{\varepsilon}, \Delta\dot{\boldsymbol{\varepsilon}}\}$, in which Δ signifies departure of strain or strain rate from values at maximum entropy.

2.5. How this relationship can be used.

From the relationship $\underline{\underline{G}} = ½.\beta.\langle \alpha\alpha^T \rangle^{-1}$, estimates of parameters $\underline{\underline{K}}$ and $\underline{\underline{D}}$ within the matrix $\underline{\underline{G}}$ can be derived from correlations in fluctuations in well flowrates with more assumptions:

a) That the flowrate fluctuation of a well is a proxy for local volumetric strain, justified for example in Manga et al., 2012; Wang & Xu, 2013; Heffer 2012a,b, 2013. For a single-phase fluid flowrates into or from a well are a function of the local permeability and the average pressure in the vicinity of the well. Due to the strong dependence of the conductivity of fractures on their apertures, changes in the strain components around a well (and it is changes that are pertinent in the theory) can alter the local permeability. I refer to volumetric strain so that there is no prior assumption about the orientation of the dominant fracture set in that location.

b) Hitherto in this paper I have applied the Jaynes method to small volumes of the reservoir, leading to this covariance relationship. I now envisage a gross upscaling over the elemental volumes surrounding each individual well in a reservoir, and assume that the same form of relationship applies to covariances between flowrates at wells with some consequent spatial averaging of the parameters of permeability and compliance. The number of variables is restricted to that of the wells, N_w; i.e. $N_V = 2.N_w$.

c) That the averages of flowrate fluctuations over available histories provide reasonable estimates of the covariances of strain fluctuations.

d) That the fluctuations in flowrates that are caused by factors extraneous to the spontaneous fluctuations of the poroelastic reservoir (operator-induced events; changes to well conditions; changes to reservoir conditions etc.) are not significantly correlated between wells when the flowrates are detrended to remove low frequencies.

e) The smoothed production or injection rate history at each well (see figure 2) approximates that which would occur at that well if the reservoir were constantly in a state of maximum entropy production. Given that this is only conjecture, in Appendix 2 it is shown that, given reasonable assumptions about the differences between smoothed history and actual maximum entropy production states, the off-diagonal terms of the covariance matrix of variables are unaffected.

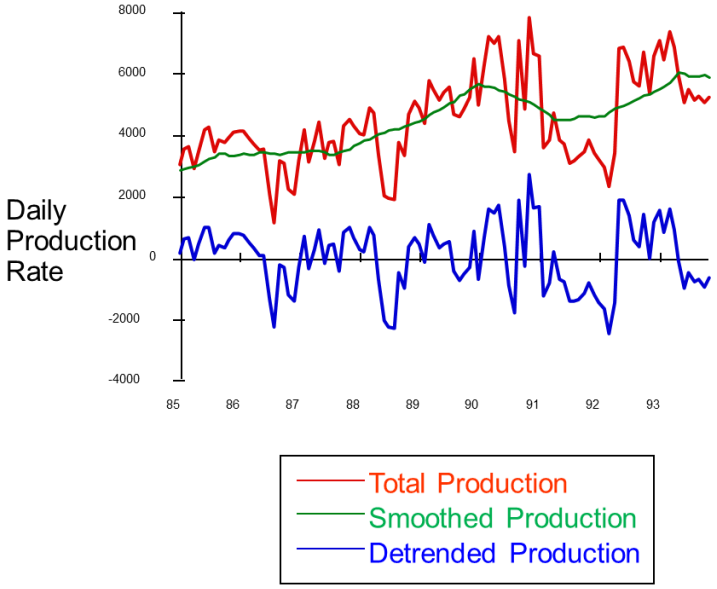


Figure 2. For one well: a typical production history (red line); its history smoothed by 12 month averaging (green line); and the departures of total from smoothed production history: (high-pass filtered or detrended - blue line). The smoothed production history is assumed to approximate that which would occur if the reservoir were constantly in a state of maximum entropy production.

Pearson correlation of two variables is equivalent to the covariance of the variables normalised by their respective standard deviations. The non-parametric Spearman rank correlation is more sensitive to non-linear but still monotonic relationships between variables, and has been found in practice to be a suitable proxy for Pearson correlation coefficient. Spearman rank correlations have the following advantages:

a) Their significance levels are readily computed without knowing the underlying distributions of flowrates.

b) Normalisation will obviate domination of the solutions by high flowrate wells

c) Giving more weight to non-linear relationships will both be of greater utility in many reservoir management situations where relative rather than absolute solutions of the parameters are required.

The physical meanings of parameters derived from components $(\underline{\underline{G}}^{-1})_{ij}$ should be interpreted according to whether covariances, or Pearson or Spearman rank correlations are used in place of $<\alpha_i.\alpha_j>$.

Earlier it was argued that quasi-static strain components are the dominant variables to be correlated at shorter timescales. *Detrending* the flowrate data will not only provide approximate departures of flowrates from the state of maximum entropy production but also emphasise the importance of shorter timescales in the correlations and so diminish the influence of:

a) interwell interferences due to slower-acting pressure signals,

b) any long period changes in flowrates due to, say, average pressure declines from general reservoir depletion, saturation changes, scale deposition, fluid composition changes etc.

In order to extract estimates of the stiffness and permeability parameters I use the full correlation matrix, $\boldsymbol{Q}$, of detrended flowrates and rates of change of flowrates (see Appendix 3).

2.6. Utilising all interactions, not just those of nearest neighbours

Individual nearest neighbour correlations provide some information on local interactions and therefore on local structure, but a much larger amount of information is available from all the long distance correlations, which are influenced by sequences of nearest-neighbour interactions. In order to use all indirect correlations to determine the values of the stress interferences between nearest neighbours (which I term "direct" interferences) the following algorithm is deployed. In this description I use the terms "stress" or "strain" as short-hand for "deviations of stress or strain from the condition of maximum entropy". Similarly, $\boldsymbol{\sigma}$ or $\boldsymbol{\varepsilon}$ are written rather than $\Delta\boldsymbol{\sigma}$ and $\Delta\boldsymbol{\varepsilon}$.

A component of the stiffness tensor between any two wells ($\underline{\underline{\mathbf{D}}}$ in the above equations), derived from the correlation matrix between wells, is relevant to geomechanical signals passing between those two wells, which I will term "indirect" interferences. However, the spatial routes through the reservoir taken by those indirect interferences are not known a priori and potentially circuitous, taking advantage of various geomechanically sensitive faults, fractures or other discontinuities. Communication between multiple well pairs might take advantage of geomechanical "highways", which could comprise individual faults, particularly their damage zones, or fracture "corridors". Support for such focussing comes from the empirical association of areal patterns derived from the most important eigensolutions or modes of flowrate correlation matrices with known faults (Heffer et al.,1995).

Although the stiffness moduli $\underline{\underline{\mathbf{D}}}$ are obtained empirically from flowrate correlations in a dynamic reservoir setting in which rates of change of variables are influential, it is assumed that the same values of stiffness moduli, $\underline{\underline{\mathbf{D}}}$ pertain in a static system in stress equilibrium. In order to keep the variables as scalars in line with the scalar nature of flowrates as the only data inexpensively available, I only consider the components of the stress and strain tensors which are *normal* to the main fracture or fault strikes at each location. Those are the components which will be most influential on flowrates in passing geomechanical signals between wells. With just single components of stress, σ_i, and strain, ε_j, at wells i and j respectively, the individual values of the stiffness tensors between the wells can be written as a matrix of scalar values, one for each well pair, D_{ij}. The stress at any well in this static setting can be written as the sum of strains at all wells multiplied by their individual indirect stiffness tensors:

$$\sigma_i = \sum_{j \in Nw} D_{ij} . \varepsilon_j \qquad \ldots(14)$$

where Nw is the full set of wells. Note that j = i is also included in this summation, such that Dii is the more conventional local component of an effective stiffness tensor at well i incorporating whatever faults or fractures are present in the domain of the well. This formulation considering indirect interferences is consistent with the use of the variables in the rate of change of strain energy in the dynamic non-equilibrium equations, according to which the values of the indirect stiffness moduli are derived.

An alternative formulation for the stresses at wells in a static setting which is in stress equilibrium is through considering only direct stress interactions between nearest neighbour wells. Accordingly, the stress at well i is equal to the sum of direct responses from stresses at its nearest neighbours: $\sigma_i = \sum_{k \in NN(i)} \varphi_{ik}.\sigma_k$ where NN(i) is the set of nearest neighbour wells to well i, and φ_{ik} is the stress induced at well i by a unit stress change at well k. Note that φ_{ik} is an influence function linking a stress perturbation to the stress changes at offset locations, and is therefore dimensionless, only a geometrical factor.

Combining these two approaches:

$$\sigma_i = \sum_{j \in Nw} D_{ij}.\varepsilon_j = \sum_{k \in NN(i)} \varphi_{ik}.\sigma_k = \sum_{k \in NN(i)} \varphi_{ik}. \sum_{j \in Nw} D_{kj}.\varepsilon_j = \sum_{j \in Nw} \sum_{k \in NN(i)} \varphi_{ik}.D_{kj}.\varepsilon_j \quad (15)$$

Given that any consistent distribution of strains, ε_j, at wells will obey this equation, then.

$$D_{ij} = \sum_{k \in NN(i)} \varphi_{ik}.D_{kj} \quad \ldots(16)$$

Estimates of D_{ij} are obtained from flowrate correlations as indicated in Appendix 3. The algorithm proceeds to maximise consistency between the empirical values of indirect interferences, represented by D_{ij}, by finding a least squares solution to the above over-determined equation, in which the direct stress interactions, φ_{ik} are the unknown variables. The least squares matrix, comprising values of D_{kj}, will be sparse, which in a field of a large number of wells calls for a sparse matrix solution of the above equation for φ_{ik}. It is assumed that φ_{ik} is symmetric; i.e. the stress interaction from well k to well i is equal to that from well i to well k.

The stress interactions between nearest neighbour wells, φ_{ik}, obtained as a solution to equation (16), can be used to infer orientations and densities of fractures at each well. In keeping with this concept I fit the theoretical orientational variation of the stress field around a single fracture, centred at each well, to the direct stress interactions, φ_{ik}, with its nearest neighbours. This fit is by adjustment of the orientation of the fracture, and of the length of the fracture, the latter taken as a proxy for the density of fracturing in the vicinity of the well. The angular variation of the stress field around a fracture of half-length a, simplified from Pollard & Segall (1987) to suit distances somewhat greater than a, is taken as $C_i.\sin(\theta).\sin(3\theta)$, where C_i is a constant for well i, and θ is the angle between the fracture strike and any radial line from well i. (Healy et al. (2006a&b) explain the 3-D geometry of the tensile stress field around a mode 1 crack, and its rôle in guiding the formation of polymodal faults). Thus, the theoretical value of the stress interaction between well i and a nearest neighbour well, k, is assumed to be $\varphi'_{ik} = C_i.\sin(\theta_{ik}).\sin(3\theta_{ik})$. The orientational variation of this form (with C_i=1) is depicted in figure 3.

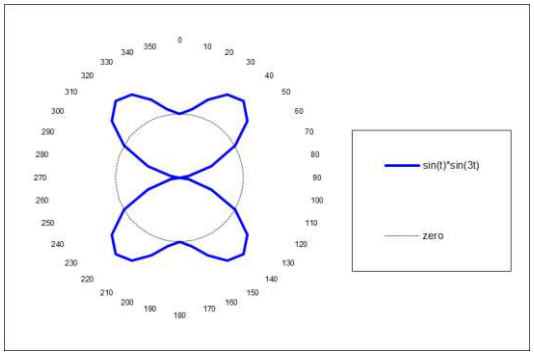


Figure 3. Simplified form of the 2D angular variation of nn tensile stress around an open crack which strikes up page . (n, fracture normal across page) (simplified from Pollard & Segall (1987))

The use of a constant C_i for a given well, i, rather than incorporation of variations in distance between wells i and k, is justified on the basis that long-range interactions are also relevant to nearest neighbour wells, in addition to the long-range strain-stress influences measured by D_{ij} .

The best-fitting fracture strike at well i and constant C_i for well i (the latter providing a proxy value for fracture density) are found by minimising the sum of squares $\sum_{k \in NN(i)}(\varphi_{ik} - \varphi'_{ik})^2$, where φ_{ik} are the interpreted values, and φ'_{ik} are the theoretical values of nearest-neighbour stress interactions.

3. Results

As is inevitable in collapsing the complexities of a heterogeneous, time-varying reservoir fluid-rock fracture system in a near-critical state into a much more tractable form for model inversion, several assumptions are involved, albeit each with justification, as described throughout the text. It is therefore only via applying the formulation to real field cases and comparing the results with independent field data (fault maps, measured orientations of maximum horizontal principal stress axes (Shmax), microseismicity, shear-wave anisotropy, surface strain measurements from InSAR surveys, etc.) that the validity of the methodology can be assessed. A start on this has been made with encouraging comparisons in the context of internationally and geologically diverse oil reservoirs under waterflood or depletion, a gas reservoir under depletion (at one time mooted for a potential Carbon Capture and Sequestration project) and a geothermal field, as well as a modelled 2D field involving coupled geomechanical changes and fluctuating flow at several wells (Heffer, 2024, 2025, Zhang, 2007). A revealing finding has been from comparisons of interpreted fracture strikes with orientations of Shmax. For most of the fields the azimuthal distributions of measured Shmax are not unique, but a distribution. To cope with the variations in interpreted fracture strikes and Shmax orientations, normalised cross-correlation coefficients were calculated between the 2 angular distributions for each case study:

$$R_{xy}(k) = \frac{\sum_i x_i \, y_{i+k} - N.\bar{x}.\bar{y}}{\sqrt{(\sum_i x_i^2 - N.\bar{x}^2) \, . \, (\sum_i y_i^2 - N.\bar{y}^2)}}$$

where x_i, y_i are the weights of fracture strikes and Shmax orientations respectively in individual azimuthal sectors, i = 1,N (N=18 for 10 degree sectors). Figure 4a shows the cross-correlations, aggregated in each sector, k = 1,N, across 6 water injection fields (5 real fields and 1 simulated 2D field). The correlation coefficients show a strong positive bias towards angles within 30 degrees of zero, and generally negative values at other angles; this is emphasised in figure 4b, by averaging the cross-correlation coefficients for the sector groups: -30 to 30 ; 40 to 50 + 130 to 140 ; 60 to 120.

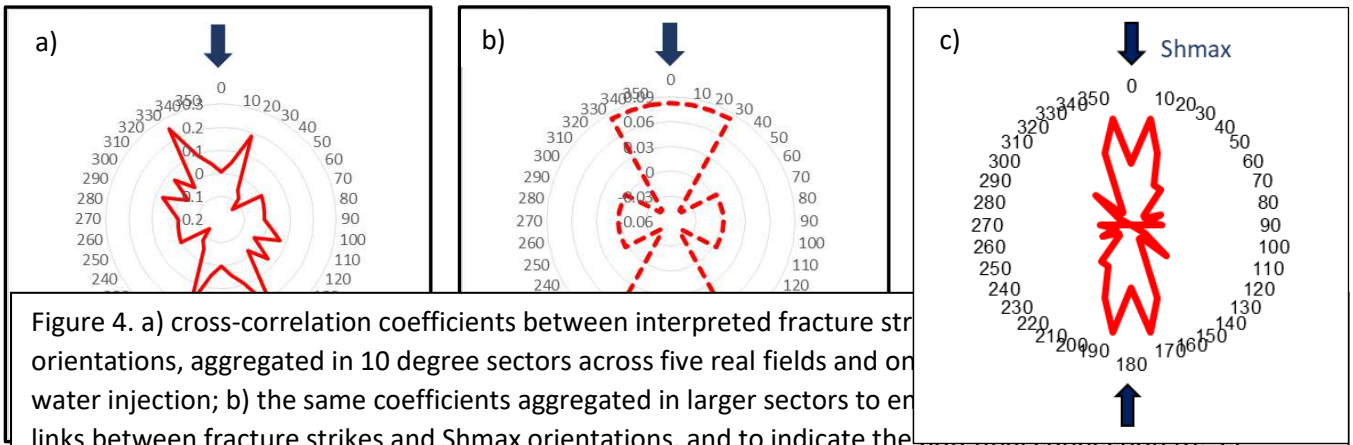


Figure 4. a) cross-correlation coefficients between interpreted fracture str
orientations, aggregated in 10 degree sectors across five real fields and on
water injection; b) the same coefficients aggregated in larger sectors to en
links between fracture strikes and Shmax orientations, and to indicate the
preferred directionalities of flooding in secondary recovery schemes relative to local Shmax orientations (Heffer & Lean, 1993).

The patterns of figure 4a) and 4b) are reasonably consistent with the empirical pattern of preferred directionalities of injected fluids in secondary recovery schemes from 47 nominally “unfractured” reservoirs relative to local orientations of Shmax (Heffer & Lean, 1993). The more variable relationship between interpreted fracture strikes and stress axes is possibly due to: i) a smaller population of cases; ii) the fact that the flood directionalities are taken as the major axes of imaginary ellipses fitted to well-to-well data ; iii) changing stress patterns during the fields’ lives; or iv) influences from heterogeneities, perhaps mainly faults, being greater for geomechanical signals than for hydraulic flow between wells.

In strong contrast, fields undergoing pressure depletion without secondary recovery show no such correlation between interpreted fracture strikes and stress (figure 5). However, in individual cases of such, there does appear to be association between fracture strikes and known fault traces, implying that faults or fault damage zones are providing conductivity paths under either in situ stress states, or stresses changed during production. In the case of the Valhall field, the pattern of interpreted fracture strikes appears to reflect the compaction bowl caused by the pressure depletion and revealed in fast shear waves just above the reservoir (Heffer, 2025).

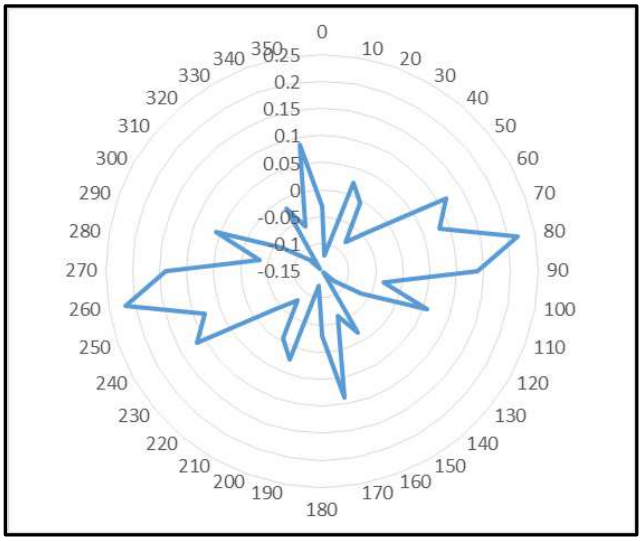


Figure 5. cross-correlation coefficients between interpreted fracture strikes and measured Shmax orientations, aggregated in 10 degree sectors across five fields undergoing pressure depletion. There appears to be no association between the fracture strikes and stress (see text for possible reasons).

Further field applications of the techniques are of course desirable to confirm or amend these tentative deductions.

4. Discussion

4.1. Consistency with the usual poroelastic equations

In Appendix 1 it is shown by the variational method that, assuming ergodicity, a maximum value of the produced entropy flux, S_{prod}, or minimum value of the average power input is consistent with both of the following equations:

a) fluid mass conservation or storage equation (Wang, 2000; his equation 4.65).

$$\mathcal{S}_{\varepsilon}.\dot{p} - \alpha.\dot{\varepsilon}_V = \nabla.(\underline{\underline{K}}/\mu.\nabla p) \qquad \ldots(7a)$$

and b) effective stress equilibrium $\frac{\partial}{\partial x_j}(\sigma_{ij} - \alpha.p.\delta_{ij}) = 0$ …(7b)

In other words, when flux entropy is maximum, the normal poroelastic equations hold, with changes in stress and strain compensated by appropriate changes in pressure to maintain stress equilibrium. The familiar equations (7a & b) also justify the choice of constant average rate of energy input, $< \dot{E}(\mathbf{A}) > = \sum_{m=1}^{s} \rho_m * \exp(-\beta.\dot{E}_m(\mathbf{A}))$ in the discrete formulation], as a constraint for maximum flux entropy and also the choice of the set of N_V variables, $\mathbf{A} \equiv \{p(\mathbf{r},t), \varepsilon(\mathbf{r},t), \dot{p}(\mathbf{r},t), \dot{\varepsilon}(\mathbf{r},t)\}$

distributed across the domain and time; it is seen that the relationships between strain, stress and fluid pressure are implicit in the chosen constraint and requirement for maximum flux entropy.

The evolution of the coupled hydro-mechanical processes in a deformable medium involving transient fluid flow perturbed by frequently changing well flowrates are likely to take considerable time to settle to a condition of effective stress equilibrium, especially if the medium is fractured with associated long-range influences, and if the geomechanical changes are in part inelastic. Speculatively, pulses or waves of dilation or compression can move over significant distances for significant times through the reservoir, guided by faults, fractures and stress, only gradually approaching stress equilibrium. The processes could be partially analogous to porosity waves (e.g. Spiegelman, 1993; Revil & Cathles, 2002). Equilibrium may not be reached before the next extraneous perturbation occurs, but even if it is, the state of maximum flux entropy may only be ephemeral. However, such fluctuating departures from the maximum flux entropy state can yield further information about the system from Einstein's Fluctuation Theory, as developed below.

The effective stress equilibrium at minimum energy input can be used to substitute stress gradients for pressure gradients in the first term of the integrand of the functional Ė given in equation (5):

$$\frac{\partial p}{\partial x_i} K_{ij} \frac{\partial p}{\partial x_j} = \frac{1}{\alpha^2} \frac{\partial (D_{imkl} \varepsilon_{kl})}{\partial x_m} K_{ij} \frac{\partial (D_{jnpq} \varepsilon_{pq})}{\partial x_n} \qquad \ldots(8)$$

Of course, when considering the system deviating from steady state and the condition of maximum entropy or minimum rate of energy input, the effective stress equilibrium no longer exactly holds. It is assumed that this has small effect on the interpretation of fracture characteristics.

4.2. Alternative explanation of long-range correlations

Section 1.2 introduced the concept that the reservoir system as a whole is near-critical, either naturally or as development and production proceed. This implies that small perturbations at one point in the system can cause significant resultant perturbations elsewhere, even at large distances. Fractures and faults will play a linking mechanism for those interactions. However the mechanism is not "action at a distance", but the passage of stresses and strains through the reservoir is envisaged to be mainly through an avalanche process, between neighbouring areas of the domain in sequences. So, a change in the strain at one well in a reservoir can lead to changes in stresses at various other wells, even those separated by large distances.

The transmission of strain via seismic waves through the reservoir occurs relatively rapidly, within the order of seconds; or by cascades of slip events (e.g. Ben Zion, 2008), occurring more slowly; or by some combination of both. In contrast pressure fields propagate according to Darcy flow over longer time periods, days or months across fieldwide scales. Although pressure is influenced by strain changes through poroelasticity, and over longer timescales will be correlated spatially across the reservoir as a consequence of the coupled fluid-geomechanics physics, nevertheless it is a reasonable assumption that quasi-static, quasi-elastic strain components are the dominant variables to be correlated over long distances at timescales of the order of a day to a month.

An alternative explanation of flowrate fluctuations is that, rather than due to fracture-influenced stress interferences, they are caused by extended pressure fields, either alone or via associated effective stress fluctuations changing permeabilities. However:

1. The correlations between flowrates at pairs of injectors and producers (Heffer et al., 1997) show an orientational dependency which is in line with strain changes corresponding to the elastic tensile stress field around a crack (Heffer, 2012). (either both dilational or both

compressional sub-parallel to the fracture strike Shmax; of opposite sense sub-parallel to Shmin).

2. Pressure fields are unlikely to be as spatially correlated as observed with flowrate fluctuations; the latter show very slow fall-off with spatial separation of well pairs and are significant even up to lag distances across the whole oilfield.
3. Furthermore, considering only fluid pressure, not strain, changes instigated at one well and passing to a nearby well (either Producers, P, or Injectors, I) in a static rock framework, only those of opposite type (I-P) would show positive correlations; e.g. an increase in reservoir pressure due to increase in injection rate at an injector would tend to cause an increase in production rate at a neighbouring producer. Between neighbouring well pairs of like type (I-I or P-P) correlations from only pressure communication would be negative. Such association of positive and negative correlations with well-pair types is not seen in the voluminous field data analysed.
4. The change of permeability with effective stress influence on fractures can be orders of magnitude, especially when they are close to being critically stressed, whereas change of flowrate with pressure alone is relatively modest.

It is also possible that solid earth tides are influencing the strains and therefore permeabilities in the reservoir. However, one would expect the volumetric strains from earth tides to change the strains and therefore the flowrates relatively homogeneously, rather than with the stress-aligned patterns observed.

4.3. Minimising the chance of spurious correlations affecting results

Spurious correlations are notorious aberrations of statistical analysis of which there are many published examples. Several various steps and factors in the methodology minimise the chance that spurious correlations affect the resulting fracture characterisation:

a) Detrending. This removes long-term, low time frequency components of the flowrate fluctuations, so that only the high frequency fluctuations are analysed.

b) Using a non-parametric correlation coefficient (Spearman Rank or Kendall's tau). This allows the significance level of correlation to be calculated. A threshold of significance level for the correlations can be imposed.

c) The variables in each correlation are not obviously unconnected in that they describe the same physical process (liquid flowrates).

d) The full matrix of correlation coefficients (for I-P, P-P and I-I, where P is a producer and I is an injector) for all wells in the field is formed, and a threshold set for the eigenvalues of the eigensolutions (independent modes of fluctuation).

e) Omission of time periods when one or both of a well-pair is not flowing in calculating their correlation.

f) Associating the correlations with coupled geomechanics-flow physics and with the general phenomena of critical systems established in a wide range of physical processes.

g) Rather than using the bare correlation coefficients, a calculation is made from them of the stress interactions between neighbouring locations of fractured/faulted rock, such that those stress interactions possess maximal consistency with all of the correlations between pairs of wells.

4.4. Conclusion

In this paper I have laid out a means of extracting fracture characteristics from readily available production data in a mature field. Application of a series of theoretical constructs, including some from standard statistical mechanics, albeit including several assumptions, has led to a template formulation for extracting estimates of fracture strikes and proxies for fracture densities at wells. Early results with input data from fields and theoretical model imply that the technique is a valid means of enhancing understanding of fracture characteristics in mature reservoirs.

Appendix 1. Variational approach

For the purposes of this approach I extend the number of possible realisations, s, to infinity with a continuum of possible values of the variables, **A**. The MaxEnt probability distribution for power input is correspondingly modified to.

$$\rho(\mathbf{A}) = \frac{1}{C.Z(\beta)} \exp\left(-\beta.\, \dot{E}(\mathbf{A})\right) \qquad \ldots(A1.1)$$

where $\rho(\mathbf{A}).d\mathbf{A}$ is the probability density of microstates with variables in the range **A** to **A**+d**A**; β is a Lagrange multiplier analogous to $1/(k_B.T)$ for equilibrium thermodynamic systems; and C is a constant inserted in the expression for entropy in order to give the correct units. With Ω as the complete continuous phase space for **A**:

$Z(\beta)$ is the partition function = $\frac{1}{C}\int_\Omega d\mathbf{A}.\exp\left(-\beta.\, \dot{E}(\mathbf{A})\right)$

The natural, or probability normalisation, constraint becomes $\int_\Omega d\mathbf{A}.\rho(\mathbf{A}) = 1$.

The average power input across all possible values of the state values is then obtained by integrating over all of phase space Ω:

$$< \dot{E} > = \int_\Omega d\mathbf{A}.\dot{E}(\mathbf{A}).\rho(\mathbf{A}) \qquad \ldots(A1.2)$$

I am looking to minimise the average rate of energy input, $< \dot{E}(\mathbf{A}) >$, in order to maximise the produced flux entropy.

Assuming ergodicity, that all possible distributions of state variables are visited, the average energy production, instead of an integral in phase space, can be alternatively expressed as an integral over the total time, t, of availability of data.
$< \dot{E} > = \frac{1}{t}\int_0^t \dot{E}(\mathbf{A}(\tau)).d\tau = \frac{1}{t}\int_V F[p(\mathbf{r},t),\boldsymbol{\varepsilon}(\mathbf{r},t)].d^3\mathbf{r}$, where commutation of the integrals over time and space has been used, and the function F is:

$$F[p,\boldsymbol{\varepsilon}] = \left\{\int_0^t \tfrac{1}{2}\, \nabla p.\frac{\underline{\underline{K}}}{\mu}.\nabla p.\, d\tau + \left[\tfrac{1}{2}\, \boldsymbol{\varepsilon}.\underline{\underline{D}}.\boldsymbol{\varepsilon} + \tfrac{1}{2}\, \boldsymbol{\mathcal{S}}_\varepsilon.\, p^2 - \alpha p \varepsilon_V\right]\right\} \qquad \ldots(A1.3)$$

in which the dependence of variables on (**r**,t) has been dropped for clarity. At its minimum, the functional, $< \dot{E} >$, is stationary with respect to the variable functions p(**r**,t) and ε(**r**,t), and their derivatives. Then, necessary, but not sufficient, conditions on the functions hold, described by the Euler-Lagrange equations (e.g. Schechter, 1967).

a) $\frac{\partial F}{\partial p} - \frac{\partial}{\partial x_i}\frac{\partial F}{\partial p_{,i}} = 0$, leading to. $\boldsymbol{\mathcal{S}}_\varepsilon.p - \alpha\varepsilon_V - \int_0^t \frac{\partial}{\partial x_i}\left(K_{ij}\frac{1}{\mu}\frac{\partial p}{\partial x_j}\right).d\tau = 0$ …(A1.4)

where I have used Einstein's summation convention, and the commutation:

$\frac{\partial}{\partial x_i}\frac{\partial F}{\partial p_{,i}} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial p_{,i}}\int_0^t \dot{F}.d\tau = \int_0^t \frac{\partial}{\partial x_i}\frac{\partial \dot{F}}{\partial p_{,i}}.d\tau$. Differentiating the Euler-Lagrange equation with respect to time, τ, we retrieve the equation of fluid mass conservation or storage equation (Wang, 2000, equation 4.65).

b) $\boldsymbol{\mathcal{S}}_\varepsilon.\dot{p} - \alpha.\dot{\varepsilon}_V = \nabla.(\underline{\underline{K}}/\mu.\nabla p)$ …(A1.5)

Notice that this equation does not contain any source or sink terms because the functional relates to the energy input to the reservoir excluding the wells and surroundings.

c) $\frac{\partial F}{\partial u_i} - \frac{\partial}{\partial x_j}\frac{\partial F}{\partial u_{i,j}} = 0$, where u_i is the i^{th} component of rock displacement and $u_{i,j}$ is its derivative in the j direction, related to components of the infinitesimal strain tensor through. $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

This leads to. $-\frac{1}{2}\frac{\partial}{\partial x_j}\left(D_{ijkl}u_{k,l} + u_{l,k}D_{lkij}\right) + \alpha\frac{\partial p}{\partial x_j}.\delta_{ij} = 0$

or, using the symmetries in the stiffness tensor.

d) $\frac{\partial}{\partial x_j}(\sigma_{ij} - \alpha.p.\delta_{ij}) = 0$ …(A1.6)

where $\sigma_{ij} = D_{ijkl}.\varepsilon_{kl}$ is the ij component of the stress tensor, and δ_{ij} is the Kronecker delta. In other words this Euler-Lagrange equation describes effective stress equilibrium at any time. Note that the equilibrium applies at the point of maximum entropy from which deviations are continuously occurring.

Appendix 2. Deviations of variables from values at maximum entropy production

If the smoothed trends of the system variables, $\bar{\mathbf{A}}$**(t)**, differ from their values at maximum entropy production, **A*(t),** by small amounts, **a(t),** then the deviations of the measured variables, **A(t)**, from the smoothed histories are: $\boldsymbol{\alpha}$**'(t)** = **A(t)** - $\bar{\mathbf{A}}$**(t)** = **A(t)** – [**A*(t)**+ **a(t)**] = $\boldsymbol{\alpha}$**(t) – a(t).**

It is assumed that the values of **a(t)**, although possibly auto-correlated in time, are not correlated with the actual deviations, $\boldsymbol{\alpha}$**(t),** from maximum entropy values; nor with each other in space; also that they have zero mean, implying that $\boldsymbol{\alpha}$**'(t)** also has zero mean. Then the covariances of the individual components of $\boldsymbol{\alpha}$**'(t)** are:

$\langle\alpha'_i(t).\alpha'_j(t)\rangle = \langle\alpha_i(t).\alpha_j(t)\rangle + \langle a_i(t).a_j(t)\rangle = \langle\alpha_i(t).\alpha_j(t)\rangle + \delta_{ij}.\text{var}(\mathbf{a}) = \frac{1}{2}.\beta.(\underline{\mathbf{G}}^{-1})_{ij} + \delta_{ij}.\text{var}(\mathbf{a})$

where var(**a**) is the variance of the values of **a(t)**.

Under these assumptions the differences, **a(t)**, enhance only the diagonal terms of the covariance matrix, $\langle\alpha_i(t).\alpha_j(t)\rangle$, which will not affect non-parametric correlations, as in the Spearman rank method.

Appendix 3. Relating flowrate correlations to system parameters

The assumption is made that the change in flowrate at a well is a monotonic function of the change in strain in the drainage (or injection) area around the well. The covariances of strain may therefore be replaced with covariances of flowrates at pairs of wells.

$$\begin{bmatrix} \langle q_1.q_1\rangle, & \langle q_1.q_2\rangle \dots & \langle q_1.q_n\rangle & \langle q_1.\dot{q}_1\rangle, & \langle q_1.\dot{q}_2\rangle \dots & \langle q_1.\dot{q}_n\rangle \\ \langle q_2.q_1\rangle, & \langle q_2.q_2\rangle \dots & \langle q_2.q_n\rangle & \langle q_2.\dot{q}_1\rangle, & \langle q_2.\dot{q}_2\rangle \dots & \langle q_2.\dot{q}_n\rangle \\ . & . & . & . & . & . \\ \langle q_n.q_1\rangle, & \langle q_n.q_2\rangle \dots & \langle q_n.q_n\rangle & \langle q_n.\dot{q}_1\rangle, & \langle q_n.\dot{q}_2\rangle \dots & \langle q_n.\dot{q}_n\rangle \\ \langle \dot{q}_1.q_1\rangle, & \langle \dot{q}_1.q_2\rangle \dots & \langle \dot{q}_1.q_n\rangle & \langle \dot{q}_1.\dot{q}_1\rangle, & \langle \dot{q}_1.\dot{q}_2\rangle \dots & \langle \dot{q}_1.\dot{q}_n\rangle \\ \langle \dot{q}_2.q_1\rangle, & \langle \dot{q}_2.q_2\rangle \dots & \langle \dot{q}_2.q_n\rangle & \langle \dot{q}_2.\dot{q}_1\rangle, & \langle \dot{q}_2.\dot{q}_2\rangle \dots & \langle \dot{q}_2.\dot{q}_n\rangle \\ . & . & . & . & . & . \\ \langle \dot{q}_n.q_1\rangle, & \langle \dot{q}_n.q_2\rangle \dots & \langle \dot{q}_n.q_n\rangle & \langle \dot{q}_n.\dot{q}_1\rangle, & \langle \dot{q}_n.\dot{q}_2\rangle \dots & \langle \dot{q}_n.\dot{q}_n\rangle \end{bmatrix}$$

or

$\boldsymbol{Q} = [\langle \boldsymbol{q}.\boldsymbol{q}^T\rangle]$, where $\boldsymbol{q}$ is the (2nx1) vector of flowrates and their time derivatives $(q_1, q_2 \dots q_n, \dot{q}_1, \dot{q}_2 \dots \dot{q}_n)$. n is the number of wells in the field.

$\boldsymbol{Q}$ is a positive semi-definite matrix as can be seen by pre- and post-multiplying by any real (2nx1) vector, **a**, to give $\mathbf{a}^{\boldsymbol{T}}.\langle \boldsymbol{q}.\boldsymbol{q}^{T}\rangle .\mathbf{a} = \langle x^{\mathbf{2}}\rangle \geq 0$, where x is the scalar $\mathbf{a}^{\boldsymbol{T}}.\mathbf{q}$.

If $\mathbf{Q}^{-1}$ is written in the form. $\begin{bmatrix}[\boldsymbol{Q}^{-1}]_1 & [\boldsymbol{Q}^{-1}]_2\\ [\boldsymbol{Q}^{-1}]_2 & [\boldsymbol{Q}^{-1}]_3\end{bmatrix}$, $\underline{\underline{\mathbf{D}}}$ is directly proportional to $[\boldsymbol{Q}^{-1}]_2$. However $[\boldsymbol{Q}^{-1}]_1$ involves the product of $\underline{\underline{\mathbf{K}}}$ and quadratic combinations of spatial gradients in $\underline{\underline{\mathbf{D}}}$ which implies that, once $\underline{\underline{\mathbf{D}}}$ has been evaluated, components of $\underline{\underline{\mathbf{K}}}$ can be extracted from a linear equation formulation; this will be the subject of a future publication.

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