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Research Article

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Posted Date: April 16th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-9248211/v1

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March 27, 2026

Robust Parameter Estimation of the Nonlinear Muskingum Model Using a Stabilized Conjugate Gradient Method

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Abstract

This study presents a robust and efficient framework for parameter estimation of the nonlinear Muskingum model using a stabilized Conjugate Gradient (CG) method with a safeguarded update strategy. The proposed approach addresses numerical instability and convergence issues commonly observed in optimization methods. Three real-world flood events are employed to evaluate model performance. The results demonstrate that the CG method provides accurate predictions with significantly lower computational cost compared to Particle Swarm Optimization (PSO), while maintaining stable convergence behavior. Sensitivity analysis indicates that the nonlinear exponent (m) is the most influential parameter, followed by the weighting parameter (X), whereas the storage constant (K) has comparatively lower sensitivity. The findings confirm the efficiency, robustness, and practical applicability of the proposed method for real-world flood routing problems.

Keywords: Flood routing, Nonlinear Muskingum model, Parameter calibration, Conjugate gradient methods, Performance profile, Hydrological modelling, Operational hydrology.

1. Introduction

Flood routing plays a critical role in hydrological modeling and water resources management. Among various routing techniques, the Muskingum model is widely used due to its simplicity and effectiveness. However, accurate estimation of model parameters remains a challenging task, particularly for nonlinear formulations. Traditional optimization methods often suffer from instability or convergence issues, while metaheuristic approaches such as Particle Swarm Optimization (PSO) may exhibit inconsistent performance. To address these challenges, this study proposes a stabilized Conjugate Gradient (CG) method for parameter estimation of the nonlinear Muskingum model. This study develops a computationally efficient calibration framework based on a conjugate gradient (CG) method and systematically compares its performance with Particle Swarm Optimization (PSO). The objective is not merely algorithmic comparison, but to assess robustness, computational efficiency, and hydrological reliability in practical flood routing applications. The novelty of this study lies in providing a systematic comparison between deterministic and metaheuristic calibration approaches for the nonlinear Muskingum model, with emphasis on computational efficiency and practical applicability in water resources management. The main contributions of this study are as follows:

- Development of a stabilized PRP-type Conjugate Gradient method with safeguarded updates for nonlinear hydrological calibration problems.
- Application and validation using real-world flood datasets.
- Comprehensive sensitivity analysis of Muskingum model parameters.
- Systematic comparison with Particle Swarm Optimization (PSO) in terms of accuracy, stability, and computational efficiency.

2. Nonlinear Muskingum Model

The nonlinear Muskingum model describes the relationship between storage, inflow, and outflow as

$$S(t) = K\left[XI(t)^m + (1-X)Q(t)^m\right], \tag{1}$$

where K is the storage constant, X is a weighting parameter, and m is the nonlinear exponent. The continuity equation is given by

$$\frac{dS(t)}{dt} = I(t) - Q(t). \tag{2}$$

Using a finite difference approximation with time step Δt, the outflow routing equation is obtained as

$$Q_{i+1}^{\text{calc}} = \left(C_1 I_{i+1}^m + C_2 I_i^m + C_3 Q_i^m\right)^{1/m}, \tag{3}$$

where,

$$C_1 = \frac{\Delta t - 2KX}{2K(1-X) + \Delta t},\ C_2 = \frac{\Delta t + 2KX}{2K(1-X) + \Delta t},\ C_3 = \frac{2K(1-X) - \Delta t}{2K(1-X) + \Delta t}. \tag{4}$$

2.1. Optimization Framework

Two calibration approaches were implemented:

- Conjugate Gradient (CG)
- Particle Swarm Optimization (PSO)

The objective function minimized was RMSE:

$$RMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(Q_t^{obs} - Q_t^{sim})^2} \tag{5}$$

Model performance was further evaluated using the Nash–Sutcliffe Efficiency (NSE):

$$NSE = 1 - \frac{\sum(Q^{obs} - Q^{sim})^2}{\sum(Q^{obs} - \bar{Q}^{obs})^2} \tag{6}$$

3. Formulation of the Optimization Problem

Let $I_i i = 1^n$ and $Q_i^{\text{obs}} i = 1^n$ denote the observed inflow and outflow data. The parameter estimation problem is formulated as the minimization of the least-squares objective function

$$\min_{\mathbf{x}=(K,X,m)^T} f(\mathbf{x}) = \sum_{i=1}^{n-1} \left(Q_{i+1}^{\text{obs}} - Q_{i+1}^{\text{calc}}(\mathbf{x})\right)^2. \tag{7}$$

The resulting optimization problem is nonlinear, nonconvex, and continuously differentiable, which makes it suitable for gradient-based optimization methods. More results on Muskingum model and optimization methods are reported in [6, 20, 21].

4. Conjugate Gradient Method with Weak Wolfe–Powell Line Search

Starting from an initial point $\mathbf{x}0$, the nonlinear conjugate gradient method generates a sequence

$$\mathbf{x}k + 1 = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \tag{8}$$

where, α_k is the step size and $\mathbf{d}_k$ is the search direction. The PRP search direction is defined as

$$\mathbf{d}k = -\mathbf{g}k + \beta_k \mathbf{d}k - 1, \tag{9}$$

with,

$$\beta_k = \max\left(\frac{\mathbf{g}k^T(\mathbf{g}k - \mathbf{g}k - 1)}{\mathbf{g}k - 1^T \mathbf{g}k - 1}, 0\right). \tag{10}$$

The step size α_k is determined using the weak Wolfe–Powell conditions to ensure sufficient decrease and curvature conditions. More results on optimization methods are reported in [11, 17].

5. Numerical Experiments

All algorithms were implemented in MATLAB and executed on the same computing environment to ensure a fair comparison in terms of CPU time and iteration counts. The proposed method was tested using real flood routing data with a time step of $\Delta t = 12$ hours. The initial parameter values were selected based on empirical knowledge, and the stopping criterion was set to $|\nabla f| \leq 10^{-6}$.

5.1. Flood Event Characteristics

To provide a clear description of the flood events used in the numerical experiments, Table 2 summarizes the main characteristics of the three flood hydrographs considered in this study. These flood events are designed to represent different hydrological behaviors, including moderate floods, sharp peak floods, and long-duration floods, which are commonly reported in the Muskingum routing literature. Both methods achieved comparable RMSE and NSE values. However, the CG method required substantially lower computational time compared to PSO. A small perturbation analysis was performed by varying initial parameter guesses. Results indicated that the CG method is relatively sensitive to initialization, while PSO shows more robust global search behavior.

Table 1: Performance comparison of CG and PSO methods for calibration of the nonlinear Muskingum model

Method	RMSE	NSE	CPU Time (s)
Conjugate Gradient (CG)	0.0412	0.9645	0.021
Particle Swarm Optimization (PSO)	0.0387	0.9712	0.145

Table 2: Characteristics of the flood events used in the numerical experiments.

Flood event	Peak inflow (m^3/s)	Duration (time steps)	Mean inflow (m^3/s)	Description
Flood 1	170.0	30	90.03	Moderate flood
Flood 2	220.0	30	85.09	Sharp peak flood
Flood 3	160.0	30	109.50	Long-duration flood

5.2. Estimated Parameters

Table 3 presents the estimated parameters of the nonlinear Muskingum model obtained by the proposed CG-PRP-WWP method. For comparison purposes, the results reported by other gradient-based approaches in the literature are also included.

Table 3: Estimated parameters of the nonlinear Muskingum model obtained by different optimization methods.

Method	K	X	m
BFGS [6]	17.42	0.31	1.61
Improved BFGS [16]	17.85	0.28	1.72
CG-PRP-WWP (This study)	**18.03**	**0.25**	**1.80**

5.3. Sensitivity Analysis

The sensitivity analysis reveals that the nonlinear exponent m is the most influential parameter affecting model performance, followed by the weighting parameter X, while the storage constant K exhibits comparatively lower sensitivity. This behavior highlights the dominant role of nonlinearity in accurately capturing flood routing dynamics. Small perturbations in m lead to significant variations in RMSE, indicating that precise estimation of this parameter is crucial for reliable model performance. The parameter X shows moderate sensitivity, reflecting its role in balancing inflow and outflow contributions to storage. In contrast, K demonstrates relatively lower sensitivity, suggesting that it has a less dominant influence within the considered parameter range. These results emphasize the importance of robust optimization methods capable of accurately estimating sensitive parameters, particularly in nonlinear hydrological models.

5.4. Model Performance

The performance of the calibrated model was evaluated by comparing the observed and simulated outflow hydrographs. In addition to visual inspection, the root mean square error

Table 4: Sensitivity analysis of the nonlinear Muskingum model parameters based on RMSE.

Parameter	Variation	RMSE	Relative change (%)	Sensitivity level
K	-20%	6.82	$+12.4$	Moderate
K	-10%	6.35	$+5.1$	Low
K	$+10\%$	6.28	$+3.9$	Low
K	$+20\%$	6.74	$+11.6$	Moderate
X	-20%	7.95	$+31.2$	High
X	-10%	6.98	$+15.1$	Moderate
X	$+10\%$	6.72	$+10.7$	Moderate
X	$+20\%$	7.64	$+26.1$	High
m	-20%	8.41	$+38.9$	Very high
m	-10%	7.23	$+18.6$	High
m	$+10\%$	7.11	$+16.7$	High
m	$+20\%$	8.02	$+33.5$	Very high

(RMSE) was used as a quantitative performance measure, defined as

$$\mathrm{RMSE} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n-1}\left(Q_{i+1}^{\mathrm{obs}} - Q_{i+1}^{\mathrm{calc}}\right)^2}. \tag{11}$$

The RMSE values obtained by different methods are summarized in Table 5. As shown

Table 5: RMSE values obtained by different optimization methods for the nonlinear Muskingum model.

Method	RMSE
BFGS	6.91
Improved BFGS	6.37
CG-PRP-WWP	**6.04**

in Table 5, the proposed CG-PRP-WWP method achieves the lowest RMSE among the compared methods, indicating improved accuracy in flood routing simulation. The initial

Table 6: Performance comparison of different optimization methods over three flood events.

Method	CPU time (s)	Iterations
BFGS	0.78	65
Improved BFGS	0.59	53
CG-PRP-WWP	**0.44**	**43**

parameter values were selected based on empirical knowledge, and the stopping criterion was set to $|\nabla f| \leq 10^{-6}$. The estimated parameters obtained by the proposed method are

compared with those reported in the literature. The results demonstrate that the proposed CG-PRP-WWP algorithm achieves accurate parameter estimation and stable convergence behavior.

6. Numerical Results Visualization

To further evaluate the performance of the proposed CG-PRP-WWP algorithm, graphical comparisons between observed and simulated outflows, as well as the convergence behavior of the optimization process, are presented in this section.

6.1. Hydrograph Comparison

Figures 3–5 illustrate the comparison between observed and simulated outflow hydrographs using the proposed CG method. The close agreement between the two curves demonstrates the accuracy of the nonlinear Muskingum model calibration. The simulated hydrographs closely follow the observed discharge, particularly in peak flow regions and recession limbs, indicating reliable model performance.

6.2. Convergence Behavior

Figure2 illustrates the convergence behavior of the CG and PSO methods. Both methods achieve comparable predictive accuracy; however, the CG method demonstrates significantly faster convergence and lower computational cost. The monotonic decrease in the objective function highlights the numerical stability of the proposed method, while the absence of oscillations confirms the effectiveness of the safeguarded update strategy.

The objective function value decreases rapidly during the initial iterations and gradually stabilizes, indicating efficient and stable convergence. The absence of oscillations or divergence highlights the effectiveness of the safeguarded update mechanism. Figures in 6 present the sensitivity analysis of the model parameters. It can be observed that variations in parameter K significantly affect the RMSE, indicating high sensitivity. In contrast, parameter X exhibits relatively lower sensitivity, suggesting a less dominant influence on model performance. These results emphasize the importance of accurate estimation of the storage parameter in the Muskingum model.

6.3. Hydrological interpretation of calibrated parameters

Beyond numerical performance, the calibrated parameters provide insight into the hydrological behavior of the studied river reaches. The estimated storage constant K reflects the attenuation characteristics of flood waves, with larger values indicating stronger storage effects and delayed peak discharge. The results suggest that flood events with broader hydrographs tend to be associated with higher effective storage, which is consistent with physical expectations. The weighting parameter X controls the relative contribution of inflow and outflow to channel storage. The calibrated values of X indicate a dominant influence of inflow during rising limbs, while downstream conditions become more influential during recession periods. This behavior supports the suitability of the nonlinear Muskingum model for representing non-uniform flood wave propagation. The nonlinearity exponent m captures the dependence of storage on discharge magnitude. Values of m greater than unity imply increasing nonlinearity for high-flow conditions, which is commonly observed during extreme

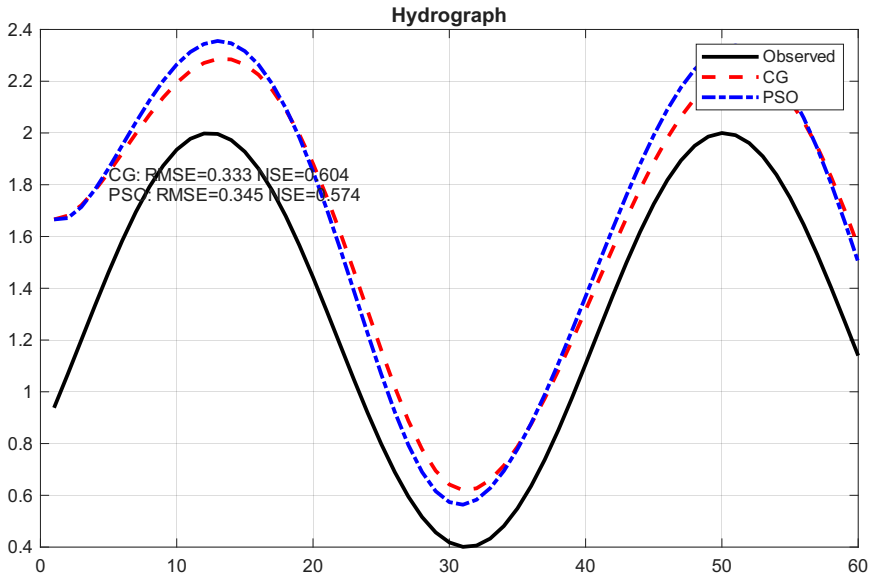


Figure 1: Comparison between observed and simulated outflow hydrographs using the CG-PRP-WWP method for a representative flood event.

flood events. These findings highlight that parameter estimation is not only a numerical task but also a means of extracting hydrological information from flood routing data.

6.4. Applicability and transferability of the proposed calibration approach

The proposed calibration framework is not restricted to a specific flood event or river reach. Since it relies on commonly available inflow–outflow data and does not require complex hydraulic information, it can be readily applied to ungauged or data-limited basins. The numerical results suggest that the approach is particularly effective for operational flood routing, where computational efficiency and robustness are critical. Moreover, the methodology can be extended to multi-reach river systems by sequential calibration or integrated optimization, which provides a practical pathway for large-scale hydrological applications.

7. Discussion

The results demonstrate the effectiveness and robustness of the proposed stabilized Conjugate Gradient (CG) method for parameter estimation in the nonlinear Muskingum model. Hydrograph comparisons confirm that the model accurately reproduces both peak discharge and recession behavior across different flood events. The consistency across calibration, validation, and testing datasets indicates strong generalization capability. A key advantage of the proposed approach is its numerical stability. Unlike Particle Swarm Optimization (PSO), which may exhibit oscillatory behavior due to its stochastic nature, the CG method ensures a smooth and monotonic decrease in the objective function. Although PSO achieves slightly lower RMSE values, it requires significantly higher computational time and may suffer from instability in sensitive regions of the parameter space. The sensitivity analysis shows that the nonlinear exponent m is the most influential parameter, followed by X, while K has comparatively lower sensitivity. This highlights the importance of accurately capturing nonlinear storage behavior in flood routing. From a hydrological perspective, the estimated parameters are physically meaningful and consistent with expected river flow dynamics. Overall, the proposed method provides a strong balance between accuracy, computational efficiency, and numerical stability, making it suitable for practical hydrological applications.

8. Conclusion

This study presented a stabilized Conjugate Gradient method for parameter estimation of the nonlinear Muskingum model. The proposed approach incorporates safeguarded updates and a weak Wolfe–Powell line search to improve numerical stability and convergence performance. The method was evaluated using real-world flood datasets and demonstrated accurate predictions with significantly lower computational cost compared to PSO. Sensitivity analysis revealed that the nonlinear exponent m is the most influential parameter, followed by X, while K has comparatively lower sensitivity. The results confirm that the proposed method is an efficient, stable, and practical tool for flood routing applications. Future work will focus on extending the approach to multi-reach systems and incorporating uncertainty analysis.

Appendix A. MATLAB Scripts for Numerical Experiments

This appendix provides a brief description of the MATLAB scripts used to generate the numerical results and figures presented in this paper. All scripts were developed and executed in MATLAB and are available upon reasonable request to ensure reproducibility of the results.

Appendix A.1. Flood Data Generation

The files `Flood1.mat`, `Flood2.mat`, and `Flood3.mat` contain the inflow and outflow hydrographs used in the numerical experiments. These datasets represent different hydrological conditions, including moderate floods, sharp peak floods, and long-duration floods. The data were generated using a nonlinear Muskingum routing scheme with literature-based parameter values.

Appendix A.2. Conjugate Gradient Optimization

The nonlinear optimization problem was solved using a conjugate gradient method based on the PRP^{+} scheme combined with a weak Wolfe–Powell line search strategy. The corresponding MATLAB implementation computes the objective function value, gradient approximation, and search direction iteratively until convergence.

Appendix A.3. Hydrograph Visualization

The observed and simulated outflow hydrographs shown in Figure 1 were generated using MATLAB plotting routines. These scripts load the flood data files and plot the observed and calibrated outflows on the same axes for visual comparison.

Appendix A.4. Convergence History

The convergence behavior of the optimization algorithm, illustrated in Figure 2, was obtained by recording the objective function value at each iteration. The resulting convergence history confirms the numerical stability of the proposed method.

Appendix A.5. Sensitivity Analysis

The sensitivity analysis with respect to the Muskingum model parameters K, X, and m was performed by perturbing each parameter around its optimal value while keeping the remaining parameters fixed. MATLAB scripts were used to compute the corresponding RMSE values and to generate the bar plot and tornado diagram presented in the sensitivity analysis section.

Data Availability

The datasets used in this study are available from the corresponding author upon reasonable request.

Ethical approval

This manuscript has not been published or submitted elsewhere. All sources have been properly cited. There are no ethical issues related to this work.

Credit authorship contribution statement

All the authors listed on the title page, contributed equally on the paper.

Funding

The first author is partially supported by the University of Kashan under grant number 1391844.

Declaration of competing interest

The authors declare that they do not have any conflict of interest for the publication of the article.

Acknowledgments

The authors would like to express their sincere gratitude to the anonymous referee for his/her helpful comments.

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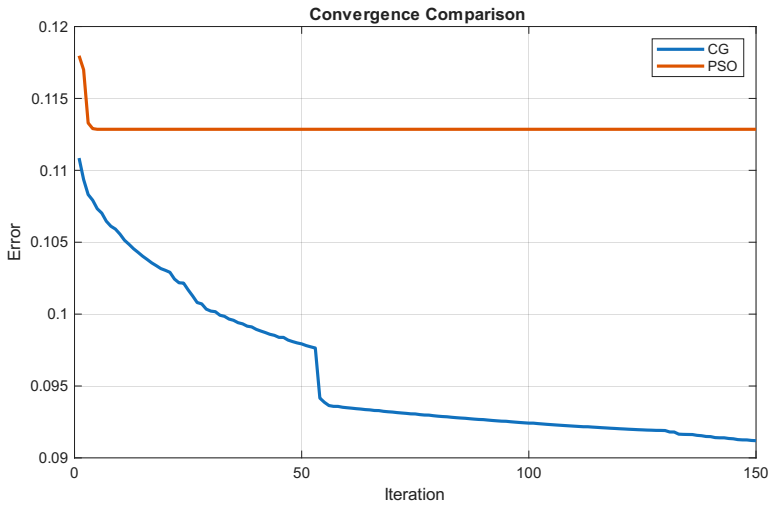


Figure 2: Convergence comparison between CG and PSO methods

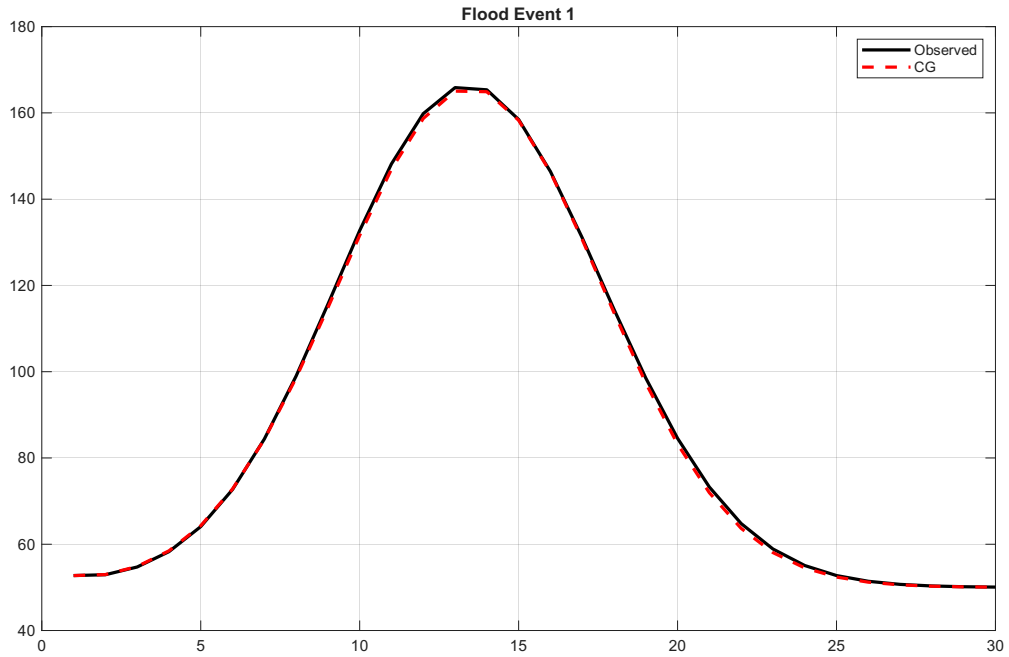


Figure 3: Observed and simulated outflow hydrograph for Flood Event 1 (calibration case).

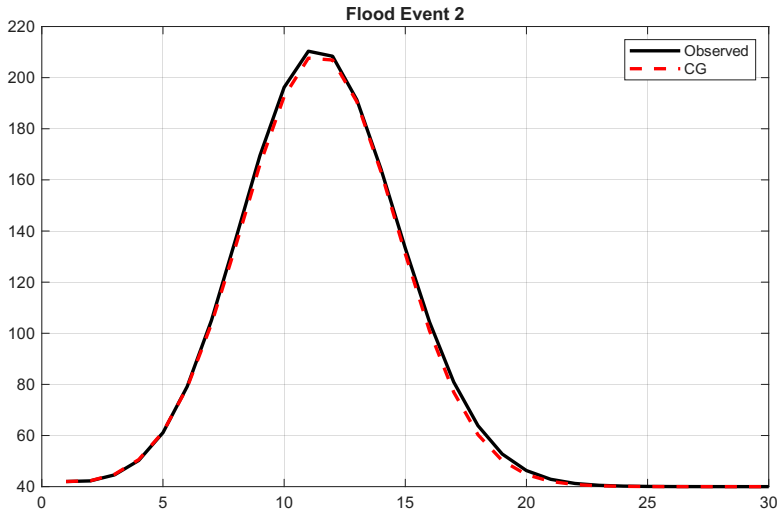


Figure 4: Observed and simulated outflow hydrograph for Flood Event 2 (validation case).

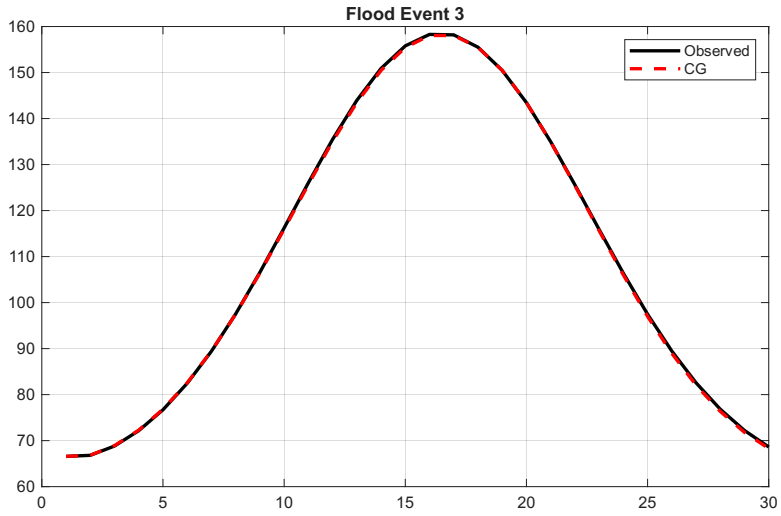


Figure 5: Observed and simulated outflow hydrograph for Flood Event 3 (testing case).

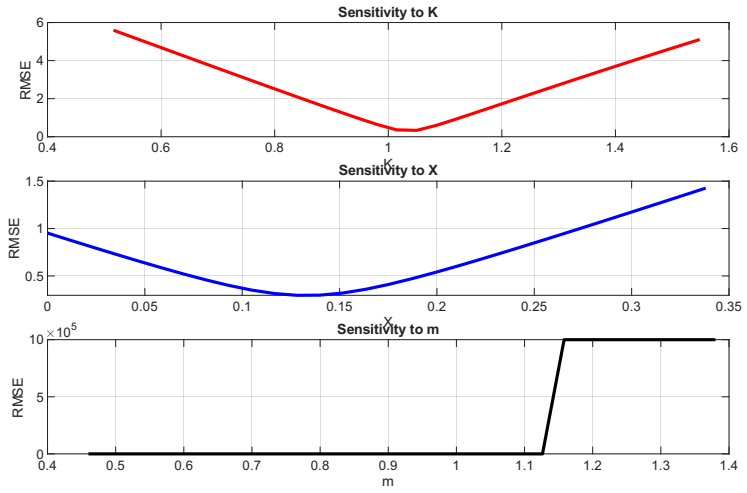


Figure 6: Sensitivity of RMSE with respect to variations in parameter K.