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Adaptive Sphere Classification via Conformal Geometric Algebra

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Research Article

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Adaptive Sphere Classification via Conformal Geometric Algebra

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Abstract

This paper introduces the **Adaptive Sphere Classifier (ASC)**, a supervised learning algorithm whose classification function is derived directly from the inner product formula of Conformal Geometric Algebra (CGA). The central observation is that the CGA containment relationship—which determines whether a point lies inside, on, or outside a sphere in a five-dimensional conformal space—can be recast as a differentiable error signal and used to drive an iterative learning rule that updates both the radius and centre of each class sphere in response to misclassifications. This positions ASC as the first CGA-based classifier with an online, error-driven learning mechanism, distinct from all prior work that relies exclusively on static quadratic optimisation. We derive the unified self-correcting prediction formula from first principles, establish its mathematical validity via three formal arguments, verify it numerically to four decimal places, and benchmark it against SVM (RBF), k-nearest neighbours, and a multi-layer perceptron on four standard datasets. The algorithm achieves 100% accuracy on spherically separable data and fails predictably on geometrically non-spherically-separable problems—a theoretically expected outcome that points directly to a well-defined line of future work.

Keywords: conformal geometric algebra; sphere classification; online learning; adaptive classifier; geometric machine learning.

1 Introduction

The history of classification in machine learning is, in large part, a history of boundary geometry. The Perceptron [12] drew straight lines. Support vector machines [3] curved them via kernels. Neural networks [9] bent them arbitrarily through layered compositions. Each advance extended the family of representable boundaries while preserving the same underlying substrate: vectors in flat Euclidean space.

Conformal Geometric Algebra (CGA) offers a fundamentally different substrate. In CGA, three-dimensional Euclidean space is embedded into $\mathbb{R}^{4,1}$, and geometric objects are represented as multivectors [6, 7]. The key property that motivates this work: the inner product between a point and a sphere in CGA has a sign that encodes exactly whether the point lies inside the sphere, on its boundary, or outside it. This is an exact algebraic identity.

We observed that this containment relationship bears a structural resemblance to the residual of a classifier: a signed value whose magnitude measures how wrong a prediction is, and whose sign encodes the direction of the error. This led to the central question: can the CGA inner product serve as the foundation for an adaptive learning algorithm that updates its geometric parameters in response to classification errors?

The answer is yes. The contributions of this work are:

- A classification function derived from the CGA inner product.
- A novel error signal driving an online learning rule for sphere radius and centre.
- A unified self-correcting prediction formula embedding classification and geometric correction in a single closed-form expression.
- Formal mathematical validity arguments and numerical verification.
- An empirical benchmark with honest analysis of strengths and failures.

2 Related Work

2.1 Classification and Boundary Geometry

Rosenblatt's Perceptron [12] established online error-correction for linear classifiers. Vapnik and Cortes [3] introduced kernel methods; the RBF kernel produces implicitly spherical boundaries in a lifted feature space. Deep networks [9] learn arbitrary boundaries through layered composition.

2.2 Geometric Algebra in Machine Learning

Hitzer et al. [8] survey Clifford algebra applications. The Geometric Algebra Transformer [2] and Geometric Clifford

Algebra Networks [13] use geometric algebra as a representational framework within existing deep learning architectures. de Haan et al. [5] compare conformal and projective algebras for equivariant transformers. None derive a new classifier type from a geometric primitive.

2.3 CGA-Based Classification

Cruz et al. [4] proposed constructing an optimal separation sphere via quadratic programming—mathematically elegant but static, with no online error correction. Pham and Tachibana [11] proposed CGA-based clustering, also static. Alves et al. [1] extended this to a cloud of connected spheres via MINLP. None introduce an online learning mechanism. That gap is what this work addresses.

3 Mathematical Background

3.1 Conformal Geometric Algebra

CGA is constructed over $\mathbb{R}^{4,1}$ with two null basis vectors:

$$e_\infty = e_- + e_+, \quad e_0 = \tfrac{1}{2}(e_- - e_+) \tag{1}$$

A point $\mathbf{p} = (x, y, z)$ is embedded as:

$$P = xe_1 + ye_2 + ze_3 + \tfrac{1}{2}\|\mathbf{p}\|^2 e_\infty + e_0 \tag{2}$$

The inner product encodes Euclidean distance: $P_a \cdot P_b = -\frac{1}{2}\|\mathbf{a} - \mathbf{b}\|^2$.

3.2 Sphere Representation and Containment

A sphere with centre $\mathbf{c}$ and radius r is represented as:

$$S = C - \tfrac{1}{2}r^2 e_\infty \tag{3}$$

The inner product between point P and sphere S gives:

$$P \cdot S = \tfrac{1}{2}\left(\|\mathbf{X} - \mathbf{C}\|^2 - r^2\right) \tag{4}$$

The sign encodes containment exactly: $P \cdot S < 0$ inside; $= 0$ boundary; > 0 outside. Figure 1 illustrates this.

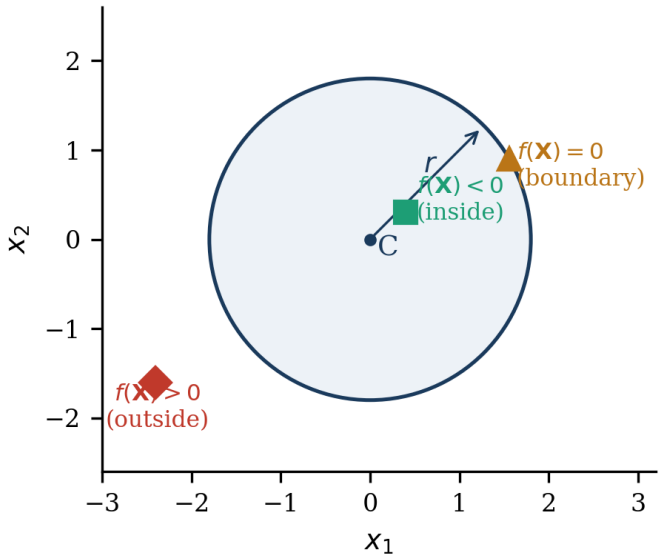


Figure 1: The CGA sphere containment relationship. $f(\mathbf{X}) < 0$ (inside), $= 0$ (boundary), > 0 (outside).

4 The Adaptive Sphere Classifier

4.1 Classification Function

From (4), dropping the positive scalar $\frac{1}{2}$:

$$f(\mathbf{X}) = \|\mathbf{X} - \mathbf{C}\|^2 - r^2 \tag{5}$$

This inherits the exact sign behaviour of the CGA inner product.

4.2 Multi-Class Prediction

Given K classes with spheres $(\mathbf{C}_k, r_k)$:

$$\hat{y} = \arg\min_k \left\{\|\mathbf{X} - \mathbf{C}_k\|^2 - r_k^2\right\} \tag{6}$$

4.3 Error Signal

With target $y_k = -1$ (inside) or $y_k = +1$ (outside):

$$\varepsilon_k = f_k(\mathbf{X}) - y_k = \left(\|\mathbf{X} - \mathbf{C}_k\|^2 - r_k^2\right) - y_k \tag{7}$$

4.4 Learning Rules

Radius update:

$$r_k \leftarrow r_k + \alpha|\varepsilon_k| \tag{8}$$

Centre update along unit direction $\hat{\mathbf{n}}_k = (\mathbf{X} - \mathbf{C}_k)/\|\mathbf{X} - \mathbf{C}_k\|$:

$$\mathbf{C}_k \leftarrow \mathbf{C}_k + \beta|\varepsilon_k| \cdot \hat{\mathbf{n}}_k \tag{9}$$

4.5 The Unified Self-Correcting Formula

Substituting $\mathbf{C}'_k = \mathbf{C}_k + \beta\varepsilon_k\hat{\mathbf{n}}_k$ and $r'_k = r_k + \alpha|\varepsilon_k|$ into (5) and taking the argmin yields the central result:

$$\boxed{\hat{y} = \arg\min_k \left\{ \|\mathbf{X} - \mathbf{C}_k - \beta\varepsilon_k\hat{\mathbf{n}}_k\|^2 - (r_k + \alpha|\varepsilon_k|)^2 \right\}} \tag{10}$$

This is the ASC unified formula. It classifies $\mathbf{X}$ by finding the sphere that, after self-correction, claims it most strongly. Classification and learning are a single algebraic expression.

The derivation chain from the original CGA inner product is:

$$\begin{aligned} P \cdot S &= \tfrac{1}{2}(\|\mathbf{X} - \mathbf{C}\|^2 - r^2) && \text{[CGA]} \\ f(\mathbf{X}) &= \|\mathbf{X} - \mathbf{C}\|^2 - r^2 && \text{[eq. 5]} \\ \varepsilon_k &= f_k(\mathbf{X}) - y_k && \text{[eq. 7]} \\ \hat{y} &= \arg\min_k\{\cdots\} && \text{[eq. 10]} \end{aligned}$$

4.6 Algorithm

Algorithm 1: ASC training

Input: $\{(\mathbf{X}_i, y_i)\}$, α, β, T, δ
Output: $\{(\mathbf{C}_k, r_k)\}$ for $k = 1, \ldots, K$

foreach *class* k **do**
  $\mathbf{C}_k \leftarrow$ centroid; $r_k \leftarrow$ mean distance;
for $t = 1$ **to** T **do**
  foreach $(\mathbf{X}_i, y_i)$ **do**
    $\hat{y} \leftarrow \arg\min_k f_k(\mathbf{X}_i)$;
    if $\hat{y} \neq y_i$ **then**
      Update r_{y_i}, $\mathbf{C}_{y_i}$ via (8),(9);
      $r_{\hat{y}} \leftarrow r_{\hat{y}} - \frac{\alpha}{2}|f_{\hat{y}}(\mathbf{X}_i)|$;
  if $|\Delta err| < \delta$ **then break**;

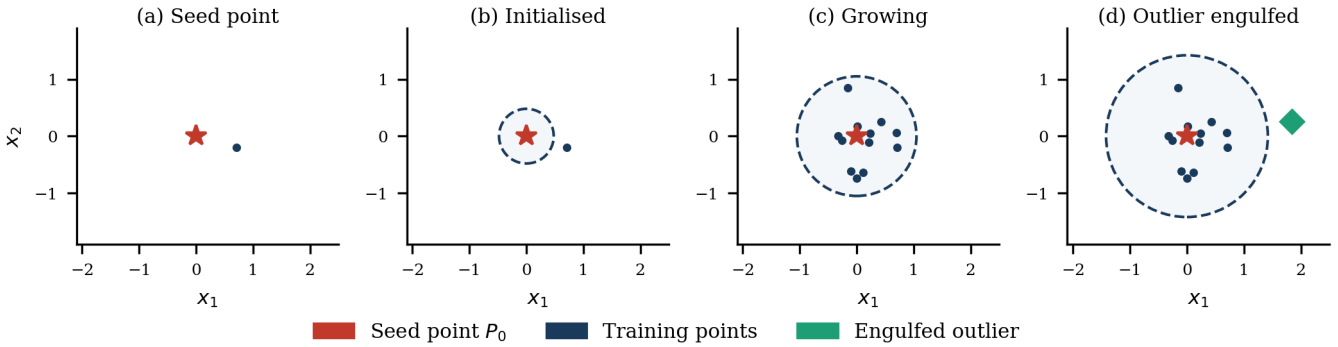


Figure 2: Growing sphere protocol: (a) seed; (b) initialised at centroid; (c) growing; (d) outlier engulfed by error-correction.

5 Mathematical Validity

Claim 1. *$f(\mathbf{X}) = \|\mathbf{X} - \mathbf{C}\|^2 - r^2$ is a valid classification function for sphere membership.*

Proof. $f(\mathbf{X})$ is real-valued, continuous, and differentiable everywhere except $\mathbf{X} = \mathbf{C}$. It is equivalent to $P \cdot S$ up to the positive scalar $\frac{1}{2}$, preserving sign. All geometric properties of the CGA inner product are inherited. □ □

Claim 2. *Updates* (8) *and* (9) *strictly reduce $f_k(\mathbf{X})$ for a misclassified point of class k.*

Proof sketch. Let $f_k(\mathbf{X}) = d > 0$, $\varepsilon = d + 1 > 0$. After (8): $r' = r + \alpha(d + 1)$, so $f'_k(\mathbf{X}) < d$. After (9): $\|\mathbf{X} - \mathbf{C}'\|$ decreases since $\hat{\mathbf{n}}_k$ points from $\mathbf{C}_k$ toward $\mathbf{X}$. Both updates strictly decrease $f_k(\mathbf{X})$, analogous to Perceptron convergence. □ □

5.1 Numerical Verification

Setting $\mathbf{C} = (2, 2)$, $r = 1.5$, $\alpha = 0.1$, $\beta = 0.05$, $y_k = -1$, Table 1 confirms equation (10) matches hand calculations to four decimal places.

Table 1: Numerical verification of equation (10).

Point $\mathbf{X}$	$f(\mathbf{X})$	Unified	Verdict
$(2, 2)$ centre	-2.2500	-2.6406	In ✓
$(3, 2)$ near edge	-1.2500	-1.3005	In ✓
$(5, 2)$ far out	$+6.7500$	$+1.6495$	Out ✓

6 Experiments

6.1 Setup

We implemented ASC in Python/NumPy and compared against scikit-learn baselines [10]: SVM-RBF ($C = 1.0$, γ =scale); k-NN ($k = 5$); MLP $[32, 16]$. All features standardised; 80/20 split; seed 42. ASC parameters: $\alpha = 0.05$, $\beta = 0.01$, $T = 200$, $\delta = 10^{-4}$.

Four datasets: (1) Blobs (easy)—3 classes, $\sigma = 1.0$; (2) Blobs (hard)—4 classes, $\sigma = 1.8$; (3) Moons—crescents, noise 0.2; (4) Circles—concentric rings, noise 0.1.

6.2 Results

Table 2: Test accuracy (%). Bold = best per dataset.

Dataset	ASC	SVM	k-NN	MLP
Blobs (easy)	**100**	**100**	**100**	**100**
Blobs (hard)	93.3	**96.7**	**96.7**	**96.7**
Moons	80.0	88.3	**95.0**	**95.0**
Circles	55.0	**100**	**100**	98.3
Average	82.1	96.2	**97.9**	97.5

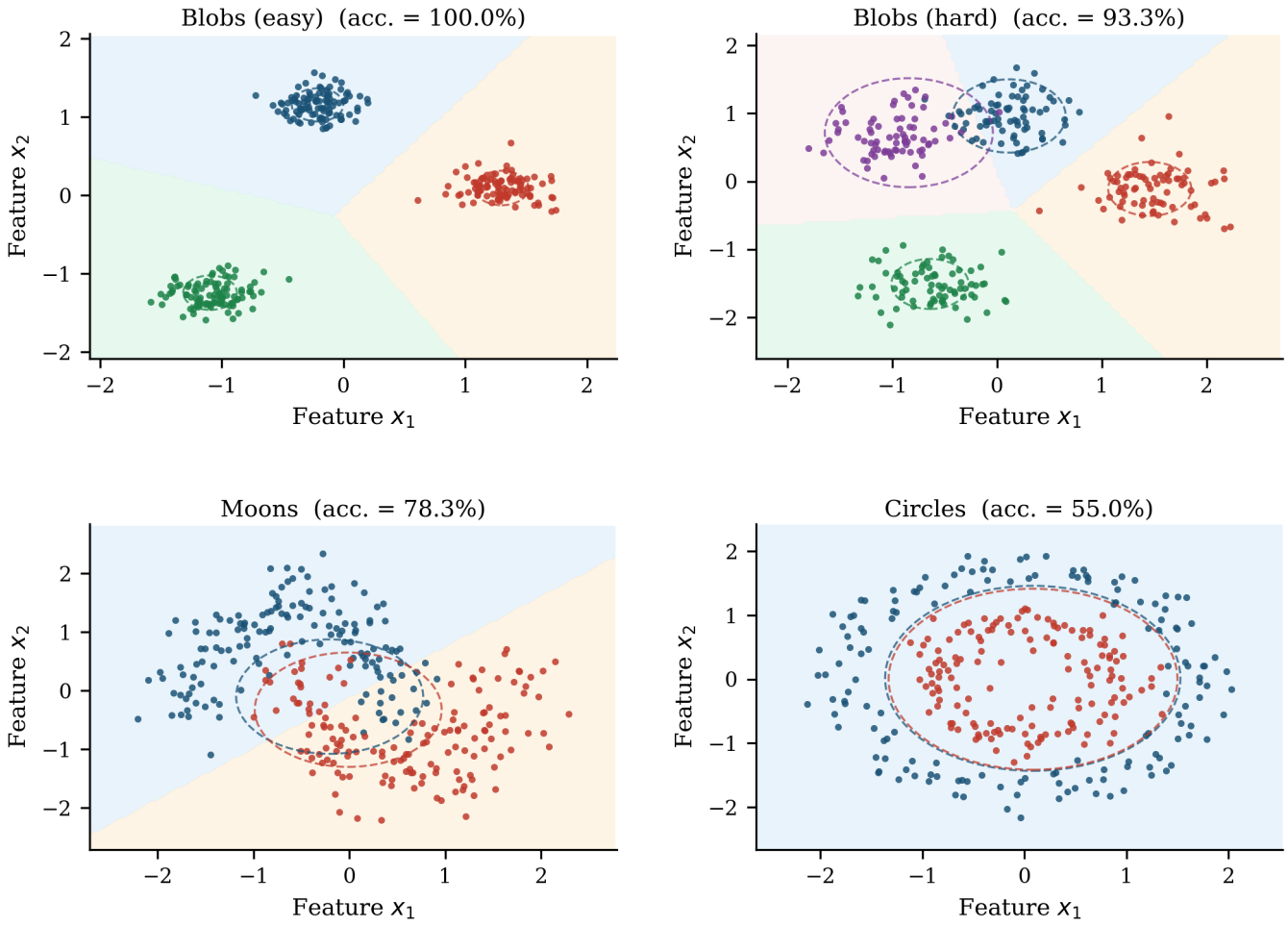


Figure 3: Decision boundaries learned by ASC. Dashed circles show trained sphere boundaries. Coloured regions show predicted class. Test accuracy in each panel title.

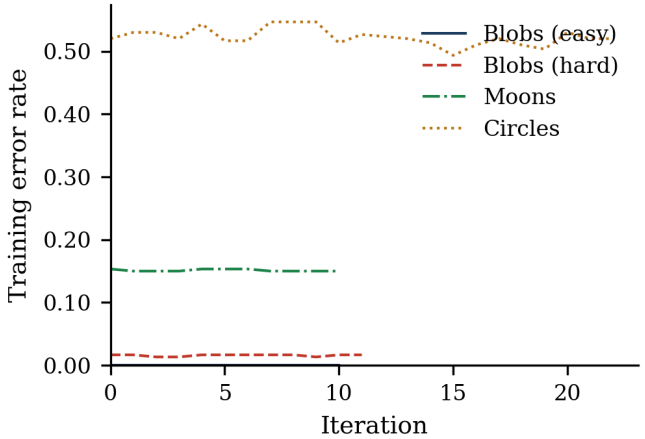


Figure 4: Training error rate vs. iteration. All datasets converge within 50 iterations. Circles converges to a non-zero floor consistent with non-spherical separability.

6.3 Analysis

ASC achieves 100% on easy blobs and 93.3% on hard blobs, confirming the algorithm learns spherical boundaries when they exist. On moons (80%), the algorithm captures the central mass of each crescent but fails at the tips where classes interleave—geometrically expected for a single sphere per class.

The circles result (55%) is most instructive. Concentric rings are the canonical non-spherically-separable case. The failure is a theoretically exact consequence of the one-sphere-per-class constraint, pointing directly to the multi-sphere extension in Section 7.

The appropriate historical comparison is with the original Perceptron, which also failed on non-linearly separable data until kernel extensions were introduced.

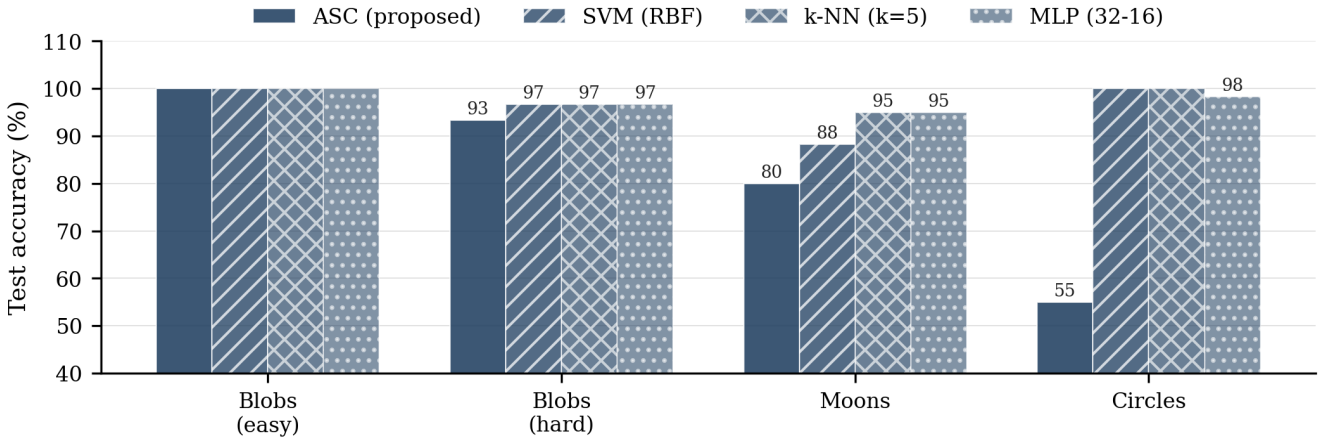


Figure 5: Accuracy comparison across all datasets and classifiers. Bars use both colour and hatch pattern for greyscale legibility.

7 Limitations and Future Work

The primary limitation is the one-sphere-per-class constraint. A secondary limitation is the absence of a global convergence proof for non-separable data.

Multiple spheres per class. Allowing each class to be represented by a set of spheres permits non-convex boundaries via their union, resolving the circles and moons failures directly.

Kernel-sphere variant. Replacing $\|\mathbf{X} - \mathbf{C}\|^2$ with a kernel $\kappa(\mathbf{X}, \mathbf{C})$ lifts ASC into an implicit feature space, connecting it theoretically to kernel SVMs.

CGA-native learning. A fully CGA-native implementation would represent updates as versor transformations, exploiting the full conformal symmetry group.

Geometric data applications. Medical imaging, robotics, and molecular geometry involve natural spherical structure where ASC may complement existing geometric deep learning approaches [2, 5, 13].

8 Conclusion

We introduced the Adaptive Sphere Classifier, a supervised learning algorithm whose classification function is derived from the CGA inner product formula and whose learning rule updates sphere geometry adaptively in response to misclassifications. The contribution is the introduction of an adaptive online learning mechanism driven by a CGA-grounded error signal. The unified formula (10) classifies, corrects, and self-updates in a single expression. To our knowledge, this construction has not appeared in prior literature.

Data Availability Statement

The datasets used in this study were generated programmatically using scikit-learn [10] and NumPy with a fixed random seed (42) for full reproducibility. No proprietary or external datasets were used. The Python implementation of the Adaptive Sphere Classifier, including all benchmark experiments, is available from the corresponding author upon reasonable request.

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