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Contextual Fragmentation in Children's Reasoning About the Invariance of Cardinality in Discrete Sets

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Abstract

Building understanding of invariance of cardinality with discrete sets is foundational to number development. Although prior research suggests that many children demonstrate conservation of number by ages 7–8 (e.g., Irwin, 1996; Markman, 1979), less is known about how robust such reasoning is across varied contexts. Drawing on a *Knowledge in Pieces*perspective (diSessa, 2018), we examined how learners ages 6–8 reason about invariance across contexts that varied in representational structure and whether totals were known or unknown. Clinical interview data revealed that while stability and identity reasoning were robust in single-group contexts, activation of these resources declined in unknown-total and base-ten contexts. Findings indicate that invariance of cardinality is not stable across representational contexts, including whether the totals were known or unknown.

Introduction

Understanding that the number of objects in a collection remains invariant despite changes in arrangement is foundational to the development of number. Early conservation research established that children must recognize cardinality as invariant under perceptual transformation (Piaget, 1952; Gelman, 1972). When children understand that spreading a row of counters does not change how many counters there are, they are attending to the countable elements of a set rather than to its perceptual appearance. This coordination is central to constructing number as an abstract property of discrete collections rather than as a perceptual impression (Sarama & Clements, 2009; Sinitsky & Ilany, 2016).

Research on conservation and related tasks demonstrated that many children, by approximately age seven years, successfully judge that rearranging objects does not change their number (e.g., Halford & Boyle, 1985; Piaget & Szeminska, 1952). Subsequent studies extended this work by examining children's reasoning in situations involving partitioning and compensation. For example, when objects are moved from one subgroup to another, the total number remains invariant, even though the quantities of the subgroups change (Irwin, 1996; Markman, 1979; Litrownik et al., 1978). Children's justifications in these contexts have been characterized using ideas such as stability ("nothing was added or removed"), identity ("you just moved them"), compensation ("one group lost what the other gained"), and reversibility ("you could put them back") (Kato et al., 2002; Sinitsky & Ilany, 2016).

While this research established important insights into children's understanding of conservation, much of it focused on relatively small quantities and tasks involving known totals. In contemporary classrooms, however, children routinely engage with larger collections, partitioned sets, and place-value groupings (e.g., bundles of ten and loose objects) as they develop understanding of multi-digit numbers and operations (Fosnot & Dolk, 2001; Sarama & Clements, 2009). These instructional contexts presume that children can coordinate invariance of cardinality across a range of representational forms, including situations in which the total number of objects is not explicitly known.

Despite longstanding assumptions that conservation of number is broadly established by the early elementary grades, relatively little recent research has revisited how children coordinate invariance of

cardinality across varied representational and contexts. In particular, we know less about how children reason when (a) the total number of objects is unknown, (b) objects are distributed across subgroups that are modified, or (c) collections are organized into base-10 groupings. Understanding children's reasoning in these contexts is important because early arithmetic relies on coordinating part–whole relations, compensation, and regrouping—ideas that depend on robust reasoning about invariance of cardinality.

In this study, we revisit conservation of number through the lens of children's reasoning about discrete sets across known and unknown total contexts. Rather than assuming conservation as stable across these contexts, we examine whether children's reasoning about invariance of cardinality is consistent across contexts or remains sensitive to task structure and representation. Specifically, we investigate how first- and second-grade children justify judgments about whether the total number of objects remains the same, increases, or decreases when objects are rearranged, partitioned, or modified.

To interpret children's reasoning, we draw on a Knowledge in Pieces (KiP) perspective (diSessa, 2018; Özdemir & Clark, 2007). In this view, conceptual understanding does not consist of fully formed, globally applicable theories. Instead, learners' knowledge is composed of locally activated resources—small, context-sensitive ideas that may be productive in some situations and not in others. In the context of invariance of cardinality, ideas such as stability, identity, compensation, and reversibility can be understood as resources that children may activate inconsistently depending on representational and epistemic features of a task. KiP perspective, conservation is not simply "present" or "absent," but may be fragmented across contexts.

By examining children's justifications across these contexts, we investigate whether reasoning about invariance of cardinality remains context-sensitive and representationally fragmented in the early elementary grades. These findings have implications for how we conceptualize early number development and for instructional practices that rely on children coordinating invariance across representations. To situate this investigation, we review prior research on invariance of cardinality in discrete sets.

Prior Research on Invariance of Cardinality with Discrete Sets

Early investigations of invariance focused on children's understanding that the number of discrete objects in a set remains constant despite changes in spatial arrangement. Piaget (1952) examined conservation of number alongside other conserved quantities such as length, area, and volume, proposing that children must come to recognize number as invariant under perceptual transformation. In conservation-of-number tasks, children compared two collections, one of which was rearranged (e.g., spread out or condensed), and judged whether the quantities remained the same.

Subsequent research refined this line of inquiry by distinguishing conservation of cardinality in discrete sets from invariance in continuous quantities (Gelman, 1972). Recognizing that 12 objects represent the quantity 12 regardless of their configuration requires coordinating attention to countable elements rather

than perceptual features. This understanding underlies the use of physical objects in early arithmetic contexts—for example, when learners regroup base-ten blocks, decompose collections, or rearrange objects to demonstrate commutativity (Fosnot & Dolk, 2001; Sarama & Clements, 2009).

While early Piagetian tasks typically involved comparing two collections, later researchers examined situations in which a single collection was transformed. Litrownik et al. (1978) and Markman (1979) described such reasoning as conserving identity or conserving equivalence. These tasks required children to judge whether the total number of objects in a single set remained invariant after rearrangement.

Irwin (1996) extended this work by investigating children's reasoning about compensation within partitioned sets. In these contexts, a collection was divided into subgroups and objects were moved from one subgroup to another. Although the quantities of the subgroups changed, the total number of objects remained invariant. Irwin also examined contexts in which objects were added to or removed from subgroups, requiring children to determine whether the total increased, decreased, or remained the same. These tasks made explicit the connection between physical transformations and symbolic arithmetic relations (e.g., 3 + 7 = 8 + 2).

Collectively, this body of research established that many children by approximately age seven years correctly judge invariance of cardinality in standard conservation contexts (Halford & Boyle, 1985; Sarama & Clements, 2009). However, much of this research involved relatively small quantities and tasks in which the total number of objects was known or perceptually accessible. Less attention has been given to how children coordinate invariance of cardinality across more varied representational conditions with discrete objects, including larger collections, unknown totals, and base-ten groupings.

Early Learners' Reasoning About Change and Invariance

Research examining children's explanations for invariance judgments suggests that learners draw on several recurring forms of reasoning. Kato et al. (2002) and Irwin (1996) documented explanations in which children justified invariance by stating that "nothing was added or taken away," that objects were "just moved," or that the original arrangement could be restored. Sinitsky and Ilany (2016) synthesized this work and identified categories in children's reasoning, including:

- **Stability**: The total remains the same because nothing was added or removed.
- **Identity**: Rearranging objects does not change what the set is.
- **Compensation**: A decrease in one subgroup is offset by an increase in another.
- **Reversibility**: The transformation can be undone to restore the original state.

Children as young as four years of age have been observed applying compensation reasoning in small-number contexts (Irwin, 1996), and by age seven many children correctly apply compensation in tasks involving ten or fewer objects. In these contexts, children often articulate reasoning such as, "You didn't

add or take away anything," or "You could put them back," demonstrating attention to cardinality rather than perceptual configuration.

Importantly, however, most prior research has examined children's reasoning in relatively constrained contexts—often with small numbers of objects and with totals that were either explicitly stated or easily countable. Less is known about whether these forms of reasoning are robust across contexts that vary (e.g., known vs. unknown totals) and representational structure (e.g., partitioned sets organized into base-ten groupings).

These findings suggest that learners draw on identifiable forms of reasoning when evaluating change and invariance. However, the consistency with which these forms of reasoning are activated across contexts remains an open question. To interpret such variability, we turn to a Knowledge in Pieces framework.

Theoretical Considerations

Prior research on conservation has often been interpreted through knowledge-as-theory perspectives (Ozdemir & Clark, 2007), in which learners are viewed as possessing coherent but potentially incorrect theories that are replaced through processes of assimilation and accommodation (Piaget, 1952). From this perspective, conceptual change involves wholesale restructuring of prior theories into more normative ones that can be broadly applied across contexts.

In contrast, we draw on diSessa's (2018) Knowledge in Pieces (KiP) perspective, which posits that learners' knowledge consists of diverse, context-sensitive elements rather than unified global theories. These "pieces" of knowledge—sometimes described as locally activated resources—may be productive in some contexts but not others. Conceptual change, from this view, involves reorganization and stabilization of these resources rather than replacement of a single misconception.

Applying this perspective to invariance of cardinality, we view stability, identity, and compensation not as components of a fully formed conservation theory, but as reasoning resources that may be activated selectively depending on task structure and representational features. In keeping with Simon's (2017) call to name the mathematical concepts under investigation, we explicitly identify and track the activation of these reasoning resources across varied contexts.

Rationale and Purpose of the Study

Prior research has shown that many children by approximately age seven years old correctly judge invariance of cardinality in standard conservation tasks involving small collections and known totals (Halford & Boyle, 1985; Irwin, 1996; Markman, 1979). These findings have often been interpreted as indicating that conservation of number is largely established in the early elementary grades.

However, contemporary classroom contexts require learners to coordinate invariance of cardinality across a broader range of situations, including partitioned collections, unknown totals, and base-ten

groupings that underpin multi-digit number understanding. We investigated how children reason about invariance of cardinality across contexts that vary in representational structure and knowledge of the total. By examining patterns in children's justifications, we aim to characterize the extent to which reasoning about invariance is context-sensitive and potentially fragmented.

Specifically, this study addresses the following research question: *How do early learners (ages 6–8 years) reason about the invariance of cardinality in discrete sets across contexts that vary in representation and in whether the total quantity is known or unknown?*

To answer this question, we designed tasks that varied systematically across single-group rearrangements, partitioned groups with known totals, partitioned groups with unknown totals, and base-ten groupings. We analyzed children's judgments and justifications to identify the reasoning resources they activated in each context.

Method

Design Overview

This study draws on data collected as part of a larger four-year research project examining early learners' reasoning about number and operations across varied contexts. The present analysis focuses specifically on children's reasoning about the invariance of cardinality in discrete sets.

We adopted a qualitative developmental design (Saldaña, 2003) using structured clinical interviews (Ginsburg, 1997) administered at multiple time points across first and second grade. The goal was not to make statistical claims about population-level differences, but to characterize patterns in the reasoning resources children activated across task contexts that varied in representation and contexts. Data analysis combined qualitative coding of children's justifications with descriptive summaries of response patterns to identify context-sensitive shifts in reasoning.

Setting and Participants

The study was conducted in two public elementary schools in the Midwestern United States within the same school district. School A enrolled approximately 275 students and was designated a Title I school, with approximately 80% of students eligible for free or reduced lunch. School B enrolled approximately 300 students, with approximately 35% of students eligible for free or reduced lunch. Both schools used the same mathematics curriculum during the study period. Seventy-eight students participated in at least one data collection point. Of these, 41 students attended School A and 37 attended School B. Thirty-three students participated in all four data collection intervals, while 45 participated in three or fewer time points.

Data were collected at four intervals across first and second grade: September of Grade 1 (beginning of first grade; ages 6–7), April of Grade 1 (end of first grade), September of Grade 2 (beginning of second

grade; ages 7–8), and April of Grade 2 (end of second grade). The present analysis includes all available student responses to invariance and change tasks at each time point.

Instrument and Data Collection Procedures

Data were collected using a researcher-developed instrument, the Early Number Battery (ENB) (Lannin et al., 2013). The ENB is a cognition-based assessment administered through structured, task-based clinical interviews (Barrett et al., 2006; Goldin, 2012). Cognition-based assessments are designed to elicit learners' reasoning in specific mathematical contexts rather than solely measure correctness (Battista, 2004).

The ENB includes tasks targeting multiple constructs related to early number and operations (e.g., cardinality, magnitude, part–whole reasoning). For this study, we analyzed a subset of tasks that assessed reasoning about invariance and change in the number of discrete physical objects.

All interviews were conducted individually and video recorded. After each task prompt, students were asked to explain their reasoning. When needed, clarifying prompts were used to ensure students attended to the total quantity rather than to subgroup comparisons.

Task Framework

To investigate context sensitivity in reasoning about invariance of cardinality, we designed tasks that varied systematically along two dimensions:

1. Representational Structure

o Single group rearrangements

o Partitioned groups (two subgroups)

o Base-ten groupings (bundles of ten and loose units)

2. Knowledge of the Total

o Known total (explicitly determined prior to transformation)

o Unknown total (not numerically specified)

Tasks were informed by prior conservation and compensation studies (Irwin, 1996, Markman, 1979) but extended in two important ways:

· Inclusion of larger quantities than typically used in earlier studies

· Inclusion of contexts involving unknown totals and base-ten groupings

For the change and the invariance tasks, students determined whether the total increased, decreased, or remained invariant when objects were added, removed or rearranged. These contexts were not treated as a developmental sequence but as varied representational conditions designed to examine whether reasoning about invariance was robust or context-sensitive. A description of the various types of tasks is included in **Figure 1** and **Figure 2**.

Analytic Framework

Guided by a Knowledge in Pieces perspective (diSessa, 2018), we analyzed children's justifications in terms of the reasoning resources they activated when evaluating invariance or change. Drawing on prior categorizations (Sinitsky & Ilany, 2016; Irwin, 1996), we identified four primary reasoning resources:

· **Stability:** The total remains invariant because nothing was added or removed.

· **Identity:** Rearranging objects does not change the set's cardinality.

· **Compensation:** A decrease in one subgroup is offset by an increase in another.

· **Reversibility:** The transformation can be undone to restore the original configuration.

Responses could receive multiple codes if children coordinated more than one reasoning resource (e.g., invoking both stability and identity). Importantly, these categories were not interpreted as fixed developmental stages. Rather, they were treated as locally activated reasoning resources that might be applied inconsistently across contexts.

Coding Procedures and Trustworthiness

All student explanations were transcribed from video recordings. An initial coding scheme was developed based on prior literature and refined inductively through iterative review of student responses. Coding proceeded in two stages: (1) one author coded all responses, and (2) a second author independently reviewed and coded a subset of responses. Discrepancies were discussed until consensus was reached. Through this iterative process, definitions of reasoning resources were clarified and operationalized. Because the study's aim was to characterize patterns in reasoning rather than to estimate population parameters, descriptive summaries of response frequencies were used to illustrate trends in resource activation across contexts. The coding scheme and examples used for the study is included in Figure 3.

Results

We present results across three representational contexts: (a) single-group rearrangements, (b) partitioned groups, and (c) base-ten groupings. Within each context, tasks varied by whether the total number of objects was known or unknown and, in some cases, whether the total remained invariant or changed. Our analysis focuses on patterns in the reasoning resources children activated when judging

invariance or change. Accuracy is reported descriptively (see Panel A tables), but our interpretation centers on shifts in resource activation across contexts (see Panel B tables).

Invariance with a Single Group of Objects

We first examined children's reasoning in a single-group invariance task in which 15 counters were rearranged (see Table 1). As shown in Table 1, Panel A, overall accuracy increased substantially from the beginning to the end of Grade 1 (64.8% to 98.1%). By the spring of first grade, nearly all learners correctly judged that the total number of objects remained invariant under rearrangement.

Patterns of reasoning shifted over time (see Table 1, Panel B). In the fall of first grade, many learners relied on counting (38.9%), suggesting uncertainty about whether rearrangement affected cardinality. Stability (18.5%) and identity (31.5%) were present but not dominant. By the spring of first grade, stability (54.7%) and identity (49.1%) were employed more frequently, while counting responses declined sharply (9.4%). When stability or identity were utilized, judgments were typically correct. These results suggest that, in single-group known-total contexts, invariance of cardinality was relatively consolidated by the end of first grade.

Table 1

Single-Group Invariance (Known Total) – Grade 1

Panel A. Overall Accuracy

Time Point	N	Correct Judgment (%)
Fall Grade 1	54	64.8
Spring Grade 1	53	98.1

Panel B. Distribution of Reasoning Resources

Time Point	Stability (%)	Identity (%)	Counting (%)	Other (%)
Fall Grade 1	18.5	31.5	38.9	24.1
Spring Grade 1	54.7	49.1	9.4	7.5

Invariance and Change with Partitioned Groups

We next examined contexts involving two groups of objects. These tasks required children to coordinate reasoning about part–whole relationships.

Two Groups – Known Total (Move Between Groups). In the spring of Grade 1, learners judged whether moving one counter between two cups changed the total number of objects (see Table 2). For this task, 88.2% of learners correctly judged that the total remained invariant. In Table 2, Panel B, stability (27.5%),

identity (41.2%), and compensation (21.6%) were all activated. Compensation reasoning—recognizing that one group lost what the other gained—was also utilized by many learners in this context. Few learners focused exclusively on subgroup changes (2.0%) or assigned imagined values (9.8%). Thus, when the total was known, many learners coordinated part–whole relationships successfully.

Table 2

Two Groups – Invariance (Known Total)

Panel A. Overall Accuracy

Grade Level	N	Correct Judgment (%)
Spring Grade 1	51	88.2

Panel B. Distribution of Reasoning Resources

Stability (%)	Identity (%)	Compensation (%)	Group Focus (%)	Assign Value (%)	Other (%)
27.5	41.2	21.6	2.0	9.8	17.6

Two Groups – Unknown Total (Move Between Groups). When the same structural task was presented with an unknown total at the beginning of Grade 2, patterns differed markedly (see Table 3). As shown in Table 3, Panel A, only 42.9% of learners correctly recognized invariance for this situation. Activation of stability (9.5%) and compensation (12.7%) declined substantially. Instead, nearly half of learners (46.0%) focused on changes within individual subgroups rather than attending to the total. Additional learners assigned values to the cups (9.5%) rather than reasoning structurally.

This contrast between known and unknown total contexts suggests that compensation and stability were not consistently activated across contexts (known vs. unknown total).

Table 3

Two Groups – Invariance (Unknown Total)

Panel A. Overall Accuracy

Grade Level	N	Correct Judgment (%)
Fall Grade 2	63	42.9

Panel B. Distribution of Reasoning Resources

Stability (%)	Identity (%)	Compensation (%)	Group Focus (%)	Assign Value (%)	Other (%)
9.5	22.2	12.7	46.0	9.5	19.0

Redistribution with Known Total (Remove and Replace Equal Quantities). In a modified task with a known total of 13 objects—where four objects were removed from one group and four different objects added to another—66.1% of learners judged invariance (see Table 4, Panel A). Notably, stability and identity were not appropriate for this task and were not observed (see Table 4, Panel B). Instead, compensation (61.3%) was the dominant reasoning resource. Learners who focused on subgroup values or assigned imagined quantities were more likely to judge incorrectly. This task further illustrates that successful reasoning required activation of compensation rather than simpler stability or identity resources.

Table 4

Two Groups – Redistribution (Known Total)

Panel A. Overall Accuracy

Grade Level	N	Correct Judgment (%)
Fall Grade 2	62	66.1

Panel B. Distribution of Reasoning Resources

Stability (%)	Identity (%)	Compensation (%)	Group Focus / Assign Value (%)	Other (%)
0.0	0.0	61.3	24.2	16.1

Change Task – Unknown Total (Net Loss). In a change task involving removal of three objects and addition of two (net loss of one), 61.9% of learners correctly judged that the total decreased (see Table 5, Panel A). Some learners activated lack-of-compensation reasoning (33.3%) or lack-of-stability reasoning (27.0%), explicitly noting that more objects were removed than added. Others focused on individual groups (28.6%) and were more likely to respond incorrectly. These results suggest that coordinating net change required integrating compensation reasoning more flexibly than in simple rearrangement tasks.

Table 5

Two Groups – Change Task (Unknown Total)

Panel A. Overall Accuracy

Grade Level	N	Correct Judgment (%)
Fall Grade 2	63	61.9

Panel B. Distribution of Reasoning Resources

Lack of Stability (%)	Lack of Compensation (%)	Single Group Focus / Assign Value (%)	Other (%)
27.0	33.3	28.6	22.2

Base-Ten Groupings: Representational Complexity

Finally, we examined invariance and change tasks involving base-ten groupings (bundles of ten and loose units).

Base-10 Invariance (Move Unit Into Bundle). When one loose unit was moved into a bundle of ten from the group of unbundled units (known total of 26), 79.4% of learners correctly judged invariance (see Table 6, Panel A). However, resource activation shifted (see Table 6, Panel B). Stability (20.6%), identity (23.5%), and compensation (8.8%) were present but less dominant than in simpler contexts. A substantial portion of learners (38.2%) attempted to recompute totals by adding parts or counting. Although many learners ultimately responded correctly, base-ten structure appeared to disrupt immediate activation of structural invariance reasoning.

Table 6

Base-10 – Invariance (Move Unit Into Bundle)

Panel A. Overall Accuracy

Grade Level	N	Correct Judgment (%)
Spring Grade 2	34	79.4

Panel B. Distribution of Reasoning Resources

Stability (%)	Identity (%)	Compensation (%)	Reversibility (%)	Add Parts / Count (%)	Other (%)
20.6	23.5	8.8	2.9	38.2	14.7

Base-10 Invariance (Move Unit Out of Bundle). When a unit was moved out of a bundle of ten and into the group of ungrouped units, accuracy declined to 70.8% (see Table 7, Panel A). Resource activation

patterns were similar: stability, identity, compensation, and reversibility were present but not dominant, and many learners relied on part-based addition (39.6%) (see Table 7, Panel B).

Table 7

Base-10 – Invariance (Move Unit Out of Bundle)

Panel A. Overall Accuracy

Grade Level	N	Correct Judgment (%)
Spring Grade 2	48	70.8

Panel B. Distribution of Reasoning Resources

Stability (%)	Identity (%)	Compensation (%)	Reversibility (%)	Add Parts / Count (%)	Other (%)
10.4	16.7	4.2	12.5	39.6	25.0

Base-10 Change (Remove One from Bundle). In a change task (removing one unit from a bundle), 65.4% of learners judged correctly that the total decreased (see Table 8, Panel A). In Table 8, Panel B, nearly half of learners (46.2%) relied on adding or counting parts, often incompletely coordinating all subcomponents of the representation. Less than half explicitly invoked lack-of-stability reasoning (30.8%).

Table 8

Base-10 – Change Task (Remove One From Bundle)

Panel A. Overall Accuracy

Grade Level	N	Correct Judgment (%)
Spring Grade 2	26	65.4

Panel B. Distribution of Reasoning Resources

Lack of Stability (%)	Lack of Compensation (%)	Lack of Reversibility (%)	Add Parts / Count (%)	Other (%)
30.8	0.0	7.7	46.2	15.4

Cross-Context Patterns: Evidence of Contextual Fragmentation

To synthesize patterns across contexts, Table 9 presents a summary of reasoning resource activation and overall accuracy across all tasks. As shown in Table 9:

· Stability and identity were most robust in single-group known-total contexts.

· Compensation was activated productively in known partitioned contexts but declined when totals were unknown.

· Counting and additive recomputation increased substantially in unknown-total and base-ten contexts.

· Accuracy declined most sharply in unknown-total partitioned contexts.

These patterns indicate that reasoning about invariance of cardinality was not stable across contexts. Instead, activation of stability, identity, and compensation resources varied depending on representational and contextual features of the task. From a Knowledge in Pieces perspective, invariance of cardinality appears context-sensitive and representationally fragmented during the early elementary grades for many learners. Table 9 summarizes reasoning resource activation across primary representational contexts. Percentages reflect the proportion of responses coded with each resource within each context, aggregated across the relevant tasks reported in Tables 1–8.

Table 9

Distribution of Reasoning Resources Across Contexts

Context	Stability (%)	Identity (%)	Compen-sation (%)	Counting / Additive Focus (%)	Overall Accuracy (%)
Single Group – Known Total (End Grade 1)	54.7	49.1	—	9.4	98.1
Two Groups – Known Total	27.5	41.2	21.6	11.8	88.2
Two Groups – Unknown Total	9.5	22.2	12.7	55.5	42.9
Base-10 – Known Total (Move Unit)	20.6	23.5	8.8	38.2	79.4
Base-10 – Remove Unit	30.8	0.0	0.0	46.2	65.4

Discussion

This study examined how early elementary learners reason about change and invariance of cardinality across varied contexts. Although conservation of number has long been interpreted as a developmental

milestone achieved by approximately age seven years (e.g., Irwin, 1996; Markman, 1979), our findings suggest a more nuanced picture. Drawing on a Knowledge in Pieces (KiP) perspective (diSessa, 2018), we interpret these findings not as evidence that learners either “have” or “lack” conservation, but as evidence that reasoning resources such as stability, identity, and compensation are activated selectively across contexts. We highlight three primary contributions in the following sections.

Invariance in the Number of Discrete Objects Is Not Fully Stabilized by Age Seven

Consistent with prior research (Irwin, 1996; Markman, 1979), learners in our study demonstrated strong performance in single-group rearrangement tasks. By the end of first grade, nearly all learners judged correctly that rearranging objects did not change the total quantity, frequently activating stability and identity reasoning.

From a Knowledge in Pieces (KiP) perspective (diSessa, 2018), however, such performance should not be interpreted as evidence that invariance of cardinality is broadly stable. Rather, these findings suggest that reasoning resources such as stability, identity, and compensation may be locally consolidated within familiar contexts while remaining sensitive to representational variation.

This interpretation is supported by the decline in performance observed in contexts involving unknown totals and base-ten groupings. In particular, fewer than half of learners correctly judged invariance when objects were moved between two groups and the total was not explicitly known. This contrasts with Irwin’s (1996) findings that most learners at similar ages demonstrated compensation reasoning in tasks involving small, known quantities. One possible explanation is that prior studies often involved small totals (e.g., fewer than 10 objects) and tasks in which totals were explicitly determined. Our findings suggest that when totals are larger, unknown, or embedded within composite base-ten representations, activation of invariance-related resources becomes less stable. For example, although nearly all learners correctly judged that rearranging 15 counters did not change the total in a single-group context, accuracy declined and additive recomputation strategies increased when the same total was represented as two bundles of ten and loose units. In these base-ten contexts, many learners counted or recombined parts rather than immediately invoking stability or compensation reasoning, suggesting that invariance resources were not automatically activated when composite units were involved.

These patterns suggest that conservation may be better understood not as a developmental endpoint achieved by age seven, but as an ongoing conceptual structure that must be reorganized across increasingly complex representational systems.

Reasoning Resources are Context-Sensitive Rather than Broadly Applied

A second contribution of this study is theoretical rather than developmental. Prior research identified stability, identity, and compensation as recurring forms of invariance reasoning (Irwin, 1996; Sinitsky &

Ilany, 2016). However, these resources have typically been documented within relatively constrained task structures. The present study extends this work by demonstrating how activation of these reasoning resources depends systematically on contexts (known versus unknown totals) and representational structure (single units versus composite base-ten units). In doing so, we move beyond identifying resources to characterizing the conditions under which they are—or are not—activated.

This pattern supports a KiP interpretation in which learners possess multiple “pieces” of knowledge that may not yet be reorganized into a stable, broadly applicable structure (diSessa, 2018). Rather than conceptual change occurring through wholesale replacement of misconceptions (as in traditional knowledge-as-theory models; Ozdemir & Clark, 2007), our findings suggest that consolidation of invariance reasoning involves reorganization and stabilization of context-sensitive resources.

Representational Complexity (Especially Base-Ten Structure) Challenges Apparent Conservation

Tasks involving base-ten groupings introduced additional representational complexity, leading learners to rely more frequently on additive recomputation strategies rather than immediate structural invariance reasoning. Invariance reasoning may appear consolidated in single-unit contexts yet remain fragile when learners must coordinate composite units. Base-ten representations require simultaneous coordination of unitary and composite units (Baroody, 2017), which may increase cognitive load and disrupt spontaneous activation of invariance resources. Thus, what appears to be stable conservation in one representational system may mask ongoing reorganization of reasoning resources across structurally different representations.

Implications for Task Design and Assessment

The present study contributes a task framework that varies representational structure (single groups, partitioned groups, and base-ten groupings) as well as contexts (known versus unknown totals). These findings imply that traditional conservation tasks may overestimate the robustness of learners’ invariance reasoning when task conditions remain limited. Learners who demonstrated stable invariance reasoning in single-group, known-total contexts did not consistently activate the same reasoning resources when totals were unknown or when quantities were embedded within composite base-ten structures.

These findings have important implications for task design and assessment. If invariance reasoning is context-sensitive rather than broadly stable, then traditional conservation tasks may overestimate the robustness of learners’ understanding. Assessments that rely exclusively on rearrangement of a single known group may obscure fragmentation that becomes visible when learners must coordinate part–whole relationships, unknown totals, or composite units. Because invariance reasoning emerges through the gradual reorganization and stabilization of context-sensitive resources (diSessa, 2018), instruction should deliberately vary contexts. For example, learners might engage with tasks involving (a) known totals that are rearranged, (b) partitioned quantities where totals are unknown, (c) compensation

scenarios requiring coordination of gains and losses, and (d) base-ten representations requiring simultaneous attention to units and composite units. Such variation may support abstraction of invariance principles across contexts rather than stabilization within a single task structure.

Designing instruction and assessment to surface when stability, identity, and compensation are—or are not—activated provides a window into learners' conceptual organization. Rather than asking whether children "have" conservation, educators should consider under what conditions invariance reasoning stabilizes across representations.

Conclusion

Our findings suggest that reasoning about invariance of cardinality in discrete sets remains context-sensitive during the early elementary years. Although many learners demonstrate correct judgments in familiar conservation tasks, their reasoning resources are not uniformly activated across contexts involving unknown totals, larger quantities, or base-ten representations. From a Knowledge in Pieces perspective (diSessa, 2018), invariance of cardinality appears to emerge through gradual reorganization and stabilization of context-sensitive reasoning resources rather than through abrupt acquisition of a global conservation theory.

These findings highlight the importance of designing instructional experiences that vary contexts so that learners abstract and stabilize invariance reasoning across contexts. Further research is needed to examine how these reasoning resources reorganize over time and how instruction can support greater generalization across representations and symbolic forms. Reframing conservation through a KiP lens shifts the question from whether children "have" conservation to how and under what conditions invariance reasoning stabilizes across representational systems.

Declarations

Ethics Approval

This study was conducted in accordance with institutional guidelines for research involving human participants. Approval was obtained through appropriate institutional review processes.

Consent for Participation and Publication

Informed consent was obtained from all participants involved in the study. Consent for publication was also obtained where applicable.

Availability of Data and Materials

The datasets generated and analyzed during the current study are not publicly available due to restrictions related to participant confidentiality and the terms of the funding agreement but may be available from the corresponding author on reasonable request.

Competing Interests

The authors declare that they have no competing interests. None of the authors are members of the editorial board, nor are they serving as editors or reviewers for this journal in relation to this manuscript.

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Authors' Contributions

JL contributed to all phases of the study, including study design, data collection, analysis, and drafting the manuscript.

TVH supported data collection, analysis, and contributed to writing the methods section.

DVG contributed to study design, data analysis, and writing of the analysis and discussion sections.

All authors read and approved the final manuscript.

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Clinical Trial Registration

Not applicable.

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Figures

Single group Recognizing that when a single group of objects (same unit) are re-arranged the total remains the same.	**Known Total** The child is shown 15 counters. After the child recognizes that there are 15 counters, the counters are arranged in a "line" or pile and the child is asked to determine the number of counters.	**Unknown Total** The child is shown an unknown number of counters. The counters are arranged in a "line" or pile of counters and the child whether there are more, less, or the same number of counters as before.
Two groups Recognizing that when two groups of objects (same unit) are re-arranged the total remains the same.	**Known Total** Thirteen counters are placed in two opaque cups. The child is asked how many counters are in the two cups. Two counters are moved from one cup to the other cup and the child is asked how many counters are in the two cups.	**Unknown Total** An unknown number of counters are placed in two opaque cups. The child is told that there are some counters altogether in the two cups. Two counters are moved from one cup to the other cup and the child is asked whether there are more, less, or the same number of counters in the two cups altogether as before.
Base-10 Groups Recognizing that when multiple groups of 10 are rearranged and/or decomposed the total remains the same.	**Known Total** Objects are placed in two groups of 10 and one group of six. The total number of objects is determined. One object is moved from one group of 10 to the group of six and the child is asked to determine how many objects.	**Unknown Total** An unknown number of objects placed in two or more groups involving multiple units. One object is moved from one group into another group and the child is asked whether there are more, less, or the same number of counters in the two groups altogether.

Figure 1

Invariant Mathematical Tasks

Two Groups Recognizing that when multiple groups of objects are altered so that objects are removed or added, that the total increases or decreases.	**Known Total Quantity** Thirteen counters are placed in two opaque cups. The child is asked how many counters are in the two cups. Two counters are removed from one cup and only one counter is put in the other cup and the child is asked to determine if the number of counters is the same, less, or more than 15.	**Unknown Total Quantity** An unknown number of counters are placed in two opaque cups. The child is told that there are some counters altogether in the two cups. Two counters are added to one cup and the child is asked whether there are more, less, or the same number of counters in the two cups altogether as before.
Base-10 Groups Recognizing that when multiple groups involving multiple units and objects are added or removed the total increases or decreases.	**Known Total Quantity** Objects are placed in two groups of 10 and one group of six. The total number of objects is determined to be 26. One object is added to one group of 10 and the child is asked to determine is the total number of objects is more, less, or the same.	**Unknown Total Quantity** An unknown number of objects placed in two or more groups involving groups of 10. One object is moved from one group of 10 into another group of 10 and the child is asked whether there are more, less, or the same number of counters in the two groups altogether.

Figure 2

Mathematical Change Tasks

Concept Name	Concept	Participant Example
Stability	If objects are added or removed from a group of objects, the total number of changes.	"(There are still the same number), you didn't take any away or didn't add any."
Identity	If objects are rearranged, the total number of objects remains invariant.	"You just moved them around. There are still 12 blocks"
Compensation	If objects are moved from one group of objects to another group of objects, the total number of objects remains invariant.	"(There are still) the same because you took 4 out (of this pile) and put (objects) back here (in this other pile)."
Reversibility	If objects can be moved back to their original position, the total number of objects remains invariant.	"There are still 26, because the one (object) here goes here (can be moved back to where it was before)."

Figure 3

Coding scheme including concept descriptor and examples