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Andres F. Guerra Riano

Budapest University of Technology and Economics

Peter L. Varkonyi

`varkonyi.peter@epk.bme.hu`

Budapest University of Technology and Economics

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Quasi-static Shape Morphing of Adaptive Columns Toward Funicular Forms

Andrés F. Guerra Riaño · Péter L. Várkonyi

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Abstract Funicular load-bearing structures achieve exceptional efficiency because their shapes follow the natural trajectories of internal stresses. However, time-varying loads can disrupt funicularity if they have fixed geometries. Recently, the authors found that funicularity may be restored through shape adaptation mediated by a simple, novel feedback control law. In this work, we investigate the limit of slow adaptation—an important special case of the new structural paradigm that is relevant to real-world applications in which external loads also evolve slowly. By considering the quasi-static limit of the coupled Newtonian and shape dynamics, we develop a discrete, chain-like multibody model of a morphing column, which enables detailed examination and visualization of shape dynamics. We show that, despite the lack of Newtonian dynamics, the shape dynamics is rich and nonlinear because of large geometric changes. Illustrative examples of shape adaptation reveal cases of successful convergence to the target shape as well as sustained shape oscillations with repeated snapping. The prominent role of elastic buckling instability in shape convergence is highlighted. Linearized evolution equations are used to verify local convergence to the target shape, and a condition of non-local convergence from almost all initial configurations to the target configuration is also developed.

Keywords Funicularity · Adaptive structures · Shape Control · Stability Analysis

A. F. Guerra Riaño

Department of Mechanics, Materials and Structures, Faculty of Architecture, Budapest University of Technology and Economics. Műegyetem rkp. 3, 1111, Budapest, Hungary. E-mail: guerra.andres@edu.bme.hu

P. L. Várkonyi (corresponding author)

Department of Mechanics, Materials and Structures, Faculty of Architecture, Budapest University of Technology and Economics. Műegyetem rkp. 3, 1111, Budapest, Hungary. E-mail: varkonyi.peter@epk.bme.hu

1 Introduction

Funicular load-bearing structures such as arches and cables are distinguished by their lightness and aesthetic appeal in architectural and structural engineering applications [33]. Their efficiency is based on the design principle that structural geometry should align with the dominant internal force flow, thereby minimizing bending moments and transferring loads predominantly through axial forces (either tension or compression) [2]. In this sense, funicular forms provide a direct geometrical representation of equilibrium, as the shape itself determines load-bearing capacity [15].

Nevertheless, efforts to reduce material consumption typically lead to slender, extremely lightweight structures that are becoming increasingly widespread in architecture, civil infrastructure, and aerospace systems [16, 8, 4, 21, 34, 23]. Such systems exhibit high flexibility and associated practical challenges such as instability, dynamic response, and high sensitivity to load variations [26]. Even small perturbations in loading, boundary conditions, or geometry may induce significant effects, potentially shifting the structure away from its preferred force-aligned configuration and activating inefficient, bending-dominated response.

Flexible funicular structures, such as cable systems, can naturally compensate for imperfections and load variations because their geometry spontaneously adjusts under changing force patterns, maintaining a predominantly tensile load path [3]. In contrast, structures designed to resist loads via compression or a combination of compression and tension require sufficient flexural stiffness to prevent elastic instability [5], which eliminates the possibility of spontaneous shape adaptation. In the case of variable loads, only *active shape control* can restore funicularity. This distinction motivates the current work, in which we investigate a control law for active shape adaptation via numerical simulation and stability analysis.

Slender structures can be modeled as discrete, pin-jointed multibody systems [14]. We examine a model of a shape-active rod, where a chain of rigid bars carries nodal loads so that internal forces are predominantly axial and joint moments are minimized. Potential applications of this concept include discrete thrust-line arches formed by hinged chains [13, 6], segmented compression ribs for lightweight roofs [9], and deployable linkages that lock into compression-arch configurations [7, 19]. Discrete models have also been studied in the context of adaptive structures, where sensors, actuators, and control laws are integrated to enable shape regulation toward various targets [27, 22, 32, 31]. In recent work by the authors, active shape-morphing chains subject to stationary loads have been investigated, where active shape adaptation drove the chain toward funicular equilibrium characterized by vanishing joint moments [30, 20, 10]. Stability and response time of the actuation were investigated in terms of the parameters of Proportional-Integral-Derivative (PID) controllers. We found a trade-off between stability and the rate of shape adaptation. These results suggest that in applications where response time is not critical, the limit of slow shape adaptation offers the highest level of robustness. Slow adaptation corresponds to quasi-static shape evolution [11], where, despite the lack of dynamics in the Newtonian sense, non-trivial shape dynamics may emerge, involving nonlinearity, hysteretic behavior, and loss of stability [25, 18].

Indeed, shape-active systems subject to stationary loads are prone to three distinct types of instability. Rapid actuation may result in significant energy input. If damping is not strong enough to absorb the excess energy, *flutter* instability (i.e.,

self-induced vibration) may occur [28]. This work is concerned with the limit of slow actuation, in which case flutter does not occur due to the negligible energy input of the actuators. Instead, structures exhibit behavior similar to passive viscoelastic systems built of rigid rods, springs, and dampers, for which the static theory of elastic stability is applicable [29]. In particular, compliant structures are prone to *buckling* and *snapping* instability, both of which can be avoided by sufficiently high stiffness. If elastic instability does not occur, the shape of the structure evolves slowly through stable static equilibria under the effect of the actuators. The last instability mechanism is the lack of convergence of the slow shape-evolution process toward the target. This paper is concerned with the last two mechanisms. We examine elastic stability and develop a simplified set of equations describing shape evolution. Our main result is a sufficient condition for almost global convergence to the unique shape ensuring load bearing through compressive forces.

In Section 2, the model components are introduced and a shape control law is formulated. In Section 3, we develop the equations of shape dynamics on two distinct time scales: a stability analysis of the fast dynamics ensuring convergence to an equilibrium manifold, and the governing equations of the slow shape evolution along the equilibrium manifold. An illustration of shape adaptation via numerical simulation in Section 4 is followed by systematic linear stability analysis of the slow dynamics in Section 5, and the development of a condition of almost global convergence in Section 6. Finally, conclusions are presented in Section 7.

2 The discrete morphing column model

We investigate a discrete, planar model of a shape-morphing column: a chain of m rigid bars of length l_i , connected by pin joints (Fig. 1(a)). Nodes are labeled by integers $0, \dots, m$, with node 0 fixed at the origin, and bars are labeled $1, 2, \dots, m$. Stationary external forces $\mathbf{f}_i$ ($i = 1, 2, \dots, m$) are applied at the free nodes. Each joint (Fig. 1(b)) is equipped with a linear torsional spring of stiffness k_i and stress-free angle δ_i , generating moment $\tau_i = k_i(\alpha_{i+1} - \alpha_i - \delta_i)$ (where $\alpha_0 = 0$ by convention), as well as a damper (with unspecified damping constant). In addition, an actuator sets the stress-free angle of the spring according to a control policy described in Sec. 2.3. As we will see, the damper is necessary to ensure quasi-static behavior; however, its exact characteristics do not affect the analysis.

2.1 Notation

- $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_m]^T$ directional angles of bars.
- $\bar{\boldsymbol{\alpha}} = [\bar{\alpha}_1, \dots, \bar{\alpha}_m]^T$ directional angles in the compression-only funicular configuration.
- $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}$; $\tilde{\alpha}_i = \alpha_i - \bar{\alpha}_i$ deviations of angles from the compression-only funicular configuration.
- $\mathbf{C}_{\mathbf{v}}$, $\mathbf{c}_{\mathbf{v}}$ diagonal matrices and vectors composed of $\cos(v_i)$ where v_i are the elements of a vector $\mathbf{v}$.

$$- \mathbf{D} = \begin{bmatrix} 1 & 0 & & \\ -1 & 1 & 0 & \\ & \ddots & \ddots & \\ & & -1 & 1 & 0 \\ & & & 0 & -1 & 1 \end{bmatrix}.$$

- $\boldsymbol{\delta} = [\delta_0, \dots, \delta_{m-1}]^T$ unstressed joint angles.
- $\mathbf{f}_i$ Stationary external force applied at node i ; $f_i = \|\mathbf{f}_i\|$.
- $\mathbf{H}_P$ Hessian matrix of P .
- $\mathbf{J}$ Jacobian matrix of the evolution equation of slow shape adaptation at a funicular configuration.
- $\mathbf{J}_\delta$ Jacobian matrix of $\boldsymbol{\delta}$ with respect to $\boldsymbol{\alpha}$ along the equilibrium set.
- $k_0, k_1, \dots, k_{m-1}$ stiffness coefficients; $\mathbf{k} = [k_0, k_1, \dots, k_{m-1}]^T$, $\mathbf{K} = \text{diag}(\mathbf{k})$.
- $l_1, l_2, \dots, l_m$ lengths of bars; $\mathbf{l} = [l_1, l_2, \dots, l_m]^T$, $\mathbf{L} = \text{diag}(\mathbf{l})$.
- m number of bars.
- $\mathbf{n}_i$ cumulative load resultant at node i ; $n_i = \|\mathbf{n}_i\|$.
- $\mathbf{n} = [n_1, \dots, n_m]^T$; $\mathbf{N} = \text{diag}(\mathbf{n})$.
- P potential energy.
- $r_0, r_1, \dots, r_{m-1}$ relaxation coefficients; $\mathbf{r} = [r_0, r_1, \dots, r_{m-1}]^T$, $\mathbf{R} = \text{diag}(\mathbf{r})$.
- $\mathbf{S}_\mathbf{v}$, $\mathbf{s}_\mathbf{v}$ diagonal matrices and vectors composed of $\sin(v_i)$, where v_i are the elements of a vector $\mathbf{v}$.
- ^{T} transpose of a matrix or a column vector.
- $\tau_0, \tau_1, \dots, \tau_{m-1}$ moments in springs; $\boldsymbol{\tau} = [\tau_0, \tau_1, \dots, \tau_{m-1}]^T$.
- $\mathbf{U}$ identity matrix.

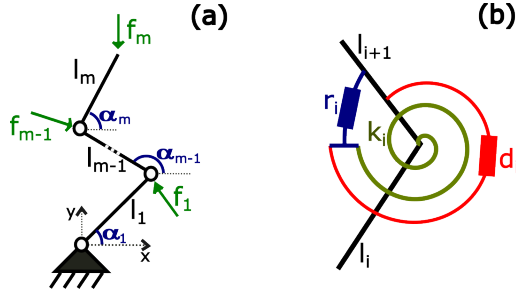


Fig. 1 (a): model of a cantilever structure with actuated pin joints and subjected to stationary nodal external loads. (b): representation of the joint actuation mechanism.

2.2 Equations of equilibrium

First, we investigate the passive, static behavior of the column without shape adaptation, i.e., we assume that the stress-free angles δ_i of the springs are stationary.

For a cantilever chain, bar i transmits the sum of all external loads applied at nodes $j \geq i$. Therefore, we introduce the cumulative load resultant vector associ-

ated with bar i :

$$\mathbf{n}_i = \sum_{j=i}^m \mathbf{f}_j, \quad (1)$$

We also introduce the notation $n_i = |\mathbf{n}_i|$, and we decompose the angles α_i as

$$\alpha_i = \bar{\alpha}_i + \tilde{\alpha}_i \quad (2)$$

where $\bar{\alpha}_i$ is the direction angle of $-\mathbf{n}_i$ relative to the x -axis. This notation will later allow us to characterize target shapes carrying the load through pure compression as $\tilde{\alpha}_i = 0$.

The equilibrium equations can be obtained by considering the potential energy of the system, which can be expressed in matrix form as

$$P = \mathbf{l}^T \mathbf{N} \mathbf{c}_{\bar{\alpha}} + \frac{1}{2} (\mathbf{D} \boldsymbol{\alpha} - \boldsymbol{\delta})^T \mathbf{K} (\mathbf{D} \boldsymbol{\alpha} - \boldsymbol{\delta}). \quad (3)$$

The first term corresponds to the potential energy associated with the loads; the second term represents the elastic energy of the springs. In particular, the vector $\mathbf{D} \boldsymbol{\alpha}$ contains relative angles $\alpha_{i+1} - \alpha_i$ between adjacent bars. The equilibrium states are the stationary points of P , where its gradient with respect to $\boldsymbol{\alpha}$ vanishes, yielding the classical equilibrium equations:

$$0 = -\mathbf{N} \mathbf{L} \sin(\tilde{\alpha}) + \mathbf{D}^T \mathbf{K} (\mathbf{D} \boldsymbol{\alpha} - \boldsymbol{\delta}). \quad (4)$$

Note the physical interpretation of the individual terms: $\mathbf{K} (\mathbf{D} \boldsymbol{\alpha} - \boldsymbol{\delta})$ is a vector containing spring moments τ_i ; multiplication by $\mathbf{D}^T$ generates the sums of moments acting on each individual bar; and the first term of the equation contains the moments of the load resultants on each bar. As we will later see, the same equilibrium equations can be used to describe quasi-static shape evolution in the case of slow shape adaptation.

2.3 A shape control law

In the quasi-static limit of the coupled mechanical and adaptation dynamics, structural equilibrium is assumed to be reached instantaneously compared to the much slower evolution of the shape parameters. Consequently, at each instant the structure satisfies the static equilibrium equations, while the variables governing shape evolve according to the adaptation law. The resulting system therefore has no inertia-driven mechanical dynamics. However, the slow evolution of the configuration parameters along the manifold of static equilibria can be modeled as a nonlinear dynamical system, which we refer to as the shape dynamics of the system.

The proposed shape adaptation mechanism is modeled as an actuator in series with the torsional spring, which tunes its unstressed joint angle δ_i . We use a control policy introduced in our recent work [32], where the rate of change of δ_i is proportional to the current moment in the spring:

$$\dot{\delta}_i(t) = r_i (\alpha_{i+1}(t) - \alpha_i(t) - \delta_i(t)). \quad (5)$$

or equivalently

$$\dot{\delta} = \mathbf{R}(\mathbf{D}\alpha - \delta). \quad (6)$$

Note that this control policy is easy to implement because it is based on geometric data measured locally at each joint. Moreover, despite relying only on local information, this policy has an important feature that makes it capable of converging to unspecified funicular shapes: the actuators keep changing the shape of the column as long as it is subject to moments τ_i ; however, shape adaptation stops, and the system enters a stationary state whenever it reaches an equilibrium where $\tau_i = 0$. These *funicular states* are the subject of the next subsection.

The scalar parameter r_i in the control policy is referred to as the *relaxation constant*. For $r_i > 0$, (5) is a standard model of linear stress relaxation—hence the name of r_i . In contrast, $r_i < 0$ represents active behavior requiring external energy input, which can be implemented using an actuator.

2.4 Funicular configurations

Funicular configurations are the targets of shape control and are of central importance to this work. Hence, we provide a brief characterization of funicular shapes for a morphing column.

By eliminating the first term from the right-hand side of (4), we obtain the condition of funicular equilibrium, $\mathbf{NL} \sin(\tilde{\alpha}) = 0$. We focus on the generic case of $n_i, l_i \neq 0$, in which funicularity implies $\sin \tilde{\alpha}_i = 0$ for all i , which means either $\tilde{\alpha}_i = 0$ or $\tilde{\alpha}_i = \pi$. For each bar, the two respective scenarios correspond to compression and tension in bar i . Hence, there are altogether 2^m distinct funicular configurations, among which $\tilde{\alpha} = [\pi, \pi, \dots, \pi]$ corresponds to pure tension in the column, and $\tilde{\alpha} = [0, 0, \dots, 0]$ to pure compression. Figure 2 shows the four possible funicular states for a chain with $m = 2$ under $\mathbf{f}_1 = [-1, -1]^T$ and $\mathbf{f}_2 = [0, -1]^T$.

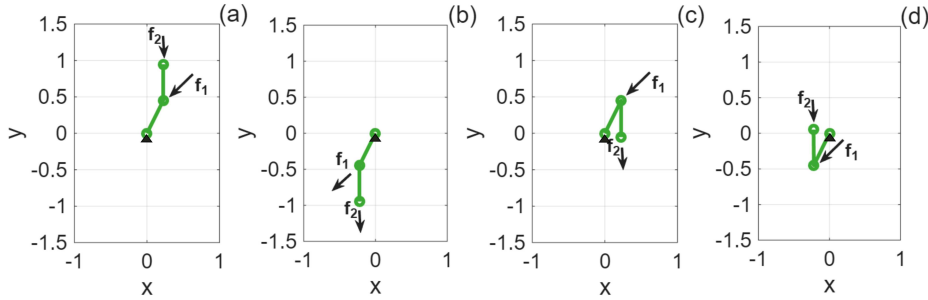


Fig. 2 An example of the four funicular configurations for $m = 2$: pure compression (a), pure tension (b), compression–tension (c), and tension–compression (d). A triangle marks the pinned node.

3 Shape evolution of the morphing column

3.1 Separation of time-scales in the limit of slow shape adaptation

Recall that the nodal loads are assumed stationary. The dynamics of a viscoelastic structure subject to stationary loads and slow shape adaptation has two separate time scales. The fast subsystem consists of the Newtonian dynamics of the angles α_i , while δ_i are frozen. Formally, this is a passive viscoelastic system. Conversely, the slow subsystem captures the quasi-static evolution of δ_i while the structure remains in static equilibrium.

As stationary loads are conservative and the fast subsystem is dissipative, initial oscillations of the system decay due to damping until the structure settles in an attractive equilibrium configuration (which is generally non-funicular, hence $\tau_i \neq 0$). Damping is essential to ensure convergence of the fast subsystem (in other words, to avoid flutter instability); nevertheless, the exact characteristics of damping are not important for the subsequent analysis, which focuses primarily on the slow dynamics.

The slow subsystem captures the slow transformation of shape along equilibrium states. A detailed characterization of the set of equilibrium states is given below in Sec. 3.2, and an evolution equation for shape is developed in Sec. 3.4. As we will see, the slow shape dynamics is determined by the interplay between shape adaptation (i.e., the evolution of δ_i) and moment-induced elastic deformation of the springs. In this context, successful shape control means convergence along the equilibrium set to the preferred funicular equilibrium.

Similarly to other slow-fast systems, slow shape adaptation may be interrupted by a sudden dynamic jump if the system drifts from the stable regime of the equilibrium set into a regime of unstable equilibria. Such jumps drive the system to a distant, stable equilibrium configuration, where slow shape evolution continues. Buckling, snap-through, and other related examples of *elastic instability* are well known in elasticity theory and structural design [1]. Normally, elastic instability is triggered by a slowly changing load parameter. In our case, shape change plays the role of the load parameter; nevertheless, the emerging instability phenomena are essentially the same. Sudden jumps due to instability invalidate the evolution equation for slow shape adaptation. Hence, the analysis of elastic stability is inevitable, and it is carried out in Sec. 3.3.

3.2 The equilibrium set

The equilibrium equations (4) are nonlinear in α due to the $\sin(\tilde{\alpha})$ term. Hence, solving the direct kinematics problem, i.e., finding α for prescribed loads and δ , is generally challenging, and the existence of a unique solution is not guaranteed. In contrast—as is usually the case for compliant structures—the inverse kinematics problem, i.e., finding δ for prescribed deformed shape α and loads, is straightforward. In particular, (4) yields

$$\delta = D\alpha - K^{-1}D^{-T}NL \sin \tilde{\alpha}. \quad (7)$$

where $\boldsymbol{\delta} = \mathbf{D}\boldsymbol{\alpha}$ corresponds to unstressed springs, and the second term on the right-hand side represents load-induced deformation, which is present whenever the system is in non-funicular equilibrium.

Multiplication by $\mathbf{K}^{-1}$ implies that smaller values of k_i lead to larger load-induced deformations, whereas highly stiff structures exhibit smaller deviations. This effect is illustrated by two examples, both representing a column with tip load.

Example 1 A structure composed of a single bar ($m = 1$) of length $l_1 = 1$, and vertical load $\mathbf{f}_1 = [0, -1]^T$.

These model parameters correspond to $\bar{\alpha}_1 = \pi/2$, and $n_1 = 1$. The equilibrium equation is of the form

$$\delta_0 = \alpha_1 - k_0^{-1} n_1 l_1 \sin \tilde{\alpha}_1 \quad (8)$$

Figure 3 shows δ_0 against $\tilde{\alpha}_1$ for three distinct values of k_0 . For high values ($k_0 \gg 1$), a linear relationship between the two quantities reflects vanishing load-induced deformations. For intermediate values ($k_0 > 1$), the equilibrium set is represented by a monotonic curve, whereas for low values ($k_0 < 1$), δ_0 is a non-monotonic (and thus non-invertible) function of α_1 .

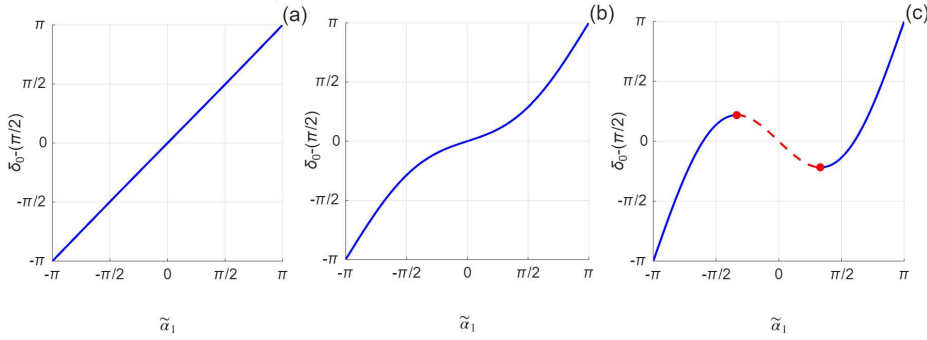


Fig. 3 Equilibrium set of Example 1 for $k_0 = 100$ (a), $k_0 = 1.5$ (b), and $k_0 = 0.5$ (c). Unstable equilibria (see Sec. 3.3) are marked by dashed curves, and fold points by solid circle markers.

Example 2 A structure with $m = 2$ bars, $\mathbf{l} = [1/2, 1/2]^T$, $\mathbf{f}_1 = 0$, and $\mathbf{f}_2 = [0, -1]^T$.

The model parameters correspond to $\bar{\alpha}_1 = \bar{\alpha}_2 = \pi/2$, and $n_1 = n_2 = 1$. This time, we illustrate the geometry of the equilibrium set in a slightly different form. Figure 4 depicts contour plots of $\delta_0(\alpha_1, \alpha_2)$ and $\delta_1(\alpha_1, \alpha_2)$ obtained by numerical evaluation of (7). Three cases are considered:

- **High stiffness** ($k_0 = k_1 = 100$, Fig. 4(a)): Both contour families are straight and evenly spaced, indicating that the mapping is approximately linear.
- **Intermediate stiffness** ($k_0 = k_1 = 1.5$, Fig. 4(b)): Contour lines are slightly curved nevertheless the two families of contour curves intersect transversally.

- **Low stiffness** ($k_0 = k_1 = 0.5$, as well as $k_0 = k_1 = 0.2$, Fig. 4(c,d)): The two contour families do not intersect transversally, indicating that the mapping from α to δ is non-invertible, and multiple configurations α may correspond to the same δ .

As we will see, non-invertibility will have a crucial effect on shape adaptation. It can be detected by considering the Jacobian of δ with respect to $\tilde{\alpha}$ obtained by differentiating (7):

$$\mathbf{J}_\delta = \mathbf{D} - \mathbf{K}^{-1} \mathbf{D}^{-T} \mathbf{N} \mathbf{L} \mathbf{C}_{\tilde{\alpha}}. \quad (9)$$

Specifically, the equation $\det \mathbf{J}_\delta = 0$ marks fold points of the functions, the existence of which is responsible for non-invertibility. These folds are indicated by red markers in Fig. 3(c), and by solid curves in Fig. 4(c,d).

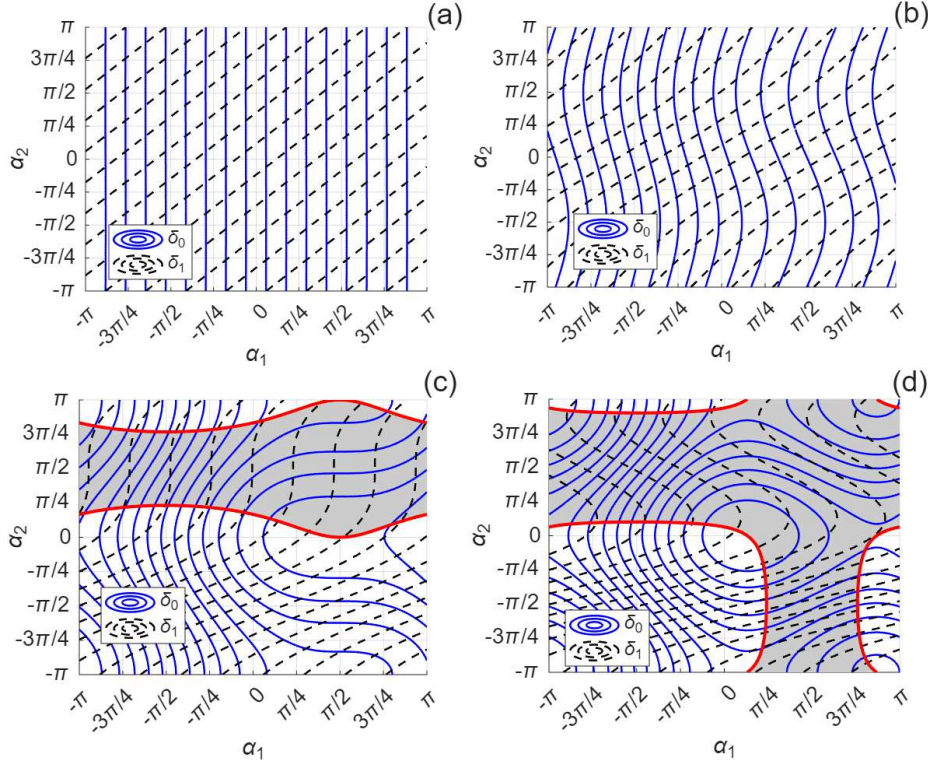


Fig. 4 Contour plots of $\delta_0(\alpha_1, \alpha_2)$ and $\delta_1(\alpha_1, \alpha_2)$ for Example 2 using different stiffness values: $k_i = 100$ (a), $k_i = 1.5$ (b), $k_i = 0.5$ (c), and $k_i = 0.2$ (d). Fold lines are depicted by thick red curves, and unstable equilibria (see Sec. 3.3) by a gray background.

3.3 Stability of equilibria under the fast dynamics

The fast subsystem is equivalent to the passive elastic column studied in Sec. 2.2. Stability analysis of this kind of system under conservative loads is a standard

m	k_i	k_i/m	m	k_i	k_i/m	m	k_i	k_i/m
1	1	1.0000	10	4.4766	0.4477	200	81.463	0.4073
2	1.309	0.6545	30	12.57	0.4190	400	162.52	0.4063
3	1.683	0.5610	50	20.673	0.4135	600	243.58	0.4060
4	2.0727	0.5182	70	28.778	0.4111	800	324.63	0.4058
5	2.4687	0.4937	90	36.883	0.4098	1000	405.69	0.4057

Table 1 Critical stiffness values $k_1 = k_2 = \dots = k_m$ for buckling if $n_i = 1$, $l_i = 1/m$, and $\tilde{\alpha}_i = 0$ for all i .

task in structural mechanics [24]. In quasi-static models, an equilibrium is stable if and only if it corresponds to a local minimum of the potential energy associated with the loads and internal springs. The Hessian of the potential energy P can be computed by differentiating the right-hand side of (4) once more, yielding

$$\mathbf{H}_P = -\mathbf{NLC}_{\tilde{\alpha}} + \mathbf{D}^T \mathbf{K} \mathbf{D} \quad (10)$$

A configuration α is stable if all eigenvalues of the symmetric matrix $\mathbf{H}_P$ evaluated at that state are positive, and unstable if it has a strictly negative eigenvalue. Critical (marginally stable) states can be found by using the condition that the smallest eigenvalue of $\mathbf{H}_P$ is 0, for which $\det \mathbf{H}_P = 0$ is a necessary condition. It is remarkable that

$$\mathbf{H}_P = \mathbf{D}^T \mathbf{K} \mathbf{J}_{\delta} \quad (11)$$

by (9) and (10). Hence, the critical points are also fold points. In the case of Example 1, the Hessian is scalar $\mathbf{H}_P = -n_1 l_1 \cos \tilde{\alpha}_1 + k_1$ implying that instability occurs if $\cos \tilde{\alpha}_1 > 0$, and $k_1 \leq n_1 l_1 \cos \tilde{\alpha}_1$. With the parameter values used in Fig. 3, a range of instability occurs if $k_0 < 1$. Unstable points within the equilibrium set are depicted in dashed line in Figure 3(c). Similarly, for Example 2 the Hessian is a 2 by 2 matrix, the eigenvalues of which have been evaluated numerically. Unstable equilibria have been marked by a grey background in Fig. 4(c,d).

For both examples, there is a region of instability around $\tilde{\alpha} = 0$, surrounded by a region of stability, which contains the pure tension configuration $\tilde{\alpha} = [\pi, \pi, \dots, \pi]^T$. Weyl's inequality [12] implies that this picture is generic: increased values of $n_i l_i \cos \tilde{\alpha}_i$ for all i always make the eigenvalues of the Hessian smaller. Thus, the existence of a region of instability is equivalent to instability of the $\tilde{\alpha} = 0$ configuration.

Figure 5 shows a contour plot of the smallest eigenvalue of the Hessian for Example 2, evaluated at the configuration $\tilde{\alpha} = 0$, as a function of the spring stiffness values k_i . Two related examples with different values of $\mathbf{n}$ are also presented in the figure. In each case, instability occurs below critical stiffness values marked by dashed curves. Additionally, Table 1 shows critical values of k_i for different m values, at which the stability of the $\tilde{\alpha} = 0$ configuration is lost, if all k_i are identical, $l_i = 1/m$, and $n_i = 1$ for all i . Each of these examples corresponds to discrete columns of unit total length subject to a tip load. It is straightforward to show that the critical value of k_i/m in the limit of $m \rightarrow \infty$ corresponds to the critical bending stiffness for Euler buckling of an elastica column under tip load.

If the springs are very stiff ($k_i \gg 1$ for all i), then the second term of (10) dominates over the first one, implying stability. This observation reflects the well-known result that stiff elastic structures do not buckle. In contrast, for very low stiffness,

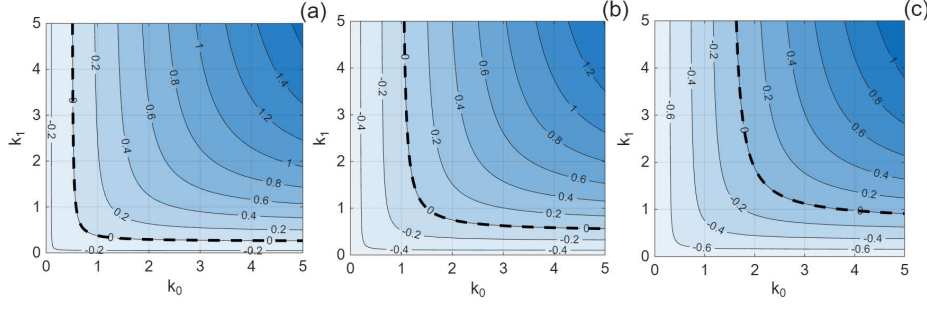


Fig. 5 Contour plot of the smallest eigenvalue of $\mathbf{H}_p$ for Example 2 as a function of stiffness parameters k_1 and k_2 (panel (b)), and similar plots for modified cumulative load resultant values including $n_1 = n_2 = 1/2$ (a), and $n_1 = n_2 = -3/2$ (c).

the first term dominates, implying stability if $\cos \tilde{\alpha}_i < 0$ for all i , and instability otherwise. Again, we recover a well-known fact: flexible chains are unstable under (partial) compression, but stable under tension.

3.4 The evolution equation of shape

If the system has settled onto the equilibrium set, and the equilibrium set is stable (see Sec. 3.3), then the shape of the column changes slowly along that set. We now develop the corresponding evolution equation in the state variables α_i . Physical parameters include the integer m , as well as l_i , $\mathbf{f}_i$, k_i and r_i , all of which are stationary. As we perform quasi-static analysis, angular velocities, nodal masses, and the amount of damping are not relevant. The equations of slow shape adaptation can be formulated by combining the evolution law of the unstressed angles (6) with the expression (7) of inverse kinematics:

$$\dot{\delta} = \mathbf{R}\mathbf{K}^{-1}\mathbf{D}^{-T}\mathbf{N}\mathbf{L}\sin \tilde{\alpha}.$$

Then, further transformation using the identities $\dot{\alpha} = \dot{\tilde{\alpha}}$, $\dot{\delta} = \mathbf{J}_{\delta}\dot{\tilde{\alpha}}$, and the expression (9) of the Jacobian yields

$$\dot{\tilde{\alpha}} = (\mathbf{D} - \mathbf{K}^{-1}\mathbf{D}^{-T}\mathbf{N}\mathbf{L}\mathbf{C}_{\tilde{\alpha}})^{-1}\mathbf{R}\mathbf{K}^{-1}\mathbf{D}^{-T}\mathbf{N}\mathbf{L}\sin \tilde{\alpha} \quad (12)$$

which can also be expressed with the help of (10) as

$$\dot{\tilde{\alpha}} = \mathbf{H}_P^{-1}\mathbf{D}^T\mathbf{R}\mathbf{D}^{-T}\mathbf{N}\mathbf{L}\sin \tilde{\alpha} \quad (13)$$

Both forms represent complete evolution equations for the shape dynamics in terms of $\tilde{\alpha}$ as state variables. The form (13) reflects that the evolution equation becomes undefined if $\mathbf{H}_P$ is singular, i.e., at fold points. We remind the reader that the quasi-static model becomes invalid at those points due to instability under the fast dynamics.

An important implication of the quasi-static formulation is that the shape evolution equation does not depend explicitly on the external load vectors or on the funicular geometry. Conversely, bar lengths l_i and the cumulative loads n_i enter the equation. This simplification is characteristic of the quasi-static approach, and

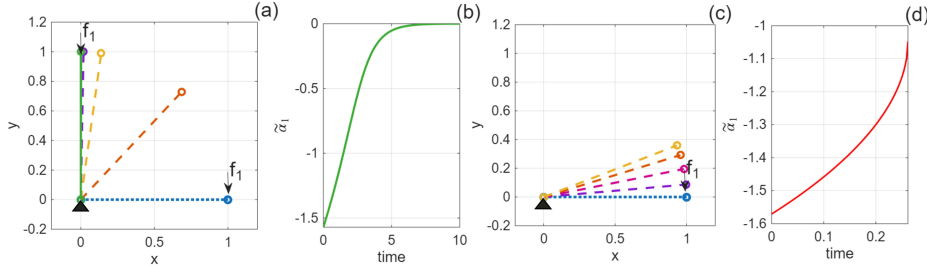


Fig. 6 Numerical simulation of Example 1: Snapshots of convergent shape dynamics for $k_1 = 1.5$ and $r_1 = -0.5$ (a), and the corresponding time evolution of $\tilde{\alpha}_1$ (b). Panels (c,d): similar plots for non-convergent shape evolution for $k_1 = 0.5$.

it does not extend to the full dynamic shape adaptation problem [10], where the response depends on the complete load distribution and the evolution of geometry due to inertial effects.

4 Numerical investigation of shape dynamics

We present numerical simulation results of the evolution equation to illustrate some features of the shape adaptation process. The differential equation (12) is integrated using MATLAB's ode45 solver. The simulations focus on the influence of the bending stiffness k_i .

For Example 1, it has been pointed out that positive values of r_0 (i.e., passive stress relaxation) induce convergence to the tension-only funicular shape $\tilde{\alpha}_1 = \pi$. Prior numerical results using the full equations of Newtonian dynamics suggest that the compression-only shape $\tilde{\alpha}_1 = 0$ can be stabilized by using negative r_0 .

This observation is illustrated by numerical simulations with the initial configuration $\tilde{\alpha}_1 = -\pi/2$. For $k_0 = 1.5$, $r_0 = -0.5$, the simulation reveals convergence to the funicular configuration $\tilde{\alpha}_1 = 0$ (Fig. 6(a,b)), as expected. Results for $k_0 = 0.5$, $r_0 = -0.5$ are depicted in Fig. 6(c,d). The chosen stiffness value is below the critical stiffness $k_0 = 1$ (Table 1). Initially, the system approaches the funicular state $\tilde{\alpha}_1 = 0$ through stable equilibria, until it hits a fold point at $\tilde{\alpha}_1 = -2\pi/3$, where the quasi-static simulation breaks down due to singularity of $\mathbf{H}_P$. The subsequent behavior of the active column model cannot be simulated with the quasi-static evolution equation; however, it can be understood by looking at the equilibrium set. Fig. 7 illustrates the shape evolution process, which involves episodes of quasi-static shape evolution repeatedly interrupted by jumps of $\tilde{\alpha}_1$ at two fold points. The periodic shape evolution fails to reach the funicular state, which is not surprising, as it belongs to the unstable portion of the equilibrium set.

Example 2 shows highly similar behavior. In this case, the slow shape evolution defines a two-dimensional vector field, which can be visualized directly in the $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ plane. Figure 8(a) exemplifies the case of positive relaxation constants with parameters $\mathbf{k} = [1.5, 1.5]^T$ and $\mathbf{r} = [0.5, 0.5]^T$. In line with expectations, all initial shapes converge to the tension-only configuration $\tilde{\alpha} = [\pi, \pi]^T$. Figure 8(b) illustrates the case of negative relaxation constants $\mathbf{r} = [-0.5, -0.5]^T$ combined with $\mathbf{k} = [1.5, 1.5]$, in which case the numerically evaluated vector field suggests

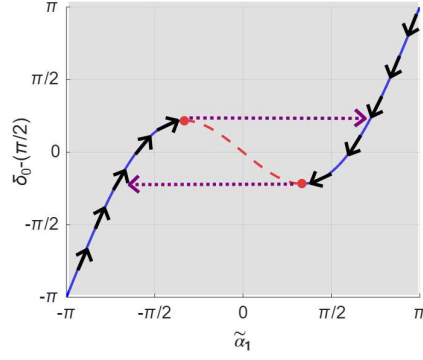


Fig. 7 Oscillating shape dynamics of Example 1 for $k_0 = 0.5$, $r_0 = -0.5$, in which case the funicular configuration $\tilde{\alpha}_1 = 0$ is unstable. Arrows and dotted lines indicate the direction of slow shape evolution and rapid snapping, respectively.

that almost all initial conditions converge to $\tilde{\alpha}_i = [0, 0]^T$. In addition, Fig. 9(a,b) shows one particular time history.

In contrast, for the same $\mathbf{r}$ combined with lower stiffness values $\mathbf{k} = [0.5, 0.5]^T$ below the buckling limit, the configuration $\tilde{\alpha}_i = [0, 0]^T$ is within the unstable regime, whose boundaries are highlighted by red curves (Fig. 8(c)). Similarly to Example 1, shape evolution hits a fold point where the quasi-static model becomes ill-defined, and the simulation breaks down, as shown in Fig. 9(c,d). Subsequent shape evolution can be deduced by combining the vector field in Fig. 8(c) with the contour-plot representation of the equilibrium set in Fig. 4. The latter allows us to deduce the termination point of the jumps: during a rapid jump, the stress-free angles δ_0, δ_1 remain unchanged, which means that the initial and terminal points of the jump must lie at the intersection of the same contour curves. This technique has been used to demonstrate oscillatory behavior with episodes of slow shape evolution interrupted by rapid snap-through events (Fig. 10). We again conclude that the system does not reach the expected funicular configuration.

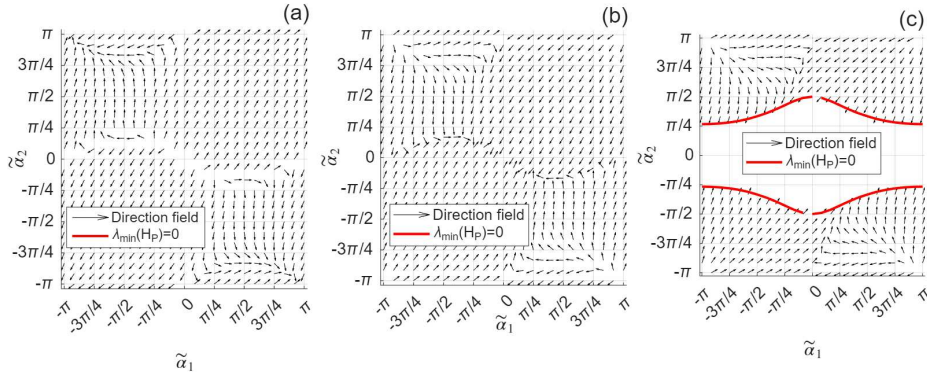


Fig. 8 Vector-flow representation of the shape dynamics for three cases of Example 2. (a): positive relaxation constants $r_i = 0.5$; (b): negative relaxation constants $r_i = -0.5$ combined with $k_i = 1.5$ above the critical level; and (c): negative relaxation constants $r_i = -0.5$ combined with $k_i = 0.5$ below the critical level

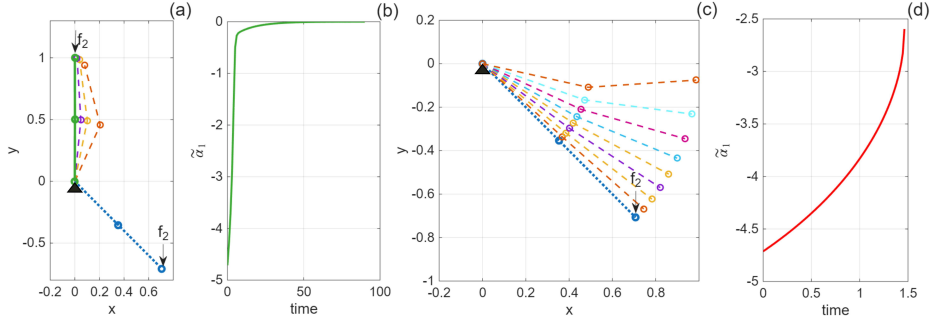


Fig. 9 Numerical simulation of a two-bar cantilever structure with $\mathbf{n} = [1, 1]^T$, $\mathbf{l} = [1/2, 1/2]^T$ from initial state $\tilde{\alpha} = [-3\pi/4, -3\pi/4]^T$ (a,b): snapshots and time history of $\tilde{\alpha}_1$ for convergent dynamics with $k_i = 2$ and $r_i = -0.5$. (c,d): snapshots and time history of $\tilde{\alpha}_1$ for unstable shape dynamics with $k_i = 0.5$.

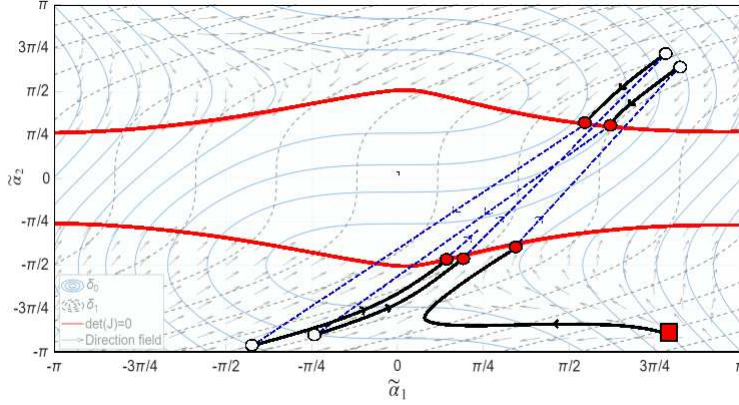


Fig. 10 Oscillatory shape evolution of Example 2 with stiffnesses $k_i = 0.5$ below the critical level for buckling. A square marks the initial configuration; filled and empty circles represent initial and terminal points of rapid jumps, which are depicted as dashed line segments.

5 Linear stability analysis of the funicular shape under the slow dynamics

Numerical simulation gives an impression about the convergence of shape dynamics to funicular forms, however a more complete understanding requires systematic stability analysis. In this section we perform linear stability analysis of funicular configurations under the slow shape adaptation dynamics. Linear stability under the slow and under the fast dynamics together ensure local convergence from nearby configurations.

5.1 The tension-only configuration

The pure tension configuration $\tilde{\alpha} = [\pi, \pi, \dots, \pi]^T$ always features linear stability (i.e., exponential convergence) if a column is flexible ($k_i = 0$) or if it is subject to stress relaxation ($k_i, r_i > 0$). This result follows from the exponential stability theorem [17, Theorem 4.5].

5.2 The compression-only configuration

The governing equation (13) is linearized at $\tilde{\alpha} = 0$ via the approximations $\mathbf{C}_{\tilde{\alpha}} \approx \mathbf{U}$, $\sin \tilde{\alpha} \approx \tilde{\alpha}$, yielding

$$\dot{\tilde{\alpha}} = \mathbf{H}_P^{*-1} \mathbf{D}^T \mathbf{R} \mathbf{D}^{-T} \mathbf{N} \mathbf{L} \tilde{\alpha} \quad (14)$$

Here $\mathbf{H}_P^* = -\mathbf{N} \mathbf{L} + \mathbf{D}^T \mathbf{K} \mathbf{D}$ is the Hessian of the potential energy at $\tilde{\alpha} = 0$. Eq. (14) is of the form $\dot{\tilde{\alpha}} = \mathbf{J} \tilde{\alpha}$, and linear stability corresponds to all eigenvalues of the Jacobian matrix $\mathbf{J}$ having negative real parts.

For Example 1, we have seen that stability of the fast dynamics requires $k_0 > 1$. At the same time, the linearized slow dynamics has a unique eigenvalue $\lambda = -\frac{r_0}{k_0-1}$ from which we obtain $k_0 > 1$ and $r_0 < 0$ as a combined stability condition. This result can be generalized for $m > 1$ as

Theorem 1 *If r_i are equal and negative for all i , and the compression-only configuration $\tilde{\alpha} = 0$ possesses linear stability under the fast dynamics then it is also linearly stable under the slow dynamics.*

Proof If all r_i are equal, then $\mathbf{J}$ can be rewritten as $\mathbf{J} = r_i \mathbf{H}_P^{*-1} \mathbf{D}^T \mathbf{D}^{-T} \mathbf{N} \mathbf{L}$. Eigenvalues of matrix products are invariant under cyclic permutation, thus we can investigate $\mathbf{J}' = r_i \mathbf{D}^{-T} \mathbf{N} \mathbf{L} \mathbf{H}_P^{*-1} \mathbf{D}^T$ instead of $\mathbf{J}$. Similarity transformations also leave eigenvalues unchanged, hence we further modify $\mathbf{J}'$ as $\mathbf{J}'' = r_i \mathbf{N} \mathbf{L} \mathbf{H}_P^{*-1}$. $\mathbf{H}_P^{*-1}$ is positive definite due to the assumption of stability under the fast dynamics. Hence, it can be decomposed as $\mathbf{H}_P^{*-1} = \mathbf{H}_P^{*-1/2} \mathbf{H}_P^{*-1/2}$; then one more cyclic permutation yields

$$\mathbf{J}''' = r_i \mathbf{H}_P^{*-1/2} \mathbf{N} \mathbf{L} \mathbf{H}_P^{*-1/2}.$$

$\mathbf{J}'''$ is congruent to the diagonal, positive definite matrix $\mathbf{N} \mathbf{L}$, hence by Sylvester's law of inertia, all eigenvalues of $\mathbf{J}'''$ are positive and real. The same goes for $\mathbf{J}$, completing the proof.

Systematic eigenvalue tests of Example 2, as well as of other examples with higher m values, suggest that the statement of Theorem 1 is also true in the more general case of unequal $r_i < 0$ values. The general statement remains an unproven conjecture; however, in Sec. 6, we demonstrate the related property of non-local convergence to $\tilde{\alpha} = 0$ in the general case.

5.3 Mixed configurations

One can also investigate the stability of mixed (tension-compression) funicular shapes $\tilde{\alpha}^*$ in a similar manner. The linearized equations use the approximations $\mathbf{C}_{\tilde{\alpha}} \approx \mathbf{C}_{\tilde{\alpha}^*}$, $\sin \tilde{\alpha} \approx \mathbf{C}_{\tilde{\alpha}^*}(\tilde{\alpha} - \tilde{\alpha}^*)$ where $\mathbf{C}_{\tilde{\alpha}^*}$ is a diagonal matrix with all diagonal entries equal to ± 1 :

$$\dot{\tilde{\alpha}} = \mathbf{H}_P^{*-1} \mathbf{D}^T \mathbf{R} \mathbf{D}^{-T} \mathbf{N} \mathbf{L} \mathbf{C}_{\tilde{\alpha}^*} (\tilde{\alpha} - \tilde{\alpha}^*) \quad (15)$$

This equation can be used for linear stability analysis of specific examples via numerical eigenvalue analysis.

In an earlier work, the authors found that a funicular state with a mixture of bars under compression and tension can often be stabilized if the signs of r_i follow an equivalent pattern ($r_i > 0$ if bar $i + 1$ is under tension, and $r_i < 0$ if it is under compression). However, this is not always the case. As a counterexample, we show numerically obtained stability charts for the four funicular configurations of Example 2 for $k_0 = k_1 = 2$. The stability regions of the slow dynamics—depicted as white regions in Fig. 11—cover the appropriate quadrants of the parameter plane in three cases, including the tension-only and compression-only configurations, but the last stability diagram shows a different pattern.

We note an implication of the irregularity of Fig. 11(d): there is a range in the (r_0, r_1) plane where two funicular shapes are simultaneously stable, hence the evolution of the system depends on its initial state. At the same time, there is another range of parameter values where none of the funicular shapes is stable, and thus shape evolution continues indefinitely without convergence to a stationary shape.

We do not develop a general linear stability condition for mixed configurations. However we prove the following statement:

Theorem 2 *If r_i are equal and negative for all i , then mixed funicular configurations are linearly unstable under the slow dynamics.*

Proof The proof is highly similar to that of Theorem 1. This time we end up with the expression

$$\mathbf{J}''' = r_i \mathbf{H}_P^{*-1/2} \mathbf{N} \mathbf{L} \mathbf{C}_{\tilde{\alpha}^*} \mathbf{H}_P^{*-1/2}.$$

and Sylvester's law of inertia implies that $\mathbf{J}$ has as many positive eigenvalues as the number of $\tilde{\alpha}_i = \pi$ values.

The same statement appears to be true in the general case of unequal $r_i < 0$ values; however, this observation remains unproven.

6 Global convergence of slow dynamics

The linear stability of funicular configuration, ensures local convergence. In general, global convergence is a stronger property - and more challenging to prove. Now we establish some elementary results relevant to the pure tension configurations, where global convergence occurs without active control. Then the same ideas are extended to the case of pure compression. In particular, it is proven that compressed funicular shapes attract almost all initial conditions if $r_i < 0$ for all i , unless elastic instability occurs owing to low spring stiffness. Mixed compression-tension configurations are not addressed in this section.

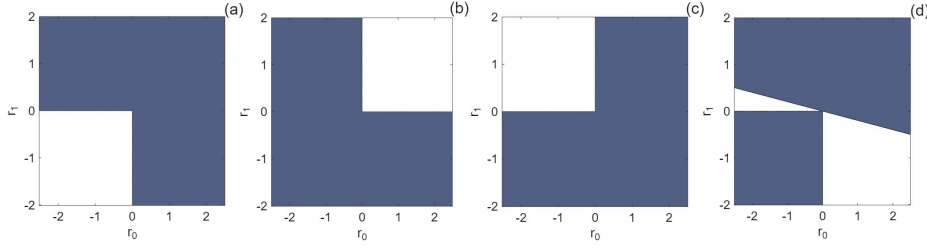


Fig. 11 Stability chart of Example 2, $k_i=2$ as function of (r_0, r_1) for different funicular configurations: $\tilde{\alpha} = [0, 0]^T$ (a), $\tilde{\alpha} = [\pi, \pi]^T$ (b), $\tilde{\alpha} = [0, \pi]^T$ (c) and $\tilde{\alpha} = [\pi, 0]^T$ (d). White regions represent stability.

6.1 The case of pure tension

Recall the well-known behavior of cables and chains fixed at one endpoint: in the presence of an appropriate damping mechanism, those structures respond to constant external loads by transient dynamic motion, settling at the pure tension funicular shape $\tilde{\alpha} = [\pi, \pi, \dots, \pi]^T$. This mechanism is robust and does not require active shape adaptation. Within the context of quasi-static shape morphing, essentially the same behavior can be observed by setting $k_i > 0$, $r_i > 0$: the structure initially resists loads via a combination of bending and internal forces due to its bending stiffness characterized by k_i . However, passive relaxation characterized by positive r_i values gradually removes bending action, and thus the structure converges to the same shape as a flexible cable.

Physical intuition suggests that all initial conditions converge to the tension-only funicular shape. Theoretically, convergence to other funicular equilibria is also possible; nevertheless, those are unstable, hence they attract only a vanishingly small (measure 0) set of initial states.

6.2 The case of pure compression

We have previously seen examples where robust convergence to the compression-only shape occurred for $r_i < 0$. The viability of this approach is now formally proved in the form of a theorem and a corollary:

Theorem 3 *If $r_i < 0$ for all i , furthermore the pure compression configuration $\tilde{\alpha} = 0$ is linearly stable under the fast dynamics then all initial conditions converge to a funicular configuration under the slow dynamics.*

Proof First, we investigate the energy balance of the system while evolving under the slow dynamics. The external loads and the springs are conservative, hence they preserve mechanical energy. Dampers dissipate energy; however, this is negligible in the quasi-static limit. According to (5), $r_i < 0$ means that the actuators shift the stress-free angle δ_i against the moment τ_i exerted by the torsional spring, thereby increasing the total potential energy of the system:

$$\dot{P} = \sum_{i=0}^{n-1} \tau_i \dot{\delta}_i = \sum_{i=0}^{n-1} -r_i k_i (\alpha_{i+1}(t) - \alpha_i(t) - \delta_i(t))^2 \geq 0 \quad (16)$$

Hence, the potential energy *monotonically increases over time*, and $\dot{P} = 0$ if $\alpha_{i+1}(t) - \alpha_i(t) - \delta_i(t) = 0$ for all i , which means that the structure is in one of its funicular configurations.

Next we show that the funicular configurations are the only stationary points of P_{eq} where its gradient vanishes. Instead of studying the expression (3) for P directly, we use the evolution equation of shape dynamics derived earlier. The total time derivative of P can be expressed by the chain rule and (12) as

$$\dot{P} = \text{grad}(P_{eq})\dot{\tilde{\alpha}} \quad (17)$$

$$= \text{grad}(P_{eq})\mathbf{H}_P^{-1}\mathbf{D}^T\mathbf{R}\mathbf{D}^{-T}\mathbf{N}\mathbf{L}\sin\tilde{\alpha} \quad (18)$$

where grad means gradient, and P_{eq} is the potential energy P evaluated along the equilibrium surface as a function of $\tilde{\alpha}$. Eq. (18) implies that $\text{grad}(P_{eq})$ cannot be zero except for the funicular configurations (where $\dot{P} = 0$), as well as for those configurations where $\mathbf{H}_P$ is singular, hence the slow dynamics is undefined. However stability of equilibria under the fast dynamics implies that $\mathbf{H}_P$ is non-singular. We conclude that the *only stationary points* of P_{eq} along the equilibrium surface are the funicular configurations.

Among the funicular configurations, those which have $\tilde{\alpha}_i = \pi$ for some i are *not local maximum points* of P because an infinitesimal perturbation of $\tilde{\alpha}_i$ with all other angles unchanged, makes the first term of (3) larger by increasing $\cos\tilde{\alpha}_i$, whereas the second term also increases from 0 to a positive value due to the loss of perfect funicularity.

At the same time, P_{eq} is periodic in $\tilde{\alpha}_i$, i.e., it is defined over a compact manifold. According to the Extreme Value Theorem, one of its stationary points must be a global maximum. We conclude that $\tilde{\alpha} = 0$ is the *global maximum point* of P_{eq} , and simultaneously the *only local maximum point*.

Let P_{eq}^* denote the potential energy for $\tilde{\alpha} = 0$. Then $P_{eq} - P_{eq}^*$ is non-positive, but its derivative with respect to time is non-negative. La Salle's invariance principle with $P_{eq} - P_{eq}^*$ as Lyapunov function implies that the shape dynamics must converge to one of the funicular configurations, which are the only invariant sets of the slow dynamics satisfying $\dot{P} = 0$. This finding completes the proof.

Theorem 3 shows that all trajectories converge to a funicular configuration; however, it does not show which funicular state they end up at. At the same time, Theorem 2 implies that all funicular configurations, except $\tilde{\alpha} = 0$, are linearly unstable, meaning that their stable manifolds form a set of measure zero. Consequently,

Corollary 1 *If all r_i are equal and negative, furthermore the pure compression configuration $\tilde{\alpha} = 0$ is linearly stable under the fast dynamics then almost all initial conditions (except for a set of measure 0) converge to it under the slow dynamics.*

In the general case of unequal r_i values, the same statement appears true, but this conjecture is left unproven.

We note that if spring are soft, violating the condition of stability under the fast dynamics, then (16) is still true. However in this case the slow dynamics is repeatedly interrupted by fast jumps, where energy dissipation by dampers becomes significant. Thus oscillatory shape evolution with zero energy balance becomes possible.

7 Conclusions

Inspired by the efficiency of funicular systems and their high sensitivity to load variations, slow shape adaptation of a pin-jointed multibody model of a shape-morphing column under quasi-stationary nodal loads was investigated. We demonstrated the viability of restoring funicular configurations through feedback control and actuation using a simple, local control law. We developed reduced equations of slow shape dynamics along an equilibrium manifold. We emphasize that slow evolution remains physically meaningful as long as the equilibrium states are not prone to elastic instability.

We used two simple examples to characterize the role of load-induced deformations, with emphasis on the emergence of multiple equilibrium configurations corresponding to the same internal state of the actuators—a typical behavior associated with buckling instability. It has been demonstrated that, for stiffness values below the buckling limit, the compression-only funicular state becomes unreachable through quasi-static shape evolution.

Local attractivity of the funicular configuration $\tilde{\alpha} = 0$ was studied by linearizing the slow shape evolution equations. For $m = 1$, an explicit stability boundary was found, whereas for more complex setups, numerical eigenvalue analysis was used to test stability. We also found an analytical condition for non-local convergence to the pure-compression funicular configuration using La Salle's invariance principle.

Our results are consistent with previous observations obtained using the full model of fast shape dynamics. However, owing to the simpler equations, the quasi-static analysis provided stronger results regarding robust convergence to the target state.

Nevertheless, future work is needed to improve the practical applicability of the presented theoretical results. Questions to be addressed include reliability, fault tolerance, the role of time delay, robustness under imperfect sensor data and actuation errors, optimal control laws, and shape morphing of structural systems with more complex kinematic constraints, among others.

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Conflict of interest

The authors declare that they have no conflict of interest.

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