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The Long-Run Evolution of Wealth Inequality under Increasing Investment Freedom, A Multi-Agent Simulation Study

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Abstract

This paper studies how access to capital markets affects wealth inequality. Most studies assume that expanding investment access improves outcomes, but they do not ask when this effect may weaken or reverse. This paper addresses that gap by focusing on the role of household consumption behavior.

We develop a simulation model in which 1,000 agents start with identical wealth and income but differ in how much they consume. Investment Freedom determines the share of the population that can invest in capital markets. To ensure that the results are not driven by a specific random path, we run the model 26 times under different market conditions, creating independent datasets. The results show a clear behavioral threshold. When annual consumption is below about 7 percent of wealth, expanding investment access reduces inequality. Above this level, the effect becomes weaker and may even reverse. This pattern appears consistently across all simulations and across three different analytical approaches. We also find that higher investment access leads to more stable inequality outcomes across different market histories.

The contribution of the paper is to show that the effect of investment access on inequality is conditional, not automatic. It depends on how households use their wealth. By identifying a simple threshold and separating an efficient range from a flat range, the paper adds a new mechanism to the literature on financial openness and inequality.

Keywords

Wealth inequality, Investment freedom, Financial access, Multi-agent simulation, Consumption behavior, Capital market access

JEL Classification

D31, G11, O15, C63, E21

1 Introduction

Over the past several decades, opening up capital markets has become a key part of economic reform worldwide. When countries reduce barriers to investment, both within their borders and across them, more people gain access to financial markets. The link between this *Investment Freedom* and economic growth is well established. What remains unclear is how *Investment Freedom* affects wealth inequality over the long run.

A central question in development economics is when access to capital markets expands, how this affects the long-term distribution of wealth. Empirical studies provide mixed results. One reason is that countries differ in their institutions, tax systems, and levels of financial development. These differences make it difficult to isolate the specific role of market participation from other factors that influence wealth accumulation.

This paper studies how expanding investment access affects wealth inequality when agents differ only in their consumption behavior and in random investment outcomes. All agents start with the same initial conditions. This allows us to isolate the dynamic effect of market participation on wealth inequality over time. Differences in wealth then emerge only through compound returns and through the way agents reinvest their gains.

This paper tackles the question through theory and multi agent simulation. We build a formal model of wealth accumulation where everyone starts on equal footing: same initial wealth, same labor income, and same ability to invest. *Investment Freedom* is modeled as a policy choice that determines what fraction of the population has access to capital markets. Beyond this access and how much people choose to consume; we assume no other differences between individuals. We hold constant things like institutions, income patterns, and investment skills[1].

The distribution of wealth evolves through an evolutionary process driven by labor income, consumption, and returns on investment. We implement the model using a multi agent simulation that tracks the evolution of many individuals' wealth over time. To verify robustness, we run the entire simulation 26 times with different random seeds, generating independent datasets with different market histories. The wealth inequality (measured by the *Gini* index), emerge naturally from these individual paths rather than being built into the model from the start.

Our analysis shows an important behavioral threshold that is robust across 26 independent simulation runs with different random return sequences. Higher φ reduces inequality when consumption propensities (α, β) are low, specifically when α is below 7%. When consumption rates become higher, this effect becomes weaker and may even reverse. This pattern shows that the distributional effect of market access depends strongly on consumption and saving decisions. The consistency of these findings across multiple independent random market histories demonstrates that the results reflect structural features of the model rather than particular random draws. These results come from the dynamics of compound returns and reinvestment. They do not arise from differences in talent, or institutional advantages. Instead,

[1] A justification for the model assumption will be presented later in the article.

the results show that how much people consume from their wealth and from their capital gains plays a central role in determining whether wider access to investment leads to more equality or more inequality over time.

This paper contributes to the literature on wealth dynamics in three ways. First, we treat investment access as a policy parameter φ that determines how many agents can invest, instead of treating it as a broad institutional variable. Second, we study how market access interacts with consumption behavior. The results show that the effect on inequality depends strongly on how much people save and reinvest. Third, we implement the model using large scale simulations, generating about 60 million observations across 600 parameter combinations. This allows us to trace the evolution of inequality over 100 periods, which would be difficult to analyze analytically. By focusing on these underlying dynamics, our analysis complements existing empirical studies and helps explain why the same *Investment Freedom* policies may lead to different inequality outcomes in different behavioral environments.

The link between investment access and inequality works through three simple channels. First, *Investment Freedom* determines what share of agents can earn returns on capital. Second, these returns accumulate over time through reinvestment. This creates compound wealth growth for those who invest. Third, consumption behavior matters. In the model this is captured by the propensity to consume from wealth α and from capital gains β. These parameters determine how much of the gains are reinvested and how much are consumed. When consumption from wealth remains moderate, extended access to investment spreads compound returns across a larger part of the population. This can reduce the concentration of wealth at the top. However, when consumption rates are high, a larger share of gains is spent instead of reinvested. In that case the equalizing effect becomes weaker and may even reverse.

This study is a theoretical simulation exercise. We refer to empirical datasets such as the World Inequality Database wealth distributions, the Heritage Investment Freedom indices, and historical S&P 500 return data mainly as motivation and for parameter calibration. We do not analyze these datasets directly. The contribution of the paper is to clarify theoretical mechanisms through controlled computational experiments rather than to measure real world effects.

The rest of the paper is organized as follows. We first present the formal model and its assumptions, then describe the simulation design and results. The final section discusses limitations and suggests directions for future research.

2 Literature Review

The relationship between financial *Investment Freedom* and economic inequality has generated extensive academic interest, particularly in the context of global capital mobility and institutional differences. Investment freedom, defined as the absence of barriers to capital allocation across sectors and borders, is a key dimension of financial openness. It has been linked to higher economic growth, foreign direct investment, and financial development (Gwartney et al., 2022; Tag and Degirmen, 2022; Shokhrukh et al., 2023). However, the distributional consequences of such *Investment Freedom* remain theoretically ambiguous and empirically contested.

Supporters of *Investment Freedom* argue that wider access to capital markets can help create a more equal distribution of wealth. The idea is that when more households are able to invest, more people can benefit from returns on assets

(Herrera-Echeverri et al., 2014; Feldmann, 2017). In this view, when investment opportunities are not limited to a small elite, compound returns can spread across the population instead of remaining concentrated among those who already hold wealth. Empirical studies that support this view often use panel data across countries. These studies usually find that higher levels of economic freedom, including investment freedom, are associated with lower levels of long-term inequality (Lawson, 2006; Gwartney et al., 2022).

Conversely, a growing body of research highlights the potential for *Investment Freedom* to reinforce or exacerbate wealth concentration. Scholars such as Furceri and Loungani (2018) and Bumann et al. (2013) document that capital account openness can increase inequality, particularly in countries with weak institutions or unequal access to investment vehicles. Piketty (2014) and Milanovic (2016) further argue that returns to capital tend to exceed returns to labor in liberalized economies, implying that *Investment Freedom* may disproportionately benefit wealthier households that are better positioned to accumulate capital. The empirical evidence is mixed and often context dependent, with some studies finding inequality-reducing effects of *Investment Freedom* only under strong institutional constraints (Bumann and Lensink, 2013; De Haan and Sturm, 2017; Chinn and Ito, 2006). Studies have also investigated how macroeconomic conditions impact specific facets of social inequality, such as gender-based unemployment patterns in European countries (Faďoš & Bohdalová, 2019).

Despite this rich literature, relatively few studies develop formal theoretical models that directly link *Investment Freedom* to long-term wealth distribution dynamics. Most existing work relies on empirical associations or static frameworks, offering limited insight into the behavioral and temporal mechanisms that drive the observed patterns. Some recent contributions emphasize the role of institutions (Islam, 2018), policy sequencing (Khan et al., 2021), or access to financial tools (Lusardi and Mitchell, 2023), but stop short of modeling the dynamic interaction between investment policy, capital returns, and inequality under conditions where agents start identically but differ in behavior.

Agent-based and simulation approaches offer a complementary perspective. Recent work using heterogeneous agent models demonstrates how individual-level decisions aggregate into macroeconomic patterns (Riccetti et al., 2015; Assenza et al., 2015). Studies in this tradition show that wealth inequality can emerge endogenously from interactions between simple behavioral rules and market structures, even without imposing ex ante differences in ability or endowments (Gallegati et al., 2017; Dosi et al., 2013). Shimizu et al. (2024) use simulation to examine how wealth visibility affects investment behavior and economic mobility, finding that information about others' wealth influences individual investment decisions. Similarly, Bruno (2024) explores macroeconomic determinants of happiness through agent-based modeling, demonstrating how inequality affects subjective wellbeing through reference group effects. These computational approaches complement traditional analytical methods by allowing researchers to trace how micro-level heterogeneity in behavior translates into macro-level distributional outcomes over extended time horizons.

This paper contributes to filling the gap between theoretical and computational approaches by proposing a formal model based on individual behavior. Wealth accumulation follows a stochastic process, and *Investment Freedom* is modeled as a policy parameter that determines market access. Agents are identical in initial conditions but differ in their consumption behavior and exposure to investment returns, enabling a clean identification of how investment access

shapes long-run wealth outcomes. The model isolates the role of behavioral responses, particularly propensities to consume from capital gains and initial wealth, in mediating the effects of *Investment Freedom* on inequality. In doing so, it bridges theoretical and empirical literatures and offers a tractable framework for understanding why financial openness may yield divergent inequality trajectories across countries.

By formally modeling these mechanisms through simulation, this study advances the literature on financial globalization, economic distribution, and the behavioral underpinnings of *Investment Freedom* outcomes (Klein and Olivei, 2008; Bekaert et al., 2005). It offers an alternative to purely institutional or structural explanations and highlights the importance of intertemporal decision making in shaping wealth inequality in open economies.

2.1 Conceptual Framework: Defining Investment Freedom

In this paper, Investment Freedom φ represents the market participation rate. In simple terms, it is the share of the population that has access to capital markets. It shows who is able to invest in risky assets.

Our definition focuses on one specific aspect of financial openness, access to investment. It does not attempt to capture other related ideas that appear in the literature, such as capital account liberalization, financial regulation, financial development, or institutional quality. By focusing only on participation, we isolate the direct effect of investment access on the evolution of wealth inequality. This simplified setup allows us to study how a larger group of investors affects wealth distribution through compound returns and through consumption behavior, while keeping institutional and regulatory conditions unchanged.

For clarity, we treat φ as an exogenous parameter in the model. In reality, participation rates emerge from several interacting factors, including regulation, financial literacy, wealth levels, and market structure. A full mapping between policy indices and participation rates would require modeling how regulations affect household investment behavior. Developing such a link would be an interesting extension, but it is beyond the scope of the present study.

3 Methodology

Our methodology combines theoretical modeling with computational simulation to examine how *Investment Freedom* affects long-run wealth inequality. We start by developing a formal stochastic model where wealth evolves over time through labor income, consumption, and capital returns. *Investment Freedom* is modeled as a policy parameter that determines what share of the population has access to investment opportunities, i.e., the share of agents who have access to invest in a risky asset. All agents start with the same wealth, income, and investment ability. This setup allows us to isolate the effects of *Investment Freedom* and consumption behavior on how wealth distribution changes over time.

Our approach intentionally simplifies many real-world complexities in order to isolate the main mechanism linking investment access, consumption behavior, and wealth accumulation. Like many theoretical models, the goal is not to reproduce reality in every detail but to clarify how the main forces interact. By holding wages, skills, and initial wealth constant, the model allows us to observe how changes in *Investment Freedom* affect wealth dynamics. In this setting, responses of wealth inequality to *Investment Freedom* can appear even when agents begin from identical conditions.

This modeling strategy follows a long scientific tradition. Weisberg (2007) explains that physicists often begin by studying motion on frictionless surfaces in order to derive the basic laws, and only later introduce real-world frictions. In economics, Friedman (1953) argues that simplified or "unrealistic" assumptions can help isolate causal mechanisms. Sugden (2000) similarly suggests that models create "credible worlds" that help researchers understand how economic mechanisms work, even if many details differ from reality.

Starting from equal initial conditions also addresses a common problem in causal analysis, namely omitted variable bias. Empirical studies are often challenged because unobserved factors may influence the results. In our framework, agents are identical at the beginning, so there are no external differences that could explain later variation in wealth. Any inequality that appears arises only from the internal dynamics of the model. In this way, the framework helps isolate the role of *Investment Freedom* and consumption behavior in shaping wealth outcomes.

We implement the model through a large-scale simulation that lets us analyze long-term patterns. The simulation tracks wealth for a hypothetical population under different levels of *Investment Freedom* and different consumption behaviors. This approach enables us to understand the long-term effects of *Investment Freedom* on the Gini index of inequality.

3.1 The Formal Model

This section presents the formal model developed to analyze how *Investment Freedom* influences long-run wealth accumulation and inequality. We model individual wealth as a stochastic process, where agents accumulate wealth over time through labor income, capital returns, and consumption. The model is designed to isolate the dynamic effects of investment freedom, modeled as the share of the population with access to capital markets, on the Gini index of wealth inequality

3.1.1 Notation

The following notation defines the key variables, parameters, and stochastic processes used in the formal model of wealth accumulation:

Agent-Level Variables and Parameters:

w_t is wealth at the end of period *t* (endogenous variable).

s_t is wage/salary in current period *t*. we assume that for any period *t* the wage/salary is identical. I.e, $s_t = s$.

Investment Return Process:

$r_{i,t}$ is stochastic return on capital in period *t* for agent *i* (stochastic element).

μ is the mean of the underlying normal component of the return distribution.

σ is the standard deviation of the underlying normal component.

r_N is normally truncated distributed return on investment. $r_N : N\left(\mu, \sigma^2\right)$, where $0 < \mu < 1$.

r_{UL} is uniformly distributed return noise on investment. $r_{UL} : U\left(\mu - k_1 \cdot \sigma, \mu - k_2 \cdot \sigma\right)$, where $k_1 > k_2 > 1$. I.e., r_{UL} is sampled from an interval to the left of the mean of the normal distribution r_N.

r_{UH} is uniformly distributed return noise on investment. $r_{UH} : U\left(\mu + k_2 \cdot \sigma, \mu + k_1 \cdot \sigma\right)$ where $k_1 > k_2 > 1$. i.e., r_{UH} is sampled from an interval to the right of the mean of the normal distribution r_N.

We assume that

$$k_1 < \frac{1+\mu}{\sigma}, \qquad (1)$$

to ensure that $\left|r_{UL}\right| < 1$ and $\left|r_{UH}\right| < 1$ (see Appendix A for proof). For example, if $\mu = 10\%$ and $\sigma = 15\%$, we assume that $k_1 < \frac{110\%}{15\%} \approx 7.33$.

$\eta \in (0,1)$ is the probability that the return on investment will be r_N.

Policy Parameter:

$\varphi \in [0,1]$ is the Market Participation Rate, meaning the share of the population that has access to invest in the risky asset. In the model this parameter represents *Investment Freedom* as the proportion of agents who are able to earn stochastic investment returns. This definition differs from empirical investment freedom indices, such as those of the Heritage Foundation or the Fraser Institute. Those indices measure policy restrictions, while φ represents the actual participation rate in capital markets.

δ_φ is a Bernoulli random variable:

$$\delta_\varphi = \begin{cases} 1 \text{ with probability } \varphi \\ 0 \text{ otherwise} \end{cases} \qquad (2)$$

In the simulation, investment access is determined once at the beginning, at period t equals 0, and then remains fixed for all 100 periods. If an agent is assigned to the investor group at the start, that agent keeps access to investment for the entire simulation. If an agent does not receive access at t equals 0, that agent never invests. As a result, two persistent groups appear in the model, investors and non-investors, and their wealth evolves differently over time. If we assume that in each period an agent may lose investment access with probability 1-φ, this would create an unrealistic situation. An agent who is able to invest in period *i* might suddenly be unable to invest in period *i* + 1. In real economies, access to capital markets usually does not change from one period to the next in such a random way. For this reason, in our model investment access is determined once at the beginning and remains fixed throughout the simulation.

Consumption parameters:

c_t is the total consumption in period *t* (endogenous variable).

α is a constant percentage of consumption from wealth at the beginning of period *t* (exogenous variable),

β is a constant percentage of consumption from capital earnings in the current period *t* (exogenous variable). We assume that $1 >> \beta > \alpha > 0$.

γ is the percentage of consumption from wage/salary in current period *t* (exogenous variable).

3.1.2 Assumptions

The model relies on several key assumptions to isolate the simulated relationship between φ and inequality. First, all agents start with identical wealth and receive the same fixed wage each period. Wages remain constant over time with no growth, unemployment, or variation across individuals. Second, individuals cannot borrow or hold negative wealth. Third, while all individuals have the same investment ability, access to capital markets is determined by a policy parameter representing what fraction of the population has investment freedom. Those without access receive no returns from capital. Fourth, agents differ only in their consumption behavior, specifically their propensities to consume from wealth (α), capital gains (β), and wage income (γ). Fifth, the model excludes inheritance and intergenerational transfers, treating wealth accumulation as purely individual. Sixth, aside from the *Investment Freedom* parameter itself, institutional factors such as taxation, legal enforcement, and credit markets are held constant to isolate the behavioral and structural effects of investment access. Finally, consumption from capital earnings occurs only when investment returns are positive. When returns are zero or negative, individuals maintain their consumption by drawing only from their existing wealth and wage income.

Consumption from capital income occurs only when investment returns are positive. When returns are zero or negative, individuals keep their consumption by drawing only from their existing wealth and from wage income. In these periods there are no capital gains to consume. This rule creates an asymmetric response to positive and negative shocks. Agents benefit from gains when markets rise, but their consumption does not collapse when markets fall. This asymmetry affects the evolution of inequality. It limits the ability of return volatility to concentrate wealth and therefore provides a baseline environment that is relatively favorable to the idea that wider access to investment can reduce inequality.

3.1.3 The return distribution

To effectively analyze wealth characteristics over time, it's essential to model investment return distribution. We've chosen the S&P 500 return distribution, which differs from the normal distribution in terms of kurtosis and skewness, indicating fatter tails and asymmetry. Studies have shown that the S&P 500 return distribution deviates from normality, with a higher likelihood of extreme events. Therefore, it's important to recognize these deviations when analyzing wealth characteristics. (Mandelbrot, 1997; McCulloch, 1996; Egan, 2007; Linden, 2001).

To capture the attributes of the S&P 500 return distribution of fatter tails and asymmetry, we model the return on the investment as comprised of three distributions, conditional on the return to be larger than -0.4 ($r_{i,t} \geq -0.4$) yielding an asymmetrical distribution[2],

$$r_{i,t} = \left\{ \begin{array}{ll} r_N & \text{with probability } \eta \\ r_{UL} & \text{with probability } \left(\frac{1-\eta}{2}\right) \\ r_{UH} & \text{with probability } \left(\frac{1-\eta}{2}\right) \end{array} \right\} \text{ conditional on } r_{i,t} \geq -0.4 \ . \qquad (3)$$

In our model, each investor receives a different random return in each period. In other words, the return is idiosyncratic across agents. This means that even among agents who all have access to the market, wealth can evolve differently over time because each one experiences a different sequence of returns.

This feature allows inequality to arise not only between investors and non-investors, but also within the group of investors itself. Some investors may receive more favorable return sequences, while others may receive weaker ones. As a result, expanding φ exposes more agents to stochastic compounding. The effect on inequality therefore depends both on access to investment and on the dispersion created by different return realizations across investors. This design captures the idea that wealth inequality can grow through differences in individual investment outcomes, even when all investors face the same market environment.

Figure 1 presents a comparison between a Normal distribution of returns and the mixed Normal and Uniform distributions with a truncation of -40% (see equation (3)). We assume that an investor does not issue stock options where his loss might be unlimited.

[2] The threshold of -40% is based on the approximate yearly minimal historical return on the S&P 500 for the years 1975-2025. Dataset is collected and prepared by the Economist Robert Shiller: https://datahub.io/core/s-and-p-500#data

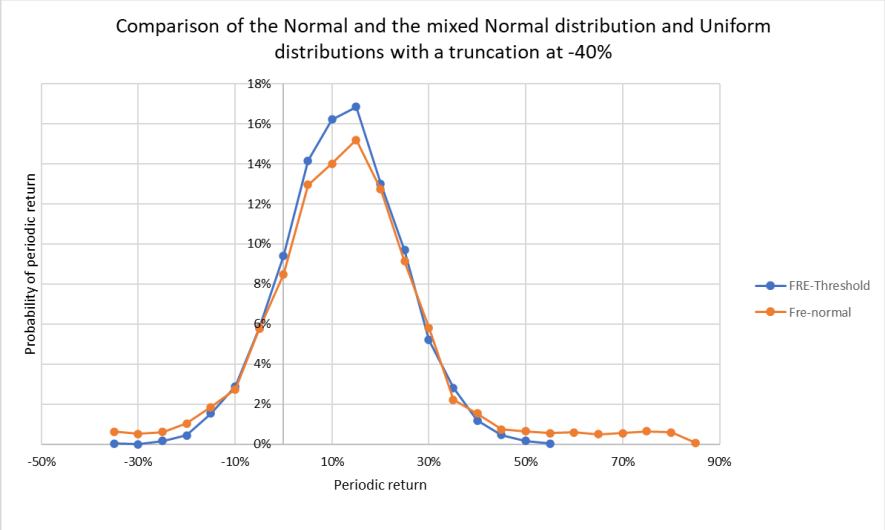

Figure 1: A comparison between a Normal distribution $(\mu = 10\%, \sigma = 12\%)$ of returns and the mixed Normal and Uniform distributions $(\eta = 90\%, k_1 = 6, k_2 = 2)$ with a truncation of -40%.

The variance of r_t, without truncation, denoted by $V(r_t)$, is (See Appendix B for the formulation of equation (4)):

$$V(r_t) = \left[\eta + \frac{1-\eta}{3}\left(k_1^2 + k_2^2 + k_1 k_2\right)\right]\sigma^2 \quad . \tag{4}$$

Since $k_1 > k_2 > 1$, we obtain that $k_1^2 + k_2^2 + k_1 k_2 > 3$, and thus $V(r_t) = A(\eta, k_1, k_2)\cdot\sigma^2$ where $A(\eta, k_1, k_2) > 1$. Therefore, the variance of r_t is always larger than the variance of $V(r_N) = \sigma^2$. For example, for $k_1 = 6$, $k_2 = 2$, η=0.9 we obtain that $V(r_t) = \left[0.9 + \frac{1-0.9}{3}\left(6^2 + 2^2 + 6\cdot 2\right)\right]\sigma^2 = 2.63\sigma^2$, and $\sigma_{r_t} = \sigma\sqrt{2.63} \approx 1.62\sigma$. Therefore, if the standard deviation of the Normal distribution is 10%, we obtain that the standard deviation of the return on the investment is about 16.2%. Due to the truncation at -40%, the variance of r_t decreases but not significantly.

Other approaches to model the return distribution of an investment in shares propose a mixture of normal distribution and exponential or binomial distribution. For example, Linden (2001) proposes a normal distribution, adapted by an assumption that the standard deviation of the normal distribution is a random variable distrusted exponentially. This approach results in most cases in a fatter tail than our approach. We chose to model the distribution as a mixture of the normal and uniform distributions (see equation (3)) since it also provides a fatter tail than the Normal distribution, but thinner than other approaches. We wanted to show that even without a very fat tail, freedom to invest leads to a higher average wealth.

3.1.4 The evolution of wealth in the model

The wealth of an agent, to the end of period *t*, is comprised of the wealth at the end of period $t-1$ plus the salary or wage in period *t* and the capital earnings from investing the wealth at the end of period $t-1$, minus the consumption in period *t*. The consumption in period *t* for an agent who is free to invest is,

$$c_t = \begin{cases} \alpha w_{t-1} + \beta r_t w_{t-1} + \gamma s & \text{when } r_t > 0 \\ \alpha w_{t-1} + \gamma s & \text{when } r_t \le 0 \end{cases}, \quad (5)$$

that is, if the return on the investment is positive, $r_t > 0$, then the consumption in period *t* is comprised of a percentage α of the wealth at the beginning of the period, plus a percentage β of the capital earnings, plus a percentage γ of the salary/wage. Otherwise, if the return on the investment is non-positive, $r_t \le 0$, meaning there is a loss on the investment or zero profit, then the consumption is a percentage α of the wealth at the beginning of the period, plus a percentage γ of the salary/wage. In other words, capital loss does not reduce consumption.

The consumption in period *t* for an agent who is not free to invest is,

$$c_t = \alpha w_{t-1} + \gamma s \ , \quad (6)$$

From equations (5) and (6) we get,

$$c_t = \begin{cases} \alpha w_{t-1} + \beta w_{t-1} \cdot Max(r_t, 0) + \gamma s & \text{for an individual who is free to invest} \\ \alpha w_{t-1} + \gamma s & \text{for an individual who is not free to invest} \end{cases}. \quad (7)$$

Notice that we have not considered the case of $w_{t-1} \le 0$ because $r_t \ge -40\%$. I.e., for each period, the absolute value of a loss does not exceed the initial wealth at the beginning of that period.

Our model includes two features that limit extreme downside risk. First, investment returns are truncated at minus 40 percent. This prevents catastrophic wealth losses. The assumption reflects institutional features such as limited liability in equity markets, where investors cannot lose more than the value of their investment. Second, when returns are negative, consumption is not decreased due to the capital losses. This creates a consumption floor that resembles real world mechanisms such as access to credit and consumption smoothing.

These assumptions influence the way inequality evolves in the model. In many models where agents can lose all their wealth or face bankruptcy, extreme losses can push some agents permanently to very low wealth levels. This often produces higher inequality over time. Our specification limits such extreme outcomes and therefore provides a conservative baseline for studying the potential equalizing effect of wider investment access. If inequality still increases in some regions of the parameter space, for example when α and β are high, this suggests that behavioral factors can dominate the potential benefits of expanding access to investment markets.

The wealth at the end of period *t* for an agent who is free to invest is,

$$w_t = w_{t-1}(1 + r_t) + s - c_t \ . \quad (8)$$

Otherwise, the wealth at the end of period *t* for an agent who is not free to invest is,

$$w_t = w_{t-1} + s - c_t \ . \tag{9}$$

Note that w_t follows a multiplicative stochastic process[3]. Substitute c_t (equation (7)) into w_t, for an agent who is free to invest, yields:

$$w_t = \begin{cases} w_{t-1}\left(1 + r_t\left(1-\beta\right) - \alpha\right) + \left(1-\gamma\right)s & \text{when } r_t > 0 \\ w_{t-1}\left(1 + r_t - \alpha\right) + \left(1-\gamma\right)s & \text{when } r_t \le 0 \end{cases} \Bigg\} \ . \tag{10}$$

Equation (10) is equivalent to,

$$w_t = \left(1 + r_t - \beta \cdot Max\left(r_t, 0\right) - \alpha\right) w_{t-1} + \left(1-\gamma\right)s \ . \tag{11}$$

For an agent who is not free to invest we obtain,

$$w_t = \left(1-\alpha\right) w_{t-1} + \left(1-\gamma\right)s \ . \tag{12}$$

Therefore, w_t follows a multiplicative stochastic process of the form:

$$w_t = m_t w_{t-1} + \left(1-\gamma\right)s \ , \tag{13}$$

where m_t, for an agent who is free to invest, is a random variable given by:

$$m_t = \left(1 + r_t - \beta \cdot Max\left(r_t, 0\right) - \alpha\right) \ , \tag{14}$$

and $\left(1-\gamma\right)s$ is a constant (if the salary does not change). For an agent who is not free to invest,

$$m_t = 1 - \alpha \ . \tag{15}$$

3.1.5 Convergence to a stable wealth distribution

Since we want to compare the *Gini* inequality index of the wealth distribution after *n* periods for different values of freedom to invest, it is important first to establish that the distribution after *n* periods converges to a specific distribution. Theorem I establishes that the wealth distribution after a given number of *n* periods has a specific distribution that depends upon the level of freedom to invest, φ, the return on investment *r,* and the parameters $\left(\alpha, \beta, \gamma, s, w_0\right)$.

Theorem I:

The wealth distribution after a given number of *n* periods has a specific distribution that depends upon the level of freedom to invest, φ, the return-on-investment *r,* and the parameters $\left(\alpha, \beta, \gamma, s, w_0\right)$.

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[3] A multiplicative stochastic process is a mathematical model that describes the evolution of a variable over time where changes occur multiplicatively, meaning the variable is subject to random multiplicative fluctuations.

Proof:

We will prove it by induction. For n=1, using equations (11) and (12), we obtain that,

$$w_1 = \delta_\varphi \left[m_1 w_0 + (1-\gamma)s \right] + (1-\delta_\varphi)\left[(1-\alpha)\cdot w_0 + (1-\gamma)s \right], \qquad (16)$$

where

$$\delta_\varphi = \begin{cases} 1 \text{ with probability } \varphi \\ 0 \text{ otherwise} \end{cases}$$

is a Bernoulli random variable, and $m_1 = \left(1 + r_1 - \beta \cdot Max(r_1, 0) - \alpha\right)$. Since we assume that at time 0 all agents have the same wage we obtain that equation (can be expressed as,

$$w_1 = f(r_1, \delta_\varphi | \alpha, \beta, \gamma, s, w_0). \qquad (17)$$

Given the distributions of (r_1, δ_φ), we obtain that w_1 has a specific distribution.

Assume that the wealth distribution after n periods, w_n, has a specific distribution. We will prove that the wealth distribution after n+1 periods has a specific distribution. Wealth w_{n+1} after n+1 periods is given by,

$$w_{n+1} = \delta_\varphi \left[m_{n+1} w_n + (1-\gamma)s \right] + (1-\delta_\varphi)\left[(1-\alpha)\cdot w_n + (1-\gamma)s \right], \qquad (18)$$

where the random variable $m_{n+1} = \left(1 + r_{n+1} - \beta \cdot Max(r_{n+1}, 0) - \alpha\right)$. Since, by the induction assumption, w_n has a specific distribution, we obtain a specific distribution of w_{n+1} that depends upon the specific distribution of w_n and the random variables $(r_{n+1}, \delta_\varphi)$, given the parameters $(\alpha, \beta, \gamma, s)$.□□ [4]

Therefore, in the simulation process, it is legitimate to examine the attributes of the wealth distribution after a specific number of n periods.

Theorem I shows that wealth follows a particular distribution at any finite period n. However, it does not prove that the system converges to a steady state or to an ergodic distribution. Whether convergence occurs depends on conditions discussed by Lux and Sornette (2002), especially the relationship between return parameters μ and σ and the consumption parameters α and β.

A visual inspection of Figures 2 to 4 suggests that the *Gini* coefficient begins to stabilize after about 60 to 80 periods for most parameter combinations. The change in the *Gini* index between period 90 and period 100 is usually less than

[4] In general, after an infinite number of periods the wealth distribution may not converge (see Appendix C for sufficient conditions to the wealth distribution to converge in probability).

about 5 percent. We interpret period 100 as an extended simulation horizon rather than as a proven steady state. This horizon is sufficient for our analysis because all parameter combinations are compared at the same simulation length.

3.2 Simulation-Based Implementation

This subsection presents the simulation-based implementation of the formal model, used to evaluate how varying levels of investment freedom φ affect the Gini index of wealth inequality over a 100-period horizon.

In the simulation, we assume a population of 1,000 agents and we track the development of the wealth per capita distribution over 100 periods. Also, we assume that only a percentage φ of the population is free to invest. Therefore, from equation (11), after embedding the level of *Investment Freedom* φ, we obtain,

$$w_t = \begin{cases} \left(1 + r_t - \beta \cdot Max\left(r_t, 0\right) - \alpha\right) w_{t-1} + \left(1-\gamma\right) s & \text{for } \varphi \text{ percent of the population} \\ \left(1-\alpha\right) w_{t-1} + \left(1-\gamma\right) s & \text{for 1-}\varphi \text{ percent of the population} \end{cases} . \qquad (19)$$

Since a percentage φ of the population is free to invest, the wealth at period *t* is given by equation (11). The rest of the population is not free to invest and therefore the return on investment r_t is zero. After substituting $r_t = 0$ into equation (11), we obtain $w_t = \left(1-\alpha\right) w_{t-1} + \left(1-\gamma\right) s$. The return on investment r_t is given by the mixed distribution,

$$r_t = \left\{ \begin{array}{l} r_N \quad \text{with probability } \eta \\ r_{UL} \quad \text{with probability } \left(\dfrac{1-\eta}{2}\right) \\ r_{UH} \quad \text{with probability } \left(\dfrac{1-\eta}{2}\right) \end{array} \right\} \text{ conditional on } r_t \geq -0.4 , \qquad (20)$$

where $\eta = 0.9$ is the probability that the return on investment will be drawn from a normal distribution r_N. The normal distribution is generated with $\mu = 10\%$, and $\sigma = 12\%$. The uniformly distributed return on investment, $r_{UL} : U\left(10\% - 6 \cdot 12\%, 10\% - 2 \cdot 12\%\right)$, and the uniformly distributed return on investment $r_{UH} : U\left(10\% + 2 \cdot 12\%, 10\% + 6 \cdot 12\%\right)$. That is, r_{UL} is sampled from an interval to the left of the mean of the normal distribution, and r_{UH} is sampled from an interval to the right of the mean of the normal distribution. We obtained that the mean value and the standard deviation of the return (*r*) are 8.35% and 17.25% respectively.

We assume that at time zero, *t* = 0, all agents have an identical wealth of $500 and their identical periodic income is $50,000. We purposely assume identical agents with the same investment capabilities, initial wealth, and periodic income to isolate the effect of changes in the level of *Investment Freedom* on the *Gini* wealth inequality index.

The values of the propensity to consume from initial wealth were set to $\alpha \in [1\%, 2\%, \ldots, 10\%]$, and the values of the propensity to consume from capital gains were set to $\beta \in [10\%, 11\%, \ldots, 15\%]$. The values of the freedom to invest parameter were set to $\varphi \in [10\%, 20\%, \ldots, 100\%]$. The value of the propensity to consume from wage/salary is set to be $\gamma = 95\%$. In total 600 combinations of α, β and φ. Each such combination produced a dataset of the evolution of wealth of 1,000 agents for 100 periods. The final database that the simulation produced contains a total of 60 million wealth data items, on which we ran the analysis.

At time $0 < t \le 100$, the wealth per capita is determined by equation (. Notice that in this model we assume that the investors are conservative in the sense they do not issue stock options but invest only in limited liability papers such as shares. Since the threshold on the return on investment is -40%, the wealth per capita at the end of each period is always positive, meaning, investors cannot lose their wealth. This is true because,

$$\left(1 + r_t - \beta \cdot Max\left(r_t, 0\right) - \alpha\right) w_{t-1} + \left(1 - \gamma\right) s > \left(1 + r_t - \alpha\right) w_{t-1} \ge 1 - 40\% - 10\% = 50\% > 0 . \quad (21)$$

3.3 Simulation Data Structure

This simulation runs the same wealth inequality model 26 times, each time with a different "random seed" (starting point for random number generation). In each run, the simulation creates a population of 1,000 agents who start with identical wealth ($500) and identical income ($50,000 per year). Over 100 time periods, these people invest their money and earn random returns that vary from person to person and from period to period. The simulation tests 600 different combinations of behavioral parameters (how much people consume from their wealth, how much they consume from investment gains, and what percentage of the population can invest). For each of the 26 runs, the simulation tracks how wealth inequality evolves and saves the results to separate files labeled with the seed number. By running the simulation 26 times with different random number sequences, we can verify that our findings about the relationship between investment freedom and inequality are stable and not just the result of one lucky set of random outcomes. Each run produces 60 million observations (1,000 people × 100 periods × 600 parameter combinations), and all 26 runs together provide strong evidence about whether expanding investment opportunities genuinely affects wealth inequality. The main figures and decision-tree results are therefore based on both a single baseline run and on a multi-seed panel that allows us to check whether the patterns we report are sensitive to a particular realization of investment returns.

4 Results

In this section, we present results for the simulation. The results indicate that an increase in *Investment Freedom* correlates with a reduction in the *Gini* coefficient of inequality.

Our analysis uses linear regression, visual analysis, and decision tree to summarize the relationships that appear in the simulated data. Because we control the data generating process, these statistical tools describe patterns implied by the model rather than estimate unknown parameters from observed data. All agents begin with the same initial wealth and

income. They differ only in their *Investment Freedom* parameter φ and in their consumption propensities α, β, and γ. Wealth accumulation then emerges from the interaction between these parameters and stochastic investment returns. The statistical relationships we report, including the regressions and the decision tree results, therefore reflect the mechanisms built directly into the model equations.

In simulation models, causality comes from the structure of the data generating process rather than from statistical identification methods. All agents start with identical initial conditions, initial wealth equals 500 dollars and income equals 50,000 dollars. Agents differ only in their *Investment Freedom* parameter φ. Because of this design, differences in wealth outcomes arise directly from the equations of the model. The parameter φ is exogenous. It is set before the simulation begins, not randomly assigned as a treatment in an experiment. As a result, the relationship we observe between φ and inequality, measured by the *Gini* coefficient, reflects the mechanisms built into the model. These mechanisms include compound returns that spread across more agents as participation increases, together with the role of consumption behavior. This design is different from empirical identification strategies such as randomized controlled trials (Stolberg et al., 2004), or Difference in Differences (Dimick & Ryan, 2014) methods. In empirical studies researchers must deal with confounding variables and selection bias in observational data. In our simulation environment these problems do not arise. All agents are identical at the start, so any variation in outcomes comes only from the interaction between φ, stochastic returns, and the consumption parameters α, β, and γ.

4.1 Cross-validated linear regression of the Gini wealth-inequality index as a function α, β and φ.

In this section we present the results of the linear regression of the *Gini* wealth inequality index variables as a function of the propensity to consume from the initial wealth α, the propensity to consume from capital gains β, and the level of *Investment Freedom* φ. The regression uses the cross section of *Gini* values at period 100 across the 600 parameter combinations. Each observation therefore represents the inequality outcome for one specific combination of φ, α, and β. Since the simulation runs on predetermined values of the variables α, β and φ, the variables are not correlated. To examine how the model parameters affect inequality after a long simulation period, we run the regression using the *Gini* coefficient calculated from the wealth distribution at period 100. To verify the robustness of our findings, we run the entire simulation 26 times using different random seeds, generating 26 independent datasets. This multi-seed design ensures that our results are not driven by a particular sequence of random returns but reflect stable relationships across different market histories.

For each of the 26 independent simulation runs, we use 5-fold cross validation to check that the relationship between parameters and outcomes is stable across different parts of the parameter space. In this procedure, the 600 parameter combinations are divided into five groups of 120 combinations each. For each step, we estimate the regression using four groups, which include 480 combinations, and then evaluate the results on the remaining group of 120 combinations. This process is repeated five times so that each group serves once as the test set. In many machine learning applications cross validation is used to prevent overfitting when the data generating process is unknown. In our simulation the data generating process is known because it is defined by the model equations. Here cross validation has a different role. It

helps verify that the linear specification summarizes the relationship between φ, α, β and the *Gini* index across the full parameter space. The similar results across the five folds, shown in Table 1, indicate that the regression captures stable patterns rather than being driven by a particular subset of parameter combinations.

The cross-validated linear regression results for the *Gini* wealth-inequality index for period 100 as a function of α, β and φ are presented in Table 1. Across the 26 independent simulation runs, the regression consistently explains approximately 81% of the variance in inequality outcomes (mean $R^2 = 0.812$, range 0.656–0.855), demonstrating that the parameter-inequality relationship is stable across different random market histories. The negative coefficients for α (mean = -5.06 ± 0.41) and β (mean = -0.84 ± 0.05) are highly significant across all seeds, indicating that higher consumption propensities reliably reduce wealth inequality. The coefficient for φ (mean = -0.062 ± 0.037) is negative in all seeds, except for seed 23 which is not statistically significance (p-value=0.6782). This pattern suggests that higher *Investment Freedom* tends to be associated with lower wealth inequality, though the magnitude varies across different random realizations.

By generating 26 complete datasets with different return histories and demonstrating consistent patterns in all of them, we provide evidence that the negative relationship between *Investment Freedom* and inequality reflects the structural mechanisms of the model rather than fortunate random outcomes. The modest variation in coefficient magnitudes across seeds (φ ranges from -0.164 to +0.001) indicates that the strength of the relationship does depend on market conditions, but the direction is stable in nearly all cases.

Table 1: Results of 5-fold linear regression for the Gini inequality index on the explaining variables: α, β, φ. Results shown independent simulation runs (seeds 1-26) and are averaged across 5-folds for each run.

Seed	RMSE	R^2	Adj. R^2	Coefficients				P-Values			
				Intercept	α	β	φ	Intercept	α	β	φ
1	0.0712	0.8092	0.8075	1.0077	-4.9821	-0.8324	-0.0726	3.56E-149	7.41E-165	0.0001	0.0000
10	0.0672	0.8335	0.8320	1.0263	-5.1235	-0.8400	-0.0643	5.73E-161	1.90E-179	0.0000	0.0000
11	0.0720	0.8085	0.8069	1.0118	-5.0285	-0.8251	-0.0741	5.07E-148	2.60E-164	0.0002	0.0000
12	0.0677	0.8313	0.8299	0.9903	-5.1437	-0.8418	-0.0491	1.98E-154	1.14E-178	0.0000	0.0002
13	0.0641	0.8550	0.8538	1.0564	-5.2483	-0.8879	-0.1030	1.38E-173	2.72E-191	0.0000	0.0000
14	0.0758	0.7958	0.7941	1.0069	-5.1309	-0.8595	-0.0310	2.80E-139	2.84E-159	0.0002	0.0389
15	0.0695	0.8346	0.8332	1.0616	-5.2815	-0.8894	-0.0943	8.96E-161	3.25E-178	0.0000	0.0000
16	0.0678	0.8363	0.8349	1.0377	-5.2342	-0.8685	-0.0634	2.27E-161	9.16E-181	0.0000	0.0000
17	0.0692	0.8321	0.8307	1.0338	-5.2428	-0.8685	-0.0703	1.46E-157	2.74E-178	0.0000	0.0000
18	0.0669	0.8278	0.8263	0.9881	-5.0273	-0.8240	-0.0438	2.20E-155	1.07E-175	0.0000	0.0008
19	0.0701	0.8213	0.8198	1.0100	-5.0987	-0.8360	-0.0836	9.74E-152	1.29E-171	0.0001	0.0000
20	0.0726	0.8111	0.8096	1.0215	-5.1301	-0.8628	-0.0620	3.91E-148	3.95E-166	0.0001	0.0000
21	0.0697	0.8350	0.8337	1.0379	-5.3639	-0.8695	-0.0563	7.88E-157	1.99E-180	0.0000	0.0000
22	0.0744	0.7987	0.7970	0.9804	-5.0904	-0.8437	-0.0213	3.41E-138	1.02E-160	0.0002	0.1334
23	0.0744	0.8091	0.8075	0.9979	-5.2564	-0.8843	0.0012	4.59E-141	1.07E-165	0.0001	0.6782
24	0.0743	0.8018	0.8001	0.9807	-5.1284	-0.8400	-0.0212	9.61E-139	2.00E-162	0.0002	0.1392
25	0.0688	0.8228	0.8213	0.9980	-5.0707	-0.8504	-0.0451	6.42E-153	2.78E-174	0.0000	0.0006
26	0.0482	0.6561	0.6499	0.9710	-3.4623	-0.6743	-0.1642	3.54E-95	8.10E-34	0.0014	0.0000

Notes. The dependent variable is the *Gini* coefficient at period 100 for each combination of φ, α, and β. All regressions include an intercept. Negative coefficients mean that higher values of φ, α, and β are associated with lower inequality at period 100. Since the data come from a simulated model, statistical significance reflects sample size and model fit rather than uncertainty about real world parameters.

In real economies, Investment Freedom can affect wealth inequality through several channels that are not included in our model, such as economic growth, entrepreneurship, and the development of small and medium sized firms (Klapper et al., 2006; Hvide and Moen, 2010; Henrekson et al., 2022). In our model the mechanism is simpler. It works through two main forces. First, when more agents are allowed to invest, a larger share of the population can earn compound returns on capital. Second, reinvestment behavior spreads these gains across the growing group of investors when consumption remains moderate. Over time, stochastic returns accumulate across agents with different consumption propensities, and this process shapes the evolution of wealth inequality.

In real life, the effect of *Investment Freedom* on wealth inequality also depends on broader conditions, such as the legal system, education, and tax policy (De Haan & Sturm, 2017; Kim et al., 2021). Greater *Investment Freedom* may help reduce inequality, but it is usually not sufficient on its own. It works best as part of a broader set of policies that address wealth disparities. The negative relationship between *Investment Freedom* and the Gini index in our results suggests that higher *Investment Freedom* can help limit inequality over time, but supportive policies remain important for inclusive growth and a more balanced distribution of wealth.

4.2 Visual Analysis

Figures 2 and 3 show the "Average Wealth Gini" on the y-axis and "Period" on the x-axis (0-100). The *Gini* coefficient starts at zero, indicating equal wealth distribution, and increases over time. Lines represent different levels of *Investment Freedom* (10%-100%), with higher freedom levels (60%-100%) generally associated with lower inequality at period 100. As time progresses (moving right along the x-axis), the *Gini* coefficients tend to increase for all levels of freedom, indicating growing wealth inequality. However, the lines for higher levels of *Investment Freedom* (60% to 100%) are generally lower on the graph compared to those for lower levels of *Investment Freedom* (20% to 50%). This suggests that higher *Investment Freedom* is associated with lower wealth inequality in the long run. It is also observable that the lines for higher levels of freedom tend to flatten or rise more slowly, while lines for lower levels of freedom show a steeper slope, indicating faster increase of inequality. The trend shown suggests that over time *Investment Freedom* may be negatively related to wealth inequality. Higher *Investment Freedom* appears to slow the increase in inequality, as measured by the Gini coefficient.

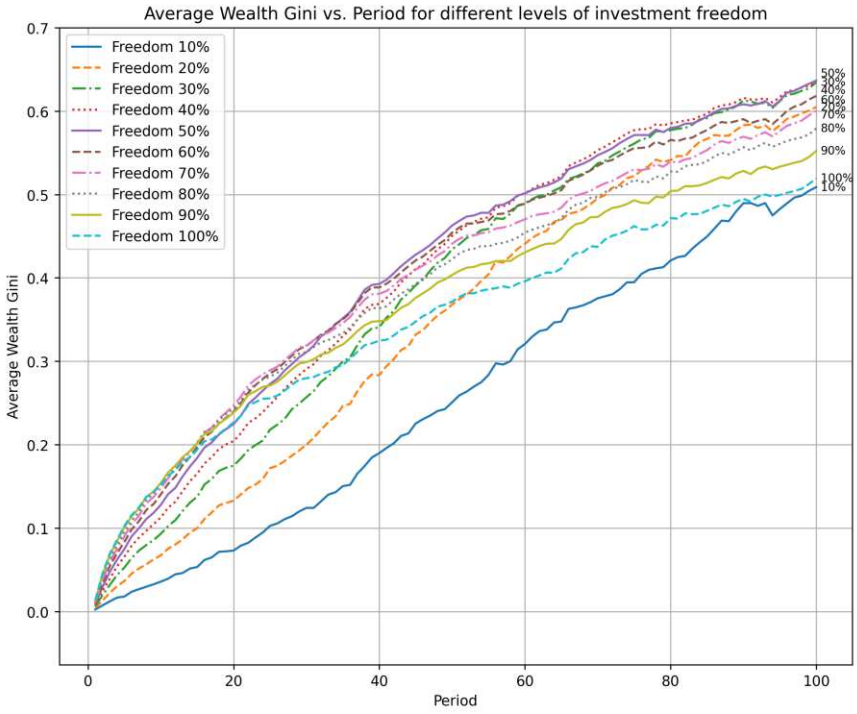

Figure 2: Average *Gini* wealth inequality index for each period, for different levels of *Investment Freedom*.

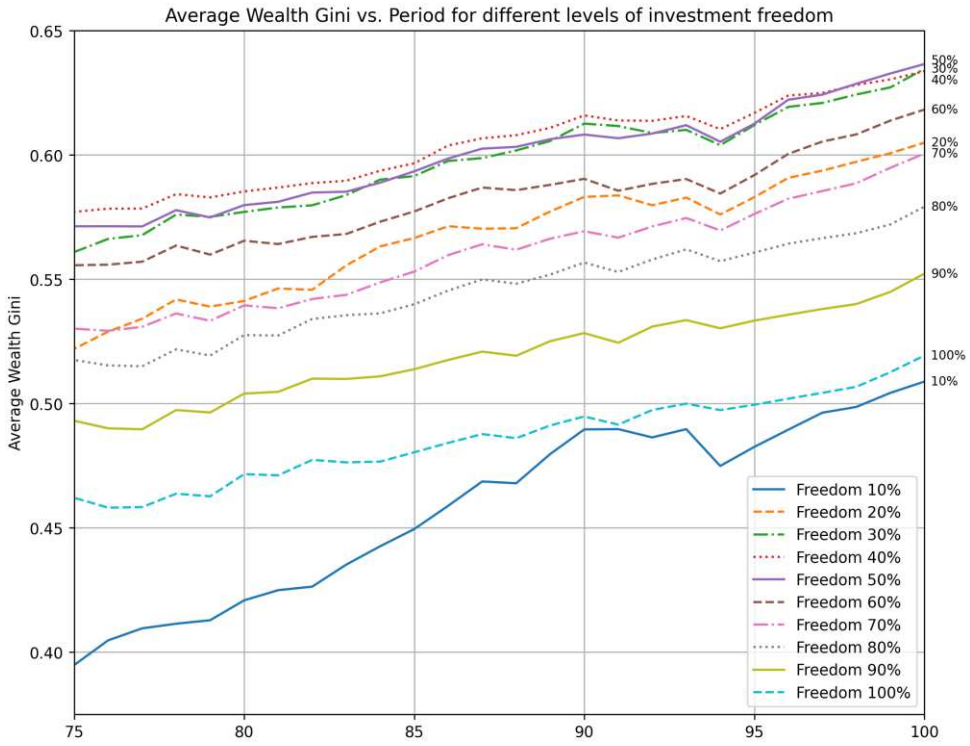

Figure 3: Zoom-in the upper right of Figure 4.

Figure 4 highlights intersections between different levels of *Investment Freedom*. Higher freedom levels (60%-100%) intersect with lower levels earlier than intermediate levels, suggesting that greater *Investment Freedom* initially accelerates changes in wealth inequality but slows its growth in the long run. While higher *Investment Freedom* can lead to rapid early capital accumulation, economic growth eventually moderates inequality. These intersections illustrate the

complex relationship between *Investment Freedom* and wealth distribution, showing that the impact on inequality is not always linear.

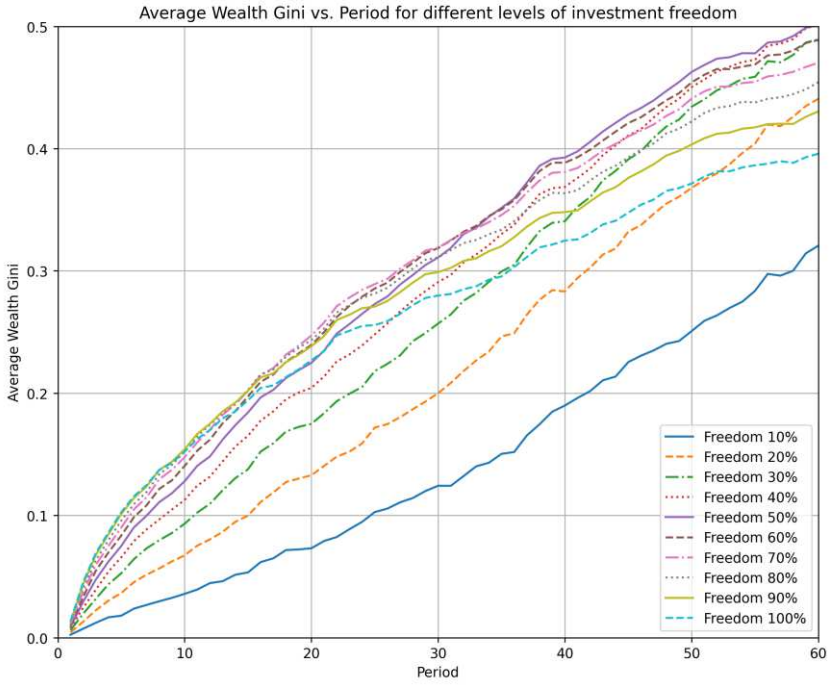

Figure 4: Zoom-in on the intersections between different invest levels.

The patterns shown in Figures 2–4 are based on a single representative simulation run. While these figures illustrate the general dynamics clearly, they represent one particular sequence of random returns. To examine whether these visual patterns hold across different market histories, Figures 5 and 6 present the evolution of inequality using data from all 26 independent simulation runs.

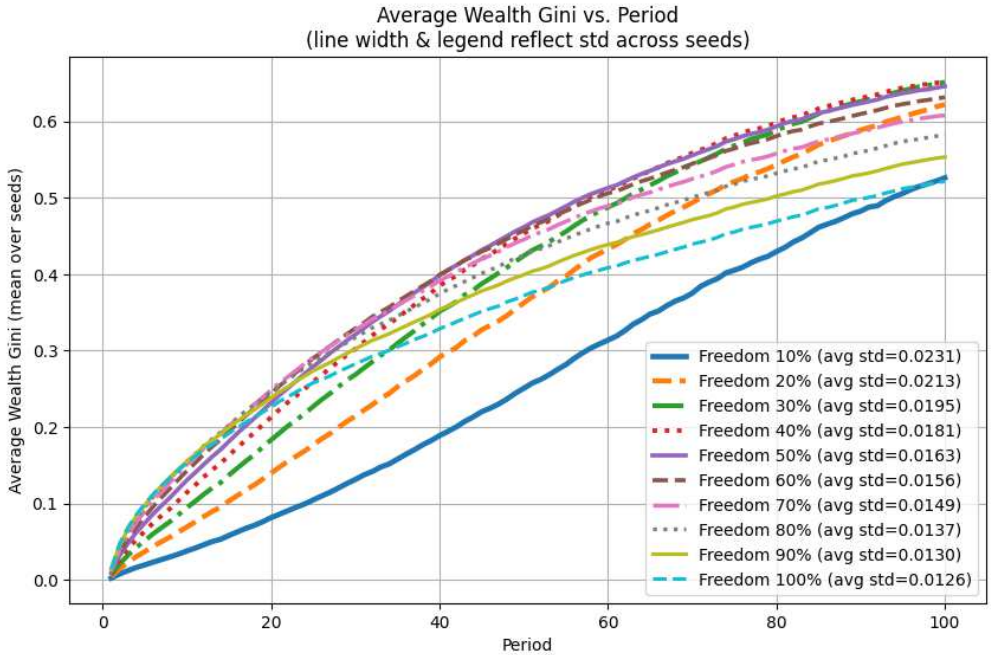

Figure 5: Average Gini wealth inequality index over 100 periods for different levels of Investment Freedom, averaged across 26 independent simulation runs. Line thickness represents variation across seeds (thicker lines = higher averge cross-seed standard deviation). Legend shows average mean standard deviation for each *Investment Freedom* level. Lower *Investment Freedom* levels (10%–30%, blue-green colors) show

consistently higher final inequality, while higher levels (60%–100%, purple-pink colors) show lower and more stable final inequality across different random market histories.

Figure 5 confirms the main result. Higher *Investment Freedom* is linked to lower inequality over time, and this result holds across different random return sequences. The figure shows the average *Gini* paths across 26 seeds. The width of each line shows how much the results vary across seeds. Several patterns are clear. First, the ranking of *Investment Freedom* levels is very stable. In almost all cases, higher φ leads to lower *Gini* at period 100. Second, the variation across seeds is small compared to the differences between φ levels. For example, when φ is 10 percent, the average *Gini* is about 0.53 with a standard deviation of 0.023. When φ is 100 percent, the average *Gini* is about 0.47 with a standard deviation of 0.013. Third, higher *Investment Freedom* is linked not only to lower inequality, but also to less variation across different return paths. This is seen in the thinner lines at higher φ. The line widths show a clear striking pattern. There is an inverse relationship between Investment Freedom and variation in outcomes. Higher φ leads not only to lower inequality but also to more stable and predictable results across different random market histories. The average standard deviation falls steadily from 0.023 when φ is 10 percent to 0.013 when φ is 100 percent, which is close to a 50 percent reduction in variation across seeds. This pattern suggests that wider investment access has a stabilizing effect. When only a small share of the population can invest, the results depend heavily on a few agents who experience very good or very bad returns. This makes inequality sensitive to the specific sequence of random outcomes. When access is wider, these individual shocks are spread across many investors and tend to average out. As a result, the distribution of wealth becomes more stable and less dependent on a particular return path. This stabilizing effect is separate from the reduction in inequality and adds to it. An economy with wide access to investment tends to have both lower inequality and more predictable outcomes.

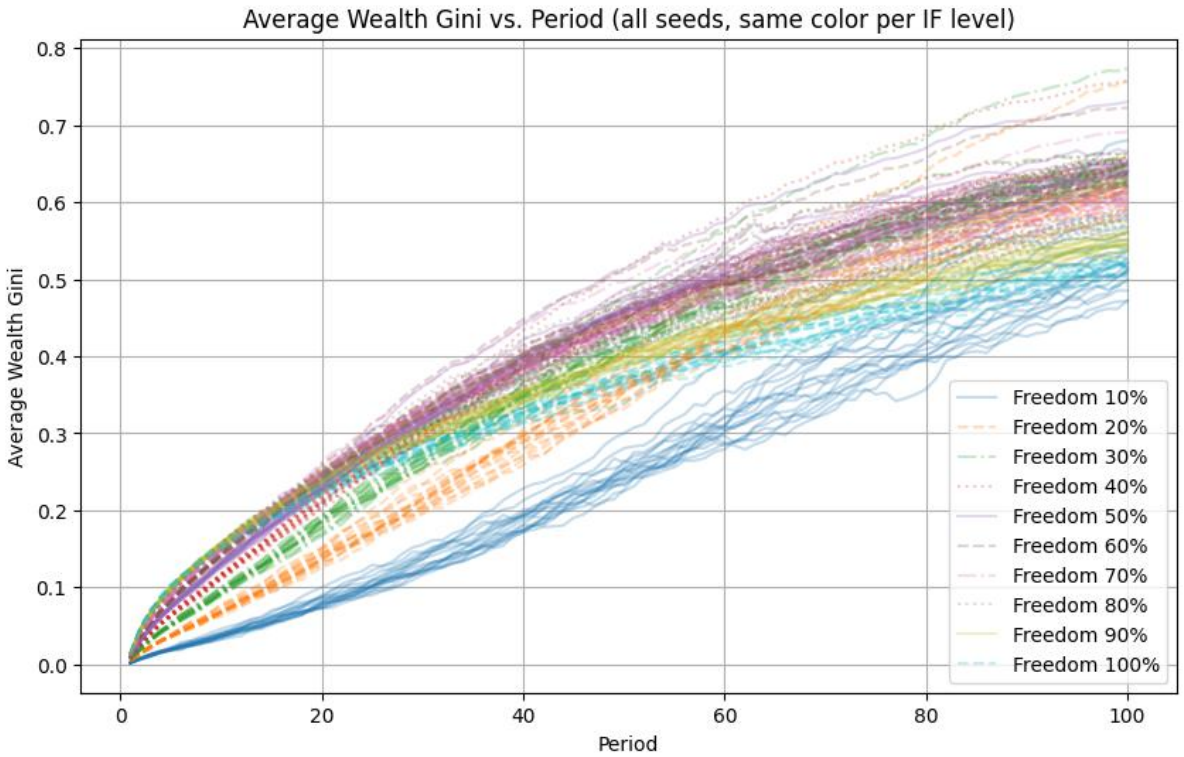

Figure 6: The figure shows the *Gini* paths for all 26 simulation runs, with each seed plotted as a separate line. Lines are colored by the level of *Investment Freedom*, using the same colors as in Figure 5. When all runs are shown together, there is some variation across seeds within each level of *Investment Freedom*. However, the separation between different φ levels is clear and consistent. Higher *Investment Freedom* leads to decreasing *Gini* in the long-run for all simulations.

Figure 6 shows each of the 26 simulations separately. Each line represents one seed, and colors indicate the level of *Investment Freedom*. This figure shows both the stability of the main result and the natural variation across different return paths. The pattern is clear. In almost every simulation, higher φ leads to lower inequality. Within each φ level, there is some variation because returns differ across seeds. However, this variation is smaller than the differences between φ levels. The separation between the color groups shows that *Investment Freedom* has a strong and consistent effect that is not driven by random luck. This supports the regression results and shows that the relationship is visible in the simulated data itself.

The single-seed figures and the multi-seed figures each show a different side of the results. The single-seed figures present the detailed dynamics and make it easier to follow how the paths evolve over time. The multi-seed figures show that these patterns are stable and do not depend on one specific random outcome. The consistency across 26 different simulations supports the conclusion that the negative relationship between *Investment Freedom* and long run inequality is a real feature of the model, and not just the result of one lucky random outcome.

The multi-seed decision-tree analysis in Section 4.3 confirms this visual evidence, showing that the same α and φ threshold regions are selected repeatedly across seeds.

4.3 Decision tree analysis

To capture the non-linear interactions and heterogeneities within the economic variables that contribute to wealth distribution, we employed a decision tree regressor with a depth of five to parse out the relationships between the variables *Investment Freedom* (φ), propensities to save and invest (represented by α and β), and their subsequent effects on the Gini inequality index after 100 periods. The Decision Tree model yields a high R^2 value of 0.9721. Such values are common in simulation settings because the underlying data-generating process is known and noise is limited (Lamperti, 2018; Grazzini and Richiardi, 2015). The model should therefore be interpreted as a descriptive tool that helps identify patterns in the simulated outcomes (Windrum et al., 2007; Fabretti, 2013).

Table 2 presents the results of the decision tree model used to predict the Gini inequality index. The decision tree uses the same cross-sectional dataset. The 600 observations include the parameters φ, α, and β as predictors and the *Gini* coefficient at period 100 as the outcome. The results show that when agents have a low propensity to consume (low α and β), they tend to save and invest a larger share of their income. When *Investment Freedom* is higher in these conditions, more of these savings are invested. This supports wealth accumulation across a broader part of the population and can reduce wealth inequality.

The decision tree reveals a pattern in the simulated data. When saving and investment are stronger than consumption, higher *Investment Freedom* is associated with a more equal distribution of wealth. Rows 1 to 8 in Table 2 illustrate this

pattern. In these rows β remains between 10 percent and 15 percent, while α is relatively low. As *Investment Freedom* increases in these ranges, the predicted *Gini* index declines.

It is important to note that decision trees identify splits in the data that reduce prediction error. These splits should not be interpreted as formally estimated structural thresholds. The split that appears around α of about 7% suggests a behavioral point in the model where consumption begins to dominate the equalizing effect of wider investment access, and the multi-seed trees recover splits in the same region. However, confirming such a threshold would require formal threshold regression methods, such as those proposed by Hansen (2000), in order to test the threshold statistically and estimate its precision.

A similar pattern appears in rows 15–16 and 25–26. In these cases, α remains below 7% and β stays between 10% and 15%. Under these conditions, higher *Investment Freedom* again corresponds to lower predicted inequality. The interpretation is that wider access to investment opportunities allows wealth gains to spread across more individuals, which limits the concentration of wealth.

However, the results also show that the relationship is not uniform. Rows (25, 26), (29, 31), and (30, 32) indicate a different pattern when α becomes larger. When α rises above about 7% while β remains between 10% and 15%, higher *Investment Freedom* is associated with higher predicted inequality. In these cases a larger share of wealth is consumed rather than reinvested, which weakens the equalizing effect of broader investment freedom.

A key finding from the decision tree analysis is that the relationship between *Investment Freedom* and inequality is not linear and depends strongly on consumption behavior. When the propensity to consume from initial wealth α remains below 7%, higher *Investment Freedom* is consistently associated with lower predicted inequality. When α rises above this level, the direction of the relationship can change. This threshold shows that the effect of *Investment Freedom* on inequality depends on whether agents reinvest enough of their returns. When reinvestment remains strong, compound growth spreads across a larger group of investors. This helps reduce the concentration of wealth. When reinvestment is weaker, higher participation in capital markets does not necessarily lead to a broader distribution of wealth.

Overall, these results highlight the close relationship between consumption behavior, investment freedom, and the evolution of wealth inequality. They suggest that the distributional impact of *Investment Freedom* depends not only on access to capital markets but also on how agents allocate their income between consumption and saving.

Table 2: Decision tree model for the prediction of the *Gini* inequality index

Row #	α	β	φ	Predicted Gini
1	[0%, 2%]	[10%, 15%]	[0%, 35%]	0.85
2			(35%, 65%]	0.79
3	(2%, 3%]		[0%, 65%]	0.77
4	(3%, 4%]			0.72
5	[1%, 3%]		(65%, 85%]	0.71
6	(3%, 4%]			0.68
7	[1%, 4%]		(85%, 95%]	0.66
8			(95%, 100%]	0.62
9	(4%, 5%]	[10%, 12%]	[0%, 15%]	0.58
10		(12%, 15%]		0.54
11	(5%, 6%]	[10%, 12%]		0.47
12		(12%, 15%]		0.42

13	(4%, 5%]	[10%, 15%]	(15%, 75%]	0.69
14	(5%, 6%]			0.62
15	(4%, 6%]	[10%, 15%]	(75%, 95%]	0.59
16			(95%, 100%]	0.54
17	(6%, 7%]	[10%, 13%]	[0%, 15%]	0.37
18		(13%, 15%]		0.32
19	(7%, 10%]	[10%, 12%]		0.29
20		(12%, 15%]		0.25
21	(9%, 10%]	[10%, 12%]		0.22
22		(12%, 15%]		0.2
23	10%	[10%, 12%]		0.18
24		(12%, 15%]		0.16
25	[1%, 7%]	[10%, 15%]	(15%, 85%]	0.55
26			(85%, 100%]	0.49
27	(7%, 10%]		(15%, 25%]	0.4
28			(25%, 100%]	0.48
29	(9%, 10%]		(15%, 25%]	0.32
30	10%			0.27
31	(9%, 10%]		(25%, 100%]	0.42
32	10%			0.37

To check that these patterns are not driven by a specific random seed, we repeat the decision tree analysis on 26 independent simulations that differ only in the sequence of investment returns. Across these runs, the trees continue to fit the period 100 *Gini* very well. The relative importance of the variables is also stable. The propensity to consume from wealth α explains most of the variation in the *Gini*, followed by *Investment Freedom* φ, while the propensity to consume from capital gains β plays a smaller role.

The split patterns are also very similar to the single run results. For α, more than 90% of the splits fall between 2 and 10%, with a strong concentration above 7%. This supports the behavioral threshold shown in Table 2. For β, all splits fall between 10% and 15%, mostly above 12%, which matches the ranges in the table. For φ, the trees repeatedly divide the data into similar regions of low, medium, and high *Investment Freedom*, around 15% to 35%, 35% to 65%, 65% to 85%, and 85% to 95%. These results show that the main decision tree patterns are stable across multi-seed simulations. In particular, the threshold around α of 7% and the different *Investment Freedom* ranges appear consistently. This suggests that these patterns reflect the structure of the model and not the result of one specific simulation.

5 Discussion and Conclusions

This paper develops a theoretical model of wealth accumulation and distribution under different levels of Investment Freedom. In the model, agents begin with the same wealth, wages, and investment ability. This setup allows us to isolate the role of Investment Freedom, which enters the model as a policy parameter. Differences between agents arise only through their consumption behavior and their exposure to investment opportunities. The model generates a clear prediction: higher *Investment Freedom* can lead to a decline in wealth inequality in the long run.

Understanding the relationship between *Investment Freedom* and inequality has become increasingly important for both researchers and policymakers. As capital markets become more integrated, questions about the distributional effects of investment openness have moved to the center of economic debate (Levine, 1997; Edison et al., 2002; Bekaert et al., 2005). *Investment Freedom* has long been associated with economic growth and capital accumulation, yet its effect on wealth inequality remains debated in both theory and empirical work (De Haan & Sturm, 2017; Bennett & Nikolaev,

2017; Furceri & Loungani, 2018). This study contributes to this discussion by presenting a micro-founded model that focuses on the behavioral mechanisms through which *Investment Freedom* affects long-run wealth outcomes.

The model is implemented through a stochastic simulation that follows the wealth paths of agents over many periods. This approach allows us to observe how wealth evolves when agents differ in their consumption behavior. By running the simulation under different levels of *Investment Freedom* and different consumption propensities, we generate a time-evolving distribution of wealth and its associated Gini index. The results show that higher *Investment Freedom* accelerates capital accumulation. When consumption from wealth and capital gains remains moderate, this process gradually produces a more equal distribution of wealth. In this way, the simulation provides a micro-level explanation for distributional patterns that are often assumed rather than explicitly modeled (Dosi et al., 2013; Riccetti et al., 2015; Lux & Sornette, 2002).

The results also help clarify different views in the literature. Some studies suggest that greater financial openness can increase inequality, especially when institutional safeguards are weak. Our model shows that the path of inequality depends strongly on consumption behavior over time. When agents reinvest a sufficient share of their returns, wider investment access can reduce inequality. When consumption from wealth becomes too high, this equalizing effect becomes weaker (Furceri & Loungani, 2018; Kim et al., 2021; De Soysa & Vadlamannati, 2023).

The findings have several implications. From a theoretical perspective, the paper shows that even a simple model with identical agents can generate meaningful distributional dynamics once access to investment is introduced. From a methodological point of view, the study shows the value of running the model more than once. We run 26 simulations with different random return sequences and find the same patterns in all of them. This shows that the results come from the structure of the model and not from one specific random outcome. The results also show that higher Investment Freedom does not only reduce average inequality, it also makes outcomes more stable. The variation across runs falls by about 46 percent when φ increases from 10 percent to 100 percent. This suggests that wider participation in financial markets reduces the noise in the value of mean wealth. From a policy perspective, the results suggest that expanding *Investment Freedom* can support long-term equity, especially when combined with policies that encourage saving and reinvestment rather than short-term consumption (Modigliani, 1986; Klapper et al., 2006; Henrekson et al., 2022).

The analysis also has limitations. The model abstracts from several institutional and market frictions, including credit constraints, taxation, and redistributive transfers. It also assumes that agents have the same investment ability, labor income, and no intergenerational transfers. However, our multi-seed simulation design addresses one important concern, in stochastic simulation studies, of path dependence and sensitivity to random draws. By demonstrating consistency across 26 independent random market histories, we show that the main findings are robust to different sequences of investment returns. These factors may influence the evolution of inequality in real life. The simplified structure is useful for isolating the main mechanism. Future research could extend the framework in several directions. One extension would be to introduce institutional heterogeneity, which may help explain why the effects of *Investment Freedom* differ across countries. Another extension would be to relax the assumption that agents have identical investment ability and

income streams. Doing so would bring the model closer to observed wealth dynamics and could help explain temporary increases in inequality during periods of economic transition.

Declaration of interest statement

1. There is no any potential conflict of interest. Our study received no funding and there are no other relationships that can influence our work.
2. Our research does not involve any human participants of animals.

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Statements and Declarations

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Author contributions

All authors contributed to the study conception and design. Material preparation, data collection, and analysis were performed by Arie Jacobi and Joseph Tzur. The first draft of the manuscript was written by Arie Jacobi and Joseph Tzur, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Data availability

The simulation code and aggregated data that support the findings of this study are available from the corresponding author upon reasonable request.

Code Availability

The simulation code that supports the findings of this study is available from the corresponding author upon reasonable request.

Appendix A: A proof that $|r_{UL}|<1$ and $|r_{UH}|<1$

Notice that,

$$|\mu-k_1\cdot\sigma|<1\Leftrightarrow(\mu-k_1\cdot\sigma)^2<1\Leftrightarrow\sigma^2{k_1}^2-2\mu\sigma k_1+\mu^2-1<0.$$

The roots of the quadratic expression are:

$$k_{1_{1,2}}=\frac{2\mu\sigma\pm\sqrt{4\mu^2\sigma^2-4\sigma^2\left(\mu^2-1\right)}}{2\sigma^2}=\frac{\mu\pm\sqrt{\mu^2-4\left(\mu^2-1\right)}}{\sigma}.$$

Since $|\mu|<1$, we obtain the roots expressions,

$$k_{1_1}=\frac{\mu+\sqrt{4-3\mu^2}}{\sigma}>0,$$
$$k_{1_2}=\frac{\mu-\sqrt{4-3\mu^2}}{\sigma}<0.$$

Therefore, $0<k_1<k_{1_1}=\frac{\mu+\sqrt{4-3\mu^2}}{\sigma}$ ensures that $|\mu-k_1\cdot\sigma|<1$. Similarly, $0<k_2<\frac{\mu+\sqrt{4-3\mu^2}}{\sigma}$ ensures that $|\mu-k_2\cdot\sigma|<1$. Therefore, $0<k_2<k_1<\frac{\mu+\sqrt{4-3\mu^2}}{\sigma}$ ensures that $|r_{UL}|<1$.

Similarly, $0<k_1<\frac{\mu+\sqrt{4-3\mu^2}}{\sigma}$ ensures that $|r_{UH}|<1$. Therefore, $k_1<\frac{\mu+\sqrt{4-3\mu^2}}{\sigma}$ ensures that both $|r_{UL}|<1$ and $|r_{UH}|<1$. Notice that $\frac{1+\mu}{\sigma}<\frac{\mu+\sqrt{4-3\mu^2}}{\sigma}$ since $\mu^2<1$. Therefore, $k_1<\frac{1+\mu}{\sigma}$ ensures that $|r_{UL}|<1$ and $|r_{UH}|<1$.

Appendix B: The formulation of equation (4)

$$V(r_t)=E(r_t^2)-(E(r_t))^2. \tag{B1}$$

Notice that,

$$E(r_t^2)=\eta E({r_N}^2)+\left(\frac{1-\eta}{2}\right)E({r_{UL}}^2)+\left(\frac{1-\eta}{2}\right)E({r_{UH}}^2) \tag{B2}$$

Substituting equation (B2) in equation (B1) yields

$$V(r_t) = \eta E(r_N{}^2) + \left(\frac{1-\eta}{2}\right) E(r_{UL}{}^2) + \left(\frac{1-\eta}{2}\right) E(r_{UH}{}^2) - \mu^2 \tag{B3}$$

$$V(r_t) = -(1-\eta) E(r_N{}^2) + \left(\frac{1-\eta}{2}\right) E(r_{UL}{}^2) + \left(\frac{1-\eta}{2}\right) E(r_{UH}{}^2) + E(r_N{}^2) - \mu^2 \tag{B4}$$

Since $\sigma^2 = E(r_N{}^2) - \mu^2$ and $E(r_N{}^2) = \sigma^2 + \mu^2$ we obtain that

$$V(r_t) = -(1-\eta)(\sigma^2 + \mu^2) + \left(\frac{1-\eta}{2}\right) E(r_{UL}{}^2) + \left(\frac{1-\eta}{2}\right) E(r_{UH}{}^2) + \sigma^2 \tag{B5}$$

Simplifying equation (B5), yields,

$$V(r_t) = \eta\sigma^2 - \mu^2(1-\eta) + \left(\frac{1-\eta}{2}\right) E(r_{UL}{}^2) + \left(\frac{1-\eta}{2}\right) E(r_{UH}{}^2) \tag{B6}$$

Since $r_{UL} \sim U(\mu - k_1 \cdot \sigma, \mu - k_2 \cdot \sigma)$ and $V(r_{UL}) = \frac{\sigma^2}{12}(k_1 - k_2)^2$ it follows that

$$E(r_{UL}{}^2) = (E(r_{UL}))^2 + V(r_{UL}) \tag{B7}$$

Substituting in $V(r_{UL}) = \frac{\sigma^2}{12}(k_1 - k_2)^2$ and $E(r_{UL}) = \mu - \frac{k_1 + k_2}{2} \cdot \sigma$ into equation (B7),

$$E(r_{UL}{}^2) = \left(\mu - \frac{k_1 + k_2}{2} \cdot \sigma\right)^2 + \frac{\sigma^2}{12}(k_1 - k_2)^2 \tag{B8}$$

In a similar fashion we have,

$$E(r_{UH}{}^2) = \left(\mu + \frac{k_1 + k_2}{2} \cdot \sigma\right)^2 + \frac{\sigma^2}{12}(k_1 - k_2)^2 \tag{B9}$$

Substituting equations (B8)-(B9) into equation (B6) we obtain,

$$\begin{aligned} V(r_t) = \eta\sigma^2 - \mu^2(1-\eta) &+ \left(\frac{1-\eta}{2}\right)\left[\left(\mu - \frac{k_1 + k_2}{2} \cdot \sigma\right)^2 + \frac{\sigma^2}{12}(k_1 - k_2)^2\right] \\ &+ \left(\frac{1-\eta}{2}\right)\left[\left(\mu + \frac{k_1 + k_2}{2} \cdot \sigma\right)^2 + \frac{\sigma^2}{12}(k_1 - k_2)^2\right] \end{aligned} \tag{B10}$$

Rearranging equation (B10) yields,

$$V(r_t) = \left[\eta + \frac{1-\eta}{3}(k_1{}^2 + k_2{}^2 + k_1 k_2)\right]\sigma^2 \tag{B11}$$

Appendix C

To determine whether the stochastic process yields converge to a stable wealth distribution, we use Theorem [1] in Lux & Sornette (2002) that proves the sufficient condition for converges: if $E\left[\ln\left(\left|m_t\right|\right)\right]<0$ then $w_t = m_t w_{t-1} + q_t$ converges in distribution and has a unique limiting distribution.

Lemma 1:

If $P_r\left\{r_t < \alpha/(1-\beta)\right\} = 1$ then $E\left[\ln\left(\left|m_t\right|\right)\right]<0$.

Proof :

It follows from equation (14),

$$\ln\left(\left|m_t\right|\right) = \begin{cases} \ln\left(\left|1+r_t(1-\beta)-\alpha\right|\right) & \text{when } r_t > 0 \\ \ln\left(\left|1+r_t-\alpha\right|\right) & \text{when } r_t \le 0 \end{cases} \qquad \text{(C1)}$$

When $r_t \le 0$ we obtain,

$$\ln\left(\left|m_t\right|\right) = \ln\left(\left|1+r_t-\alpha\right|\right) \qquad \text{(C2)}$$

We would like to show that $\ln\left(\left|m_t\right|\right)<0$ for the case that $-1<r_t\le 0$. Since $0<\alpha<1$ and $-1<r_t\le 0$, it follows that $-1<1+r_t-\alpha<1$. Thus $\left|1+r_t-\alpha\right|<1$, and in turn, $\ln\left(\left|1+r_t-\alpha\right|\right)<0$. Consequently, $E\left[\ln\left(\left|1+r_t-\alpha\right|\right)\middle| -1<r_t\le 0\right]<0$.

When $r_t>0$ we obtain, $\ln\left(\left|m_t\right|\right)=\ln\left(\left|1+r_t(1-\beta)-\alpha\right|\right)$. We would like to show that $\ln\left(\left|m_t\right|\right)<0$ for the case that $0<r_t<1$. Therefore, if $0<r_t<\alpha/(1-\beta)$ then $r_t(1-\beta)<\alpha$ and $1+r_t(1-\beta)-\alpha<1$. In addition, notice that $1-\alpha>0$ and $r_t(1-\beta)>0$.

Since $0<1+r_t(1-\beta)-\alpha<1$, $\left|1+r_t(1-\beta)-\alpha\right| = 1+r_t(1-\beta)-\alpha$. Therefore, $\ln\left(\left|1+r_t(1-\beta)-\alpha\right|\right)<0$. Consequently, $r_t<\alpha/(1-\beta)$ implies that $E\left[\ln\left(\left|1+r_t(1-\beta)-\alpha\right|\right)\middle| r_t>0\right]<0$ □□ .

Lemma 2:

There exists a mean value of the return distribution μ^* such that if $\mu<\mu^*$ then $E\left[\ln\left(\left|m_t\right|\right)\right]<0$.

Proof :

Clearly, if the $P_r\{r_t < \alpha/(1-\beta)\} = 1$, we obtain that $E\left[\ln\left(|m_t|\right)\right] < 0$. If $P_r\{r_t > \alpha/(1-\beta)\} = 1$ then $1 + r_t(1-\beta) - \alpha > 1$ and $E\left[\ln\left(|m_t|\right)\right] > 0$. Hence, as the mean value of the return distribution, μ, increases, while holding the standard deviation, σ, constant, the probability increases that $E\left[\ln\left(|m_t|\right)\right] > 0$ and then w_t does not converge. Therefore, there exist a mean value of the return distribution μ^* such that $E\left[\ln\left(|m_t|\right) \middle| for\ \mu = \mu^*\right] = 0$, and if $\mu < \mu^*$ then. □□

By theorem [1] in Lux & Sornette (2002), if $E\left[\ln\left(|m_t|\right)\right] < 0$ then w_t converges in distribution and has a unique limiting distribution. Therefore, by *lemma 2*, if $\mu < \mu^*$ then w_t converges in distribution and has a unique limiting distribution.

The threshold μ^* depends upon the parameters of α, β, k_1, k_2 , and σ.

Lemma 3:

The threshold μ^* is positively related to σ. The higher is σ the higher is μ^*.

Proof :

Suppose that for $\sigma = \sigma_1$ there exists μ_1^* such that for $\mu = \mu_1^*$, the expected value $E\left[\ln\left(|m_t|\right)\right] = 0$. Let $\sigma = \sigma_2 > \sigma_1$, and keep α, β, k_1, k_2 are unchanged. Since the distribution of r_t is symmetrical around its mean $\mu = \mu_1^*$, an increase in the standard deviation of r_N does not affect the mean value of the distribution $\mu = \mu_1^*$, but it increases the standard deviation of r_t (see equation (4)).

Since the slope of $\ln\left(|m_t|\right)$ function is decreasing in $|m_t|$, the increase of the variance of r_t has a lower impact on values of $|m_t| > 1$ than the absolute (and negative) impact on values of $|m_t| < 1$. Therefore, for $\mu = \mu_1^*$ and $\sigma = \sigma_2 > \sigma_1$, we obtain that $E\left[\ln\left(|m_t|\right)\right] < 0$. Hence, in order to obtain $E\left[\ln\left(|m_t|\right)\right] = 0$ we must increase $\mu = \mu_1^*$ to $\mu = \mu_2^*$ when $\mu_2^* > \mu_1^*$. □□