# Supplementary Material: Entropy-Based Indicators of Critical Transitions in Heavy-Tailed Networks Under Progressive Node Removal

## S1. Fisher Information of the Residual Degree Distribution

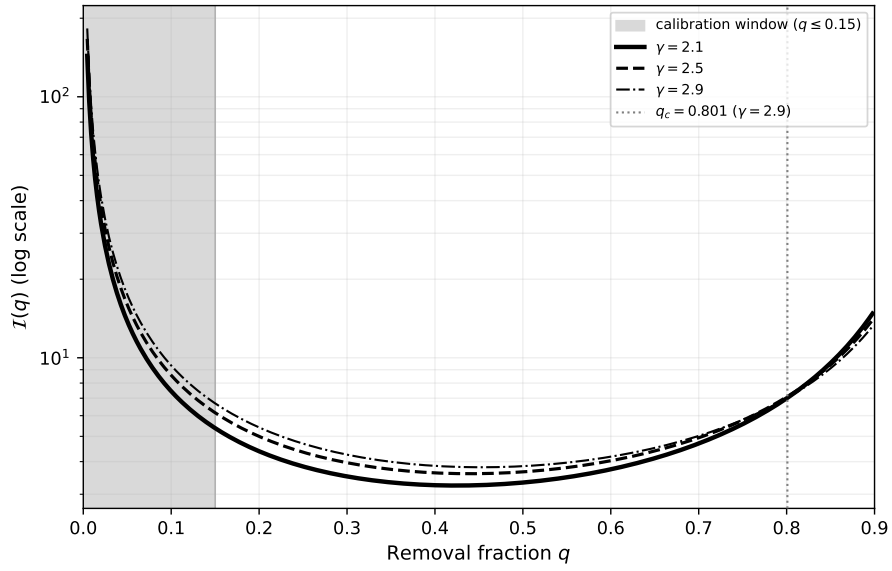


Figure S1: Fisher information $\mathcal{I}(q)$ of the residual degree distribution $\tilde{P}(k'; q)$ under random node removal, computed via central finite differences on the binomial-thinning residual (Eq. 3 of the main text), for $\gamma \in \{2.1, 2.5, 2.9\}$ (log $y$-axis). Shaded region: calibration window ($q \leq 0.15$); gray dotted vertical line: analytical Molloy–Reed threshold $q_c$ for $\gamma = 2.9$ (the only case with $q_c < q_{\max} = 0.9$). $\mathcal{I}(q)$ is large at small $q$, reaches a minimum in the mid-damage range, then grows toward $q_c$, consistent with signal acceleration as the percolation threshold is approached in the population-level model. This growth toward $q_c$ was evaluated under the discrete power-law reference $P_0$ specifically; it is not asserted as a general property of all heavy-tailed initial degree distributions.

# S2. Reproducibility

## One-command reproduction

All results can be regenerated from the repository root with:

`bash run_docker.sh`

This builds a Docker image with pinned dependencies and runs the full pipeline, producing `paper.pdf` and all intermediate artifacts in `paper/`. Requires Docker. The CAIDA data file (`caida_as_edges.txt`) must be obtained separately; see `README.md` in the repository for download instructions. The pipeline will skip the CAIDA step gracefully if the file is absent.

## Without Docker

Individual steps can be run directly after installing dependencies (`pip install -r requirements.txt`)

```
# Chung-Lu gamma sweep
.venv/bin/python main.py

# Sensitivity table
.venv/bin/python main.py --sensitivity-only

# Configuration-model sweep (exponent-matched)
.venv/bin/python main.py --graph-model config

# Degree-sequence-matched CM diagnostic
.venv/bin/python -m runner.degree_matched_config_check
.venv/bin/python -m runner.make_degree_matched_table

# Baseline-window sensitivity analysis
.venv/bin/python -m runner.baseline_window_sensitivity

# CAIDA figure
.venv/bin/python -m runner.make_caida_fig --edges caida_as_edges.txt

# Compile PDF
latexmk -pdf paper.tex
```

# S3. Sensitivity to Smoothing and Threshold

This table reports sensitivity of warning timing to $\alpha$ and $z$ under the fixed decision rule ($\gamma = 2.5$, 40 seeds). Values are mean $\pm$ std with detection counts $n_{\text{det}}/n$.

Table S1: Sensitivity of random-failure warning timing to EWMA smoothing $\alpha$ and baseline threshold $z$ in the rule $\tilde{D}_{\mathrm{KL}} > \mu_0 + z\sigma_0$. Values report mean ± std across seeds (with $n_{\mathrm{det}}/n$ counts).

| $\alpha$ | $z$ | $\gamma$ | $q_{\mathrm{warn}}$ (mean ± std) $[n_{\mathrm{det}}/n]$ | $\Delta_{\mathrm{warn}}$ (mean ± std) $[n_{\Delta}/n]$ |
|---|---|---|---|---|
| 0.10 | 1.5 | 2.5 | 0.174 ± 0.023 [40/40] | 0.653 ± 0.034 [40/40] |
| 0.10 | 2.0 | 2.5 | 0.215 ± 0.044 [40/40] | 0.612 ± 0.049 [40/40] |
| 0.10 | 2.5 | 2.5 | 0.284 ± 0.094 [40/40] | 0.542 ± 0.094 [40/40] |
| 0.20 | 1.5 | 2.5 | 0.216 ± 0.056 [40/40] | 0.610 ± 0.059 [40/40] |
| 0.20 | 2.0 | 2.5 | 0.297 ± 0.111 [40/40] | 0.529 ± 0.111 [40/40] |
| 0.20 | 2.5 | 2.5 | 0.443 ± 0.179 [39/40] | 0.382 ± 0.179 [39/40] |
| 0.30 | 1.5 | 2.5 | 0.230 ± 0.077 [40/40] | 0.596 ± 0.077 [40/40] |
| 0.30 | 2.0 | 2.5 | 0.306 ± 0.132 [40/40] | 0.520 ± 0.130 [40/40] |
| 0.30 | 2.5 | 2.5 | 0.447 ± 0.168 [39/40] | 0.379 ± 0.168 [39/40] |

# S4. Moment-Based Benchmark: Baseline Deviation on $\kappa(q)$

This benchmark applies the same EWMA smoothing ($\alpha = 0.20$) and baseline-deviation rule ($z = 2$, $q_0 = 0.15$) to the Molloy–Reed branching factor $\kappa(q) = (\langle k^2 \rangle_q - \langle k \rangle_q)/\langle k \rangle_q$ as a low-order-moment comparator. Results show 0/40 detections at $\gamma = 2.5$, confirming that matching the first two moments is insufficient to detect the pre-collapse distributional reorganization captured by successive KL divergence.

Table S2: Moment-based benchmark under random failure. We apply the baseline-deviation rule to the EWMA-smoothed Molloy–Reed branching factor $\tilde{\kappa}(q)$ (baseline window $q \leq 0.15$, $\alpha = 0.20$). We report $q^{\kappa}_{\mathrm{warn}}$, $\Delta^{\kappa}_{\mathrm{warn}} = q_{\mathrm{collapse}} - q^{\kappa}_{\mathrm{warn}}$, detection counts $(n_{\mathrm{det}}/n)$, and the baseline-referenced $Z^{\kappa}_{\mathrm{max}} = \max_q \left(\tilde{\kappa}(q) - \mu_0\right)/\sigma_0$ as mean ± std across seeds (Chung–Lu, $N = 10{,}000$, $\gamma = 2.5$).

| $\gamma$ | $q^{\kappa}_{\mathrm{warn}}$ (mean ± std) | $(n_{\mathrm{det}}/n)$ | $\Delta^{\kappa}_{\mathrm{warn}}$ (mean ± std) | $Z^{\kappa}_{\mathrm{max}}$ (mean ± std) |
|---|---|---|---|---|
| 2.5 | – | (0/40) | – | 1.219 ± 0.284 |

# S5. Configuration-Model Graph Construction Details

Detection results and mechanistic interpretation for the configuration-model experiments are reported in the main Results section. Construction details are provided here.

## Exponent-matched CM (main experiment)

A target degree sequence is drawn from a discrete power-law distribution with exponent $\gamma$ and minimum degree $k_{\mathrm{min}} = 1$, truncated to $N = 10{,}000$ nodes. The sequence is passed to the configuration-model stub-matching procedure; self-loops and parallel edges are removed; and the experiment is conducted on the largest connected component. Each ($\gamma$, seed) run uses an independently sampled degree sequence.

## Sequence-matched CM (degree-matched diagnostic)

For each ($\gamma$, cl_seed) pair, a Chung–Lu graph $G_{\mathrm{CL}}$ is generated and its realized degree sequence $\mathbf{d}$ is extracted. Multiple CM wiring replicates are then constructed from $\mathbf{d}$ (varying only the stub-pairing seed), producing CM graphs with the exact same degree sequence as the CL source. Self-loops and parallel edges are removed and the experiment is conducted on the largest connected component. This design controls for the degree sequence and isolates the effect of the hard-constraint wiring mechanism.