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Conditions for Quantum Violation of Macrorealism in Large-spin Limit

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Abstract

This study investigates the emergence of macroscopic classical behavior from quantum foundations via the entropic Leggett–Garg inequality. We introduce a geometric framework for deriving entropic Leggett–Garg inequalities with higher-order temporal correlations and demonstrate their advantages over conventional formulations. Numerical analyses show that entropic Leggett–Garg inequalities offer a robust and complementary criterion to standard approaches, providing a transparent information theoretic interpretation that facilitates the characterization of coherent quantum processes. By applying the WKB approximation, we prove that violations for maximally mixed states remain bounded by a constant in the macroscopic limit, indicating that macrorealism dominates in generic parameter regimes. We further explain previously reported maximal violations at specific parameter regimes as a consequence of the breakdown of the WKB approximation. Our findings indicate that quantum and classical descriptions remain macroscopically incompatible, while violations persist only in fine-tuned regimes, clarifying the conditions for detecting macroscopic quantum phenomena.

Keywords: Leggett–Garg Inequality, The Correspondence Principle, Macroscopic Quantum Phenomenon, Wigner d-Matrix, WKB Approximation

1 Introduction

Quantum mechanics (QM) exhibits counterintuitive features, such as nonlocality [1] and contextuality [2], which are fundamentally incompatible with the classical description of macroscopic systems. Despite the success of QM at atomic and subatomic scales, characterizing the crossover between the quantum and classical regimes remains a fundamental challenge. A central question [3] is whether quantum phenomena intrinsically vanish with increasing system size, or persist at the macroscopic scale under specific conditions. Although superconductivity [4] and superfluidity [5] are frequently cited as paradigms of “macroscopic quantum phenomena”, Leggett [6] argued that these are merely macroscopic accumulations of microscopic quantum effects, and hence insufficient to determine whether the quantum mechanical description can be extrapolated to macroscopic systems. Instead, Leggett proposed that the superposition of macroscopically distinguishable states could serve as more convincing experimental evidence.

To this end, a method capable of distinguishing genuine quantum superposition from classical statistical mixtures is needed. Leggett and Garg [7, 8] introduced a definition of “classical worldview” known as macrorealism, consisting of two principal assumptions: (1) Macroscopic realism: A macroscopic system with two or more macroscopically distinct states available to it will at all times be in one or the other of these states; (2) Noninvasive measurability at the macroscopic level: It is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics. Analogous to Bell inequalities [9], Leggett and Garg [7] showed that the temporal correlations must obey a class of inequalities under macrorealism, for instance,

$$1 + C_{12} + C_{23} + C_{13} \geq 0 , \tag{1}$$

$$|C_{12} + C_{23} + C_{14} - C_{24}| \leq 2 , \tag{2}$$

where $C_{ij} \equiv \langle Q(t_i)Q(t_j) \rangle$ denote the temporal correlation functions. Violations of such Leggett–Garg inequalities (LGIs) sufficiently (not necessarily) indicate the failure of macrorealism.

Such violations have been observed in a variety of experiments employing diverse measurement protocols, ranging from pre-diagonalized measurements [10, 11] and weak measurements [12–14] to the more rigorous ideal negative measurements (INM) [15]. The INM approach enforces non-invasiveness via null-result postselection. If no detector click is observed, one infers that the system occupied a state not coupled to the probe, implying that the measurement interaction left the system effectively undisturbed. Implementations of this principle generally fall into two categories: (i) ancilla-assisted schemes [16–18], where the system is entangled with auxiliary systems at the intermediate time and the projective readout is performed only at the end; and (ii) interferometric null-result schemes [19, 20], which exploit interaction-free measurement principles similar to the Elitzur–Vaidman bomb test [21]. Notably, these non-invasive protocols have been successfully extended to multi-outcome scenarios [22], demonstrating their feasibility for the high-dimensional systems.

Beyond the conventional application in testing macrorealism, LGIs also serve as indicators of quantum coherence [23–27], certifying dynamical non-classicality even in scenarios where static, state-based measures [28] prove inadequate. For instance, a maximally mixed state can yield LGI violations during unitary evolution, even though it remains incoherent throughout the process [29, 30]. To probe such quantum violations across broader physical scenarios, various LGI formulations have been developed [31, 32], most notably the standard correlator-based forms [33–36] and the probability-based Wigner forms [37, 38]. Nevertheless, these approaches are typically tailored to a fixed number of outcomes, thus hindering adaptation to high-dimensional systems or experimental setups with detection inefficiencies [39].

To address these limitations, an information-theoretic approach based on Shannon entropy, known as the entropic Leggett–Garg inequality (ELGI), was introduced [40, 41]. Whereas, hitherto the existing ELGIs are all confined to two-point joint entropies. It is thus of considerable interest to explore ELGIs involving higher-order temporal correlations, which would offer enhanced detection sensitivity and manifest novel phenomena. Furthermore, the precise relationship between this entropic formulation and other established methods, such as standard LGIs (SLGIs) [33–36] and Wigner form of LGIs (WLGIs) [37, 38], remains unclear.

In this study, we investigate the persistence of violations of entropic Leggett–Garg inequalities (ELGIs) at the macroscopic scale and their utility in characterizing quantum dynamics. To this end, we introduce a systematic approach to derive ELGIs with higher-order temporal correlations. We then demonstrate the operational advantages of ELGIs, highlighting their complementarity to conventional LGIs and their ability to detect non-Markovian memory effects directly through observable statistics. Furthermore, we analyze ELGI violations in spin- j systems driven by a constant external field and examine their asymptotic behavior in the limit $j \rightarrow \infty$ through the WKB approximation. Our analysis reveals that, in the regime where the WKB approximation is valid, quantum violations persist but are negligible relative to the system size; conversely, when the WKB approximation breaks down, the system rapidly reaches the maximal quantum violation of $\ln(2j+1)$. These findings delineate clear boundaries for the experimental testing of entropic Leggett–Garg inequalities at the macroscopic scale.

2 Results

2.1 Geometric framework for entropic Leggett–Garg inequalities

We begin by recounting the experimental scheme proposed by Leggett and Garg [7] in deriving the LGIs. Given an observable Q , which can be treated as a variable mathematically, to measure it in an experiment non-invasively, one performs a series of measurements at some or all points of n potential times $(t_1, \dots, t_n)$. In the α -th individual experiment, measurements are conducted at d points of those n times in sequence to acquire multipoint joint probabilities $p(q^{(\alpha)}(t_i), q^{(\alpha)}(t_j), \dots, q^{(\alpha)}(t_l))$, while the system remains undisturbed at the other times. Here, $q^{(\alpha)}(t_i)$ denotes the measurement outcome of Q at time t_i , the indices $i, j, \dots, l \in \{1, 2, \dots, n\}$, and $\text{card}\{i, j, \dots, l\} = d$, as illustrated in fig. 1.

Suppose M rounds of experiments are performed in total, each of which provides a joint entropy

$$H(Q^{(\alpha)}(t_i), \dots, Q^{(\alpha)}(t_l)) = - \sum_p p(q^{(\alpha)}(t_i), \dots, q^{(\alpha)}(t_l)) \ln p(q^{(\alpha)}(t_i), \dots, q^{(\alpha)}(t_l)). \quad (3)$$

Here, $Q^{(\alpha)}(t_i)$ represents the variable Q being measured in the α -th experiment at time t_i . The M joint entropies can then be arranged into a vector $\vec{h}_{\text{obs}} \in \mathbb{R}^M$. It should be noted that $Q^{(\alpha)}(t_i)$ and $Q^{(\beta)}(t_i)$ are distinct random variables due to the potential disturbance induced in measurement, even if they are measured at the same time t_i . Consequently, there are $d_1 + \dots + d_M$ random variables in total. In contrast, macrorealism posits that measurements are non-invasive, such that any measurement merely reveals the pre-existing value $q_{\text{pre}}(t)$ at time t without perturbation, which leads to

$$H(Q^{(\alpha)}(t_i), \dots, Q^{(\alpha)}(t_l)) = H(Q_{\text{pre}}(t_i), \dots, Q_{\text{pre}}(t_l)), \quad (4)$$

for all M joint entropies. It should be noted that $Q_{\text{pre}}(t_i)$ remains identical across all M rounds of experiments; consequently, a macrorealistic model involves no more than n random variables. Yeung [42] proved that entropic vectors $\vec{h}_{\text{obs}}$ satisfying eq. (4) are confined within a convex cone $\Gamma_E \subset \mathbb{R}^M$. Therefore, observation of $\vec{h}_{\text{obs}}$ lying outside Γ_E indicates violation of macrorealism.

Since a complete characterization of Γ_E remains an open problem in information theory, we employ the Shannon cone Γ_{Sh} defined by the non-negativity of conditional entropy and mutual information as an outer approximation, from which we derive two types of entropic Leggett–Garg inequalities:

$$\mathcal{D}_i := H(Q_i|Q_{\text{All}} - \{Q_i\}) = H(Q_{\text{All}}) - H(Q_{\text{All}} - \{Q_i\}) \geq 0 , \quad (5)$$

$$\begin{aligned} \mathcal{D}_{i,j} &:= I(Q_i; Q_j|Q_{\text{All}} - \{Q_i, Q_j\}) \\ &= H(Q_{\text{All}} - \{Q_i\}) + H(Q_{\text{All}} - \{Q_j\}) - H(Q_{\text{All}}) - H(Q_{\text{All}} - \{Q_i, Q_j\}) \geq 0 . \end{aligned} \quad (6)$$

Here, $Q_{\text{All}} = \{Q_1, \dots, Q_n\}$ with Q_i denoting $Q^\alpha(t_i)$ for simplicity. Yeung [42] demonstrated that these two types of inequalities provide a complete characterization of the Shannon cone; thus, $\mathcal{D}_i$ and $\mathcal{D}_{i,j}$ constitute two fundamental basis elements from which all other Shannon-type ELGIs can be systematically derived. For instance, the $n = 3$ case of ELGI proposed by Devi *et al.* [40], though originally derived through a different method, can be expressed as $H(Q_2|Q_1) + H(Q_3|Q_2) - H(Q_3|Q_1) = \mathcal{D}_{1,3} + \mathcal{D}_2 \geq 0$. This geometric approach parallels the method introduced by Chaves and Fritz for entropic Bell and contextuality inequalities [39, 43].

A key advantage of our approach is its capacity for deriving ELGIs in higher-order temporal correlation. As shown in fig. 2, higher-order ELGIs show larger violations and over a broader parameter range, compared to Devi’s second-order ELGIs. Similar enhancements obtained from higher-order correlations have also been observed in

other Leggett–Garg criteria, such as standard LGIs [35, 44] and no-signaling in time (NSIT) conditions [45, 46]. The enhanced sensitivity facilitates more effective detection of quantum behavior in temporal correlations, particularly in systems with intricate dynamics where two-point correlations might fail to capture subtle characteristics, such as non-Markovian processes [47]. Notably, the measurement of these higher-order correlations can be achieved by extending existing ideal negative-measurement schemes [16–20], as illustrated by a simple example in the Supplementary Information.

2.2 Advantages of Entropic Leggett–Garg Inequalities

While ELGIs are derived from elementary information-theoretic constraints, SLGIs and WLGIs originate from linear positivity conditions imposed on an underlying quasiprobability distribution [33, 37]. Consequently, ELGIs cannot be obtained from SLGIs or WLGIs, nor can SLGIs or WLGIs be inferred from ELGIs. This formal distinction is further illustrated through numerical analyses of spin systems.

As shown in fig. 3, the violation regions of ELGIs, SLGIs, and WLGIs do not exhibit a strict inclusion relation. Although SLGIs and WLGIs display substantially overlapping violation regions owing to their shared algebraic structure, the ELGI violation region overlaps only partially with either family and is largely complementary to both. This complementarity indicates that ELGIs can detect temporal quantum correlations in parameter regimes where conventional LGI tests show no violations. Notably, ELGIs require neither additional experimental observables nor modified measurement protocols, since all three LGI families are evaluated from the same multi-time probabilities obtained using an identical measurement sequence. Thus, ELGIs provide an alternative, information-theoretic consistency test for macrorealism, enhancing detection capability without adding experimental complexity.

Under dissipative evolution, ELGIs maintain detectable violations over longer time intervals and under stronger dissipation compared with SLGIs and WLGIs. This

robustness to decoherence makes the entropic test particularly suitable for probing quantum behavior in noisy or open-system scenarios. Further investigation is needed to clarify the mechanisms underlying this robustness.

Furthermore, the ELGI framework offers a clear information-theoretic interpretation, being directly grounded in the non-negativity of conditional entropy and conditional mutual information. This property allows any Shannon-type information equality or inequality to be mapped onto a corresponding ELGI, providing a systematic method to distinguish information disturbances arising from invasive quantum measurements from those due to other physical effects.

As an illustrative application, we employ ELGIs to characterize non-Markovian behavior in quantum dynamics. Specifically, we consider repeated measurements of an observable $\hat{Q}$ at three time points $t_1 < t_2 < t_3$, yielding outcomes Q_1, Q_2, Q_3 . In a classical Markov process, the conditional independence implied by Markovianity requires

$$H(Q_3|Q_2) = H(Q_3|Q_1, Q_2), \quad (7)$$

which corresponds to the ELGI

$$\mathcal{D}_{1,3} = H(Q_2, Q_3) + H(Q_1, Q_2) - H(Q_1, Q_2, Q_3) - H(Q_2) \geq 0. \quad (8)$$

Any deviation from the classical Markovianity condition eq. (7) has a clear physical interpretation. Positive violations indicate non-Markovian memory effects in the system or environment, whereas negative violations typically arise from the invasiveness of quantum measurements. Therefore, in scenarios where a Markovian approximation predicts no quantum violation, any observed positive deviation serves as a clear signature of genuine non-Markovian dynamics.

To illustrate the applicability of our approach, we consider a dissipative qubit system with decay to a sink state $|g\rangle$, designed to effectively simulate a non-Hermitian

qubit dynamics [48, 49]. In practice, the qubit state $|1\rangle$ decays to $|g\rangle$ via an intermediate state $|e\rangle$, with near-Markovian behavior achieved when the dissipation rate γ is much larger than the coupling strength ω between $|1\rangle$ and $|e\rangle$ ($\gamma/\omega \gg 1$); otherwise, non-Markovian effects become significant. As shown in fig. 4, numerical results reveal clear positive violations of the classical Markov condition for moderate dissipation, which gradually weaken as γ increases, in agreement with theoretical expectations. Such analyses rely on the premise that ELGIs exhibit negligible quantum violations under the Markov approximation, demonstrating that ELGIs provide a practical and directly measurable tool for probing environmental memory effects in open quantum systems.

More generally, this approach can be naturally extended to detect higher-order non-Markovianity. Following the framework developed by Taranto *et al.* [47], an $(n+1)$ -step classical stochastic process can be decomposed into future $F = \{t_n, \dots, t_k\}$, memory $M = \{t_{k-1}, \dots, t_{k-l}\}$, and history $H = \{t_{k-l-1}, \dots, t_0\}$ components, with the corresponding outcomes denoted by $Q^{(F)}$, $Q^{(M)}$, and $Q^{(H)}$. Markovianity requires the Shannon-type equalities

$$I(Q^{(F)}; Q^{(H)} | Q^{(M)}) = H(Q^{(F)} | Q^{(M)}) - H(Q^{(F)} | Q^{(H)}, Q^{(M)}) = 0, \quad (9)$$

each of which can be directly mapped onto a corresponding ELGI. In regimes where the Markovian approximation predicts no violation, any observed positive violation of this equality constitutes a clear signature of multi-step memory effects. We expect that future research will further extend the application of ELGIs to a broader range of scenarios, enabling the characterization of additional dynamical features in quantum systems.

2.3 Entropic Leggett–Garg inequalities in large-spin systems

Based on the general framework of entropic Leggett–Garg inequalities, we now turn to the spin- j system and investigate the asymptotic behavior of ELGI violations in the limit $j \rightarrow +\infty$, to examine whether quantum properties are inevitably excluded by macrorealism at the macroscopic scale.

Consider a spin system with Hamiltonian $\hat{H} = \omega \hat{J}_y$ and observable $\hat{Q} = \hat{J}_z$. In the Heisenberg picture, the projection operator corresponding to outcome m_i measured at time t_i is given by $\hat{\Pi}_{m_i}(t_i) = \hat{U}^\dagger(t_i) |j, m_i\rangle \langle j, m_i| \hat{U}(t_i)$, where $\hat{U}(t) = \exp(-i\omega \hat{J}_y t/\hbar)$ denotes the unitary evolution operator. According to Leggett’s experimental protocol, one conducts a series of measurements at time $t_1, \dots, t_k$, resulting in outcomes $m_1, \dots, m_k$ respectively. It is important to note that, within the standard quantum formalism, such sequential projective measurements necessarily disturb the system state, which directly violates the assumption of non-invasive measurability postulated by macrorealism. Quantum mechanics predicts the joint probabilities to be

$$\begin{aligned} p(m_1, \dots, m_k) &= \text{Tr} (\hat{\Pi}_{m_{k-1}}(t_{k-1}) \cdots \hat{\Pi}_{m_1}(t_1) \hat{\rho}_0 \hat{\Pi}_{m_1}(t_1) \cdots \hat{\Pi}_{m_{k-1}}(t_{k-1}) \hat{\Pi}_{m_k}(t_k)) \\ &= \langle j, m_1 | \hat{\rho}_0 | j, m_1 \rangle \prod_{i=1}^{k-1} |d_{m_{i+1}m_i}^j(\beta_{i,i+1})|^2, \end{aligned} \quad (10)$$

where $\beta_{ij} = \omega(t_j - t_i)$ denotes the rotation angle about y -axis, $\hat{\rho}_0$ is the density operator of initial state, and $d_{mn}^j(\beta) = \langle j, n | \exp(-i\beta \hat{J}_y/\hbar) | j, m \rangle$ are the Wigner d -matrix elements.

Previous works have reported LGI violations in the large-spin limit, with maximal violations observed for pure states [50–52], and have demonstrated that violations can also occur for maximally mixed states [29, 30]. Nevertheless, these investigations concentrate on particular measurement settings and parameter regimes, and the scaling behavior of both the magnitude and the range of LGI violations with system

size has yet to be addressed. Here, we extend the analysis to arbitrary experimental settings, systematically investigating how the violation scales as the system size increases. Specifically, we consider the case where the initial state is maximally mixed, $\hat{\rho} = \hat{I}/(2j+1)$, ensuring that the system initially possesses no quantum coherence. In this case, quantum mechanical joint entropies are simplified as

$$H(Q^{(\alpha)}(t_i), \dots, Q^{(\alpha)}(t_l)) = \ln(2j+1) + \sum_{s,s'} H_j(\beta_{s,s'}) , \quad (11)$$

where s and s' are adjacent measurements in the α -th experiment, and $H_j(\beta)$ is the “entropy” of the Wigner matrix element, defined as

$$H_j(\beta) := -\frac{1}{2j+1} \sum_{m,n=-j}^j |d_{nm}^j(\beta)|^2 \ln |d_{nm}^j(\beta)|^2 . \quad (12)$$

The ELGI parameters $\mathcal{D}_i$ and $\mathcal{D}_{i,k}$ can thus be expressed as linear combinations of the terms $H_j(\beta)$, with explicit formulas for all cases detailed in the Methods section.

Rigorous calculation of ELGIs in spin system involves inevitable computation of $H_j(\beta)$, which raises significant challenges to both numerical and analytical treatments, due to the mathematical complexity of the Wigner d -matrix elements $d_{mn}^j(\beta)$ [53–55]. Instead, we employ the WKB method [56] for sufficiently large j . For most parameter settings, we obtain the asymptotic expression

$$H_j(\beta) = \ln(2j+1)\pi + \ln|\sin\beta| - \frac{5}{2} + o(1) . \quad (13)$$

Note that the above asymptotic expression diverges as $\beta \rightarrow k\pi$ ($k \in \mathbb{Z}$), indicating the semiclassical approximation tends to be invalid, which will be addressed in the next

section. Substituting $H_j(\beta)$ into the elementary ELGIs, we obtain

$$\mathcal{D}_i = \ln(2j+1)\pi + \ln|1 + \tan\beta_{i-1,i}\tan\beta_{i,i+1}| - \frac{5}{2} + o(1) , \quad (14)$$

$$\mathcal{D}_{i,i+1} = \ln \left| \frac{\sin\beta_{i-1,i+1}\sin\beta_{i,i+2}}{\sin\beta_{i-1,i+2}\sin\beta_{i,i+1}} \right| + o(1) , \quad (15)$$

which demonstrates that the maximal violation of the ELGI is of order $O(1)$. More precisely, eq. (14) reveals that $\mathcal{D}_i$ diverges to positive infinity in the macroscopic limit, and thus remains non-negative. Consequently, only ELGIs of the form $\mathcal{D}_{i,i+1} \geq 0$ are capable of detecting macroscopic quantum violations. As $j \rightarrow +\infty$, the absolute magnitude of ELGI violation converges to a value determined solely by the measurement time intervals, as specified by the rotation angles $\beta_{i,i+1}$. However, since the von Neumann entropy $S(\hat{\rho})$ of the system grows logarithmically with j as $S(\hat{\rho}) = \ln(2j+1)$, the relative violation, expressed as the ratio of the violation magnitude to the system entropy, declines logarithmically to zero as j increases.

Although the derivation of eq. (15) employs the Shannon cone approximation, considering that the distance between the boundaries of Γ_{Sh} and Γ_{E} is a constant independent of j , the discussion above can be extended to any entropic Leggett–Garg inequalities, not only the Shannon-type ones. From experimental perspective, quantum violation tends progressively to be more challenging to detect at larger scales. We observe that the magnitude of the violation remains bounded, while the background entropy expands logarithmically. This scaling behavior implies that the violation signal is effectively submerged by the macroscopic uncertainty, like searching for tiny shells in a rising tide. Notably, this result suggests a complementary mechanism for the emergence of macrorealism. Unlike the coarse-graining restriction proposed by Kofler and Brukner [29, 50], which attributes classicality to limited measurement resolution, our findings indicate that even under ideal sharp measurements, quantum

features can become operationally negligible due to intrinsic information-theoretic scaling constraints.

2.4 Macroscopic quantum violations beyond the WKB approximation

Although the numerical results generally agree with the asymptotic limit across most parameter ranges, as illustrated in fig. 5, there are exceptions identified in eq. (15) exhibiting troublesome divergences. Kofler and Brukner [29] provided a relevant example, showing that quantum violation of the maximally mixed state near $\beta = 0$ persist in the $j \rightarrow +\infty$ limit. We further demonstrate that such a phenomenon applies to the discrete sequence $\beta = n\pi$ ($n \in \mathbb{Z}$), arising from the failure of the WKB approximation (see also Methods for details of the asymptotic analysis).

As detailed in the Methods section, the WKB approximation relies on the evaluation of the action $S(\beta)$ associated with the nutation angle. The quantum action can be real or purely imaginary, depending on the sign of the discriminant $R(m, n, \beta) = (j + 1/2)^2 \sin^2 \beta - m^2 - n^2 + 2mn \cos \beta$. Imaginary action corresponds to a trajectory in phase space denied by classical mechanics, as the canonical variables are complex [57–59]. Hence, for a fixed rotation angle β , the Wigner d -matrix elements $d_{mn}^j(\beta)$ can be divided into two regions, as shown in fig. 6: a “classically allowed” region with real action, characterized by rapid oscillations; and a “classically forbidden” region with imaginary action, where the matrix elements exponentially decay.

Though the WKB approximation proves accurate in both domains, it diverges near the classical-quantum boundary, which, for most parameter settings, proves to be inconsequential in section 2.3. For instance, in the calculation of $H_j(\beta)$, analysis indicates that the semi-classical approximation fails within a boundary layer of width $\delta \sim j^{-1/3}$, where the number of matrix elements is an infinitesimal compared to the

classically allowed region, and hence the WKB approximation is legitimate to apply to the leading-order asymptotic terms.

Nevertheless, when $\beta \rightarrow n\pi$ ($n \in \mathbb{Z}$), the classically allowed region degenerates from an ellipse to a line, causing the WKB approximation to fail entirely within the neighborhood of width $\Delta\beta \sim j^{-1}$. In this regime, we obtain

$$\mathcal{D}_{i,i+1} \sim \begin{cases} -\ln(2j+1) + O(\beta^2), & \beta_{i,i+1} \rightarrow n\pi ; \\ \ln(2j+1) + O(\beta^2), & \beta_{i-1,i} \rightarrow n\pi \text{ or } \beta_{i+1,i+2} \rightarrow n\pi; \end{cases} \quad (16)$$

which indicates when $\beta_{i,i+1} \rightarrow n\pi$ ($n \in \mathbb{Z}$), the quantum violation approaches the maximum rapidly. This result provides an experimental evidence of contradiction between quantum mechanics and the classical worldview. Remarkably, the magnitude of the quantum violation increases with the scale of the system, while the quantum resources required do not scale accordingly, as the conclusion is drawn from maximally mixed state.

This counterintuitive conclusion arises from the simple fact that, the ensemble average of the time correlation is not equal to the time correlation of the ensemble average. Consequently, the correlations observed from sequential measurements provide a more comprehensive characterization of quantum phenomena, which is inaccessible per analysis of the dynamical evolution of density matrix alone.

3 Discussion

In this study, we established a systematic geometric framework for entropic Leggett–Garg inequalities, providing a unified methodology for deriving inequalities that include higher-order temporal correlations. Numerical results demonstrate that ELGIs exhibit quantum violations across broader parameter regimes than conventional LGIs.

ELGIs also exhibit enhanced robustness against decoherence, maintaining observable violations under stronger dissipation and over extended time intervals. These characteristics position ELGIs as a robust, information-theoretically grounded tool for certifying quantum coherence in noisy or open quantum systems. Notably, the information-theoretic foundation of our framework provides a transparent physical interpretation, enabling the operational distinction between measurement invasiveness and environmental memory effects in open quantum systems.

The WKB analysis in large-spin limit reveals that, for an initially maximally mixed system, quantum violation occurs exclusively in ELGIs of the form $\mathcal{D}_{i,i+1} \geq 0$. In generic parameter regimes, the magnitude of these violations saturates at a finite value, making them increasingly negligible relative to the logarithmically growing system entropy. Nevertheless, in the degenerate case where the rotation angle $\beta_{i,i+1} \rightarrow k\pi$, the quantum violation rapidly approaches the maximal value $\mathcal{D}_{i,i+1} \rightarrow -\ln(2j+1)$. This observation provides a clear pathway for quantum violation detection in large-scale systems and for the detection of fundamental boundaries of quantum mechanics.

These findings offer a complementary perspective on the emergence of classicality at the macroscopic scale. While the fundamental incompatibility between quantum and classical descriptions persists at the macroscopic scale, the prevalence of classical behavior implies a robust suppression mechanism for quantum effects. Unlike the coarse-graining restriction [29], which attributes this transition to limited measurement resolution, our results demonstrate that even under ideal sharp measurements, quantum violations remain effectively negligible in most parameter regimes due to intrinsic information-theoretic scaling constraints. Consequently, macroscopic quantum phenomena do not vanish but persist only in specific, fine-tuned regimes. Our results provide a clear pathway for detecting quantum violations in large-scale systems and for probing the fundamental boundaries of quantum mechanics.

Finally, since the analysis is limited to unitary evolution, a natural question arises as to whether macroscopic quantum violations can still persist in open system subject to dissipation or not. If not, how do such violations diminish as the strength of dissipation increases? Further investigation of this open question will advance our understanding of the decoherence mechanism, by which quantum systems become effectively classical at macroscopic scales, and will clarify the potential of the ELGI framework.

4 Methods

4.1 Construction of higher-order entropic Leggett–Garg inequalities

The derivation of the elementary ELGIs (5) and (6) relies on the geometric structure of the Shannon cone $\Gamma_{\text{Sh}} \subset \mathbb{R}^{2^n - 1}$, which is defined in the full variable space spanned by the joint entropies of all possible subsets of the n random variables. The elementary inequalities together determine the facets of this high-dimensional polyhedral cone. Nevertheless, in experimental settings, only a restricted subset of joint entropies is accessible, typically those of a given correlation order. This limitation corresponds to a projection of Γ_{Sh} onto a lower-dimensional subspace of observable coordinates. To obtain the explicit inequalities that characterize these marginal scenarios, we apply the Fourier–Motzkin elimination procedure [60].

We illustrate this procedure using the $n = 3$ case. Here, the third-order joint entropy $H(Q_1, Q_2, Q_3)$ is unobservable in standard Leggett–Garg scenarios, which involve only sequential pairwise measurements. To obtain constraints on the accessible marginals, we eliminate this term by combining elementary inequalities where it appears with opposite signs. Specifically, we consider the non-negativity of conditional

entropy and conditional mutual information:

$$\mathcal{D}_k = H(Q_1, Q_2, Q_3) - H(Q_i, Q_j) \geq 0, \quad (17)$$

$$\mathcal{D}_{i,j} = H(Q_i, Q_k) + H(Q_j, Q_k) - H(Q_1, Q_2, Q_3) - H(Q_k) \geq 0, \quad (18)$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$. Summing these eliminates the third-order term, yielding the second-order ELGI:

$$\mathcal{D}_{i,j|k} = H(Q_i, Q_k) + H(Q_j, Q_k) - H(Q_k) - H(Q_i, Q_j) \geq 0. \quad (19)$$

This elimination process yields a complete set of constraints on the second-order joint entropies. It recovers the inequality derived by Devi *et al.* [40] as a special case. We have applied this method to derive ELGI of each order for $n = 3$ and $n = 4$ scenarios in fig. 2. The complete list of derived inequalities and the details of the elimination steps are provided in the Supplementary Information.

4.2 Numerical simulation of dissipative dynamics

To investigate environmental memory effects, we simulate a trapped-ion qubit system with tunable dissipation [48, 49]. The full dynamics of this four-level system is described by the Lindblad master equation

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\{\hat{L}^\dagger\hat{L}, \hat{\rho}\}, \quad (20)$$

where the Hamiltonian takes the form $\hat{H} = \omega_0 |0\rangle\langle 1| + \omega |1\rangle\langle e| + \text{h.c.}$ (in the rotating frame), and the dissipative process is captured by the Lindblad jump operator $\hat{L} = \sqrt{\gamma} |g\rangle\langle e|$. In the regime where the intermediate state $|e\rangle$ decays much faster than the coherent couplings ($\gamma \gg \omega_0, \omega$), the population of $|e\rangle$ can be adiabatically eliminated,

giving rise to an effective three-level model. The effective Hamiltonian restricted to the qubit subspace is $\hat{H}_{\text{eff}} = \omega_0 |0\rangle \langle 1| + \text{h.c.}$, and the effective decay from $|1\rangle$ to the sink state $|g\rangle$ is described by

$$\hat{L}_{\text{eff}} = \sqrt{\Gamma_{\text{eff}}} |g\rangle \langle 1|, \quad \Gamma_{\text{eff}} = \frac{|\omega|^2 \gamma}{\Delta^2 + (\gamma/2)^2}, \quad (21)$$

where $\Delta = E_e - E_1$ denotes the detuning. We numerically integrate the master equation to compute the sequential measurement probabilities $p(q_1, q_2, q_3)$ for the observable $\hat{Q}$ at times t_1, t_2, t_3 . These probabilities are used to evaluate the ELGI parameter $\mathcal{D}_{1,3}$ in fig. 4 across varying dissipation strengths γ/ω and time intervals Δt .

All numerical simulations were performed using custom Python code developed for this work. The code comprises a numerical framework for spin- j dynamics based on the generalized Bloch representation with irreducible tensor operators, together with dedicated routines for constructing and evaluating entropic, standard, and Wigner-type Leggett-Garg inequalities. The complete code, including all scripts used to generate the figures is available in the accompanying repository (see Code Availability).

4.3 Simplification of ELGIs for maximally mixed states

For a spin- j system initialized in the maximally mixed state $\hat{\rho}_0 = \hat{I}/(2j+1)$, the quantum mechanical joint probabilities simplify considerably due to the $SU(2)$ invariance of the initial state. As shown in section 2.3, the joint entropy for a sequence of measurements at times $t_i, \dots, t_l$ reduces to

$$H(Q^{(\alpha)}(t_i), \dots, Q^{(\alpha)}(t_l)) = \ln(2j+1) + \sum_{s,s'} H_j(\beta_{s,s'}) , \quad (22)$$

where the sum runs over adjacent measurement times in the α -th experiment, and $H_j(\beta)$ denotes the entropy of the Wigner d -matrix defined in eq. (12).

Substituting these expressions into the elementary ELGI parameters $\mathcal{D}_i$ and $\mathcal{D}_{i,k}$ given by eqs. (5) and (6), we obtain simplified forms that depend solely on the rotation angles $\beta_{i,j}$. For interior measurement times ($1 < i < n$), the parameters are

$$\mathcal{D}_i = H_j(\beta_{i-1,i}) + H_j(\beta_{i,i+1}) - H_j(\beta_{i-1,i+1}) ; \quad (23)$$

$$\mathcal{D}_{i,k} = \begin{cases} H_j(\beta_{i,i+2}) + H_j(\beta_{i-1,i+1}) - H_j(\beta_{i,i+1}) - H_j(\beta_{i-1,i+2}), & k = i + 1 ; \\ 0, & k > i + 1 ; \end{cases} \quad (24)$$

where the condition $k \neq n$ is assumed. Notably, $\mathcal{D}_{i,k}$ vanishes identically for non-adjacent indices ($k > i + 1$), reflecting the Markovian structure of the sequential measurement statistics for the maximally mixed state.

For boundary measurement times, the expressions simplify further:

$$\mathcal{D}_i = \begin{cases} H_j(\beta_{1,2}), & i = 1 ; \\ H_j(\beta_{n-1,n}), & i = n . \end{cases} \quad (25)$$

$$\mathcal{D}_{i,i+1} = \begin{cases} H_j(\beta_{1,3}) - H_j(\beta_{1,2}), & i = 1, n \neq 2 ; \\ H_j(\beta_{n-2,n}) - H_j(\beta_{n-1,n}), & i = n - 1, n \neq 2 ; \\ -H_j(\beta_{1,2}), & i = 1, n = 2 . \end{cases} \quad (26)$$

These explicit formulas enable efficient numerical evaluation of ELGIs for arbitrary measurement configurations.

4.4 WKB analysis in the macroscopic limit

The asymptotic behavior of ELGI violations in the large-spin limit ($j \rightarrow \infty$) is analyzed using the WKB approximation for the Wigner d -matrix elements $d_{mn}^j(\beta)$. As noted by Wigner [61], the spin system under consideration admits the symmetric top

as a classical analogue, and the Wigner d -matrix elements satisfy the second-order differential equation [62]

$$\left(\frac{d^2}{d\beta^2} + \frac{\cos \beta}{\sin \beta} \frac{d}{d\beta} + j(j+1) - \frac{m^2 - 2mn \cos \beta + n^2}{\sin^2 \beta} \right) d_{mn}^j(\beta) = 0. \quad (27)$$

To facilitate the WKB analysis, we introduce the scaled variables $J = j + 1/2$, $\mu = m/J$, $\nu = n/J$. We employ the uniform asymptotic approximation based on Airy functions [63] to properly handle the turning points $\beta_{\pm}$, where the discriminant $R(\beta) = \sin^2 \beta - \mu^2 - \nu^2 + 2\mu\nu \cos \beta = 0$. The approximate solution takes the form

$$d_{nm}^j(\beta) = (-1)^{\max\{0, n-m\}} \sqrt{\frac{2}{J}} \cdot \text{Ai}(\zeta) \left(\frac{-\zeta}{R(\beta)} \right)^{1/4}, \quad (28)$$

where the auxiliary variable $\zeta(\beta)$ is determined implicitly by

$$\zeta \left(\frac{d\zeta}{d\beta} \right)^2 = -\frac{J^2}{\sin^2 \beta} R(\beta). \quad (29)$$

Explicit solutions for $\zeta(\beta)$, expressed in terms of the action function $S(\beta)$, are provided in the Supplementary Information. This analytical framework enables us to approximate the Wigner d -matrix elements $d_{nm}^j(\beta)$ as functions of continuous phase-space coordinates (μ, ν) , effectively replacing the discrete lattice of quantum numbers (m, n) .

The entropic quantity $H_j(\beta)$ is evaluated by partitioning the sum over magnetic quantum numbers into three regions: the classically allowed region (where $R(\mu, \nu) > 0$), the classically forbidden region (where $R(\mu, \nu) < 0$), and a boundary layer of width $\varepsilon \sim O(J^{-2/3})$. In the large-spin limit, contributions from the boundary and forbidden regions become negligible relative to the allowed region, and the discrete sums can be

approximated by continuous phase-space integrals. This yields the asymptotic result:

$$H_j(\beta) = \ln(2\pi J) + \ln|\sin \beta| - \frac{5}{2} + o(1). \quad (30)$$

In the singular regime where $\beta \rightarrow n\pi$ ($n \in \mathbb{Z}$), the standard WKB approximation fails due to the coalescence of turning points and the degeneration of the classically allowed region from an ellipse to a line. For these cases, we employ a small-angle asymptotic expansion based on Wigner's series representation [61]:

$$d_{nm}^j(\beta) = \sqrt{(j+m)!(j-m)!(j+n)!(j-n)!} \sum_{s=s_{\min}}^{s_{\max}} \frac{(-1)^{n-m+s} (\cos(\beta/2))^{2j+m-n-2s} (\sin(\beta/2))^{n-m+2s}}{(j+m-s)!s!(n-m+s)!(j-n-s)!}, \quad (31)$$

where $s_{\min} = \max\{0, m-n\}$ and $s_{\max} = \min\{j+m, j-n\}$. Retaining only the leading-order term, this refined analysis captures the logarithmic divergence of the ELGI violation in the singular regime. Detailed derivations and error bounds are presented in the Supplementary Information.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Code Availability. The code used to generate the results presented in this study is openly available on GitHub at: [PolarisMegrez/Entropic-Leggett-Garg-Inequalities](https://github.com/PolarisMegrez/Entropic-Leggett-Garg-Inequalities).

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Q.H.C., M.C.Y., A.X.L., and C.F.Q. performed the numerical calculation. All authors contributed to discussing, editing, and revising the manuscript.

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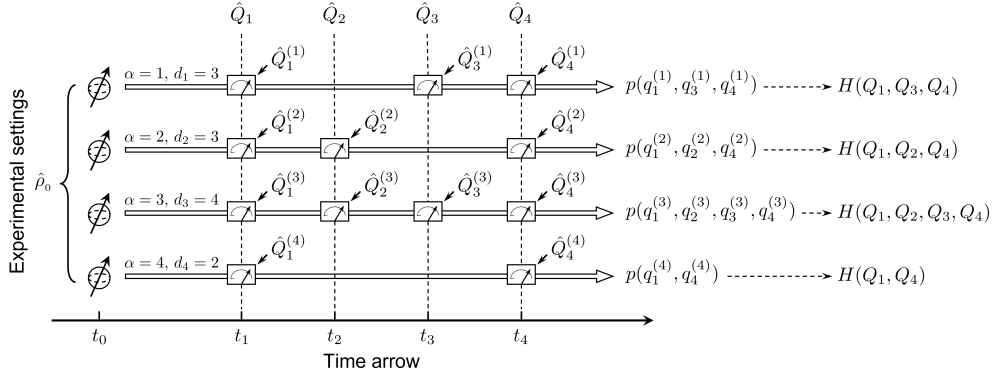


Fig. 1 Schematic representation of the experimental setting for ELGI. The elementary type ELGI $\mathcal{D}_{2,3} = H(Q_1, Q_3, Q_4) + H(Q_1, Q_2, Q_4) - H(Q_1, Q_2, Q_3, Q_4) - H(Q_1, Q_4) \geq 0$ is taken as an example, where Q_i denotes $Q(t_i)$ for simplicity. All the M experiments begin with the same initial state $\hat{\rho}_0$. The joint entropy must be obtained from independent experiments rather than derived from the marginal probability distribution of a single experiment with measurements at all time points, as later measurements could be affected by earlier ones.

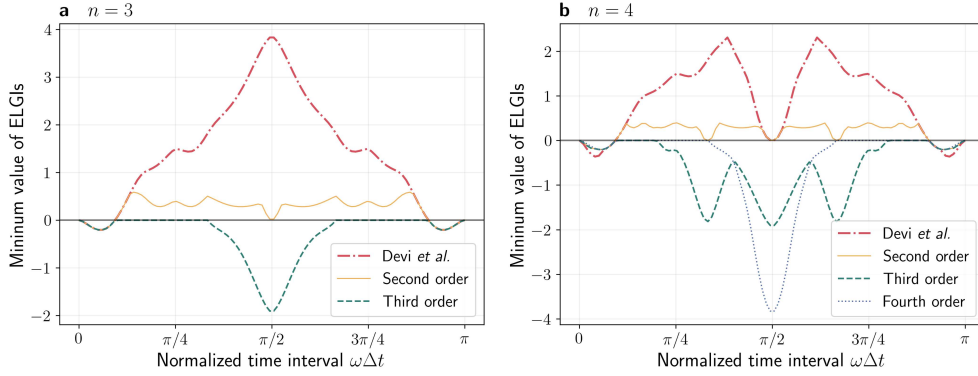


Fig. 2 Comparison of quantum violations of ELGIs involving different orders of temporal correlations. All possible Shannon-type ELGIs are grouped according to their order. ELGIs that can be written as sums of other ELGIs of the same order are excluded, since they do not provide any additional information about quantum violations. Numerical calculations are performed for a spin-2 system with Hamiltonian $\hat{H} = \omega \hat{J}_y$ and observable $\hat{Q} = \hat{J}_z$, considering (a) $n = 3$ and (b) $n = 4$ equally spaced measurement time points, $t_{i+1} - t_i = \Delta t$. The complete list of inequalities involved in the figure is provided in the Supplementary Information.

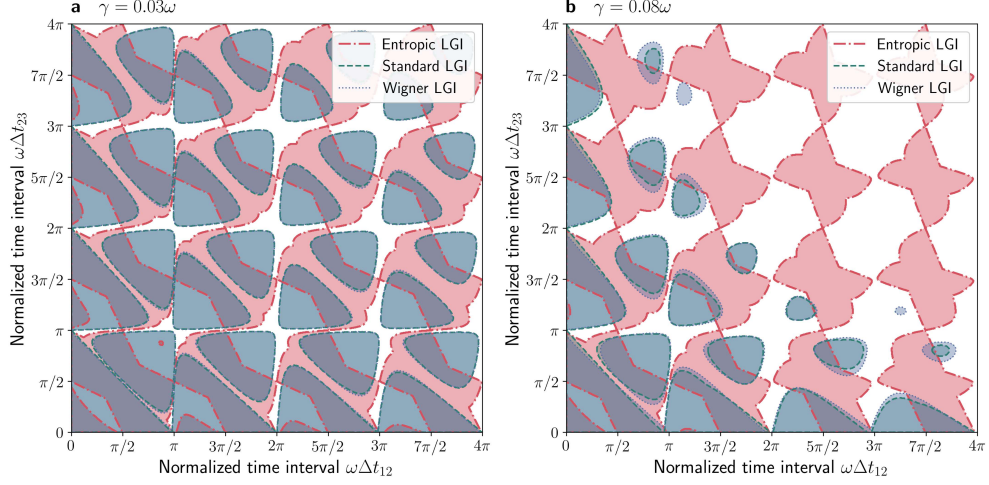


Fig. 3 Comparison of quantum-violation regions for ELGIs, SLGIs, and WLGIs. Shaded regions indicate parameter regimes where at least one inequality in each family is violated: red for ELGIs, green for SLGIs, and blue for WLGIs. Numerical results are obtained for a spin-1 system governed by the Lindblad master equation $d\hat{\rho}/dt = -i[\omega\hat{J}_y, \hat{\rho}] + \gamma(\hat{J}_-\hat{\rho}\hat{J}_+ - \frac{1}{2}\{\hat{J}_+\hat{J}_-, \hat{\rho}\})$ with the observable $\hat{Q} = \hat{J}_z$ and $n = 3$ measurement times. The panels correspond to different decay rates: (a) $\gamma = 0.03\omega$ and (b) $\gamma = 0.08\omega$, demonstrating that ELGI violations persist under stronger dissipation, whereas violations in the other families may disappear.

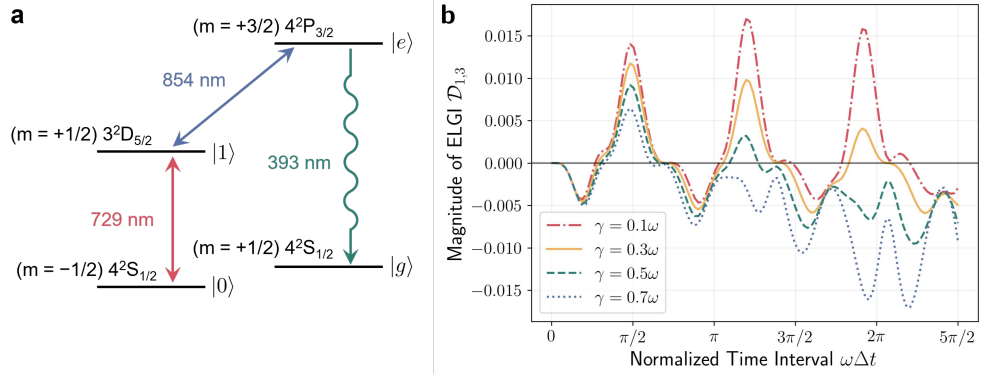


Fig. 4 Probing non-Markovianity via ELGI violations. (a) Schematic of the experimental setup for a trapped $^{40}\text{Ca}^+$ qubit with tunable dissipation [49]. The qubit subspace is formed by Zeeman levels $|0\rangle = |4^2S_{1/2}, m = -1/2\rangle$, $|1\rangle = |3^2D_{5/2}, m = +1/2\rangle$, and the sink state $|g\rangle = |4^2S_{1/2}, m = +1/2\rangle$. The $|0\rangle \leftrightarrow |1\rangle$ transition is driven by a 729-nm laser, while a tunable loss from $|1\rangle$ to $|g\rangle$ is induced via the short-lived intermediate state $|e\rangle$ using an 854-nm laser. (b) Numerical ELGI parameter $\mathcal{D}_{1,3}$ for the dissipative qubit system at different dissipation rates $\gamma/\omega = 0.1, 0.3, 0.5, 0.7$ (as shown in legend), with $n = 3$ equally spaced measurement times $t_{i+1} - t_i = \Delta t$. Positive violations of the classical Markovianity condition are observed, which gradually weaken as γ increases. Dynamical equations and simulation details are provided in the Supplementary Information.

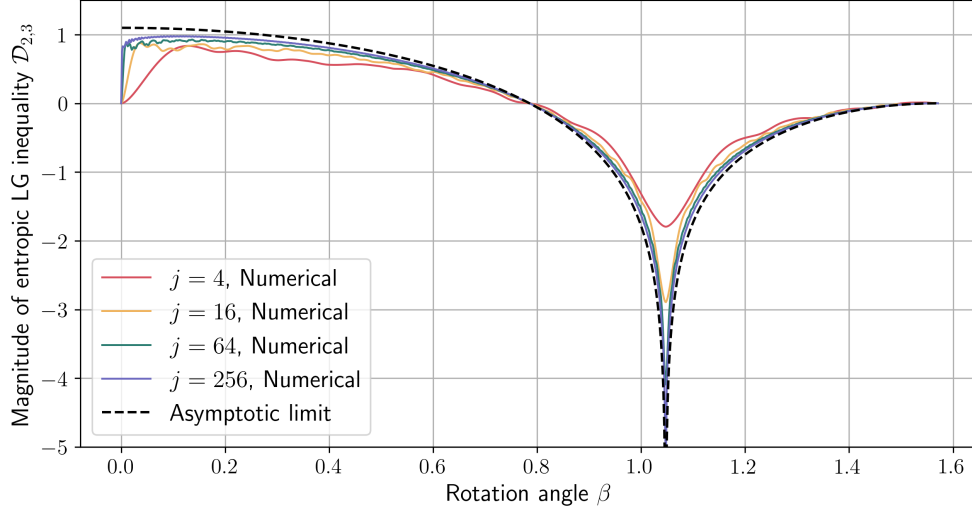


Fig. 5 Comparison of numerical results with asymptotic limits. $n = 3$ measurement time points are considered, and the rotation angles are set as $\beta_{1,2} = 2\beta$ and $\beta_{2,3} = \beta$, yielding an asymptotic limit of $\mathcal{D}_{2,3} = -\ln|\sin 3\beta/\sin \beta|$. Because of the symmetry of the Wigner d -matrix, calculations are restricted to $0 \leq \beta \leq \pi$ without loss of generality. Apart from the divergence at $\beta = \pi/3$, where $\beta_{1,3} \rightarrow \pi$, the numerical results closely match the asymptotic limit as the spin j increases.

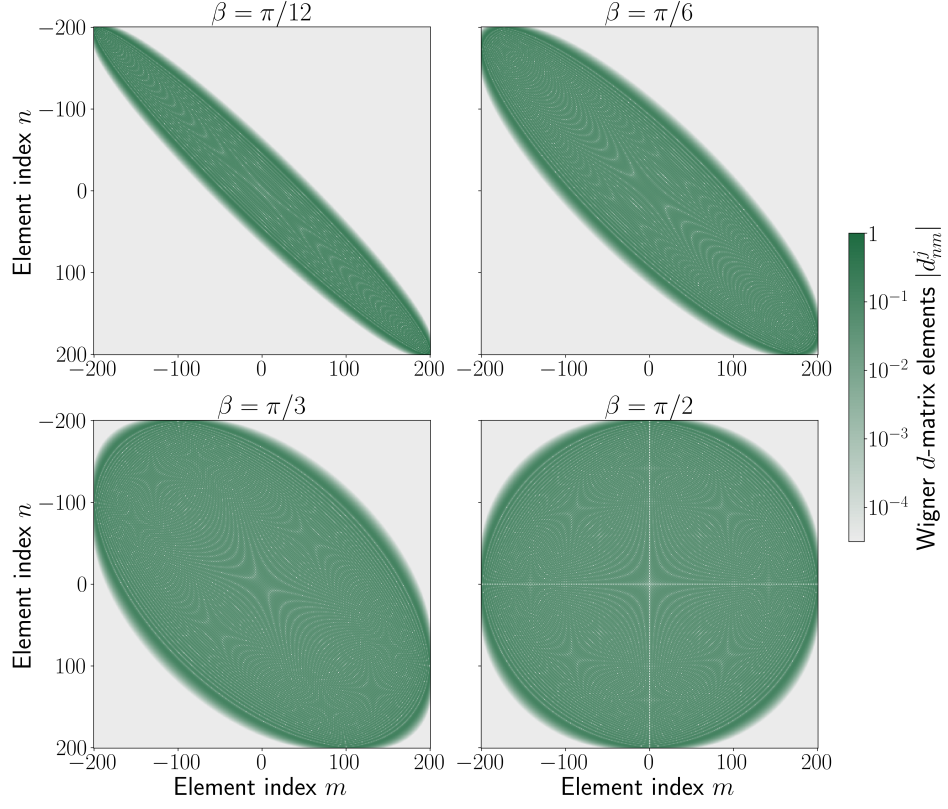


Fig. 6 Numerical calculation of Wigner d -matrix elements. Wigner d -matrix elements $d_{mn}^j(\beta)$ are calculated at rotation angles $\beta = \pi/12, \pi/6, \pi/3, \pi/2$ for $j = 200$, and plotted on semi-logarithmic axes. The elliptical region with rapid oscillations corresponds to the classically allowed region, while outside the boundary the matrix elements exhibit exponential decay in classically forbidden region. As β decreases from $\pi/2$ to zero, the ellipse's eccentricity increases until it degenerates into a line at $\beta = 0$ (not shown).

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