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Zhenfeng Cai
Hubei University of Technology
Kit Ian Kou
kikou@umac.mo
University of Macau
Weinian Zhang
Sichuan University
Yanglin Zhang
Guangxi University

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Overcoming Non-Commutativity: New Methods for Linear Quaternion Differential Equations

Zhenfeng Cai[1], Kit Ian Kou *[2], Weinian Zhang[3], and Yanglin Zhang[4]

[1]School of Science, Hubei University of Technology, Wuhan, Hubei, 430000, China.
[2]Department of Mathematics, Faculty of Science and Technology, University of Macau, Taipa, Macao, 999078, China.
[3]School of Mathematics, Sichuan University, Chengdu, Sichuan, 611130, China.
[4]School of Mathematics, Guangxi University, Nanning, Guangxi, 530004, China.

Abstract

The non-commutativity of quaternion multiplication presents a fundamental obstacle in analyzing linear quaternion-valued differential equations (QDEs). While the exponential solution for homogeneous linear QDEs by Campos and Mawhin is a cornerstone of the field, its reliance on a restrictive commutativity condition limits it to a narrow, complex-like subclass of functions. This work overcomes this limitation by introducing a novel algorithmic framework that solves the homogeneous initial value problem without any commutativity assumptions. We first demonstrate that the commutativity condition is equivalent to confining the dynamics to a complex-valued subspace. Our primary contribution is a method that systematically reduces the QDE to a solvable real nonlinear differential equation. We further derive closed-form solutions for key non-commutative cases. These results dramatically expand the solvable landscape of linear QDEs, with direct applications in control theory, quantum mechanics, and hypercomplex signal processing, where non-commutative dynamics are intrinsic. We demonstrate the power of our approach by applying it to Robinson's Quaternion Kinematical Differential Equations and a problem in medical image communication security.

*Corresponding author: kikou@umac.mo

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1 Introduction

Since their discovery by Sir William Rowan Hamilton, quaternions have proven indispensable for representing rotations and orientations in three-dimensional space. Their utility extends far beyond pure mathematics, forming the computational backbone of modern aerospace engineering [1], computer graphics [2], and robotics [3], where they are prized for their numerical stability and computational efficiency. More recently, the hypercomplex structure of quaternions has enabled significant advances in signal and image processing, providing a holistic framework for analyzing vector-valued data such as color images [4] and signals from vector sensors [5].

The natural progression to dynamical systems formulated directly in the quaternion domain leads to Quaternion-Valued Differential Equations (QDEs). These equations offer a powerful framework for modeling physical phenomena with inherent three-dimensional dynamics, including rigid body motion [6], quantum systems with non-Abelian gauge structures [7], and advanced filtering processes [8]. The appeal of QDEs derives not only from their analogy to complex formulations of planar vector fields but also from the mathematical convenience of quaternions in applications ranging from robotic manipulation [29] to fluid mechanics [19] and differential geometry [23].

However, the analytical treatment of QDEs confronts a fundamental obstacle: the **non-commutativity of quaternion multiplication**. The property that $q_1 q_2 \neq q_2 q_1$ in general invalidates many analytical techniques standard in the theory of real and complex-valued differential equations, thereby significantly impeding the development of a comprehensive theory for QDEs [14, 15, 16, 17, 26, 27, 32, 39].

A pivotal advancement was achieved by Campos and Mawhin [12], who established a foundational theory for linear QDEs of the form:

$$q'(t) = a(t)q(t) + f(t), \tag{1.1}$$

where $a, f : \mathbb{R} \to \mathbb{H}$ are continuous functions. For the homogeneous equation $q' = a(t)q$, they provided a fundamental solution of exponential form, $\exp(A(t))q(0)$, under the critical **restrictive commutativity condition**:

$$a(t)A(t) = A(t)a(t), \quad \forall t, \tag{1.2}$$

where $A(t) = \int_0^t a(s)\,ds$. While naturally satisfied for constant coefficients, this condition proves highly restrictive for time-varying systems. Nevertheless, it has served as the cornerstone for subsequent work, including the analysis of periodic solutions [12, 13, 34, 35] and two-dimensional systems [24, 28].

This dependence on the commutativity condition creates a significant theoretical gap, limiting applicability to a narrow, "complex-like" subset of quaternionic functions and leaving the vast landscape of genuinely non-commutative linear QDEs largely unexplored. This paper addresses two fundamental questions arising from this limitation:

1. What is the precise mathematical nature of the commutativity condition, and what class of quaternionic functions does it actually permit?

2. Can we develop systematic methods to solve homogeneous linear QDEs without imposing this condition?

Our work directly confronts the challenge of non-commutativity. We first provide a definitive resolution to question (Q1) by proving that the commutativity condition necessarily constrains the coefficient function $a(t)$ to complex-like behavior, effectively restricting system dynamics to a two-dimensional subspace within $\mathbb{H}$. Our primary and most significant contribution addresses question (Q2) through the introduction of a novel algorithmic framework for solving initial value problems for homogeneous linear QDEs **without any commutativity assumptions**. This method systematically reduces the QDE to a solvable real, nonautonomous nonlinear differential equation. Furthermore, we derive explicit closed-form solutions for several key non-commutative cases.

These results substantially expand the solvable landscape of linear QDEs, advancing the field beyond the limitations of the commutativity condition. Our work not only deepens the theoretical understanding of quaternionic systems but also opens new avenues for practical applications in advanced control theory, quantum mechanics, and hypercomplex signal processing, where full non-commutativity is intrinsic rather than exceptional.

The paper is organized as follows: Section 2 reviews essential mathematical preliminaries on quaternions and linear QDEs. Section 3 provides a critical analysis of the commutativity condition and characterizes its solutions. Section 4 presents our main contributions, including the general solution algorithm and closed-form solutions. Section 5 presented the exact solutions in the special cases of the constructive algorithm via Picard iteration for solving the decisive equation. Section 6 illustrates the theory with representative examples, while Section 7 demonstrates the application of the proposal theory to Robinson's Quaternion Kinematical Differential equation and Section 8 present the communication security during the transmission of medical images. That is encryption and decryption of medical image with proposal theory. Section 9 discusses potential applications. Concluding remarks are presented in Section 10.

2 Preliminaries on Quaternions and Linear QDEs

This section establishes the foundational concepts and notation for quaternions and linear quaternion-valued differential equations (QDEs) that are central to the results presented in this paper.

2.1 Algebraic Structure of Quaternions

The quaternion algebra $\mathbb{H}$ extends the complex numbers to a four-dimensional associative but non-commutative division algebra. Any quaternion $q \in \mathbb{H}$ is uniquely expressed as

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ are its components. The basis elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ obey the fundamental Hamiltonian relations [1]:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

which directly imply the anti-commutative rules:

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

We define the scalar part of q as $S(q) = q_0$ and the vector part as $\underline{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. This allows the quaternion to be written concisely as $q = S(q) + \underline{q}$. Key algebraic operations include:

- **Conjugate:** $\overline{q} = S(q) - \underline{q}$.
- **Norm:** $|q| = \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.
- **Inverse:** For $q \neq 0$, $q^{-1} = \overline{q}/|q|^2$.

The norm is multiplicative, i.e., $|pq| = |p||q|$ for any $p, q \in \mathbb{H}$. The inner product of two quaternions is given by $\langle p, q\rangle = S(p\overline{q}) = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3$, which satisfies $|q| = \sqrt{\langle q, q\rangle}$.

The non-commutativity of multiplication, exemplified by $\mathbf{ij} \neq \mathbf{ji}$, is the defining feature of $\mathbb{H}$ and the primary source of analytical challenges in the study of QDEs [7, 25].

2.2 Linear Quaternion-Valued Differential Equations

We consider the first-order linear nonhomogeneous QDE of the form

$$q'(t) = a(t)q(t) + f(t), \tag{2.1}$$

where $a, f : I \subset \mathbb{R} \to \mathbb{H}$ are continuous functions on an interval I, and $q : I \to \mathbb{H}$ is the unknown differentiable function. The corresponding homogeneous equation is

$$q'(t) = a(t)q(t). \tag{2.2}$$

A cornerstone in the theory of linear QDEs was established by Campos and Mawhin [9]. Their approach relies on the quaternion exponential function, defined for $q \in \mathbb{H}$ by the absolutely convergent series

$$\exp(q) := \sum_{n=0}^{\infty} \frac{q^n}{n!}.$$

A fundamental solution to the homogeneous equation (2.2) can be constructed using this exponential, but with a critical constraint. Let $A(t) = \int_{t_0}^{t} a(s)\,ds$. If the following *commutativity condition* holds:

$$a(t)A(t) = A(t)a(t) \quad \text{for all } t \in I, \tag{2.3}$$

then the function $\exp(A(t))$ is a solution of (2.2) and the initial value problem with $q(t_0) = q_0$ has the unique solution $q(t) = \exp(A(t))q_0$.

This formalism further leads to a *Variation of Constants Formula* for the nonhomogeneous equation (2.1):

$$q(t) = \exp(A(t)) \left[q_0 + \int_{t_0}^{t} \exp(-A(s)) f(s)\,ds \right].$$

The condition (2.3) is trivially satisfied for constant coefficients but represents a significant restriction for time-dependent $a(t)$, as it effectively confines the dynamics to a complex subspace within $\mathbb{H}$ [25]. The primary objective of this work is to develop a solution theory for linear QDEs that moves beyond this commutativity constraint.

3 A Critical Analysis of the Commutativity Condition

The commutativity condition $a(t)A(t) = A(t)a(t)$, where $A(t) = \int_0^t a(s)\,ds$, has been a cornerstone of the existing theory for linear quaternion-valued differential equations (QDEs). While it enables the construction of an elegant exponential solution, this condition is highly restrictive. This section moves beyond its application to provide a rigorous deconstruction, precisely characterizing the mathematical constraints it imposes and revealing the severe limitations these constraints place on modeling quaternionic dynamics.

3.1 A Necessary and Sufficient Condition for Commutativity

The analysis begins with a fundamental lemma that characterizes commutativity between two quaternion-valued functions. This result is the key to unlocking the structure of the commutativity condition for QDEs.

Lemma 1. *Let $p(t) = p_0(t) + \underline{p}(t)$ and $q(t) = q_0(t) + \underline{q}(t)$ be quaternion-valued functions, where $\underline{p}(t)$ and $\underline{q}(t)$ denote their vector parts. If $|\underline{q}(t)| \neq 0$ for all t, then $p(t)q(t) = q(t)p(t)$ for all t if and only if their vector parts are parallel. That is, there exists a real-valued function $\lambda(t)$ such that $\underline{p}(t) = \lambda(t)\underline{q}(t)$.*

Proof. The quaternion product, expressed in vector form [30, §1.13, §1.14], is given by:

$$\begin{aligned} p(t)q(t) &= (p_0(t)q_0(t) - \underline{p}(t)\cdot\underline{q}(t)) + p_0(t)\underline{q}(t) + q_0(t)\underline{p}(t) + \underline{p}(t)\times\underline{q}(t), \\ q(t)p(t) &= (q_0(t)p_0(t) - \underline{q}(t)\cdot\underline{p}(t)) + q_0(t)\underline{p}(t) + p_0(t)\underline{q}(t) + \underline{q}(t)\times\underline{p}(t). \end{aligned}$$

The equality $p(t)q(t) = q(t)p(t)$ holds if and only if the vector parts of both expressions are equal. Since the dot product is commutative and the cross product is anti-commutative, this vector equality simplifies to the condition:

$$\underline{p}(t) \times \underline{q}(t) \equiv 0. \tag{3.1}$$

This cross product vanishes if and only if the vectors $\underline{p}(t)$ and $\underline{q}(t)$ are parallel for all t. Given $|\underline{q}(t)| \neq 0$, this is equivalent to the existence of a real-valued function $\lambda(t)$ such that $\underline{p}(t) = \lambda(t)\underline{q}(t)$. □

Equipped with this lemma, we now present the main theorem, which provides a complete characterization of the commutativity condition for linear QDEs.

Theorem 3.1 (Characterization of the Commutativity Condition)**.** *Let $a(t) = a_0(t) + a_1(t)\mathbf{i} + a_2(t)\mathbf{j} + a_3(t)\mathbf{k}$ be a continuous quaternion-valued function, and define $B(t) = B(0) + \int_0^t a(s)\,ds$, where $B(0) = B_0(0) + B_1(0)\mathbf{i} + B_2(0)\mathbf{j} + B_3(0)\mathbf{k}$ is an arbitrary constant quaternion. Then, the commutativity condition*

$$a(t)B(t) = B(t)a(t) \quad \textit{for all } t \in [0, T]$$

holds if and only if the vector part of $a(t)$ is pointwise parallel to the vector part of $B(0)$. Specifically, there exists a real-valued function $\kappa(t)$ such that

$$(a_1(t), a_2(t), a_3(t)) = \kappa(t)(B_1(0), B_2(0), B_3(0)), \quad \forall t \in [0, T]. \tag{3.2}$$

Furthermore, if $a(t)$ is differentiable, this condition implies that $a(t)$ commutes with its own derivative, i.e., $a(t)a'(t) = a'(t)a(t)$ for all t.

Proof. Let $A(t) = \int_0^t a(s)\,ds = A_0(t) + A_1(t)\mathbf{i} + A_2(t)\mathbf{j} + A_3(t)\mathbf{k}$. Then $B(t) = B(0) + A(t)$, and its components satisfy $B'_\ell(t) = a_\ell(t)$ for $\ell = 0, 1, 2, 3$.

The proof proceeds by analyzing the parallelism condition from Lemma 1 across all possibilities for $\underline{B}(t)$.

Case 1: $\underline{B}(t) \neq 0$ for all $t \in [0, T]$.
By Lemma 1, $a(t)B(t) = B(t)a(t)$ holds if and only if $\underline{a}(t) = \lambda(t)\underline{B}(t)$ for some real function $\lambda(t)$. Component-wise, this means:

$$a_i(t) = \lambda(t)B_i(t), \quad \text{for } i = 1, 2, 3.$$

Substituting $a_i(t) = B'_i(t)$ yields the system $B'_i(t) = \lambda(t)B_i(t)$. Solving these differential equations gives:

$$B_i(t) = B_i(0) \exp\left(\int_0^t \lambda(s)\,ds\right), \quad \text{and thus} \quad a_i(t) = B_i(0)\lambda(t) \exp\left(\int_0^t \lambda(s)\,ds\right).$$

This establishes the proportionality $a_1(t) : a_2(t) : a_3(t) \equiv B_1(0) : B_2(0) : B_3(0)$.

Case 2: $\underline{B}(t_0) = 0$ for some isolated t_0.
The argument from Case 1 holds on $[0, T] \setminus \{t_0\}$. By the continuity of $a(t)$, the proportionality extends to the point t_0.

Case 3: $\underline{B}(t) \equiv 0$.
Then $B_i(t) \equiv 0$ and thus $a_i(t) \equiv 0$ for $i = 1, 2, 3$, trivially satisfying the proportionality condition.

Converse: If (3.2) holds, then $\underline{a}(t)$ is parallel to the fixed vector $(B_1(0), B_2(0), B_3(0))$. A direct computation shows that $\underline{B}(t)$ is also parallel to this same vector. Therefore, $\underline{a}(t)$ and $\underline{B}(t)$ are parallel for all t, and by Lemma 1, $a(t)B(t) = B(t)a(t)$.

Commutation with Derivative: The proportionality condition (3.2) implies that both $\underline{a}(t)$ and $\underline{a}'(t)$ are parallel to the fixed direction $(B_1(0), B_2(0), B_3(0))$. Consequently, they are parallel to each other, ensuring $a(t)a'(t) = a'(t)a(t)$. □

3.2 Structural Implications and the Path Forward

The characterization in Theorem 3.1 reveals that the commutativity condition is not a mild technicality but a severe structural constraint. Its most profound consequence is that it reduces the quaternionic system to a complex-like one.

Corollary 1. *If $B_1^2(0) + B_2^2(0) + B_3^2(0) \neq 0$, the commutativity condition forces $a(t)$ to be a* complex-like function. *It can be written as*

$$a(t) = a_0(t) + g(t)\mathbf{I}, \tag{3.3}$$

where $g(t)$ is a real-valued function and $\mathbf{I}$ is the constant unit pure quaternion

$$\mathbf{I} = \frac{B_1(0)\mathbf{i} + B_2(0)\mathbf{j} + B_3(0)\mathbf{k}}{\sqrt{B_1^2(0) + B_2^2(0) + B_3^2(0)}}.$$

The system's dynamics are confined to the two-dimensional subspace $\mathbb{C}_{\mathbf{I}} = \{x + y\mathbf{I} : x, y \in \mathbb{R}\} \subset \mathbb{H}$, which is isomorphic to the complex plane.

This reduction has two critical implications:

- **Loss of Generality:** It excludes the vast majority of time-dependent quaternion functions, particularly those whose vector part changes direction—a common scenario in physical systems like rotations with time-varying axes or complex quantum dynamics.
- **Trivialization of Non-Commutativity:** It nullifies the essential feature that makes quaternions powerful for three-dimensional modeling, reducing the rich, four-dimensional non-commutative algebra to a commutative subalgebra isomorphic to $\mathbb{C}$.

In conclusion, while the commutativity condition provides an elegant solution theory for a narrow class of problems, it fundamentally limits the expressive power of QDEs. This critical analysis underscores the necessity of developing solution methods that do not rely on this condition. The following sections are dedicated to this goal, presenting a general framework for solving linear QDEs that fully embraces their non-commutative nature.

4 Solution Structure for Pure Quaternion Differential Equations

4.1 Superposition Principle and Solution Characterization

Linear quaternion-valued differential equations (QDEs) exhibit a distinctive superposition principle that provides a powerful analytical framework, though it differs fundamentally from the classical vector space structure of real and complex systems. As established in [18, 24], solutions form a right module over the quaternion ring, yielding the general solution structure for the pure QDE:

$$y'(t) = \underline{a}(t)y(t). \tag{4.1}$$

The general solution of (4.1) admits the representation:

$$y(t) = q(t)\mathbf{C}, \tag{4.2}$$

where $q(t)$ is any nontrivial solution and $\mathbf{C}$ is an arbitrary quaternionic constant. This formulation underscores the right-linear algebraic structure inherent to quaternionic systems.

4.2 Norm Preservation in Pure QDEs

A fundamental property of pure QDEs is the conservation of magnitude, as formalized in the following result.

Lemma 2 (Norm Preservation for Pure QDEs)**.** *Let* $y(t) = y_0(t) + y_1(t)\mathbf{i} + y_2(t)\mathbf{j} + y_3(t)\mathbf{k}$ *be a solution of the pure QDE:*

$$y'(t) = \underline{a}(t)y(t), \tag{4.3}$$

where $\underline{a}(t) = a_1(t)\mathbf{i} + a_2(t)\mathbf{j} + a_3(t)\mathbf{k}$ *is continuous. Then the Euclidean norm* $|y(t)|$ *remains constant for all* t*:*

$$\frac{d}{dt}|y(t)|^2 = 0. \tag{4.4}$$

Proof. The system expands component-wise as:

$$\begin{aligned} y_0' &= -a_1 y_1 - a_2 y_2 - a_3 y_3, \\ y_1' &= a_1 y_0 - a_3 y_2 + a_2 y_3, \\ y_2' &= a_2 y_0 + a_3 y_1 - a_1 y_3, \\ y_3' &= a_3 y_0 - a_2 y_1 + a_1 y_2. \end{aligned}$$

Differentiating $|y(t)|^2$ yields:

$$\begin{aligned} \frac{d}{dt}|y(t)|^2 &= 2(y_0 y_0' + y_1 y_1' + y_2 y_2' + y_3 y_3') \\ &= 2[-a_1 y_0 y_1 - a_2 y_0 y_2 - a_3 y_0 y_3 \\ &\quad + a_1 y_0 y_1 - a_3 y_1 y_2 + a_2 y_1 y_3 \\ &\quad + a_2 y_0 y_2 + a_3 y_1 y_2 - a_1 y_2 y_3 \\ &\quad + a_3 y_0 y_3 - a_2 y_1 y_3 + a_1 y_2 y_3] = 0. \end{aligned}$$

The complete cancellation establishes norm preservation. □

Remark 1. *This conservation property is specific to pure QDEs. For general QDEs* $q'(t) = a(t)q(t)$ *with* $a(t) = a_0(t) + \underline{a}(t)$*, the scalar component* $a_0(t)$ *generally induces norm evolution.*

4.3 Normalization and Problem Reduction

Lemma 2 enables normalization of solutions to unit magnitude, yielding an elegant reformulation of the initial value problem:

$$y'(t) = \underline{a}(t)y(t), \quad y(t_0) = \mathbf{C}_0. \tag{4.5}$$

The solution factors as:

$$y(t) = \phi(t)\mathbf{C}_0, \tag{4.6}$$

where $\phi(t)$ satisfies the normalized problem:

$$\phi'(t) = \underline{a}(t)\phi(t), \quad \phi(t_0) = 1. \tag{4.7}$$

This reduction demonstrates that the general solution constructively derives from a single normalized solution, thereby transforming the core analytical challenge to solving (4.7).

4.4 Exponential Parameterization of Unit Quaternions

To address the normalized problem, we employ an exponential parameterization that provides the mathematical foundation for our solution methodology.

Lemma 3 (Corollary 2 of [11]). *Every unit quaternion* $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$ *admits the Euler-angle representation:*

$$q = e^{\mathbf{i}\theta_1} e^{\mathbf{j}\theta_2} e^{\mathbf{k}\theta_3}, \tag{4.8}$$

with phase angles $(\theta_1, \theta_2, \theta_3) \in [-\pi, \pi) \times [-\pi/4, \pi/4] \times [-\pi/2, \pi/2)$ *determined as follows:*

Case 1: Regular Case *(*$\theta_2 \in (-\pi/4, \pi/4)$*)*

$$\begin{aligned} \theta_2 &= \frac{1}{2}\arcsin(2(q_0q_2 + q_1q_3)), \\ \widetilde{\theta}_1 &= \frac{1}{2}\mathrm{atan2}(2(q_0q_1 - q_2q_3), q_0^2 - q_1^2 - q_2^2 + q_3^2), \\ \theta_3 &= \frac{1}{2}\mathrm{atan2}(2(q_0q_3 - q_1q_2), q_0^2 + q_1^2 - q_2^2 - q_3^2). \end{aligned} \tag{4.9}$$

Case 2: Singular Case *(*$\theta_2 = \pi/4$*)*

$$\widetilde{\theta}_1 = \frac{1}{2}\mathrm{atan2}\left(\frac{2(q_0q_1 + q_2q_3)}{q_0^2 - q_1^2 + q_2^2 - q_3^2}\right) - \theta_3, \quad \theta_3 \in (-\pi/2, \pi/2]. \tag{4.10}$$

Case 3: Singular Case *(*$\theta_2 = -\pi/4$*)*

$$\widetilde{\theta}_1 = \frac{1}{2}\mathrm{atan2}\left(\frac{2(q_0q_1 + q_2q_3)}{q_0^2 - q_1^2 + q_2^2 - q_3^2}\right) + \theta_3, \quad \theta_3 \in (-\pi/2, \pi/2]. \tag{4.11}$$

The expression $e^{\mathbf{i}\widetilde{\theta}_1} e^{\mathbf{j}\theta_2} e^{\mathbf{k}\theta_3}$ *can take one of the values* q *or* $-q$*. If* $e^{\mathbf{i}\widetilde{\theta}_1} e^{\mathbf{j}\theta_2} e^{\mathbf{k}\theta_3} = -q$ *and* $\widetilde{\theta}_1 \geq 0$*, set* $\theta_1 = \widetilde{\theta}_1 - \pi$*. Else if* $e^{\mathbf{i}\widetilde{\theta}_1} e^{\mathbf{j}\theta_2} e^{\mathbf{k}\theta_3} = -q$ *and* $\widetilde{\theta}_1 < 0$*, set* $\theta_1 = \widetilde{\theta}_1 + \pi$*. Else* $\theta_1 = \widetilde{\theta}_1$*.*

Remark 2. *The representation exhibits several important characteristics:*

(i) Phase Ambiguity: *The* 2π*-periodicity of* $e^{\mathbf{i}\theta_1}$ *permits normalization of* θ_1 *to* $[-\pi, \pi)$ *via integer multiples of* 2π*.*

(ii) Quadrant Determination: *The function* $\mathrm{atan2}(b, a)$ *is defined as:*

$$\mathrm{atan2}(b, a) := \begin{cases} \arctan(b/a) & a > 0, \\ \arctan(b/a) + \pi & a < 0, b \geq 0, \\ \arctan(b/a) - \pi & a < 0, b < 0, \\ \pi/2 & a = 0, b > 0, \\ -\pi/2 & a = 0, b < 0. \end{cases}$$

(iii) Uniqueness Properties: *The representation is "almost unique" with singularities at* $\theta_2 = \pm\pi/4$*. For* $\theta_2 \in (-\pi/4, \pi/4)$*, equations (4.9) yield unique* $(\theta_1, \theta_2, \theta_3)$*. At* $\theta_2 = \pm\pi/4$*, infinite families satisfy* $\theta_1 \pm \theta_3 =$ *constant. Engineering convention [10, 11] restores uniqueness by setting* $\theta_3 = 0$*.*

4.5 Existence and Uniqueness for the Decisive Equation

The exponential parameterization leads to a decisive ordinary differential equation which can be expressed as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \tag{4.12}$$

where

$$\mathbf{x} = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{pmatrix}, \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} a_1(t) + \sin(2\theta_1)\tan(2\theta_2)a_2(t) - \cos(2\theta_1)\tan(2\theta_2)a_3(t) \\ \cos(2\theta_1)a_2(t) + \sin(2\theta_1)a_3(t) \\ -\frac{\sin(2\theta_1)}{\cos(2\theta_2)}a_2(t) + \frac{\cos(2\theta_1)}{\cos(2\theta_2)}a_3(t) \end{pmatrix}.$$

The solvability is guaranteed by the following result.

Theorem 4.1. *Let $a_1(t), a_2(t), a_3(t)$ be continuous on $[t_0-a, t_0+a]$. The decisive equation*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \tag{4.13}$$

with initial value $\mathbf{x}(t_0) = (0, 0, 0)^\top$, admits a unique solution on $I = [t_0 - h, t_0 + h]$, where $h = \min\{a, b/M\}$, with $M = \max\{|\mathbf{f}(t, \mathbf{x})| : |t - t_0| \le a, |\mathbf{x}| \le b\}$ and b a positive constant less than $\pi/4$.

Proof. Define the domain $D := \{(t, \mathbf{x}) : |t - t_0| \le a, |\mathbf{x}(t)| \le b\}$, a closed subset of $\mathbb{R} \times (-\pi, \pi] \times [-\pi/4, \pi/4] \times (-\pi/2, \pi/2]$ excluding points where $\theta_2(t) = \pm\pi/4$.

The function $\mathbf{f}(t, \mathbf{x})$ is continuous on D. To establish Lipschitz continuity, consider $(t, \mathbf{x}_1) = (t, \theta_1^1, \theta_2^1, \theta_3^1)$ and $(t, \mathbf{x}_2) = (t, \theta_1^2, \theta_2^2, \theta_3^2)$ in D. The difference satisfies:

$$\begin{aligned} |\mathbf{f}(t, \mathbf{x}_2) - \mathbf{f}(t, \mathbf{x}_1)| \le\ & |(\sin(2\theta_1^2)\tan(2\theta_2^2) - \sin(2\theta_1^1)\tan(2\theta_2^1))a_2(t)| \\ & + |(\cos(2\theta_1^2)\tan(2\theta_2^2) - \cos(2\theta_1^1)\tan(2\theta_2^1))a_3(t)| \\ & + |(\cos(2\theta_1^2) - \cos(2\theta_1^1))a_2(t)| + |(\sin(2\theta_1^2) - \sin(2\theta_1^1))a_3(t)| \\ & + \left| \left(-\frac{\sin(2\theta_1^2)}{\cos(2\theta_2^2)} + \frac{\sin(2\theta_1^1)}{\cos(2\theta_2^1)} \right) a_2(t) \right| \\ & + \left| \left(\frac{\cos(2\theta_1^2)}{\cos(2\theta_2^2)} - \frac{\cos(2\theta_1^1)}{\cos(2\theta_2^1)} \right) a_3(t) \right|. \end{aligned}$$

Each trigonometric difference admits a Lipschitz bound. For instance:

$$\begin{aligned} & |\sin(2\theta_1^2)\tan(2\theta_2^2) - \sin(2\theta_1^1)\tan(2\theta_2^1)| \\ & \le k_1|\theta_1^2 - \theta_1^1| + 2k_1|\theta_2^2 - \theta_2^1|, \end{aligned}$$

for some $k_1 > 0$. Similar bounds hold for the remaining terms with constants $k_2, k_3, k_4 > 0$.

Since $a_i(t)$ are continuous on the compact set D, there exists $M > 0$ such that $|a_i(t)| \le M$. Combining bounds yields:

$$|\mathbf{f}(t, \mathbf{x}_2) - \mathbf{f}(t, \mathbf{x}_1)| \le L|\mathbf{x}_2 - \mathbf{x}_1|,$$

where $L = (3k_1 + 3k_2 + 4 + 2k_3 + 2k_4)M$. This establishes Lipschitz continuity.

By Theorem 3.1 in Chapter 1 of [22], the decisive equation admits a unique solution on the specified interval. □

Remark 3. *Analogous to Theorem 2.1 in [22], the solution* $\mathbf{x}$ *can be extended to a maximal interval. If* (a, b) *is such a maximal interval, then* $\mathbf{x}(t)$ *approaches the boundary of* D *as* $t \to a^+$ *and* $t \to b^-$.

5 Exact Solutions in Special Cases

While Theorem 4.1 provides a constructive algorithm via Picard iteration for solving the decisive equation (4.13), we now pursue direct computation of exact solutions in three analytically tractable scenarios:

Case I: Vanishing θ_1 component,

Case II: Vanishing θ_2 component,

Case III: Vanishing θ_3 component.

For each case, we derive explicit solutions by imposing specific structural constraints on the exponential representation and identifying the corresponding consistency conditions on the coefficient functions.

5.1 The Decisive System

The exponential parameterization of unit quaternions leads to a crucial system of ordinary differential equations that we term the *decisive system*.

Definition 1 (Decisive System)**.** *For a pure quaternion-valued differential equation* $y'(t) = \underline{a}(t)y(t)$ *with* $\underline{a}(t) = a_1(t)\mathbf{i} + a_2(t)\mathbf{j} + a_3(t)\mathbf{k}$*, the corresponding* decisive system *is given by:*

$$\begin{aligned} \theta_1'(t) &= a_1(t) + \sin(2\theta_1)\tan(2\theta_2)a_2(t) - \cos(2\theta_1)\tan(2\theta_2)a_3(t), \\ \theta_2'(t) &= \cos(2\theta_1)a_2(t) + \sin(2\theta_1)a_3(t), \\ \theta_3'(t) &= -\frac{\sin(2\theta_1)}{\cos(2\theta_2)}a_2(t) + \frac{\cos(2\theta_1)}{\cos(2\theta_2)}a_3(t), \end{aligned} \tag{5.1}$$

where $(\theta_1(t), \theta_2(t), \theta_3(t))$ *are the phase angles in the exponential representation* $q(t) = e^{\mathbf{i}\theta_1(t)}e^{\mathbf{j}\theta_2(t)}e^{\mathbf{k}\theta_3(t)}$.

This system is derived by substituting the exponential representation into the original QDE and equating coefficients, as established in Theorem 4.1. The decisive system provides the fundamental connection between the quaternionic dynamics and the real-valued phase angles, enabling the solution of pure QDEs through conventional ODE methods.

Remark 4. *The decisive system is valid for $\theta_2(t) \neq \pm\pi/4$, where the trigonometric functions remain well-defined. At the singular boundaries $\theta_2(t) = \pm\pi/4$, alternative parameterizations or limiting procedures are required.*

5.2 Case I: Vanishing θ_1 Component

Corollary 2. *Assume the coefficient functions satisfy $a_1(t) = a_3(t)\tan(2A_2(t))$, where $A_2(t) = \int_{t_0}^{t} a_2(s)\,ds$. Then the pure QDE (4.5) admits an exact solution of the form:*

$$q(t) = e^{\mathbf{j}A_2(t)} e^{\mathbf{k}\theta_3(t)} \mathbf{C}_0,$$

where $\theta_3(t) = \int_{t_0}^{t} \frac{a_3(s)}{\cos(2A_2(s))}\,ds$ and $\mathbf{C}_0$ is an arbitrary quaternionic constant.

Proof. Consider solutions parameterized as $q(t) = e^{\mathbf{j}\theta_2(t)} e^{\mathbf{k}\theta_3(t)}$, corresponding to $\theta_1(t) \equiv 0$. Substitution into the decisive system (5.1) yields the simplified equations:

$$\begin{aligned}
\theta_1'(t) &= a_1(t) - \tan(2\theta_2) a_3(t) = 0, \\
\theta_2'(t) &= a_2(t), \\
\theta_3'(t) &= \frac{1}{\cos(2\theta_2)} a_3(t).
\end{aligned}$$

Integration of the second equation gives $\theta_2(t) = \int_{t_0}^{t} a_2(s)\,ds = A_2(t)$. The first equation then imposes the consistency condition $a_1(t) = a_3(t)\tan(2A_2(t))$. Under this constraint, the third equation integrates to:

$$\theta_3(t) = \int_{t_0}^{t} \frac{a_3(s)}{\cos(2A_2(s))}\,ds.$$

The triple $(\theta_1(t), \theta_2(t), \theta_3(t)) = (0, A_2(t), \theta_3(t))$ thus satisfies the decisive system, and the corresponding quaternion $q(t) = e^{\mathbf{j}A_2(t)} e^{\mathbf{k}\theta_3(t)} \mathbf{C}_0$ provides an exact solution to the QDE. □

5.3 Case II: Vanishing θ_2 Component

Corollary 3. *Assume the coefficient functions satisfy $a_2(t) = -a_3(t)\tan(2A_1(t))$, where $A_1(t) = \int_{t_0}^{t} a_1(s)\,ds$. Then the pure QDE (4.5) admits an exact solution of the form:*

$$q(t) = e^{\mathbf{i}A_1(t)} e^{\mathbf{k}\theta_3(t)} \mathbf{C}_0,$$

where $\theta_3(t) = \int_{t_0}^{t} \frac{a_3(s)}{\cos(2A_1(s))}\,ds$ and $\mathbf{C}_0$ is an arbitrary quaternionic constant.

Proof. Consider the parameterization $q(t) = e^{\mathbf{i}\theta_1(t)} e^{\mathbf{k}\theta_3(t)}$ with $\theta_2(t) \equiv 0$. The decisive system reduces to:

$$\begin{aligned}
\theta_1'(t) &= a_1(t), \\
\theta_2'(t) &= \cos(2\theta_1) a_2(t) + \sin(2\theta_1) a_3(t) = 0, \\
\theta_3'(t) &= -\sin(2\theta_1) a_2(t) + \cos(2\theta_1) a_3(t).
\end{aligned}$$

Integration yields $\theta_1(t) = \int_{t_0}^{t} a_1(s)\,ds = A_1(t)$. The second equation then requires:

$$a_2(t) = -a_3(t)\tan(2A_1(t)).$$

Under this condition, the third equation simplifies to $\theta_3'(t) = a_3(t)/\cos(2A_1(t))$, integrating to:

$$\theta_3(t) = \int_{t_0}^{t} \frac{a_3(s)}{\cos(2A_1(s))}\,ds.$$

The solution $(\theta_1(t), \theta_2(t), \theta_3(t)) = (A_1(t), 0, \theta_3(t))$ generates the quaternion solution $q(t) = e^{\mathbf{i}A_1(t)}e^{\mathbf{k}\theta_3(t)}\mathbf{C}_0$. □

5.4 Case III: Vanishing θ_3 Component

Corollary 4. *Assume the coefficient functions satisfy $a_3(t) = a_2(t)\tan(2A_1(t))$, where $A_1(t) = \int_{t_0}^{t} a_1(s)\,ds$. Then the pure QDE (4.5) admits an exact solution of the form:*

$$q(t) = e^{\mathbf{i}A_1(t)}e^{\mathbf{j}\theta_2(t)}\mathbf{C}_0,$$

where $\theta_2(t) = \int_{t_0}^{t} \frac{a_2(s)}{\cos(2A_1(s))}\,ds$ and $\mathbf{C}_0$ is an arbitrary quaternionic constant.

Proof. Adopt the parameterization $q(t) = e^{\mathbf{i}\theta_1(t)}e^{\mathbf{j}\theta_2(t)}$ with $\theta_3(t) \equiv 0$. The decisive system becomes:

$$\begin{aligned}
\theta_1'(t) &= a_1(t) + \sin(2\theta_1)\tan(2\theta_2)a_2(t) - \cos(2\theta_1)\tan(2\theta_2)a_3(t),\\
\theta_2'(t) &= \cos(2\theta_1)a_2(t) + \sin(2\theta_1)a_3(t),\\
\theta_3'(t) &= -\frac{\sin(2\theta_1)}{\cos(2\theta_2)}a_2(t) + \frac{\cos(2\theta_1)}{\cos(2\theta_2)}a_3(t) = 0.
\end{aligned}$$

The third equation dictates the consistency condition:

$$a_3(t) = a_2(t)\tan(2\theta_1(t)).$$

Substitution into the remaining equations yields:

$$\theta_1(t) = \int_{t_0}^{t} a_1(s)\,ds = A_1(t), \quad \theta_2(t) = \int_{t_0}^{t} \frac{a_2(s)}{\cos(2A_1(s))}\,ds.$$

The solution $(\theta_1(t), \theta_2(t), \theta_3(t)) = (A_1(t), \theta_2(t), 0)$ produces the quaternion solution $q(t) = e^{\mathbf{i}A_1(t)}e^{\mathbf{j}\theta_2(t)}\mathbf{C}_0$. □

6 Illustrative Examples

This section presents three examples demonstrating the application of the corollaries derived in Section 5.

Example 1. *Solve the quaternion-valued differential equation:*

$$q'(t) = (\sin(2t)\mathbf{i} + \mathbf{j} + \cos(2t)\mathbf{k})q(t).$$

Identifying the coefficients:

$$a_1(t) = \sin(2t), \quad a_2(t) = 1, \quad a_3(t) = \cos(2t),$$

we observe that $a(t) = \underline{a}(t)$ (purely vectorial) and does not satisfy the commutativity condition (3.2). Therefore, the standard exponential solution approach is inapplicable.

Computing $A_2(t) = \int_0^t a_2(s)\,ds = t$, we verify the condition from Corollary 2:

$$a_1(t) = \sin(2t) = \cos(2t)\tan(2t) = a_3(t)\tan(2A_2(t)).$$

The decisive equation (5.1) admits the solution:

$$\theta_1(t) = 0, \quad \theta_2(t) = t, \quad \theta_3(t) = \int_0^t \frac{\cos(2s)}{\cos(2s)}\,ds = t.$$

Thus, the general solution takes the form:

$$q(t) = e^{\mathbf{j}t}e^{\mathbf{k}t}\mathbf{C},$$

where $\mathbf{C}$ is an arbitrary quaternionic constant.

Verification:

Computing the derivative:

$$\begin{aligned}
q'(t) &= \mathbf{j}e^{\mathbf{j}t}e^{\mathbf{k}t} + e^{\mathbf{j}t}\mathbf{k}e^{\mathbf{k}t} \\
&= \mathbf{j}(\cos t + \mathbf{j}\sin t)(\cos t + \mathbf{k}\sin t) + (\cos t + \mathbf{j}\sin t)\mathbf{k}(\cos t + \mathbf{k}\sin t) \\
&= (-\cos t\sin t + \mathbf{i}\cos t\sin t + \mathbf{j}\cos^2 t - \mathbf{k}\sin^2 t) \\
&\quad + (-\cos t\sin t + \mathbf{i}\cos t\sin t - \mathbf{j}\sin^2 t + \mathbf{k}\cos^2 t) \\
&= (-\sin(2t) + \mathbf{i}\sin(2t) + \mathbf{j}\cos(2t) + \mathbf{k}\cos(2t)).
\end{aligned}$$

Computing the right-hand side:

$$\begin{aligned}
a(t)q(t) &= (\sin(2t)\mathbf{i} + \mathbf{j} + \cos(2t)\mathbf{k})e^{\mathbf{j}t}e^{\mathbf{k}t} \\
&= (\sin(2t)\mathbf{i} + \mathbf{j} + \cos(2t)\mathbf{k})(\cos t + \mathbf{j}\sin t)(\cos t + \mathbf{k}\sin t) \\
&= (\sin(2t)\mathbf{i} + \mathbf{j} + \cos(2t)\mathbf{k})(\cos^2 t + \mathbf{i}\sin^2 t + \mathbf{j}\cos t\sin t + \mathbf{k}\cos t\sin t) \\
&= (-\sin(2t)\sin^2 t - \sin t\cos t - \cos(2t)\sin t\cos t) \\
&\quad + \mathbf{i}(\sin(2t)\cos^2 t + \sin t\cos t - \cos(2t)\sin t\cos t) \\
&\quad + \mathbf{j}(-\sin(2t)\sin t\cos t + \cos^2 t + \cos(2t)\sin^2 t) \\
&\quad + \mathbf{k}(\sin(2t)\sin t\cos t - \sin^2 t + \cos(2t)\cos^2 t).
\end{aligned}$$

Simplifying using trigonometric identities:

$$\begin{aligned}
&\text{Scalar part: } -2\sin^3 t\cos t - \sin t\cos t - 2\sin t\cos^3 t + \sin t\cos t = -\sin(2t),\\
&\mathbf{i} \text{ part: } 2\sin t\cos^3 t + \sin t\cos t - 2\sin t\cos^3 t + \sin t\cos t = \sin(2t),\\
&\mathbf{j} \text{ part: } -2\sin^2 t\cos^2 t + \cos^2 t + \sin^2 t\cos^2 t - \sin^4 t = \cos(2t),\\
&\mathbf{k} \text{ part: } 2\sin^2 t\cos^2 t - \sin^2 t + \cos^4 t - \sin^2 t\cos^2 t = \cos(2t).
\end{aligned}$$

Thus, $a(t)q(t) = (-\sin(2t) + \mathbf{i}\sin(2t) + \mathbf{j}\cos(2t) + \mathbf{k}\cos(2t))$, confirming that $q(t)$ satisfies the QDE.

Example 2. *Solve the quaternion-valued differential equation:*

$$q'(t) = (\mathbf{i} + t\sin(2t)\mathbf{j} - t\cos(2t)\mathbf{k})q(t).$$

Identifying the coefficients:

$$a_1(t) = 1, \quad a_2(t) = t\sin(2t), \quad a_3(t) = -t\cos(2t),$$

we observe that $a(t) = \underline{a}(t)$ is purely vectorial and does not satisfy the commutativity condition.

Computing $A_1(t) = \int_0^t a_1(s)\,ds = t$, we verify the condition from Corollary 3:

$$a_2(t) = t\sin(2t) = -(-t\cos(2t))\tan(2t) = -a_3(t)\tan(2A_1(t)).$$

The decisive equation (5.1) admits the solution:

$$\theta_1(t) = t, \quad \theta_2(t) = 0, \quad \theta_3(t) = \int_0^t \frac{-s\cos(2s)}{\cos(2s)}\,ds = -\int_0^t s\,ds = -\frac{t^2}{2}.$$

Thus, the general solution takes the form:

$$q(t) = e^{\mathbf{i}t}e^{\mathbf{k}(-t^2/2)}\mathbf{C} = e^{\mathbf{i}t}e^{-\mathbf{k}t^2/2}\mathbf{C},$$

where $\mathbf{C}$ is an arbitrary quaternionic constant.

This solution can be verified by direct computation, confirming it satisfies the original QDE.

Example 3. *Solve the quaternion-valued differential equation:*

$$q'(t) = (\mathbf{i} + t\sin(2t)\mathbf{j} - t\cos(2t)\mathbf{k})q(t).$$

The coefficient function $a(t) = \underline{a}(t) = \mathbf{i} + t\sin(2t)\mathbf{j} - t\cos(2t)\mathbf{k}$ is purely vectorial and fails to satisfy the commutativity condition (3.2). Consequently, the standard exponential solution approach is inapplicable.

Identifying the components:

$$a_1(t) = 1, \quad a_2(t) = t\sin(2t), \quad a_3(t) = -t\cos(2t),$$

and computing $A_1(t) = \int_0^t a_1(s)\,ds = t$, we observe that:

$$a_2(t) = t\sin(2t) = -(-t\cos(2t))\tan(2t) = -a_3(t)\tan(2A_1(t)).$$

This satisfies the condition of Corollary 3. The decisive system (5.1) admits the solution:

$$\theta_1(t) = t, \quad \theta_2(t) = 0, \quad \theta_3(t) = \int_0^t \frac{-s\cos(2s)}{\cos(2s)}\,ds = -\frac{t^2}{2}.$$

Thus, the general solution is:

$$q(t) = e^{\mathbf{i}t}e^{-\mathbf{k}t^2/2}\mathbf{C},$$

where $\mathbf{C}$ is an arbitrary quaternionic constant.

Verification: Computing the derivative:

$$\begin{aligned} q'(t) = &\left[-\sin t\cos\left(\frac{t^2}{2}\right) - t\cos t\sin\left(\frac{t^2}{2}\right)\right] \\ &+ \left[\cos t\cos\left(\frac{t^2}{2}\right) - t\sin t\sin\left(\frac{t^2}{2}\right)\right]\mathbf{i} \\ &+ \left[\cos t\sin\left(\frac{t^2}{2}\right) + t\sin t\cos\left(\frac{t^2}{2}\right)\right]\mathbf{j} \\ &+ \left[\sin t\sin\left(\frac{t^2}{2}\right) - t\cos t\cos\left(\frac{t^2}{2}\right)\right]\mathbf{k} \\ = &\,(\mathbf{i} + t\sin(2t)\mathbf{j} - t\cos(2t)\mathbf{k})q(t), \end{aligned}$$

confirming that $q(t)$ satisfies the original QDE.

Example 4. *Solve the quaternion-valued differential equation:*

$$q'(t) = (\mathbf{i} + t\cos(2t)\mathbf{j} + t\sin(2t)\mathbf{k})q(t).$$

The coefficient function $a(t) = \underline{a}(t) = \mathbf{i} + t\cos(2t)\mathbf{j} + t\sin(2t)\mathbf{k}$ is purely vectorial and violates the commutativity condition (3.2), precluding the standard exponential solution approach.

Identifying the components:

$$a_1(t) = 1, \quad a_2(t) = t\cos(2t), \quad a_3(t) = t\sin(2t),$$

and computing $A_1(t) = \int_0^t a_1(s)\,ds = t$, we observe that:

$$a_3(t) = t\sin(2t) = t\cos(2t)\tan(2t) = a_2(t)\tan(2A_1(t)).$$

This satisfies the condition of Corollary 4. The decisive system (5.1) admits the solution:

$$\theta_1(t) = t, \quad \theta_2(t) = \int_0^t \frac{s\cos(2s)}{\cos(2s)}\,ds = \frac{t^2}{2}, \quad \theta_3(t) = 0.$$

Thus, the general solution is:

$$q(t) = e^{\mathbf{i}t} e^{\mathbf{j}t^2/2} \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary quaternionic constant.

Verification: Computing the derivative:

$$\begin{aligned} q'(t) = &\left[-\sin t \cos\left(\frac{t^2}{2}\right) - t\cos t \sin\left(\frac{t^2}{2}\right) \right] \\ &+ \left[\cos t \cos\left(\frac{t^2}{2}\right) - t \sin t \sin\left(\frac{t^2}{2}\right) \right] \mathbf{i} \\ &+ \left[-\sin t \sin\left(\frac{t^2}{2}\right) + t\cos t \cos\left(\frac{t^2}{2}\right) \right] \mathbf{j} \\ &+ \left[\cos t \sin\left(\frac{t^2}{2}\right) + t \sin t \cos\left(\frac{t^2}{2}\right) \right] \mathbf{k} \\ = &(\mathbf{i} + t\cos(2t)\mathbf{j} + t\sin(2t)\mathbf{k}) q(t), \end{aligned}$$

confirming that $q(t)$ satisfies the original QDE.

7 Applications to Robinson's Equation

This section demonstrates the application of our theoretical results to Robinson's Quaternion Kinematical Differential Equation (QKDE), a fundamental equation in automation, control, and guidance systems [31].

7.1 Background and Problem Formulation

The QKDE, originally presented by Robinson [31], takes the form:

$$\begin{cases} \mathbf{q}'(t) = \frac{1}{2}\mathbf{A}(\mathbf{a}(t))\mathbf{q}(t), \quad t \in [t_0, t_f], \\ \mathbf{q}(t_0) = \mathbf{q}_0, \end{cases} \tag{7.1}$$

where $\mathbf{a}(t) = [a_1(t), a_2(t), a_3(t)]^\top$ represents the angular velocity measured by an inertial measurement unit, $\mathbf{q}(t) = [q_0(t), q_1(t), q_2(t), q_3(t)]^\top \in \mathbb{R}^4$ is the quaternion state vector, $[t_0, t_f]$ is the time interval, and

$$\mathbf{A}(\mathbf{a}(t)) = \begin{bmatrix} 0 & -\mathbf{a}^\top(t) \\ \mathbf{a}(t) & -[\mathbf{a}(t)]_\times \end{bmatrix}, \quad [\mathbf{a}(t)]_\times = \begin{bmatrix} 0 & -a_3(t) & a_2(t) \\ a_3(t) & 0 & -a_1(t) \\ -a_2(t) & a_1(t) & 0 \end{bmatrix}.$$

Extensive research has been devoted to solving the QKDE [20, 21, 31, 33, 36, 37]. Notable contributions include the algorithms proposed by Zhang et al. [37], which achieve long-term precision with linear time and constant space computational complexity, and their subsequent development of explicit symplectic geometric algorithms [36].

7.2 Quaternion Formulation and Theoretical Validation

To validate the existence and uniqueness of Theorem 4.1, we reformulate the QKDE as a quaternion-valued differential equation:

$$\begin{cases} q'(t) = \frac{1}{2}a(t)q(t), \quad t \in [t_0, t_f], \\ q(t_0) = q_0, \end{cases} \tag{7.2}$$

where $q(t) = q_0(t) + q_1(t)\mathbf{i} + q_2(t)\mathbf{j} + q_3(t)\mathbf{k}$ and $a(t) = a_1(t)\mathbf{i} + a_2(t)\mathbf{j} + a_3(t)\mathbf{k} \in \mathbb{H}$.

Consider the test case from [36] with:

$$\begin{aligned} a(t) &= -\omega(1-\cos\xi)\mathbf{i} - \omega\sin\xi\sin(\omega t)\mathbf{j} - \omega\sin\xi\cos(\omega t)\mathbf{k}, \\ q(0) &= \cos(\xi/2) + \sin(\xi/2)\mathbf{j}, \\ q(t) &= \cos(\xi/2) + \sin(\xi/2)\cos(\omega t)\mathbf{j} - \sin(\xi/2)\sin(\omega t)\mathbf{k}, \end{aligned}$$

where $\omega, \xi \in \mathbb{R}$ are parameters.

To satisfy the conditions of Theorem 4.1, we set $\omega = 0$ while keeping $\xi \in \mathbb{R}$ arbitrary. Under this parameterization, Lemma 3 yields:

$$\theta_1 = 0, \quad \theta_2 = 0, \quad \theta_3 = \frac{1}{2}\text{atan2}(\sin\xi, \cos\xi). \tag{7.3}$$

This solution satisfies the decisive equation (5.1), thereby validating Theorem 4.1 for the QKDE framework.

7.3 Computational Remarks

While the original parameters ($\omega = 2\pi$, $\xi = \pi/80$) from [36] do not satisfy the decisive equation (5.1), our modified parameterization demonstrates the applicability of our theoretical framework. This highlights the importance of parameter selection when applying exact solution methods to quaternion-valued differential equations in practical guidance and control applications.

8 Application to Color Image Encryption

With the advancement of science and technology, information dissemination methods have become increasingly diverse. Consequently, issues concerning information security and privacy protection require urgent attention. For example, communication security is crucial during the transmission of medical images (such as CT and PET scans) when patients seek medical treatment [38]. Thus, quaternion-valued neural networks, which integrate quaternions with differential dynamics, are introduced for image encryption and decryption research. In this study, we selected four medical images ("Ori1-Ori4") with dimensions of $256 \times 256 \times 3$ pixels for analysis.

Based on the above research and discussion, we select the following quaternion-valued differential equation:

$$\begin{cases} \mathbf{q}'(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}f(\mathbf{q}(t)) + \mathbf{C}f(\mathbf{q}(t-\tau)), \\ \mathbf{q}(t_0) = \mathbf{q}_0, \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{8.1}$$

where $\mathbf{q}(t) \in \mathbb{H}^{256}$, coefficients $\mathbf{A} = (a_{pq}), \mathbf{B} = (b_{pq}), \mathbf{C} = (c_{pq}) \in \mathbb{H}^{256\times 256}$, and the activation function satisfies $f(q(t)) = 0.5\big(\tanh(q_0(t)) + \tanh(q_1(t))\mathbf{i} + \tanh(q_2(t))\mathbf{j} + \tanh(q_3(t))\mathbf{k}\big)$ for $q(t) = q_0(t) + q_1(t)\mathbf{i} + q_2(t)\mathbf{j} + q_3(t)\mathbf{k}$.

The encryption process is outlined as follows: Initially, the pixel matrices of the original image are read to extract its red I_r, green I_g, and blue I_b components. Subsequently, a chaotic sequence is generated by iterating a quaternion differential equation (8.1), followed by scrambling and reshaping operations. A final encryption secret key is derived from this sequence through modular arithmetic. Finally, the ciphertext is obtained by performing a bit-XOR operation between the secret key and the color component matrices (I_r, I_g, I_b).

The original images of the four medical images and the corresponding encrypted and decrypted images are shown in Figure 1. The security and effectiveness of the encrypted graph obtained under our encryption method are quantified through indicators such as histogram, correlation coefficient, information entropy, PSNR, and SSIM.

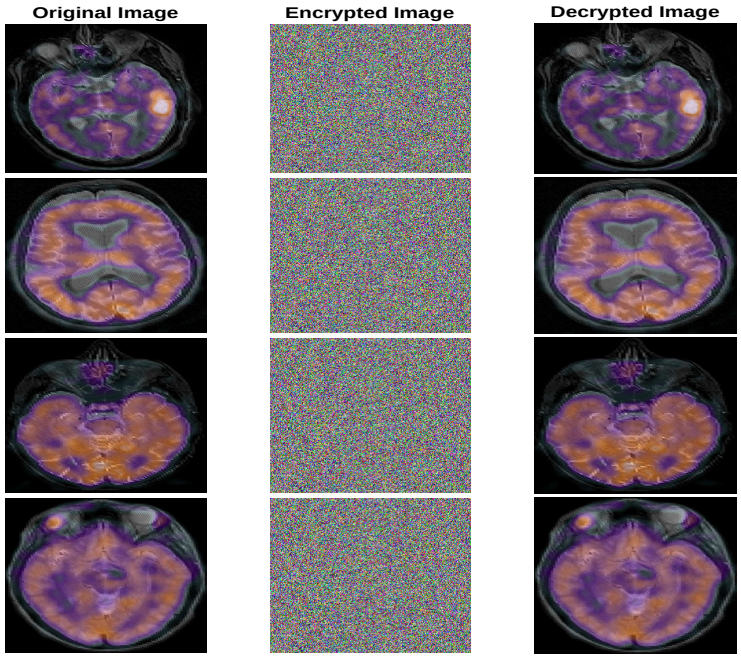
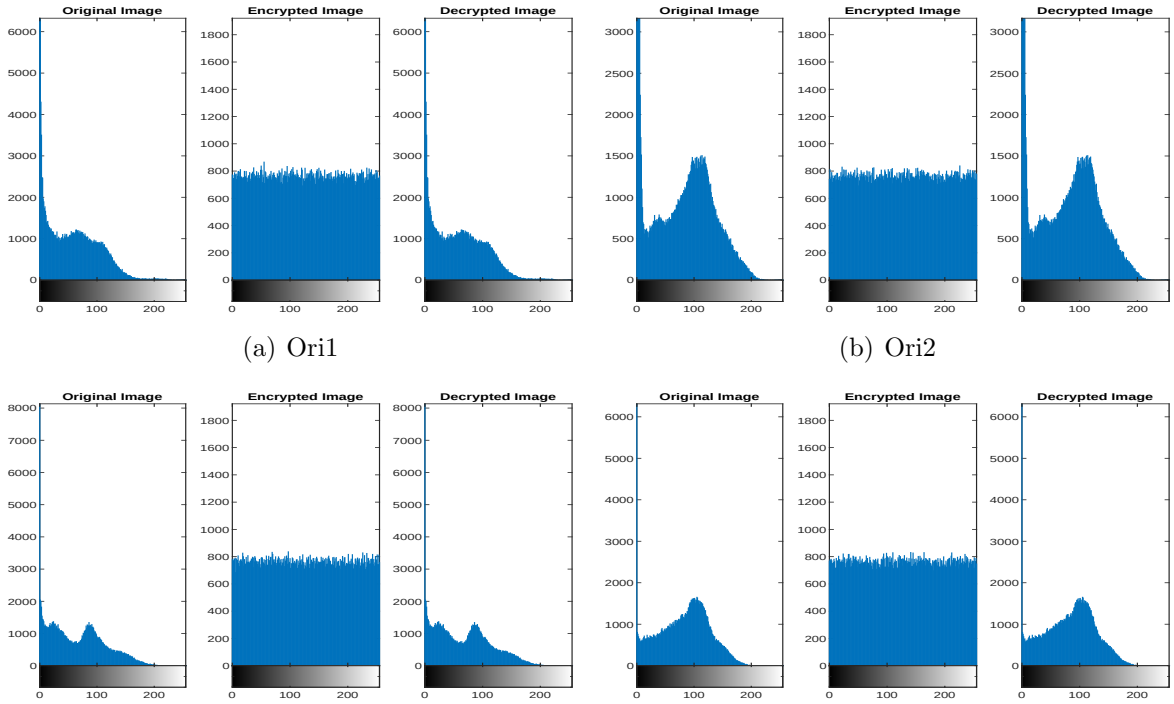


Figure 1: Experimental results. The original image, encrypted image, and decrypted image of "Ori1-Ori4".

In the field of image encryption, the histogram is an extremely important and fundamental analytical tool. The histogram reveals the information and statistical characteristics of the image and is a core indicator for measuring the performance of encryption



(a) Ori1 (b) Ori2

(c) Ori3 (d) Ori4

Figure 2: Histograms of images "Ori1-Ori4".

algorithms. As can be seen from Figure 2, compared with the original image and the decrypted image, the histogram distribution of the encrypted image is more uniform. Therefore, attackers cannot obtain any information about the original image by analyzing the distribution of pixel values.

Randomly select 5,000 pixels from the original image and the encrypted image respectively to obtain the corresponding correlation coefficient graphs in the horizontal, vertical and diagonal directions. As shown in Figures 3 and 4, the pixels of the encrypted graphs corresponding to "Ori1-Ori4" are evenly distributed throughout the entire area, indicating that the encryption algorithm has excellent diffusion characteristics. This algorithm can effectively resist statistical analysis attacks based on pixel correlation.

Table 1: The results about SSIM and PSNR of "Ori1-Ori4".

		Ciphertext			
		Ori1	Ori2	Ori3	Ori4
SSIM	$\mathcal{R}$	0.0046	0.0061	0.0063	0.0067
	$\mathcal{G}$	0.0033	0.0069	0.0064	0.0070
	$\mathcal{B}$	0.0072	0.0082	0.0052	0.0074
PSNR	$\mathcal{R}$	6.6762	7.5405	6.7537	7.5743
	$\mathcal{G}$	6.5098	7.5870	6.6036	7.2588
	$\mathcal{B}$	6.8971	7.8398	6.8079	7.7942

The effectiveness of the encryption algorithm is evidenced by the PSNR and SSIM

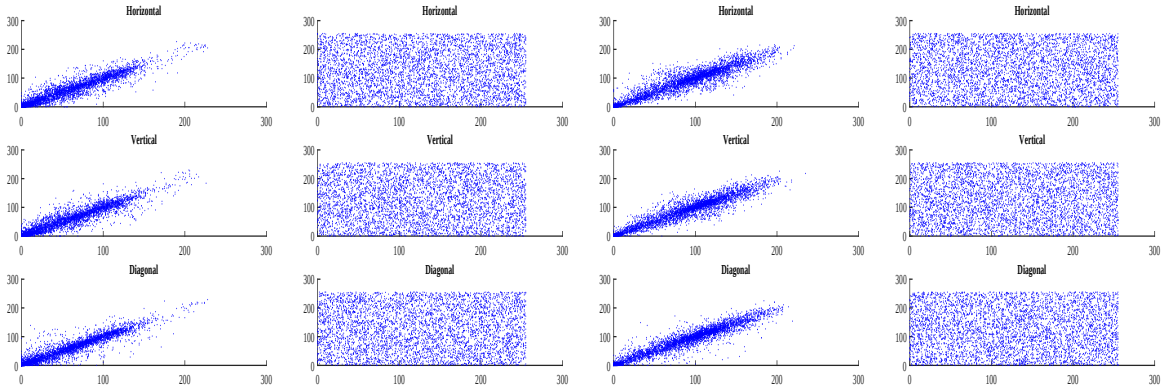


Figure 3: Correlations of two adjacent pixels in horizontal, vertical, and diagonal directions of "Ori1" and "Ori2". (a) and (c) Original image. (b) and (d) Encrypted image.

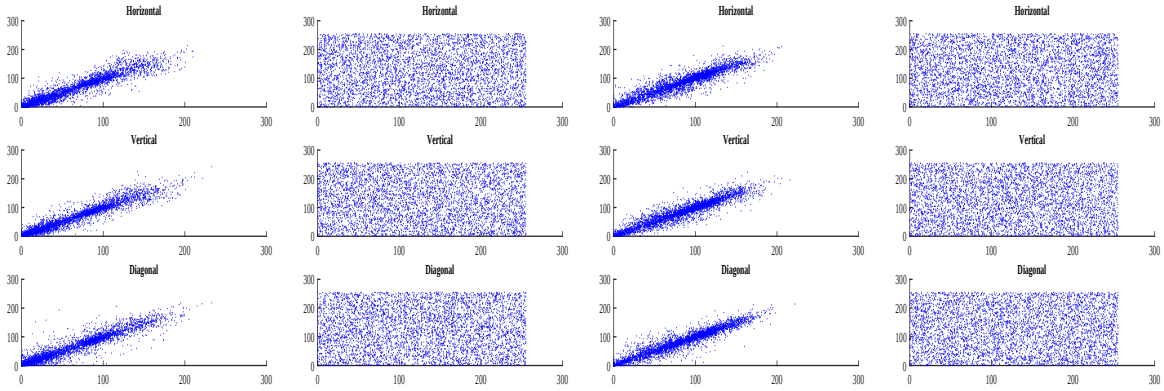


Figure 4: A comparison of adjacent pixel correlations in the horizontal, vertical, and diagonal directions for images "Ori3" and "Ori4." Panels (a) and (c) display the original images, while (b) and (d) show their encrypted versions.

results in Table 1:

- Structural Destruction ($SSIM << 0.01$): The encrypted images are structurally and visually completely dissimilar from the originals.
- Pixel-level Alteration (PSNR: $6.5 - 8.0$ dB): The original images have been effectively transformed into noise-like ciphertext.

These results confirm the algorithm's excellent confusion and diffusion characteristics. Table 2 shows the information entropy and correlation coefficient of the original image

Table 2: Information entropy and correlation coefficient of "Ori1-Ori4".

		Plaintext				Ciphertext			
		Ori1	Ori2	Ori3	Ori4	Ori1	Ori2	Ori3	Ori4
IE	$\mathcal{R}$	6.5020	7.2228	6.3152	6.6562	7.9898	7.9891	7.9891	7.9897
	$\mathcal{G}$	6.5015	7.2253	6.3152	6.6560	7.9891	7.9898	7.9896	7.9892
	$\mathcal{B}$	6.5017	7.2243	6.3152	6.6560	7.9897	7.9893	7.9886	7.9892
CC	$\mathcal{H}$	0.9404	0.9447	0.9580	0.9569	-0.0100	-0.0112	0.0269	0.0113
	$\mathcal{V}$	0.9473	0.9552	0.9673	0.9664	-0.0087	0.0019	-0.0127	0.0022
	$\mathcal{D}$	0.9676	0.9581	0.9570	0.9664	-0.0043	0.0037	-0.0041	-0.0096

and the encrypted image. The near-zero correlation coefficient of the encrypted image indicates that the spatial correlation between pixels has been disrupted, which can effectively resist external attacks. An information entropy of approximately 8 indicates that the encrypted image has achieved the maximum degree of randomness and uncertainty at the information level, and can effectively resist attacks based on entropy analysis.

All these indicators jointly prove that our algorithm successfully transforms a meaningful original image into a ciphertext image that is visually, statistically and numerically indistinguishable from random noise. The security during the transmission of medical images has been well guaranteed.

9 Potential Applications and Future Work

The solution methods developed in this paper, which explicitly address the non-commutative nature of quaternions, open up new avenues for modeling and analysis in several fields. The ability to handle genuine quaternionic dynamics, beyond the restrictive complex-like case, allows for a more natural and powerful application of quaternion algebra [1].

9.1 Systems Involving 3D Rotations and Attitude Control

Quaternions are the preferred representation for 3D rotations in computer graphics, robotics, and aerospace engineering due to their computational efficiency and lack of

gimbal lock [2]. The dynamics of a rotating rigid body or the error kinematics of an attitude control system often lead to linear QDEs where the quaternion elements do not commute. Our general solution algorithm provides a new mathematical framework to analytically model and solve these systems, potentially leading to more stable and robust control laws for satellites, drones, and autonomous vehicles [3].

9.2 Quantum Mechanics with Non-Abelian Gauge Fields

In quantum mechanics, quaternions can model systems with specific symmetries and non-Abelian gauge fields [7]. The time evolution of such systems, described by Schrödinger-like equations with quaternion-valued potentials, inherently involves non-commutative operations. Our work provides a new tool for tackling these equations analytically, which could offer deeper insight into the dynamics of particles in complex field configurations beyond what is possible with standard complex-number formulations.

9.3 Hypercomplex Signal Processing

Our results can be directly applied to the analysis and design of linear filters for quaternion-valued signals. This is crucial in applications like array processing, where quaternions can compactly represent signals from vector sensors (e.g., combining acoustic velocity and pressure) [5]. The non-commutative solutions we have derived allow for the creation of a broader class of adaptive filters that can more effectively exploit the full four-dimensional structure of the data.

9.4 Image Processing and Color Analysis

A particularly promising application lies in the domain of image processing. Color images can be represented in a quaternion framework by encoding the RGB channels into the imaginary parts of a pure quaternion, i.e., a pixel at location (x, y) is represented as $q(x, y) = r(x, y)\mathbf{i} + g(x, y)\mathbf{j} + b(x, y)\mathbf{k}$. This representation treats a color image as a holistic, hypercomplex-valued function rather than three separate channels [39].

Our work on non-commutative linear QDEs enables several advanced image processing techniques:

- **Anisotropic Color Diffusion:** Traditional linear diffusion filters often blur edges and disregard color correlations. By defining a linear QDE where the diffusion coefficients are quaternion-valued and non-commutative, one can create anisotropic diffusion filters that smooth image noise while simultaneously preserving color edges and textures in a directionally sensitive manner. Our solution algorithm provides the mathematical basis for implementing and analyzing such sophisticated filters [8].

- **Holistic Color Image Filtering:** Many conventional filters operate on each color channel independently, which can lead to color artifacts. The ability to solve general linear QDEs allows for the design of holistic linear filters that operate on the color vector as a single unit. This can be used for tasks like color enhancement, stylization, and feature detection while maintaining the intrinsic correlation between color channels [4].

- **Frequency-Domain Analysis:** The solution forms we have derived, particularly the closed-form solutions for specific cases, can be used to design novel quaternion-valued convolution kernels in the spatial domain. These kernels, based on a principled differential equation model, can achieve specific effects like seamless color shifting or illumination invariance, which are difficult to achieve with ad-hoc filter design.

9.5 Future Work

Future work will focus on translating these theoretical solution methods into concrete algorithms for the application domains discussed above, particularly in developing a new class of quaternion-based partial differential equations for color image inpainting and restoration. Experimental validation of the proposed methods in color image processing (Section 9.4) will be a primary immediate goal.

10 Conclusion

This paper has undertaken a systematic investigation of linear quaternion-valued differential equations (QDEs), confronting the fundamental challenge posed by the non-commutativity of quaternion multiplication. Our work has provided both a critical analysis of the existing theoretical foundation and a novel framework that significantly expands the scope of solvable problems.

Our primary contributions are threefold. First, we have rigorously characterized the commutativity condition introduced by Campos and Mawhin, proving that it is not a mild technical assumption but a severe structural constraint that forces the system dynamics into a complex-like subspace of the quaternions. This result precisely delineates the limitations of the existing exponential solution formalism. Second, and most significantly, we have moved beyond this limitation by developing a general algorithmic framework for solving the homogeneous linear QDE without any commutativity assumptions. This method provides a systematic procedure for reducing the QDE to a real, nonautonomous nonlinear differential equation—the decisive equation—for which existence and uniqueness theorems are established. Third, we have derived explicit closed-form solutions for several key non-commutative cases, demonstrating the practical utility of our approach and providing valuable benchmarks for future work.

The implications of this research are substantial. By providing a solution pathway for genuinely non-commutative QDEs, we have opened the door to more accurate and sophisticated modeling in fields where three-dimensional dynamics are intrinsic. Our methods are directly applicable to advanced problems in control theory, such as the attitude control of spacecraft with time-varying angular velocities; in quantum mechanics, for systems with non-Abelian gauge structures; and in hypercomplex signal processing, for the design of novel filters that fully exploit the quaternion domain.

Several promising directions for future research emerge from this work:

- **Extension to Nonhomogeneous and Nonlinear QDEs:** The framework developed here for homogeneous linear equations can be extended to investigate nonhomogeneous and certain classes of nonlinear QDEs, potentially through iterative or perturbative methods.
- **Numerical Algorithm Development:** The theoretical reduction to a real nonlinear system invites the creation of specialized, high-efficiency numerical solvers tailored to the structure of the decisive equation, which could outperform generic ODE solvers.
- **Applications in Emerging Fields:** The potential applications in color image processing, quantum computing, and neural networks warrant deeper exploration, potentially leading to new algorithms in these domains.

In conclusion, this work bridges a critical gap in the theory of quaternion-valued differential equations. By moving beyond the restrictive paradigm of commutativity, we have provided a more comprehensive toolkit for analyzing systems governed by non-commutative dynamics, thereby enriching both the mathematical theory and its practical applications.

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