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## Research Article

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# Instability of double-diffusive convection in a nanofluid layer with vertical throughflow and slip boundary conditions

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## Abstract

In this study, the onset of double-diffusive convection in a nanofluid layer is analyzed when a uniform vertical throughflow is applied, with the velocity of the fluid at the confining boundaries described by slip conditions. The model also incorporates the effects of buoyancy forces arising from temperature and concentration variations in the nanofluid, leading to a coupled system of linear eigenvalue equations governing perturbations about the throughflow basic state. The steady-state temperature and nanoparticle concentration are obtained using the Adomian decomposition method, while the stability problem is solved numerically via the Chebyshev collocation method. Neutral stability curves and critical values are computed to examine the influence of parameters such as the nanofluid Lewis number, nanoparticle and solutal Rayleigh numbers, throughflow velocity, Brownian motion parameter, thermophoretic parameter, and slip coefficient. The results indicate that stationary instability dominates over the parameter range considered, and the role of each parameter on the critical conditions is discussed. In particular, the sign of the nanoparticle Rayleigh number plays a key role in determining how nanoparticle-induced buoyancy affects the onset of instability, as well as how throughflow and slip conditions influence the critical thresholds. Overall, the study presents a detailed analysis of the onset of double, diffusive convection in a nanofluid model of higher complexity than those based on simplified slip and no-slip formulations.

**Keywords:** double-diffusive convection; slip conditions; throughflow; Nanofluid; Brownian motion; Chebyshev collocation

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<sup>☆</sup>This is draft manuscript

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## 1. Introduction

Nanofluids are suspensions of solid particles or fibrous materials of a typical nanometre scale size in a base liquid. The term "nanofluid" was used for the first time by Choi in 1995 [1]. Initially, Masuda et al. conducted experiments on these types of suspensions, suggesting the possibility of increasing the thermal conductivities of these suspensions compared to the base fluid. Later, several explanations of the possible causes of the enhancement of thermal conductivity of nanofluids compared to the base fluid, using classical theories of heat transfer as well as non-classical models of the microstructure of matter, have been proposed; for instance, [2, 3]. Because of the special thermal properties of nanofluids, they have been proposed for several possible applications in future technologies, including nuclear energy [4], while the problem of heat transfer in nanofluids has been discussed in several review papers; for instance, [5, 6].

In the case of natural convection in nanofluids, the scientific community has come to realize that depending on the effective property models alone may be quite limiting in character. This is due to the fact that the migration of the nanoparticles may cause changes in the concentration fields, which in turn cause changes in the buoyancy and stability characteristics. Buongiorno [7] critically evaluated the mechanisms that may cause slip and proposed a model in which the migration of the nanoparticles is due to the effects of the Brownian diffusion and the thermophoresis effects, i.e., the migration due to the temperature gradient. This concept has greatly supported the study of the classical Rayleigh-Bénard problem in nanofluids, including the study of thermal and convective stability [8–13]. There are many recent studies that have included studying the properties of nanofluids due to the importance and applications of this topic in heat transfer topics [14–21].

A key extension of classical convective fluid dynamics is *double-diffusive convection*, in which density variations are driven by both temperature gradients and concentration gradients that diffuse at different rates. In nanofluids, this setting becomes richer because nanoparticle transport (Brownian diffusion and thermophoresis) couples to the concentration field and hence to buoyancy, so that the combined effects of thermal diffusion, solutal diffusion, Brownian motion and thermophoresis can significantly influence both the onset conditions and the ensuing convection patterns. For example, thermal stratification and diffusion/concentration effects were shown to strongly affect velocity, temperature and nanoparticle volume-fraction profiles in flow over a vertical plate in a porous medium saturated with a nanofluid [22]. Recent work continues to emphasise the importance of such coupled mechanisms in hydrodynamic and convective stability assessments [23].

In many practical settings, the fluid layer is not quiescent but is subject to an imposed vertical flow (throughflow), and wall–fluid interactions may deviate from the ideal no-slip condition. Vertical throughflow arises naturally in microfluidic devices, filtration and throughflow processes, and thermal-management systems, while slip boundary conditions can represent micro-scale interfacial effects, surface treatments or wall interactions [24]. Both throughflow and slip can modify dissipation, transport rates and instability thresholds, and these influences can be non-trivial when combined with double diffusion and nanoparticle migration [25, 26]. Consistent with this practical relevance, a substantial

recent literature has investigated convection with throughflow effects; see, for instance, [27]. Moreover, slip may affect not only near-wall behaviour but also global measures such as skin friction and heat/mass transfer rates: for nanofluid flow over stretching surfaces, momentum, thermal and solute slip can alter local Nusselt and Sherwood numbers nonlinearly and may yield dual (multiple) solutions in the slip parameters [24]; in peristaltic channel flows with double-diffusive convection and slip walls, slip has been shown to substantially influence velocity, concentration and nanoparticle volume-fraction distributions [25]. From an engineering-design perspective, the resulting parameter-dependent transport characteristics can inform applications where nanofluids are used to enhance heat and/or mass transfer in micro/nanofluidic devices, electronic cooling, heat exchangers, and flows through porous media. As an illustration, a recent computational study considered DDMC in a BFSC laden with nanofluid and posed the Nusselt and Sherwood numbers as objective functions, using wall shear stress as a control variable to shape the flow and nanoparticle distributions [28]; transient behaviours in vertical channels under mixed convection filled with nanofluid have also been reported, highlighting the system sensitivity to boundary conditions [29].

The question of stability is crucial in this case, and slight changes in the velocity, temperature, concentration of the solute, and concentration of the nanoparticles can either decay or increase, depending on the relative strength of the effects of buoyancy, advection, diffusion, and the boundaries themselves. If the intensity of the perturbations is greater, the flow could change from a steady to an oscillatory, chaotic, or even a turbulent flow with non-uniform distributions of the nanoparticles. In other nanofluids flow cases, the existence of multiple solution branches and the impact of the stability selection on the final solution have been proven, for example, in the case of the mixed convection nanofluid flow past a permeable vertical cylinder embedded in a porous medium [30] and the case of the magnetised hybrid nanofluid flow with slip and joule heating [31]. In general, non-classical boundary conditions can fundamentally change the dissipation and flow pattern, even in classical double-diffusive convection flow. In the case of the vertically bounded double-diffusive fingering convection flow, for example, the free slip conditions exhibit greater vertical velocities and wall dissipation compared to the no-slip conditions [26]. This, in turn, calls for detailed analysis of the stability of such nanofluid double-diffusive flows with partial slip boundary conditions, and recent works are focused on investigating the impact of slip phenomena on convection and hydrodynamic flows [32, 33]. The continuous interest in stability/instability phenomena for all types of convection flows is seen from the recent literature [34–42].

The novelty of the current work is based on the fact that it is the first work in the field that performs a comprehensive linear stability analysis on a double diffusive nanofluid layer, accounting for: (i) the effects of Brownian diffusion and thermophoresis following the Buongiorno model, (ii) the effects of both solutal and nanoparticle buoyancy, (iii) the effects of a vertically throughflowing fluid at finite speed, and (iv) the effects of slip boundary conditions on both surfaces simultaneously. Indeed, there are a few earlier studies in the field that have examined each of these effects individually or in combinations. However, as far as the current authors are aware, no earlier work has examined the effects of both throughflow and slip on the stability of a double diffusive nanofluid layer in a single spectral, eigenvalue problem. Moreover, the current work is also useful in that it provides reference

solutions for the stationary case  $\tilde{Q} = 0$  and also a Chebyshev collocation method for detailed parametric studies on the variations of the critical Rayleigh number with parameters and the corresponding critical wavenumber.

The article is organised as follows. Section 2 presents the basic equations and their nondimensionalisation. Section 3 computes the simple (throughflow) solution. Sections 4, 6 describe the linear stability analysis, the numerical method, and the analytical benchmark results. Section 7 presents the results of the computation. Section 8 concludes the paper by highlighting the main results.

## 2. Mathematical model

Following the balance laws introduced in [7], we consider a dilute nanofluid in which particle agglomeration is neglected and the suspension remains dynamically stable. Under these assumptions, the nanofluid velocity field is incompressible, so that the mass-conservation relation can be written as:

$$\nabla \cdot \mathbf{h} = 0. \quad (2.1)$$

In this expression,  $\mathbf{h}$  denotes the velocity field of the nanofluid. The governing equation for nanoparticle conservation can be written as follows when thermophoretic effects are considered and no chemical reactions take place:

$$\frac{\partial \phi}{\partial t} + \mathbf{h} \cdot \nabla \phi = \nabla \cdot \left[ K_B \nabla \phi + K_T \frac{\nabla \mathcal{T}}{\mathcal{T}} \right]. \quad (2.2)$$

In the formulation above,  $\phi$  stands for the volumetric concentration of nanoparticles,  $\mathcal{T}$  for the temperature field,  $K_B$  for the Brownian diffusion coefficient, and  $K_T$  for the thermophoretic diffusivity. The research also makes the assumption that the transport pathways of nanoparticles are unaffected by the presence of solute species. The governing equation for momentum is as follows when buoyancy effects are taken into account and the Boussinesq approximation is applied:

$$\rho_0 \left( \frac{\partial \mathbf{h}}{\partial t} + \mathbf{h} \cdot \nabla \mathbf{h} \right) = -\nabla p + \mu \nabla^2 \mathbf{h} + \rho \mathbf{k} \mathbf{g}, \quad (2.3)$$

where  $\mathbf{k} = (0, 0, 1)$  denotes the unit vector in the vertical direction, and  $\rho$  represents the effective density of the nanofluid, which is assumed to be specified by the following relation:

$$\rho = \phi \rho_p + (1 - \phi) \rho_f [1 - \alpha_t (\mathcal{T} - \mathcal{T}_0) - \alpha_c (\mathcal{C} - \mathcal{C}_0)]. \quad (2.4)$$

The velocity field's components are shown by  $\mathbf{h} = (h_1, h_2, h_3)$ . The densities of the scattered nanoparticles and a reference density of the base fluid are represented by the values  $\rho_p$  and  $\rho_f$ , respectively. The coefficient of volumetric thermal expansion is represented by the parameter  $\alpha_t$ , and the equivalent coefficient related to solute concentration,  $\alpha_c$ , usually assumes negative values when the solute is denser than the solvent.

The following is the governing equation for thermal energy flow in a nanofluid:

$$\rho c \left[ \frac{\partial \mathcal{T}}{\partial t} + \mathbf{h} \cdot \nabla \mathcal{T} \right] = \kappa_t \nabla^2 \mathcal{T} + \rho_p c_p \left[ K_B \nabla \phi \cdot \nabla \mathcal{T} + K_T \frac{\nabla \mathcal{T} \cdot \nabla \mathcal{T}}{\mathcal{T}} \right]. \quad (2.5)$$

where  $c$  is the base fluid's specific heat at constant pressure,  $\kappa_t$  is the nanofluid's thermal conductivity, and  $c_p$  is the nanoparticle material's specific heat. A conservation equation for the solute is also included in the system and is expressed as follows:

$$\frac{\partial \mathcal{C}}{\partial t} + \mathbf{h} \cdot \nabla \mathcal{C} = \kappa_c \nabla^2 \mathcal{C} \quad (2.6)$$

where the solute diffusion coefficient is represented by  $\kappa_c$ . The formulation makes the assumption that the solute's transport behaviour is unaffected by the presence of nanoparticles.

The vertical direction is taken along the  $z$ -axis within a Cartesian coordinate framework. The configuration under consideration consists of a horizontal fluid layer confined between the rigid planes  $z = 0$  and  $z = d$ , as shown in Fig. 1. In contrast to earlier formulations in which all quantities were expressed in dimensional form, dimensional variables are herein distinguished by an asterisk superscript. Both bounding surfaces are assumed to be perfectly thermally conducting. The corresponding solute concentrations are set to  $\mathcal{C}_0 + \Delta \mathcal{C}$  and  $\mathcal{C}_0$ , respectively, while the temperature at the lower boundary is prescribed as  $\mathcal{T}_0 + \Delta \mathcal{T}$  and at the upper boundary as  $\mathcal{T}_0$ . The fluid is taken to be homogenous and the Oberbeck-Boussinesq approximation is used. The temperature and solute concentration fluctuations are regarded as minor within the linear stability framework used here when compared to their respective reference values,  $\mathcal{T}_0$  and  $\mathcal{C}_0$ . The dominating conservation equations under these presumptions are as follows:

$$\begin{aligned} \rho_0 (\mathbf{h}_{,t} + \mathbf{h} \cdot \nabla \mathbf{h}) &= -\nabla p + \mu \nabla^2 \mathbf{h} - \mathbf{k} g \left[ \rho_p \phi + \rho_0 (1 - \phi) (1 - \alpha_t (\mathcal{T} - \mathcal{T}_0) - \alpha_c (\mathcal{C} - \mathcal{C}_0)) \right], \\ h_{i,i} &= 0, \\ \rho c [\mathcal{T}_{,t} + \mathbf{h} \cdot \nabla \mathcal{T}] &= \kappa_t \nabla^2 \mathcal{T} + (\rho c)_p \left[ K_B \nabla \phi \cdot \nabla \mathcal{T} + \frac{K_T}{\mathcal{T}} \nabla \mathcal{T} \cdot \nabla \mathcal{T} \right], \\ \phi_{,t} + \frac{1}{\varepsilon} \mathbf{h} \cdot \nabla \phi &= K_B \nabla^2 \phi + \frac{K_T}{\mathcal{T}} \nabla^2 \mathcal{T}, \\ \mathcal{C}_{,t} + \frac{1}{\varepsilon} \mathbf{h} \cdot \nabla \mathcal{C} &= \kappa_c \nabla^2 \mathcal{C}. \end{aligned} \quad (2.7)$$

At the boundary surfaces, it is supposed that the temperature, solute concentration, and nanoparticle volume fraction stay constant. Furthermore, it is assumed that the throughflow has a uniform velocity of  $(0, 0, Q)$ . Consequently, the following is an expression for the boundary conditions:

$$\begin{aligned} h_3 &= Q, & h_{3,z} - d\lambda_L h_{3,zz} &= 0, & \mathcal{T} &= \mathcal{T}_0 + \Delta \mathcal{T}, & \mathcal{C} &= \mathcal{C}_0 + \Delta \mathcal{C}, & \phi &= \phi_0, & z &= 0, \\ h_3 &= Q, & h_{3,z} + \lambda_U h_{3,zz} &= 0, & \mathcal{T} &= \mathcal{T}_0, & \mathcal{C} &= \mathcal{C}_0, & \phi &= \phi_1, & z &= d. \end{aligned} \quad (2.8)$$

The parameters  $\lambda_L$  and  $\lambda_U$  recover the standard limiting cases:  $\lambda_{L,U} = 0$  corresponds to the no-slip (rigid) limit, whereas  $\lambda_{L,U} \rightarrow \infty$  approaches the free-slip limit [43, 44]. It

should be noted that the boundary conditions imposed on  $\phi$  involve a certain degree of arbitrariness. One could defend the position that imposing zero nanoparticle flux at the edges is physically more relevant. However, this approach leads to a difficulty: attempts to obtain a steady-state solution for the base conduction problem under such conditions result in a contradiction. To allow analytical progress, it is therefore necessary to fix the basic profile of  $\phi$ , at which point the chosen boundary conditions can be regarded as a reasonable and practical approximation.

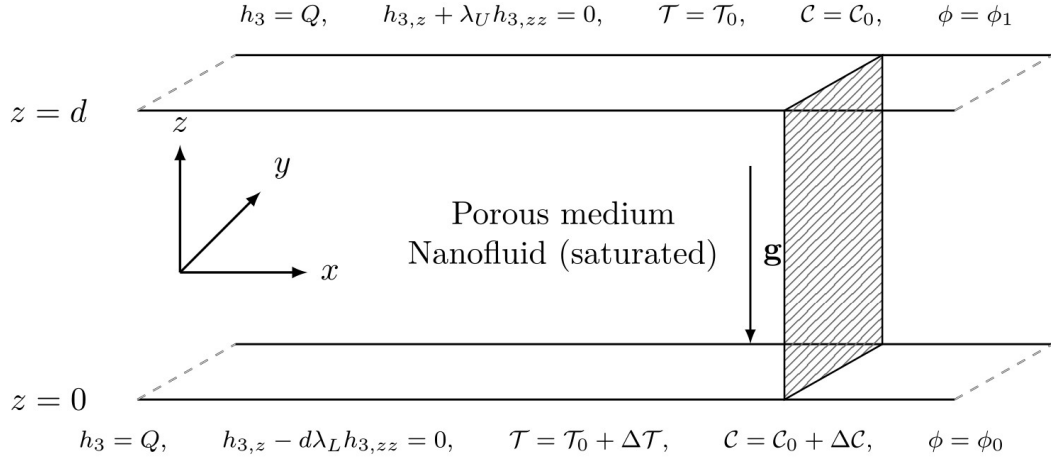


**Fig. 1.** Schematic representation of the system.

The following dimensionless variables are introduced. We define them as:

$$\begin{aligned} (x^*, y^*, z^*) &= (x, y, z)/d, & t^* &= t\alpha_f/d^2, & (u^*, v^*, w^*) &= (u, v, w)d/\alpha_f, & \tilde{Q} &= Qd/\alpha_f \\ p^* &= pd^2/(\mu\alpha_f), & \phi^* &= \frac{\phi - \phi_0}{\phi_1 - \phi_0}, & \mathcal{T}^* &= \frac{\mathcal{T} - \mathcal{T}_0}{\Delta\mathcal{T}}, & \mathcal{C}^* &= \frac{\mathcal{C} - \mathcal{C}_0}{\Delta\mathcal{C}}, \end{aligned} \quad (2.9)$$

where,

$$\alpha_f = \frac{\kappa_t}{\rho c}. \quad (2.10)$$

After omitting the asterisk, the resulting equations are given by

$$\begin{aligned} \nabla \cdot \mathbf{h} &= 0, \\ \frac{1}{\text{Pr}} \left( \frac{\partial \mathbf{h}}{\partial t} + \mathbf{h} \cdot \nabla \mathbf{h} \right) &= -\nabla p + \nabla^2 \mathbf{h} - \text{Rm} \mathbf{k} + \text{Ra} \mathcal{T} \mathbf{k} + \frac{\text{Rc}}{\text{Le}} \mathcal{C} \mathbf{k} - \text{Rn} \phi \mathbf{k}, \\ \frac{\partial \mathcal{T}}{\partial t} + \mathbf{h} \cdot \nabla \mathcal{T} &= \nabla^2 \mathcal{T} + \frac{\text{N}_B}{\text{Ln}} \nabla \phi \cdot \nabla \mathcal{T} + \frac{\text{N}_A \text{N}_B}{\text{Ln}} (\nabla \mathcal{T})^2, \\ \frac{\partial \mathcal{C}}{\partial t} + \mathbf{h} \cdot \nabla \mathcal{C} &= \frac{1}{\text{Le}} \nabla^2 \mathcal{C}, \\ \frac{\partial \phi}{\partial t} + \mathbf{h} \cdot \nabla \phi &= \frac{1}{\text{Ln}} \nabla^2 \phi + \frac{\text{N}_A}{\text{Ln}} \nabla^2 \mathcal{T}. \end{aligned} \quad (2.11)$$

The boundary conditions in terms of these dimensionless variables are given by

$$\begin{aligned} h_3 &= \tilde{Q}, & h_{3,z} - \lambda_L h_{3,zz} &= 0, & \mathcal{T} &= 1, & \mathcal{C} &= 1, & \phi &= 0, & z &= 0, \\ h_3 &= \tilde{Q}, & h_{3,z} + \lambda_U h_{3,zz} &= 0, & \mathcal{T} &= 0, & \mathcal{C} &= 0, & \phi &= 1, & z &= 1. \end{aligned} \quad (2.12)$$

The definitions of the dimensionless parameters introduced above are as follows:

$$\begin{aligned} \text{Pr} &= \frac{\mu}{\rho\alpha_f}, \\ \text{Ln} &= \frac{\alpha_f}{K_B}, \\ \text{Ra} &= \frac{\rho g \beta_f d^3 \Delta \mathcal{T}}{\mu \alpha_f}, \\ \text{Rc} &= \frac{\rho g \alpha_c d^3 \Delta \mathcal{C}}{\mu \kappa_c}, \\ \text{Rm} &= \frac{[\rho_p \phi_0 + \rho(1 - \phi_0)] g d^3}{\mu \alpha_f}, \\ \text{Rn} &= \frac{(\rho_p - \rho)(\phi_1 - \phi_0) g d^3}{\mu \alpha_f}, \\ \text{N}_A &= \frac{K_{\mathcal{T}} \Delta \mathcal{T}}{K_B \mathcal{T}_0 (\phi_1 - \phi_0)}, \\ \text{N}_B &= \frac{(\rho c)_p}{(\rho c)_f} (\phi_1 - \phi_0), \\ \text{Le} &= \frac{\alpha_f}{\kappa_c}. \end{aligned}$$

Here,  $\text{Ln}$  is defined as the ratio of the thermal diffusivity to the nanoparticle mass diffusivity of the nanofluid. The parameter  $\text{Le}$  keeps its usual meaning and is given by the Schmidt-to-Prandtl ratio,  $\text{Sc}/\text{Pr}$ . In standard notation,  $\text{Ra}$  and  $\text{Rc}$  denote the thermal and solutal Rayleigh numbers, respectively. The parameters  $\text{Rm}$  and  $\text{Rn}$  measure buoyancy effects caused by overall density variations and by nanoparticle concentration gradients. In addition,  $\text{N}_A$  represents an effective thermophoretic (cross-diffusion) diffusivity ratio, similar in role to a Soret-type coefficient, whereas  $\text{N}_B$  provides a modified measure of the particle-density contribution.

Within the Oberbeck–Boussinesq approximation, Eq. (2.11)<sub>1</sub> is linearized by neglecting the nonlinear interaction terms proportional to  $\phi \mathcal{T}$  and  $\phi \mathcal{C}$ . This assumption is suitable for dilute nanofluids when temperature and solute-concentration variations are small.

### 3. Basic Solution

The linear stability problem is set up by adding small disturbances to a chosen basic (unperturbed) state and then linearizing the governing equations about that state. The onset

of convection is obtained from the resulting eigenvalue problem by examining the growth or decay of infinitesimal perturbations. Specifying the basic state is therefore necessary to obtain a well-posed formulation and to compute accurate instability thresholds for Rayleigh–Bénard-type systems [44].

We consider a steady base flow featuring a constant imposed vertical throughflow and vanishing lateral motion. Under this assumption, the temperature, solute concentration, and nanoparticle volume fraction are functions of the vertical coordinate only. Hence, we take

$$\mathbf{h}_b = (0, 0, \tilde{Q}), \quad p = p_b(z), \quad \mathcal{T} = \mathcal{T}_b(z), \quad \mathcal{C} = \mathcal{C}_b(z), \quad \phi = \phi_b(z). \quad (3.1)$$

Substitution into Eqs. (2.11)<sub>1</sub>–(2.11)<sub>4</sub> yields the boundary-value problem

$$\begin{aligned} \frac{d^2 \mathcal{T}_b}{dz^2} + \frac{N_B}{\text{Ln}} \frac{d\mathcal{T}_b}{dz} \frac{d\phi_b}{dz} + \frac{N_A N_B}{\text{Ln}} \left( \frac{d\mathcal{T}_b}{dz} \right)^2 &= \tilde{Q} \frac{d\mathcal{T}_b}{dz}, \\ \frac{d^2 \phi_b}{dz^2} + N_A \frac{d^2 \mathcal{T}_b}{dz^2} &= \tilde{Q} \text{Ln} \frac{d\phi_b}{dz}, \\ \frac{d^2 \mathcal{C}_b}{dz^2} &= \tilde{Q} \text{Le} \frac{d\mathcal{C}_b}{dz}. \end{aligned} \quad (3.2)$$

To compute the coupled basic profiles, we employ the Adomian decomposition method (ADM) to solve (3.2)<sub>1</sub> and (3.2)<sub>2</sub> subject to the Dirichlet boundary conditions

$$\mathcal{T}_b(0) = 1, \quad \mathcal{T}_b(1) = 0, \quad \phi_b(0) = 0, \quad \phi_b(1) = 1. \quad (3.3)$$

For subsequent ADM construction, (3.2)<sub>1</sub>–(3.2)<sub>2</sub> are rewritten in operator form as

$$\mathcal{T}_b''(z) = \tilde{Q} \mathcal{T}_b'(z) - \frac{N_B}{\text{Ln}} \mathcal{T}_b'(z) \phi_b'(z) - \frac{N_A N_B}{\text{Ln}} \left( \mathcal{T}_b'(z) \right)^2, \quad (3.4)$$

$$\phi_b''(z) = \tilde{Q} \text{Ln} \phi_b'(z) - N_A \mathcal{T}_b''(z). \quad (3.5)$$

Let  $\frac{d^2 u}{dz^2} = f(z)$  on  $[0, 1]$  with Dirichlet data  $u(0) = a$  and  $u(1) = b$ . The corresponding Green's function is

$$G(z, s) = \begin{cases} s(1-z), & 0 \leq s \leq z \leq 1, \\ z(1-s), & 0 \leq z \leq s \leq 1, \end{cases} \quad (3.6)$$

and yields the representation

$$u(z) = a + (b-a)z + \int_0^1 G(z, s) f(s) ds. \quad (3.7)$$

Applying (3.7) to (3.4)–(3.5) together with (3.3) gives the equivalent integral system

$$\mathcal{T}_b(z) = 1 - z + \int_0^1 G(z, s) \left[ \tilde{Q} \mathcal{T}_b'(s) - \frac{N_B}{\text{Ln}} \mathcal{T}_b'(s) \phi_b'(s) - \frac{N_A N_B}{\text{Ln}} \left( \mathcal{T}_b'(s) \right)^2 \right] ds, \quad (3.8)$$

$$\phi_b(z) = z + \int_0^1 G(z, s) \left[ \tilde{Q} \text{Ln} \phi_b'(s) - N_A \mathcal{T}_b''(s) \right] ds. \quad (3.9)$$

Assume that  $T_b$  and  $\phi_b$  admit the ADM expansions

$$\mathcal{T}_b(z) = \sum_{n=0}^{\infty} T_n(z), \quad \phi_b(z) = \sum_{n=0}^{\infty} \phi_n(z), \quad (3.10)$$

with zeroth-order components chosen to satisfy (3.3):

$$T_0(z) = 1 - z, \quad \phi_0(z) = z. \quad (3.11)$$

The quadratic nonlinearities are decomposed into Adomian polynomials,

$$T'_b(z) \phi'_b(z) = \sum_{n=0}^{\infty} A_n(z), \quad \left(T'_b(z)\right)^2 = \sum_{n=0}^{\infty} B_n(z), \quad (3.12)$$

where, for the present product-type terms,

$$A_n(z) = \sum_{k=0}^n \phi'_k(z) T'_{n-k}(z), \quad B_n(z) = \sum_{k=0}^n T'_k(z) T'_{n-k}(z), \quad n \geq 0. \quad (3.13)$$

Substituting (3.10)–(3.13) into (3.8)–(3.9) yields the ADM recursion

$$T_{n+1}(z) = \int_0^1 G(z, s) \left[ \tilde{Q} T'_n(s) - \frac{N_B}{\text{Ln}} A_n(s) - \frac{N_A N_B}{\text{Ln}} B_n(s) \right] ds, \quad (3.14)$$

$$\phi_{n+1}(z) = \int_0^1 G(z, s) \left[ \tilde{Q} \text{Ln} \phi'_n(s) - N_A T''_n(s) \right] ds, \quad n = 0, 1, 2, \dots \quad (3.15)$$

By construction of the Dirichlet Green kernel (3.6), the higher-order components  $T_{n \geq 1}$  and  $\phi_{n \geq 1}$  satisfy homogeneous endpoint conditions. Therefore, the truncated ADM approximations

$$\mathcal{T}_b^{(m)}(z) = \sum_{n=0}^m T_n(z), \quad \phi_b^{(m)}(z) = \sum_{n=0}^m \phi_n(z), \quad (3.16)$$

preserve the boundary conditions (3.3) identically for any truncation order  $m$ .

The first-order corrections are obtained as

$$T_1(z) = \frac{\tilde{Q} \text{Ln} - \tilde{Q} - N_A \tilde{Q} \text{Ln}}{2} z(1 - z), \quad \phi_1(z) = \frac{\tilde{Q} \text{Ln}}{2} z(1 - z).$$

The second-order terms read

$$T_2(z) = \frac{\mathcal{K}}{12} z(1 - z)(2z - 1),$$

where

$$\mathcal{K} = 2N_A^2 \tilde{Q} \text{Ln}^2 + 3N_A \tilde{Q} \tilde{Q} \text{Ln} - 3N_A \tilde{Q} \text{Ln}^2 + \tilde{Q}^2 - 2\tilde{Q} \tilde{Q} \text{Ln} - \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} + \tilde{Q} \text{Ln}^2.$$

$$\phi_2(z) = -\frac{N_A(N_A \tilde{Q} \text{Ln} + \tilde{Q} - \tilde{Q} \text{Ln})}{2} z(1 - z) - \frac{\tilde{Q} \text{Ln}^2}{12} z(1 - z)(2z - 1),$$

The third-order components may be expressed compactly as

$$T_3(z) = -\frac{z(1-z)}{24} \left( p_2 z^2 + p_1 z + p_0 \right), \quad \phi_3(z) = \frac{z(1-z)}{24} \left( r_2 z^2 + r_1 z + r_0 \right),$$

with coefficients

$$\begin{aligned} p_2 &= 6N_A^3 \tilde{Q} \text{Ln}^3 + 12N_A^2 \tilde{Q} \tilde{Q} \text{Ln}^2 - 12N_A^2 \tilde{Q} \text{Ln}^3 + 7N_A \tilde{Q}^2 \tilde{Q} \text{Ln} - 14N_A \tilde{Q} \tilde{Q} \text{Ln}^2 - 4N_A \tilde{Q} \text{Ln} \tilde{Q} \text{Ln}^2 \\ &\quad + 7N_A \tilde{Q} \text{Ln}^3 + \tilde{Q}^3 - 3\tilde{Q}^2 \tilde{Q} \text{Ln} - 3\tilde{Q} \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} + 3\tilde{Q} \tilde{Q} \text{Ln}^2 - \tilde{Q} \text{Ln}^2 \tilde{Q} \text{Ln} + 3\tilde{Q} \text{Ln} \tilde{Q} \text{Ln}^2 - \tilde{Q} \text{Ln}^3, \\ p_1 &= -6N_A^3 \tilde{Q} \text{Ln}^3 - 12N_A^2 \tilde{Q} \tilde{Q} \text{Ln}^2 + 12N_A^2 \tilde{Q} \text{Ln}^3 - 4N_A^2 \tilde{Q} \text{Ln}^2 - 7N_A \tilde{Q}^2 \tilde{Q} \text{Ln} + 14N_A \tilde{Q} \tilde{Q} \text{Ln}^2 \\ &\quad - 4N_A \tilde{Q} \tilde{Q} \text{Ln} + 4N_A \tilde{Q} \text{Ln} \tilde{Q} \text{Ln}^2 - 7N_A \tilde{Q} \text{Ln}^3 + 4N_A \tilde{Q} \text{Ln}^2 - \tilde{Q}^3 + 3\tilde{Q}^2 \tilde{Q} \text{Ln} + 3\tilde{Q} \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} \\ &\quad - 3\tilde{Q} \tilde{Q} \text{Ln}^2 + \tilde{Q} \text{Ln}^2 \tilde{Q} \text{Ln} - 3\tilde{Q} \text{Ln} \tilde{Q} \text{Ln}^2 + \tilde{Q} \text{Ln}^3, \\ p_0 &= N_A^3 \tilde{Q} \text{Ln}^3 + 2N_A^2 \tilde{Q} \tilde{Q} \text{Ln}^2 - 2N_A^2 \tilde{Q} \text{Ln}^3 + 2N_A^2 \tilde{Q} \text{Ln}^2 + N_A \tilde{Q}^2 \tilde{Q} \text{Ln} - 2N_A \tilde{Q} \tilde{Q} \text{Ln}^2 + 2N_A \tilde{Q} \tilde{Q} \text{Ln} \\ &\quad - N_A \tilde{Q} \text{Ln} \tilde{Q} \text{Ln}^2 + N_A \tilde{Q} \text{Ln}^3 - 2N_A \tilde{Q} \text{Ln}^2 - \tilde{Q} \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} + \tilde{Q} \text{Ln} \tilde{Q} \text{Ln}^2, \end{aligned}$$

and

$$\begin{aligned} r_2 &= \tilde{Q} \text{Ln}^3, \\ r_1 &= 8N_A^3 \tilde{Q} \text{Ln}^2 + 12N_A^2 \tilde{Q} \tilde{Q} \text{Ln} + 4N_A^2 \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} - 12N_A^2 \tilde{Q} \text{Ln}^2 + 4N_A \tilde{Q}^2 + 4N_A \tilde{Q} \tilde{Q} \text{Ln} \\ &\quad - 8N_A \tilde{Q} \tilde{Q} \text{Ln} - 8N_A \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} + 4N_A \tilde{Q} \text{Ln}^2 - \tilde{Q} \text{Ln}^3, \\ r_0 &= -4N_A^3 \tilde{Q} \text{Ln}^2 - 6N_A^2 \tilde{Q} \tilde{Q} \text{Ln} - 2N_A^2 \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} + 6N_A^2 \tilde{Q} \text{Ln}^2 - 2N_A \tilde{Q}^2 - 2N_A \tilde{Q} \tilde{Q} \text{Ln} \\ &\quad + 4N_A \tilde{Q} \tilde{Q} \text{Ln} + 4N_A \tilde{Q} \text{Ln} \tilde{Q} \text{Ln} - 2N_A \tilde{Q} \text{Ln}^2. \end{aligned}$$

Consequently, the third-order truncated ADM approximations are

$$\begin{aligned} \mathcal{T}_b^{(3)}(z) &= \sum_{n=0}^3 T_n(z) = T_0(z) + T_1(z) + T_2(z) + T_3(z), \\ \phi_b^{(3)}(z) &= \sum_{n=0}^3 \phi_n(z) = \phi_0(z) + \phi_1(z) + \phi_2(z) + \phi_3(z). \end{aligned} \tag{3.17}$$

Since  $T_{n \geq 1}$  and  $\phi_{n \geq 1}$  satisfy homogeneous endpoint conditions by virtue of (3.6), the approximations (3.17) enforce the boundary conditions (3.3) exactly.

To assess the adequacy of the truncated ADM representation, we also constructed higher-order approximations and repeated the stability computations. Specifically, the ADM series for  $\mathcal{T}_b(z)$  and  $\phi_b(z)$  was extended to fourth and fifth order, and the resulting base profiles were used to recompute the neutral curves and the critical quantities  $(\text{Ra}_c, \delta_c)$ . For all parameter sets examined in this study, the inclusion of the fourth- and fifth-order terms produced no discernible change in the base profiles and led to only negligible variations in  $(\text{Ra}_c, \delta_c)$ , within the numerical tolerance of the Chebyshev eigenvalue solver. Accordingly, the third-order truncation is adopted throughout as it provides a converged and computationally efficient approximation of the basic state. Higher-order ADM terms are not listed because their expressions become prohibitively lengthy while offering no additional quantitative benefit or physical insight for the present analysis.

#### 4. Linear instability analysis

Convective motion in fluid systems can be triggered in response to small disturbances. The first approximation to explain that behaviour is Rayleigh-Bénard convection, which explains the Rayleigh number. If the Rayleigh number is below a critical value, then a gravitational fluid becomes stable. If it is above that value, the fluid becomes unstable as it develops convection [44]. Understanding the linearity of the system allows for a thorough understanding of the stability of the system. It can be used to split stable systems from unstable systems before the system reaches a non-linear stage [45, 46]. In stability theory, linear theory serves as a base from which to analyze a system in a non-linear, complex manner. In systems where convection is present, vertical motion from the system can be a driving force for buoyancy energy [47, 48]. This brings energy to the upper regions of the fluid, thus forming convection cells. This type of energy transfer serves as the primary reason for the instability of the system. As such, it is this linear instability which forms the mathematical foundation of convection systems.

We now superimpose small disturbances onto the basic (steady-state) solution, assuming them to have the form:

$$\mathbf{h} = \mathbf{h}_b + \tilde{\mathbf{h}}, \quad p = p_b(z) + \tilde{p}, \quad \mathcal{T} = \mathcal{T}_b(z) + \tilde{\mathcal{T}}, \quad \phi = \phi_b(z) + \tilde{\phi}, \quad \mathcal{C} = \mathcal{C}_b(z) + \tilde{\mathcal{C}}. \quad (4.1)$$

Upon inserting these representations into (2.11) and performing a linearization of the ensuing system thereby discarding all nonlinear contributions arising from products of the perturbed (primed) quantities one obtains the governing equations in the form:

$$\begin{aligned} \frac{1}{\text{Pr}}(\tilde{\mathbf{h}}_{,t} + \tilde{\mathbf{h}} \cdot \nabla \tilde{\mathbf{h}} + \tilde{Q} \partial_z \tilde{\mathbf{h}}) &= -\nabla \tilde{p} + \nabla^2 \tilde{\mathbf{h}} + \text{Ra} \tilde{\mathcal{T}} \mathbf{k} - \text{Rn} \tilde{\phi} \mathbf{k} + (\text{Rc}/\text{Le}) \tilde{\mathcal{C}} \mathbf{k}, \\ \frac{\partial \tilde{\mathcal{T}}}{\partial t} + \tilde{Q} \frac{\partial \tilde{\mathcal{T}}}{\partial z} + \tilde{h}_3 \frac{d\mathcal{T}_b}{dz} &= \nabla^2 \tilde{\mathcal{T}} + \frac{\text{N}_B}{\text{Ln}} \frac{d\phi_b}{dz} \frac{\partial \tilde{\mathcal{T}}}{\partial z} + \frac{\text{N}_B}{\text{Ln}} \frac{d\mathcal{T}_b}{dz} \frac{\partial \tilde{\phi}}{\partial z} + \frac{2\text{N}_A \text{N}_B}{\text{Ln}} \frac{d\mathcal{T}_b}{dz} \frac{\partial \tilde{\mathcal{T}}}{\partial z}, \\ \frac{\partial \tilde{\phi}}{\partial t} + \tilde{Q} \frac{\partial \tilde{\phi}}{\partial z} + \tilde{h}_3 \frac{d\phi_b}{dz} &= \frac{1}{\text{Ln}} \nabla^2 \tilde{\phi} + \frac{\text{N}_A}{\text{Ln}} \nabla^2 \tilde{\mathcal{T}}, \\ \frac{\partial \tilde{\mathcal{C}}}{\partial t} + \tilde{Q} \frac{\partial \tilde{\mathcal{C}}}{\partial z} - \tilde{h}_3 &= \frac{1}{\text{Le}} \nabla^2 \tilde{\mathcal{C}}, \\ \frac{\partial \tilde{h}_3}{\partial z} - \lambda_L \frac{\partial^2 \tilde{h}_3}{\partial z^2} &= 0, \quad \tilde{h}_3 = 0, \quad \text{at } z = 0, \\ \frac{\partial \tilde{h}_3}{\partial z} + \lambda_U \frac{\partial^2 \tilde{h}_3}{\partial z^2} &= 0, \quad \tilde{h}_3 = 0, \quad \text{at } z = 1, \\ \tilde{\mathcal{C}} = 0, \quad \tilde{\mathcal{T}} = 0, \quad \tilde{\phi} = 0, &\quad \text{at } z = 0, \\ \tilde{\mathcal{C}} = 0, \quad \tilde{\mathcal{T}} = 0, \quad \tilde{\phi} = 0, &\quad \text{at } z = 1, \end{aligned} \quad (4.2)$$

where  $\tilde{\mathbf{h}} = (\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$ .

The next step consists of removing the pressure term by taking the curl of Equation (4.2)<sub>1</sub> twice, then isolating the vertical (third) component of the resulting expression and

using (4.2)<sub>2</sub>. This procedure leads to:

$$\frac{1}{\text{Pr}} \frac{\partial}{\partial t} \nabla^2 \tilde{h}_3 = \nabla^4 \tilde{h}_3 + \text{Ra} \nabla_\star^2 \tilde{\mathcal{T}} - \text{Rn} \nabla_\star^2 \tilde{\phi} + (\text{Rc}/\text{Le}) \nabla_\star^2 \tilde{\mathcal{C}}, \quad (4.3)$$

where  $\nabla_\star^2$  is the two-dimensional Laplacian operator on the horizontal plane.

The set of equations (4.2)<sub>3</sub>-(4.3) constitutes a linear boundary-value problem that can be addressed using the normal-mode approach. Accordingly, the solution is expressed as:

$$(\tilde{h}_3, \tilde{\mathcal{T}}, \tilde{\phi}, \tilde{\mathcal{C}}) = [H_3(z), \Theta(z), \Phi(z), \Upsilon(z)]; e^{\nu t + i(k_x x + k_y y)}. \quad (4.4)$$

Substituting these forms into the governing differential equations leads to the following system:

$$\begin{aligned} (D^2 - \delta^2)^2 H_3 - \frac{\tilde{Q}}{\text{Pr}} (D^3 - \delta^2 D) H_3 - \text{Ra} \delta^2 \Theta + \text{Rn} \delta^2 \Phi - (\text{Rc}/\text{Le}) \delta^2 \Upsilon &= \frac{\nu}{\text{Pr}} (D^2 - \delta^2) H_3, \\ -H_3 \frac{d\mathcal{T}_b}{dz} + \left( D^2 - \delta^2 + \frac{\text{N}_B}{\text{Ln}} \frac{d\Phi_b}{dz} D + \frac{2\text{N}_A \text{N}_B}{\text{Ln}} \frac{d\mathcal{T}_b}{dz} D - \tilde{Q} D \right) \Theta + \frac{\text{N}_B}{\text{Ln}} \frac{d\mathcal{T}_b}{dz} D \Phi &= \nu \Theta, \\ H_3 \frac{d\Phi_b}{dz} - \frac{\text{N}_A}{\text{Ln}} (D^2 - \delta^2) \Theta - \left( \frac{1}{\text{Ln}} (D^2 - \delta^2) - \tilde{Q} D \right) \Phi &= -\nu \Phi, \\ H_3 + \left( \frac{1}{\text{Le}} (D^2 - \delta^2) - \tilde{Q} D \right) \Upsilon &= \nu \Upsilon, \\ DH_3 - \lambda_L D^2 H_3 = 0, \quad H_3 = 0, \quad \Upsilon = 0, \quad \Theta = 0, \quad \Phi = 0, \quad &\text{at } z = 0, \\ DH_3 + \lambda_U D^2 H_3 = 0, \quad H_3 = 0, \quad \Upsilon = 0, \quad \Theta = 0, \quad \Phi = 0, \quad &\text{at } z = 1, \end{aligned} \quad (4.5)$$

where

$$\delta^2 = k_x^2 + k_y^2, \text{ and } D = \frac{d}{dz}. \quad (4.6)$$

## 5. Approximate analytical solution at $\tilde{Q} = 0$

While the eigenvalue problem (4.5) is solved numerically in the general case, an explicit analytical description can be obtained when  $\tilde{Q} = 0$ . In this limit the basic state assumes the form [9]

$$\phi_b(z) = -\text{N}_A \mathcal{T}_b(z) + (1 - \text{N}_A)z + \text{N}_A, \quad (5.1)$$

where

$$\mathcal{T}_b(z) = \frac{1 - e^{-(1-\text{N}_A)\text{N}_B(1-z)/\text{Ln}}}{1 - e^{-(1-\text{N}_A)\text{N}_B/\text{Ln}}}. \quad (5.2)$$

For nanofluids, the Lewis number is typically very large,  $\text{Ln} = O(10^5 - 10^6)$ , while  $\text{N}_A$  is usually  $O(1-10)$  [7]. Consequently, the exponential terms in (5.2) are small and a first-order expansion yields the linear base profiles

$$\mathcal{T}_b(z) = 1 - z, \quad \phi_b(z) = z. \quad (5.3)$$

Substitution of (5.3) into the linearized disturbance equations leads to

$$\begin{aligned}
(D^2 - \delta^2)^2 H_3 - \text{Ra} \delta^2 \Theta + \text{Rn} \delta^2 \Phi - (\text{Rc}/\text{Le}) \delta^2 \Upsilon &= \frac{\nu}{\text{Pr}} (D^2 - \delta^2) H_3, \\
H_3 + \left( D^2 - \delta^2 + \frac{\text{N}_B}{\text{Ln}} D - \frac{2\text{N}_A \text{N}_B}{\text{Ln}} D \right) \Theta - \frac{\text{N}_B}{\text{Ln}} D \Phi &= \nu \Theta, \\
H_3 - \frac{\text{N}_A}{\text{Ln}} (D^2 - \delta^2) \Theta - \frac{1}{\text{Ln}} (D^2 - \delta^2) \Phi &= -\nu \Phi, \\
H_3 + \frac{1}{\text{Le}} (D^2 - \delta^2) \Upsilon &= \nu \Upsilon,
\end{aligned} \tag{5.4}$$

subject to

$$\begin{aligned}
H_3 = 0, \quad DH_3 - \lambda_L D^2 H_3 = 0, \quad \Theta = \Phi = \Upsilon = 0 \quad \text{at } z = 0, \\
H_3 = 0, \quad DH_3 + \lambda_U D^2 H_3 = 0, \quad \Theta = \Phi = \Upsilon = 0 \quad \text{at } z = 1.
\end{aligned}$$

Instability may arise through stationary or oscillatory modes depending on whether the critical growth rate  $\nu$  is real or complex. Here we focus on the stationary boundary, obtained under the exchange-of-stabilities condition  $\nu = 0$ . Moreover, when  $\text{N}_B = 0$  the stationary threshold becomes explicitly tractable. Introducing normal modes

$$(H_3, \Theta, \Phi, \Upsilon) = (W(z), \vartheta(z), \varphi(z), \psi(z)) \exp[i(\ell x + m y) + \nu t], \quad \delta^2 = \ell^2 + m^2,$$

and employing a one-term Galerkin approximation consistent with the homogeneous scalar boundary conditions,

$$W(z) = W_0 \sin(\pi z), \quad \vartheta(z) = \vartheta_0 \sin(\pi z), \quad \varphi(z) = \varphi_0 \sin(\pi z), \quad \psi(z) = \psi_0 \sin(\pi z),$$

yields an algebraic system  $\mathbf{A}\mathbf{X} = 0$  for  $\mathbf{X} = (W_0, \vartheta_0, \varphi_0, \psi_0)^T$ , with the coefficient matrix (as obtained in the Maple reduction)

$$\mathbf{A} = \begin{pmatrix} (\pi^2 + \delta^2)^2 & -\text{Ra} \delta^2 & \text{Rn} \delta^2 & -\frac{\text{Rc}}{\text{Le}} \delta^2 \\ 1 & -(\pi^2 + \delta^2) & 0 & 0 \\ 1 & \frac{\text{N}_A}{\text{Ln}} (\pi^2 + \delta^2) & \frac{\pi^2 + \delta^2}{\text{Ln}} & 0 \\ 1 & 0 & 0 & -\frac{\pi^2 + \delta^2}{\text{Le}} \end{pmatrix}.$$

The stationary neutral curve follows from  $\det(\mathbf{A}) = 0$ , giving

$$\text{Ra}_s(\delta) = -\text{Rc} + \frac{\delta^6 + 3\pi^2 \delta^4 + \left( 3\pi^4 - (\text{Ln} + \text{N}_A) \text{Rn} \right) \delta^2 + \pi^6}{\delta^2}. \tag{5.5}$$

Minimization of  $\text{Ra}_s(\delta)$  over  $\delta > 0$  determines the critical wavenumber. Writing  $x = \delta^2$  gives

$$\text{Ra}_s(x) = x^2 + 3\pi^2 x + 3\pi^4 + \frac{\pi^6}{x} - (\text{Ln} + \text{N}_A) \text{Rn} - \text{Rc},$$

and the optimality condition  $d\text{Ra}_s/dx = 0$  yields

$$-\frac{\pi^6}{x^2} + 3\pi^2 + 2x = 0,$$

with the admissible solution  $x = \pi^2/2$ , i.e.

$$\delta_c = \frac{\pi}{\sqrt{2}}. \quad (5.6)$$

Substituting  $\delta = \delta_c$  into (5.5) produces the closed-form stationary threshold

$$\text{Ra}_{sc} = \frac{27}{4}\pi^4 - (\text{Ln} + \text{N}_A)\text{Rn} - \text{Rc}. \quad (5.7)$$

Equation (5.7) shows that, in this stationary limit, the compositional Rayleigh number  $\text{Rc}$  shifts the onset linearly, whereas the solutal/nanofluid contribution enters through  $(\text{Ln} + \text{N}_A)\text{Rn}$ . Furthermore,  $\text{Pr}$  and  $\text{Le}$  do not influence the stationary threshold here, consistent with their primary role in the temporal (oscillatory) dynamics. In the classical limit  $\text{Rn} = \text{Rc} = 0$ , (5.7) reduces to  $\text{Ra}_{sc} = \frac{27}{4}\pi^4$ , recovering the standard stress-free stationary criterion.

## 6. Numerical approximation of the eigenvalue problem

The linearised equations of motion and energy of the governing equations are then transformed into an eigenvalue problem in order to determine the critical Rayleigh number ( $\text{Ra}_c$ ) and critical wavenumber ( $\delta_c^2$ ) in convection situations. Spectral methods like the Galerkin method and Chebyshev collocation are frequently used to achieve this. Due to the fact that the Chebyshev collocation method yields results with very high accuracy and spectrum convergence, it has become a very popular and essential tool in the numerical investigation of hydrodynamic stability and convection problems. It has been shown that the Chebyshev method can be used with very high precision to the eigenvalue problem of linear stability analysis for convection problems such as Rayleigh, Briggard and double, diffusive convection [49–53].

The method works extraordinarily well when it is based on Chebyshev polynomials as basis functions. It is particularly beneficial in cases of boundary layers since, in such situations, it remarkably depicts the flow gradients. This equally holds true even if the geometry is complicated or if the problem's diffusivity ratio varies significantly [54]. In comparison with standard finite, difference or finite, element methods, this technique enables scientists to predict the moment when convection sets in, to get the critical Rayleigh numbers, and to identify and analyze the formation of patterns using an extremely reduced number of grid points [55].

The accuracy of the Chebyshev collocation method is unmatched due to its ability to achieve spectral accuracy, where the error reduces exponentially with an increase in the number of collocation points [56–61]. Stability in matrix conditions and in the calculation of eigenvalues due to the use of Chebyshev polynomials makes the method suitable for the

treatment of hydrodynamic systems [62]. The method deals with non-uniform grids by allowing the collocation points to be dispersed densely in the regions of the flow where the gradient is large and in the vicinity of the boundaries, a feature that is handy for flows involving convection and the presence of a boundary layer [63]. The Chebyshev collocation method is, therefore, able to perform better than other low-order discretization methods in resolving the Navier-Stokes and energy equations in hydrodynamic stability and thus makes an impeccable choice for his field of [64].

To approximate system (4.5), we first employ the Chebyshev collocation method. This approach allows the system to be reformulated so that it involves only second- and third-order derivatives. By introducing the substitution  $\Pi = DH_3$ , the system is further reduced to four equations containing solely second-order derivatives. The physical domain is then mapped onto the standard Chebyshev interval  $(-1, 1)$ , and the variables  $H_3$ ,  $\Pi$ ,  $\Theta$ ,  $\Phi$ , and  $\Upsilon$  are expanded in series of Chebyshev polynomials as follows:

$$H_3 = \sum_{\gamma=0}^{\Gamma} \tilde{H}_{3\gamma} T_{\gamma}(z), \quad \Pi = \sum_{\gamma=0}^{\Gamma} \tilde{\Pi}_{\gamma} T_{\gamma}(z), \quad \Theta = \sum_{\gamma=0}^{\Gamma} \tilde{\Theta}_{\gamma} T_{\gamma}(z), \quad \Phi = \sum_{\gamma=0}^{\Gamma} \tilde{\Phi}_{\gamma} T_{\gamma}(z), \quad \Upsilon = \sum_{\gamma=0}^{\Gamma} \tilde{\Upsilon}_{\gamma} T_{\gamma}(z), \quad (6.1)$$

where

$$T_{\gamma}(z) = \cos(\gamma \arccos(z)), \quad -1 < z < 1.$$

Next, the expansion given in equation (6.1) is substituted into equation (4.5), and the resulting system is enforced at the Gauss-Lobatto collocation points, defined as:

$$y_{\gamma} = \cos\left(\frac{\pi\gamma}{\Gamma-2}\right), \quad \gamma = 0, \dots, \Gamma-2. \quad (6.2)$$

Consequently, this procedure yields a system of  $5\Gamma - 5$  equations in  $5\Gamma + 5$  unknowns, specifically  $\tilde{H}_{30}, \dots, \tilde{H}_{3\Gamma}$ ,  $\tilde{\Pi}_0, \dots, \tilde{\Pi}_{\Gamma}$ ,  $\tilde{\Theta}_0, \dots, \tilde{\Theta}_{\Gamma}$ ,  $\tilde{\Phi}_0, \dots, \tilde{\Phi}_{\Gamma}$ ,  $\tilde{\Upsilon}_0, \dots, \tilde{\Upsilon}_{\Gamma}$ . Approximating the boundary conditions using Chebyshev polynomial expansions then gives:

$$\begin{aligned} \widehat{BC}_1 : \sum_{\gamma=0}^{\Gamma} \tilde{H}_{3\gamma} &= 0, \quad \widehat{BC}_2 : \sum_{\gamma=0}^{\Gamma} (-1)^{\gamma} \tilde{H}_{3\gamma} = 0, \quad \widehat{BC}_3 : \sum_{\gamma=0}^{\Gamma} \gamma^2 \lambda_U \tilde{\Pi}_{\gamma} + \tilde{\Pi}_{\gamma} = 0, \\ \widehat{BC}_4 : \sum_{\gamma=0}^{\Gamma} (-1)^{\gamma+1} \gamma^2 \lambda_U \tilde{\Pi}_{\gamma} - (-1)^{\gamma} \tilde{\Pi}_{\gamma} &= 0, \quad \widehat{BC}_5 : \sum_{\gamma=0}^{\Gamma+2} \Theta_{\gamma} = 0, \\ \widehat{BC}_6 : \sum_{\gamma=0}^{\Gamma+2} (-1)^{\gamma} \Theta_{\gamma} &= 0, \quad \widehat{BC}_7 : \sum_{\gamma=0}^{\Gamma} \tilde{\Phi}_{\gamma} = 0, \quad \widehat{BC}_8 : \sum_{\gamma=0}^{\Gamma} (-1)^{\gamma} \tilde{\Phi}_{\gamma} = 0, \\ \widehat{BC}_9 : \sum_{\gamma=0}^{\Gamma+2} \Upsilon_{\gamma} &= 0, \quad \widehat{BC}_{10} : \sum_{\gamma=0}^{\Gamma+2} (-1)^{\gamma} \Upsilon_{\gamma} = 0, \end{aligned} \quad (6.3)$$

Ultimately, the procedure results in the following eigenvalue problem:

$$\begin{pmatrix} 2\mathbf{D} & -\mathbf{I} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} \\ \widehat{BC}_1 & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{BC}_2 & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \Lambda_0 & \Lambda_1 & -\delta^2 \text{Ra} \mathbf{I} & \delta^2 \text{Rn} \mathbf{I} & -\delta^2 (\text{Rc/Le}) \mathbf{I} \\ \widehat{RO} & \widehat{BC}_3 & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{RO} & \widehat{BC}_4 & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \zeta_1 & \widehat{\mathbf{O}} & \Lambda_2 & \Lambda_3 & \widehat{\mathbf{O}} \\ \widehat{RO} & \widehat{RO} & \widehat{BC}_5 & \widehat{RO} & \widehat{RO} \\ \widehat{RO} & \widehat{RO} & \widehat{BC}_6 & \widehat{RO} & \widehat{RO} \\ \zeta_2 & \widehat{\mathbf{O}} & \Lambda_4 & \Lambda_5 & \widehat{\mathbf{O}} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{BC}_7 & \widehat{RO} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{BC}_8 & \widehat{RO} \\ \zeta_3 & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \Lambda_6 \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{BC}_9 \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{BC}_{10} \end{pmatrix} X \quad (6.4)$$

$$= \nu \begin{pmatrix} \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ -(\delta^2/\text{Pr}) \mathbf{I} & (2/\text{Pr}) \mathbf{D} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \mathbf{I} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & -\mathbf{I} & \widehat{\mathbf{O}} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & \widehat{\mathbf{O}} & -\mathbf{I} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \\ \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} & \widehat{RO} \end{pmatrix} X, \quad (6.5)$$

where  $\widehat{RO}$  is the zero row,

$X = (\widetilde{H}_{30}, \dots, \widetilde{H}_{3\Gamma}, \widetilde{\Pi}_0, \dots, \widetilde{\Pi}_\Gamma, \widetilde{\Theta}_0, \dots, \widetilde{\Theta}_\Gamma, \widetilde{\Phi}_0, \dots, \widetilde{\Phi}_\Gamma, \widetilde{\Upsilon}_0, \dots, \widetilde{\Upsilon}_\Gamma)$ ,

$\widehat{\mathbf{O}}$  is the zero matrix,

$\mathbf{I}(\gamma_1, \gamma_2) = T_{\gamma_2}(z_{\gamma_1})$ ,

$\mathbf{D}(\gamma_1, \gamma_2) = T'_{\gamma_2}(z_{\gamma_1})$ ,

$\mathbf{D}^2(\gamma_1, \gamma_2) = T''_{\gamma_2}(z_{\gamma_1})$ ,

$\mathbf{D}^3(\gamma_1, \gamma_2) = T'''_{\gamma_2}(z_{\gamma_1})$ ,

$\zeta_1(\gamma_1, \gamma_2) = -\frac{dT_b}{dz}(z_{\gamma_1}) \mathbf{I}(\gamma_1, \gamma_2)$ ,

$$\begin{aligned}
\zeta_2(\gamma_1, \gamma_2) &= \frac{d\Phi_b}{dz}(z_{\gamma_1})\mathbf{I}(\gamma_1, \gamma_2), \\
\zeta_3(\gamma_1, \gamma_2) &= \mathbf{I}(\gamma_1, \gamma_2), \\
\Lambda_0 &= \delta^4 \mathbf{I}(\gamma_1, \gamma_2), \\
\Lambda_1 &= 8\mathbf{D}^3(\gamma_1, \gamma_2) - 4\delta^2 \mathbf{D}(\gamma_1, \gamma_2) - \frac{\tilde{Q}}{\text{Pr}}(D^2(\gamma_1, \gamma_2) - \delta^2 I(\gamma_1, \gamma_2)), \\
\Lambda_2 &= 4\mathbf{D}^2(\gamma_1, \gamma_2) - \delta^2 \mathbf{I}(\gamma_1, \gamma_2) + \frac{2N_B}{\text{Ln}} \frac{d\Phi_b}{dz}(z_{\gamma_1})\mathbf{D}(\gamma_1, \gamma_2) + \frac{4N_A N_B}{\text{Ln}} \frac{dT_b}{dz}(z_{\gamma_1})\mathbf{D}(\gamma_1, \gamma_2) - 2\tilde{Q}\mathbf{D}(\gamma_1, \gamma_2), \\
\Lambda_3 &= \frac{2N_B}{\text{Ln}} \frac{dT_b}{dz}(z_{\gamma_1})\mathbf{D}(\gamma_1, \gamma_2), \\
\Lambda_4 &= -\frac{N_A}{\text{Ln}} \left( 4\mathbf{D}^2(\gamma_1, \gamma_2) - \delta^2 \mathbf{I}(\gamma_1, \gamma_2) \right), \\
\Lambda_5 &= -\frac{1}{\text{Ln}} \left( 4\mathbf{D}^2(\gamma_1, \gamma_2) - \delta^2 \mathbf{I}(\gamma_1, \gamma_2) \right) - 2\tilde{Q}\mathbf{D}(\gamma_1, \gamma_2), \\
\Lambda_6 &= -\frac{1}{\text{Le}} \left( 4\mathbf{D}^2(\gamma_1, \gamma_2) - \delta^2 \mathbf{I}(\gamma_1, \gamma_2) \right) - 2\tilde{Q}\mathbf{D}(\gamma_1, \gamma_2), \\
\gamma_1 &= 0, \dots, \Gamma - 2, \gamma_2 = 0, \dots, \Gamma.
\end{aligned}$$

Standard Chebyshev differentiation matrices were used because, for the parameter ranges and collocation orders considered here, they produced numerically stable generalized eigenvalue problems and mesh-refined eigenvalues. Additional tests with increasing  $\Gamma$  showed no qualitative changes in the neutral curves and only negligible shifts in  $(Ra, \delta_c)$  after convergence, so no further scaling or preconditioning was required.

Rayleigh number,  $Ra$ , and wave number,  $\delta^2$ , are trial parameters for which the eigenvalue  $\nu$  (growth rate) is computed. The curve of neutral stability is reached when the eigenvalue has real part  $\text{Re}(\nu) = 0$  [65]; this curve has a  $Ra$  and  $\delta_c^2$ , minimum which is  $Ra_c$  and  $\delta_c^2$  and represents the convection onset. Convection motions are stationary if the critical eigenvalue  $\nu_c$  is real, meaning that the flow is time-independent. However, if  $\nu_c$  has a non-zero imaginary part, then there are time-periodic (or oscillatory) motions, which indicates that the instability being classified is the oscillatory type, with wave-like convection patterns [66].

One of the areas of interest is the numerical algorithm that usually undergoes an iterative procedure. This is when we select trial wave numbers  $\delta$ , and then we can solve the eigenvalue problem associated with the matrix  $\nu(Ra, \delta)$  and then determine  $Rac$  as the minimum  $Ra$  such that  $\text{Re}(\nu) = 0$ . This algorithm detects if it is in stationary and oscillatory modes by the imaginary part of  $\nu_c$  zero for stationary and for nonzero in the oscillatory part [67]. This is how the accurate mapping of marginal stability is done and the prediction of the flow behaviour is done when it is near the convection onset and provides the basis for understanding the nonlinear and turbulent extensions [68].

## 7. Numerical results

In this section, the critical Rayleigh number, denoted by  $Ra$ , is determined through numerical computations. All simulations assume equal lower and upper boundary parameters, i.e.,  $\lambda_L = \lambda_U = \lambda_0$ . Additionally, results corresponding to the minimization of  $Ra$  with respect to the wavenumber  $a$  are reported. It is indeed a fact that  $Ra$  actually depends on  $a$ , as well as on the set of dimensionless parameters:  $\text{Rn}$ ,  $\text{Rc}$ ,  $N_A$ ,  $\tilde{Q}$ ,  $N_B$ ,  $\text{Pr}$ ,  $\text{Le}$ ,  $\text{Ln}$ , and  $\lambda_0$ . Our approach thus makes it viable to perform a thorough investigation of the relationship between the critical Rayleigh number and the governing nondimensional parameters.

It was found that the dominating mode is always the stationary one, and an oscillatory mode is not present for any of the parameter values taken into account. As a result, the influence of the Prandtl number becomes completely negligible. Actually, before we start and share our findings, we would like to clarify the reasons for the absence of oscillatory mode just at the onset of convection. Through a linear stability analysis, a generalized eigenvalue problem is obtained by the linearized perturbation equations. Normal modes of the form are introduced as:

$$\Psi(x, y, z, t) = \psi(z), e^{i(k_x x + k_y y) + \nu t},$$

with the complex growth rate  $\nu = \nu_r + i\nu_i$ , leads directly to an eigenvalue problem for  $\nu$ . Stationary convection, corresponding to the exchange of stabilities, occurs when  $\nu_i = 0$  at marginal stability  $\nu_r = 0$ , so that the most unstable mode satisfies  $\nu = 0$  as a real eigenvalue. In contrast, oscillatory convection (overstability) is characterized by  $\nu_r = 0$  with  $\nu_i \neq 0$ , which signals a Hopf bifurcation.

In many double-diffusive systems, oscillatory behavior arises when two buoyancy mechanisms oppose one another, typically one stabilizing and one destabilizing, while possessing markedly different diffusivities. In the present configuration, although  $Rn$  may take either sign, the chosen parameter regime (with  $Rc > 0$  and varying  $Rn$ ) can cause the solutal and nanoparticle buoyancy effects to act in the same direction as the thermal buoyancy for the most unstable mode. Without an effective restoring mechanism that can support oscillatory motions, no buoyancy contribution can oscillate.

The main equations for  $\tilde{\phi}$  and  $\tilde{C}$  have strong diffusive terms in them, which are  $\nabla^2 \tilde{\phi}/Ln$  and  $\nabla^2 \tilde{C}/Le$ , with large  $Ln$  and moderate  $Le$ . These terms lead to significant damping of nanoparticle and solute perturbations, thus, no phase, lag mechanism for oscillations can be set up. Mathematically, the two concentration fields eigenvalue problems give solutions with large negative real parts, which means the fields follow the temperature and velocity perturbations almost instantly, i.e.,  $\partial_t \tilde{\phi} \approx 0$  and  $\partial_t \tilde{C} \approx 0$ .

Physically, the lack of oscillatory convection is that the destabilizing thermal buoyancy keeps on growing monotonically without hitting a restoring force that can bring overshoot and decay. By diffusion, the nanoparticle and solute components can either be too slow (large  $Ln$ ) or too strongly damped (moderate  $Le$ ) to produce the necessary lag in phase. The vertical throughflow  $\tilde{Q}$  does break symmetry but it is not enough to reverse the regime of dominating dissipative effects.

The closed-form stationary threshold derived in Section 5 provides a rigorous *reference solution* for evaluating the numerical accuracy of the Chebyshev collocation scheme in the corresponding limiting case ( $\tilde{Q} = 0$ ). In the stationary regime ( $\nu = 0$ ) with  $N_B = 0$ , the stability thresholds computed from Eq. (6.4) are systematically compared with the analytical results reported in Section 5.

Tables 1 and 2 list the absolute errors in the critical wavenumber  $\delta_c$  and the stationary critical Rayleigh number  $Ra_{sc}$ , respectively. The results are shown for several parameter sets and truncation orders  $N$  (with  $Rn = 1$ ). The data demonstrate rapid convergence as  $N$  increases, confirming the high accuracy of the proposed numerical implementation.

Table 1: Absolute error of  $\delta$  for  $Rn = 1$  for different parameter sets and truncation levels  $N$ .

| $(NA, Rc, Le, Ln)$ | $N = 16$                | $N = 18$                 | $N = 20$                 | $N = 22$                 |
|--------------------|-------------------------|--------------------------|--------------------------|--------------------------|
| (2, 5, 1, 10)      | $1.1863 \times 10^{-8}$ | $1.1690 \times 10^{-10}$ | $1.1774 \times 10^{-11}$ | $6.2078 \times 10^{-12}$ |
| (1, 5, 1, 10)      | $9.8014 \times 10^{-9}$ | $5.6950 \times 10^{-10}$ | $7.8295 \times 10^{-11}$ | $1.0266 \times 10^{-11}$ |
| (2, 4, 1, 10)      | $3.9558 \times 10^{-9}$ | $4.6099 \times 10^{-10}$ | $1.1233 \times 10^{-10}$ | $3.2824 \times 10^{-12}$ |
| (2, 5, 5, 10)      | $5.9626 \times 10^{-9}$ | $2.7216 \times 10^{-10}$ | $2.9518 \times 10^{-11}$ | $1.1735 \times 10^{-11}$ |
| (2, 5, 1, 15)      | $4.0844 \times 10^{-9}$ | $2.2450 \times 10^{-11}$ | $1.2284 \times 10^{-11}$ | $1.1921 \times 10^{-11}$ |
| (1, 10, 2, 10)     | $6.8472 \times 10^{-9}$ | $3.2521 \times 10^{-10}$ | $5.5835 \times 10^{-11}$ | $5.3765 \times 10^{-12}$ |
| (1, 15, 1, 10)     | $8.9171 \times 10^{-9}$ | $5.6950 \times 10^{-10}$ | $2.1977 \times 10^{-11}$ | $3.5917 \times 10^{-11}$ |
| (1, 12, 1, 50)     | $1.1963 \times 10^{-9}$ | $6.5719 \times 10^{-10}$ | $2.4309 \times 10^{-11}$ | $1.1871 \times 10^{-11}$ |
| (2, 50, 1, 100)    | $5.8127 \times 10^{-9}$ | $6.1354 \times 10^{-10}$ | $2.7023 \times 10^{-11}$ | $1.2567 \times 10^{-11}$ |

Table 2: Absolute error of  $Ra_L$  for  $Rn = 1$  for different parameter sets and truncation levels  $N$ .

| $(NA, Rc, Le, Ln)$ | $N = 16$                | $N = 18$                | $N = 20$                 | $N = 22$                 |
|--------------------|-------------------------|-------------------------|--------------------------|--------------------------|
| (2, 5, 1, 10)      | $3.4039 \times 10^{-6}$ | $1.8429 \times 10^{-8}$ | $7.0392 \times 10^{-11}$ | $2.0352 \times 10^{-13}$ |
| (1, 5, 1, 10)      | $3.4042 \times 10^{-6}$ | $1.8320 \times 10^{-8}$ | $7.0389 \times 10^{-11}$ | $2.0195 \times 10^{-13}$ |
| (2, 4, 1, 10)      | $3.4048 \times 10^{-6}$ | $1.8326 \times 10^{-8}$ | $7.0378 \times 10^{-11}$ | $2.0344 \times 10^{-13}$ |
| (2, 5, 5, 10)      | $3.4042 \times 10^{-6}$ | $1.8425 \times 10^{-8}$ | $7.0395 \times 10^{-11}$ | $2.0488 \times 10^{-13}$ |
| (2, 5, 1, 15)      | $3.4044 \times 10^{-6}$ | $1.8230 \times 10^{-8}$ | $7.0389 \times 10^{-11}$ | $2.0282 \times 10^{-13}$ |
| (1, 10, 2, 10)     | $3.4048 \times 10^{-6}$ | $1.8450 \times 10^{-8}$ | $7.0340 \times 10^{-11}$ | $2.0313 \times 10^{-13}$ |
| (1, 15, 1, 10)     | $3.4051 \times 10^{-6}$ | $1.8536 \times 10^{-8}$ | $7.0338 \times 10^{-11}$ | $2.0386 \times 10^{-13}$ |
| (1, 12, 1, 50)     | $3.4035 \times 10^{-6}$ | $1.8586 \times 10^{-8}$ | $7.0338 \times 10^{-11}$ | $2.0080 \times 10^{-13}$ |
| (2, 50, 1, 100)    | $3.4040 \times 10^{-6}$ | $1.8423 \times 10^{-8}$ | $7.0341 \times 10^{-11}$ | $2.0030 \times 10^{-13}$ |

The results reveal a near, perfect agreement between the numerical and analytical solutions, thus reflecting the strong power of the numerical scheme. The errors continuously get smaller as the polynomial order increases until a saturation level is reached, after which further refinement results in almost no improvement. Moreover, these results prove that one can obtain very accurate results with relatively low polynomial orders, i.e., the method gives dependable predictions at a lower computational cost than other numerical methods. Lastly, the errors in the critical wavenumber are, at first, smaller than those in the Rayleigh number. However, unlike the latter, they only gradually improve with the increasing number of Chebyshev polynomials.

The linear instability thresholds are depicted in Figure 2, where the dependence of the critical Rayleigh number  $Ra_c$  on the Lewis number  $Ln$  is presented for different values of the nanoparticle Rayleigh number  $Rn$ . In panel (a), the  $Rn$  values are negative, and in panel (b), they are positive. The rest of the parameters remain unchanged at  $N_A = 2$ ,  $\tilde{Q} = N_B = 0.1$ ,

$Rc = 50$ ,  $Pr = Le = 1$ , and  $\lambda_0 = 0.3$ . According to the data, the critical linear instability threshold is very much influenced by both  $Ln$  and the sign of  $Rn$ . For  $Rn < 0$ ,  $Ra_c$  strongly drops with an increase in  $Ln$ , and the decrease becomes more intense with more negative  $Rn$ . This means that a negative value of  $Rn$  corresponds to a destabilizing effect of nanoparticle buoyancy force which, in turn, enhances the density inversion and convection initiation; the greater  $Ln$  (i.e. stronger thermal diffusion in comparison with nanoparticle diffusion) further destabilizes the system, thus, the lower the thermal Rayleigh number for the onset of instability.

On the other hand, if  $Rn > 0$ , then  $Ra_c$  is an increasing function of  $Ln$  and the stabilizing effect becomes stronger as  $Rn$  gets larger. In this case, nanoparticle buoyancy acts against thermal buoyancy, thus, convection is suppressed; increasing  $Ln$  limits nanoparticle diffusion, so concentration gradients that have a stabilizing effect are maintained and a higher thermal Rayleigh number is needed to cause linear instability. The reversal in curvature of the graphs on the two panels reflects the opposing effects of thermal and nanoparticle diffusion on the linear instability threshold.

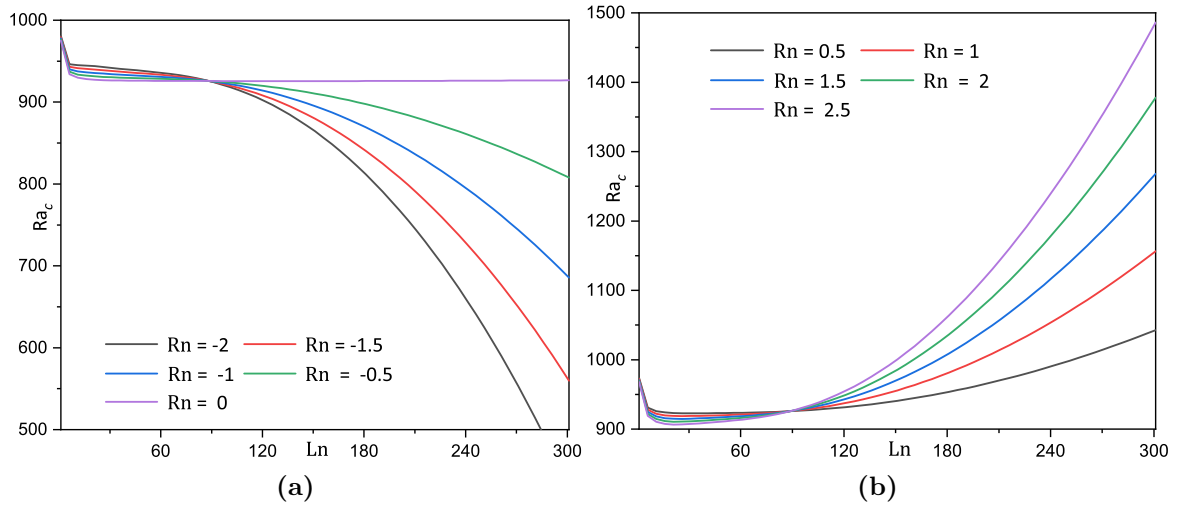


**Fig. 2.** Critical Rayleigh number  $Ra_c$  against  $Ln$  for  $Rn = -2, -1.5, -1, -0.5, 0$  in (a) and  $Rn = 0.5, 1, 1.5, 2, 2.5$  in (b), with  $N_A = 2, \tilde{Q} = N_B = 0.1, Rc = 50, Pr = Le = 1$  &  $\lambda_0 = 0.3$ .

Figure 3 presents the linear critical Rayleigh number  $Ra_c$  with respect to the Lewis number  $Ln$  for different solutal Rayleigh numbers  $Rc$ , given that  $N_A = 2, \tilde{Q} = N_B = 0.1, Pr = Le = 1$ , and  $\lambda_0 = 0.3$ . Panel (a) shows the case of  $Rn = -1$ , whereas panel (b) corresponds to  $Rn = 1$ . It can be seen in both panels that the neutral stability curves move downwards as  $Rc$  increases, pointing to the lowering of the linear critical Rayleigh number  $Ra_c$ . At  $Rn = -1$ ,  $Ra_c$  goes down as  $Ln$  increases and the destabilizing impact of  $Rc$  becomes more obvious at larger  $Ln$ . The reason is that higher  $Rc$  results in stronger solutal buoyancy due to concentration gradients, which intensifies the overall density inversion and makes convective motion easier; as  $Ln$  gets higher, thermal diffusion takes over nanoparticle

diffusion, hence, concentration gradients can still exist and the solutal part to instability is increased.

When  $Rn = 1$ , the graphs indicate a negligible dip at small  $Ln$  followed by a steep increase at larger  $Ln$ , however,  $Rc$  still plays a role in destabilizing the system. Even though the nanoparticle buoyancy tends to stabilize here, the solutal buoyancy coming from  $Rc$  negates this stabilizing effect, thus the linear instability threshold is lowered and convection is promoted. The differences in curvature between the two panels are a consequence of the influences of thermal, solutal, and nanoparticle diffusion competing with each other at the linear instability limit.

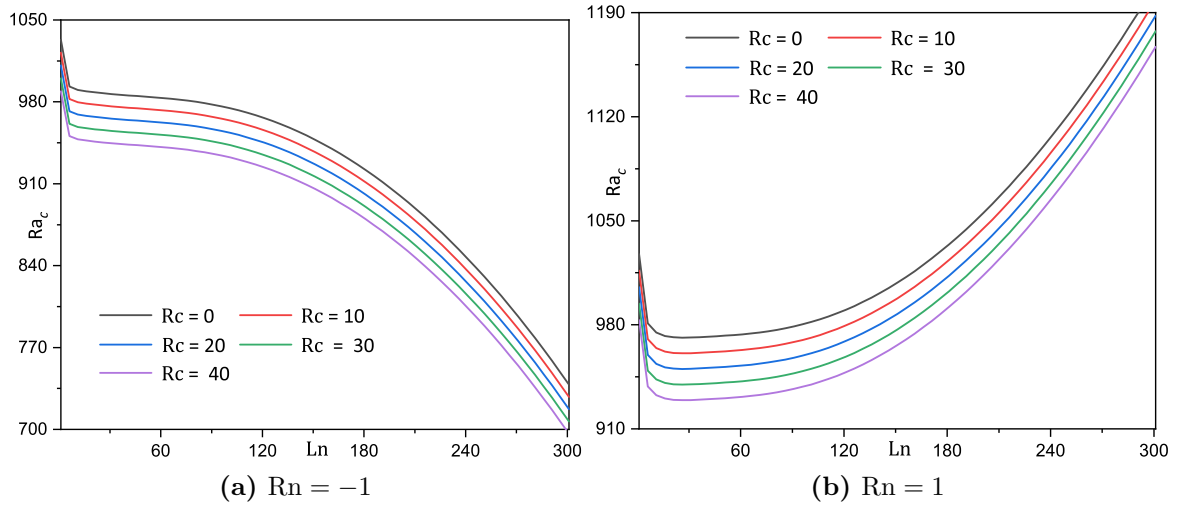


**Fig. 3.** Critical Rayleigh number  $Ra_c$  against  $Ln$  for  $Rc = 0, 10, 20, 30, 40$  with  $N_A = 2, \tilde{Q} = N_B = 0.1, Pr = Le = 1 \& \lambda_0 = 0.3$ .

Figure 4 illustrates the change of the linear critical Rayleigh number  $Ra_c$  with respect to the Lewis number  $Ln$  for different values of a vertical throughflow parameter  $\tilde{Q}$ , while  $N_A = 2, N_B = 0.1, Rc = 50, Pr = Le = 1$ , and  $\lambda_0 = 0.3$  are fixed. The figures (a) and (b) represent  $Rn = -1$ , whereas (c) and (d) represent  $Rn = 1$ , with two intervals of  $\tilde{Q}$  being considered in each case. For  $Rn = -1$ , small  $\tilde{Q}$  values (panel (a)) lead to the situation where  $Ra_c$  increases substantially with  $Ln$ , thus indicating the stabilizing effect of a weak throughflow that somewhat neutralizes the destabilizing nanoparticle buoyancy. Upon further increase of  $\tilde{Q}$  (panel (b)), the trend changes: at high  $Ln$  the curves are downward bending and  $Ra_c$  sharply falls, i.e., a strong throughflow will facilitate convective transport and thus reduce the linear instability threshold.

For the case of  $Rn = 1$  (Fig. (c) and (d)), indeed small throughflow (Fig. (c)) lowers  $Ra_c$  in comparison with no throughflow, particularly at large  $Ln$ , thus it appears that advection enhances the effect of the destabilizing thermalsolutal coupling in the presence of stabilizing nanoparticle effects. At higher  $\tilde{Q}$  (panel (d)), the  $Ra_c$  value increases dramatically with  $Ln$ , and the curves become very convex upwards, which means that the strong

throughflow impedes the formation of the buoyancy, driven cells because it advects thermal and concentration perturbations along the vertical. In short, the critical stability limit is very sensitive to the size of  $\tilde{Q}$  and to whether  $Rn$  is positive or negative. This shows that throughflow advection, double diffusion, and nanoparticle-driven buoyancy work together to set the point where convection starts.

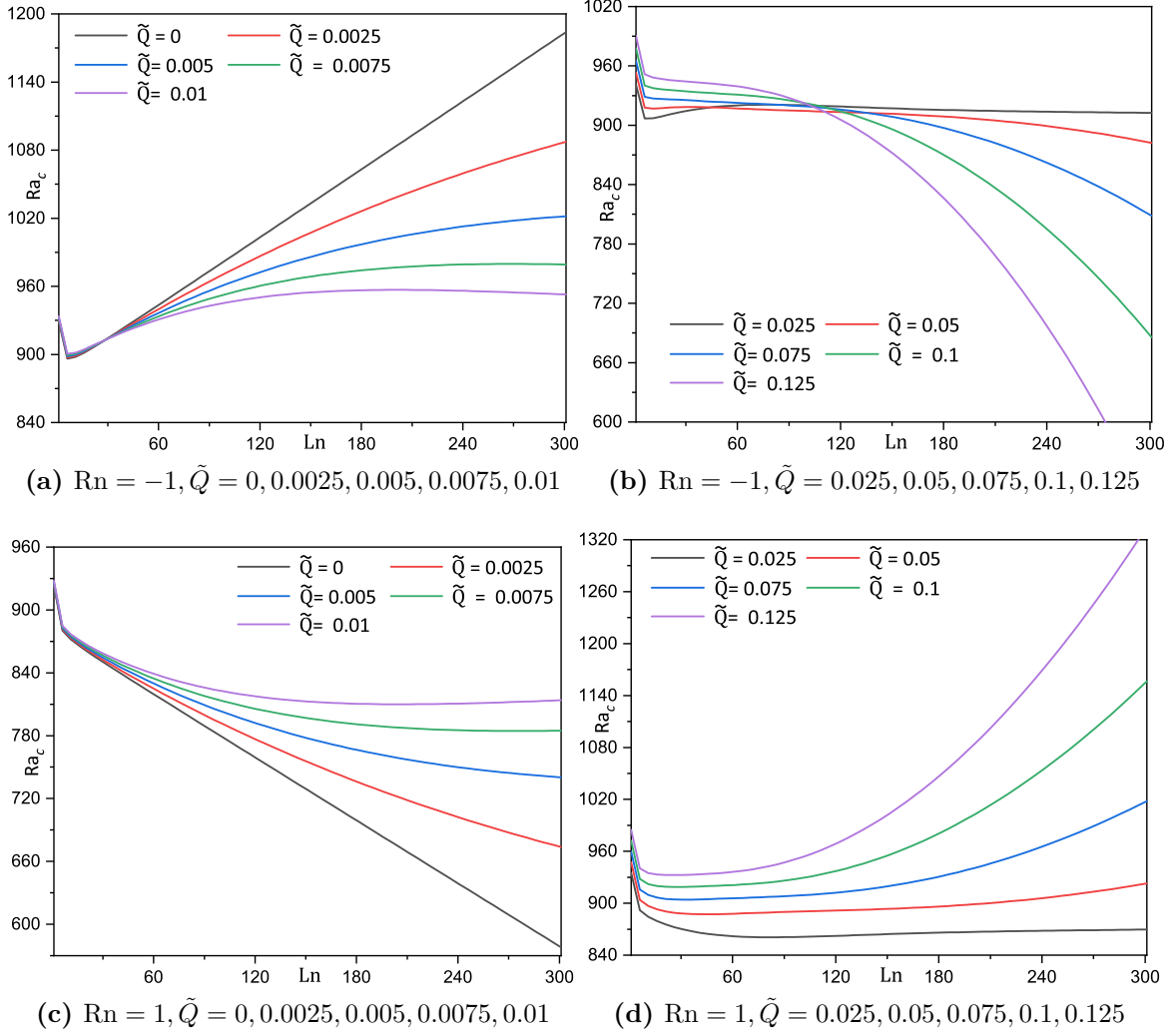


**Fig. 4.** Critical Rayleigh number  $Ra_c$  against  $Ln$  with  $N_A = 2, N_B = 0.1, Rc = 50, Pr = Le = 1$  &  $\lambda_0 = 0.3$ .

Figure 3 illustrates the variation of the linear critical Rayleigh number  $Ra_c$  as a function of the Lewis number  $Ln$  for several values of the solutal Rayleigh number  $Rc$ . The computations are carried out for fixed parameter values  $N_A = 2, \tilde{Q} = N_B = 0.1, Pr = Le = 1$ , and  $\lambda_0 = 0.3$ . Panel (a) shows the case of  $Rn = -1$ , whereas panel (b) corresponds to  $Rn = 1$ . It can be seen in both panels that the neutral stability curves move downwards as  $Rc$  increases,

pointing to the lowering of the linear critical Rayleigh number  $Ra_c$ . At  $Rn = -1$ ,  $Ra_c$  goes down as  $Ln$  increases and the destabilizing impact of  $Rc$  becomes more obvious at larger  $Ln$ . The reason is that higher  $Rc$  results in stronger solutal buoyancy due to concentration gradients, which intensifies the overall density inversion and makes convective motion easier; as  $Ln$  gets higher, thermal diffusion takes over nanoparticle diffusion, hence, concentration gradients can still exist and the solutal part to instability is increased.

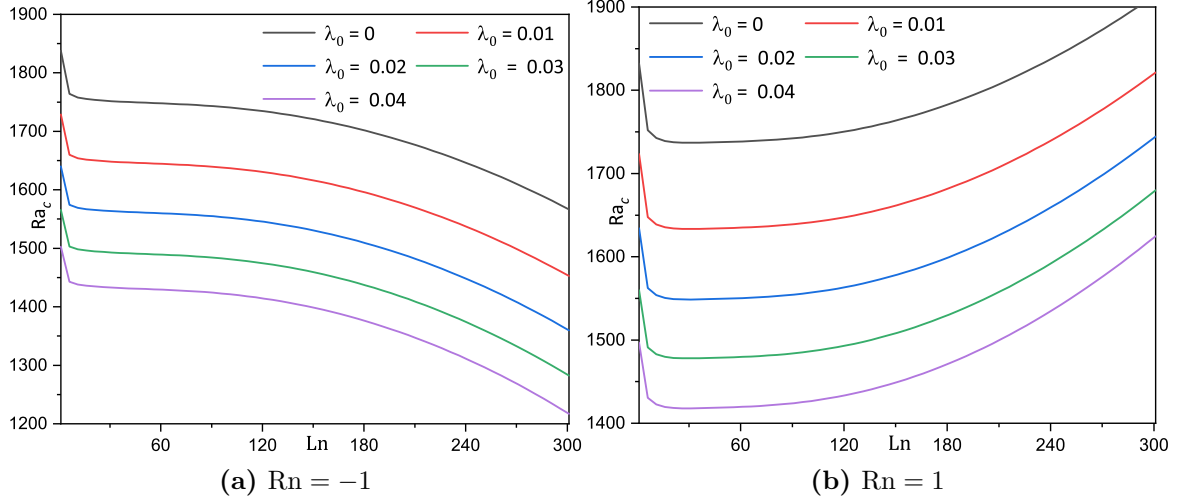


**Fig. 5.** Critical Rayleigh number  $Ra_c$  against  $Ln$  for  $\lambda_0 = 0, 0.01, 0.02, 0.03, 0.04$  with  $N_A = 2, \tilde{Q} = N_B = 0.1, Rc = 50, \&Pr = Le = 1$ .

Figure 6 shows the linear critical Rayleigh number  $Ra_c$  versus the Lewis number  $Ln$  for different values of the Brownian motion parameter  $N_B$ , where  $N_A = 2, \tilde{Q} = 0.1, Rc = 50, Pr = Le = 1$ , and  $\lambda_0 = 0.3$ . The left panel is for  $Rn = -1$ , and the right panel for  $Rn = 1$ . For  $Rn = -1$ ,  $Ra_c$  drops steadily with  $Ln$  for all  $N_B$ , and as  $N_B$  grows, the curves go down showing less instability threshold. This means that as Brownian diffusion becomes stronger, the coupling of temperature and concentration perturbations gets stronger; thus, if nanoparticle buoyancy is destabilizing ( $Rn < 0$ ), then this stronger coupling leads to density differences that make convection easier. If  $Rn = 1$ , the graphs have a sharp dip at moderate  $Ln$ , and  $N_B$  impact becomes quite significant at larger  $Ln$ . In this case, Brownian motion changes the way nanoparticles are distributed so that the stabilizing effect of the stratification is weakened at diffusivity ratios corresponding to the intermediate range, thus the value  $Ra_c$  decreases near the minimum. On the contrary, at large  $Ln$ , the combination of thermal diffusion and Brownian transport leads to a quick growth in  $Ra_c$ . To conclude, Brownian motion drastically changes the linear instability features through the competition between the thermal and the nanoparticle diffusion.

The critical Rayleigh number  $Ra_c$  has been plotted against the Lewis number  $Ln$  for progressively higher values of the modified diffusivity ratio,  $N_A = 0.5, 1, 2, 3, 4$ , while the throughflow  $\tilde{Q} = 0.1$ , nanoparticle parameter  $N_B = 2$ , solute Rayleigh number  $Rc = 50$ ,

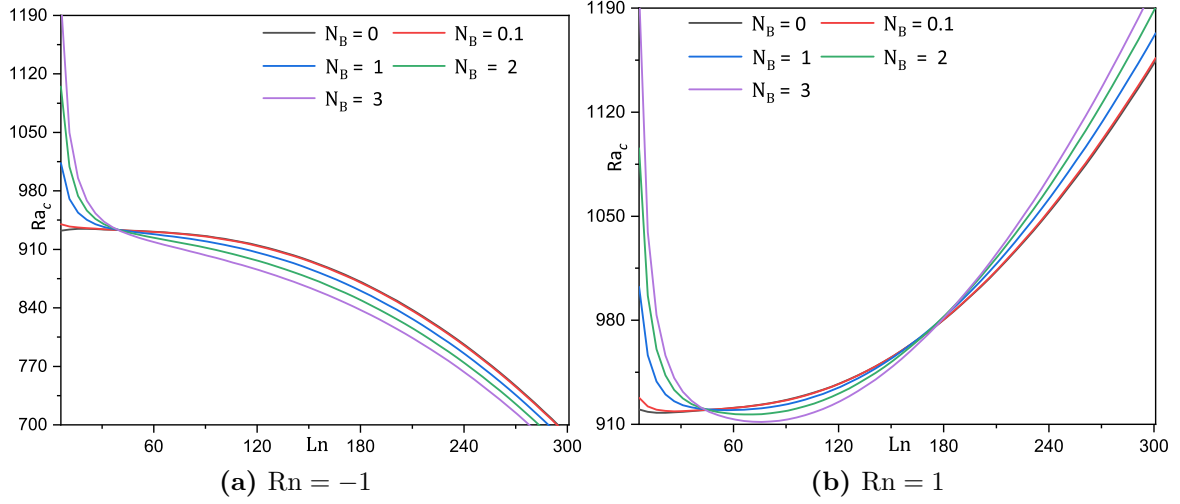


**Fig. 6.** Critical Rayleigh number  $Ra_c$  against  $Ln$  for  $N_B = 0, 0.1, 1, 2, 3$  with  $N_A = 2, \tilde{Q} = 0.1, Rc = 50, Pr = Le = 1 \& \lambda_0 = 0.3$ .

and slip parameter  $\lambda_0 = 0.3$  are kept constant, as evidenced in Figure 7. The diagrams clearly show that for a certain  $Ln$ , elevating  $N_A$  results in a steady and continuous increase of  $Ra_c$ , thereby accentuating a stabilizing effect. The sign of the nanoparticle Rayleigh number  $Rn$  controls this behavior. When  $\phi = 0$  at the lower boundary and  $\phi = 1$  at the upper boundary,  $Rn > 0$  (heavier nanoparticles) means that the nanoparticle field is stably stratified. Thermophoresis, measured by  $N_A$ , links temperature differences to the movement of nanoparticles. When  $N_A$  goes up, this link gets stronger, stabilizing stratification, and the threshold for convection goes up. Also, for any given  $N_A$ ,  $Ra_c$  slowly goes down as  $Ln$  goes up. This means that faster nanoparticle diffusion (higher  $Ln$ ) makes the stabilizing gradient weaker, which lowers the instability threshold.

Figure 8 shows the linear critical Rayleigh number  $Ra_c$  versus Lewis number  $Ln$  for different Prandtl number  $Pr$  which were computed at  $N_A = 2, \tilde{Q} = N_B = 0.1, Pr = Le = 1$ , and  $\lambda_0 = 0.1$ . Panel (a) displays the situation of  $Rn = -1$ , while panel (b) is for  $Rn = 1$ . It is obvious in both panels that the neutral stability curves shift downward as  $Rc$  increases, thus the lowering of the linear critical Rayleigh number  $Ra_c$  is indicated. At  $Rn = -1$ ,  $Ra_c$  decreases with the increase in  $Ln$  and the destabilizing effect of  $Pr$  becomes quite apparent at larger  $Ln$ . On the other hand, at  $Rn = 1$ , the figures show a very small decrease at low  $Ln$ , followed by a sharp increase at high  $Ln$ , nevertheless,  $Pr$  still have a role in destabilizing the system.

## 8. Conclusions

The onset of double-diffusive convection in a horizontal nanofluid layer is analyzed numerically, with particular emphasis on the role of vertical throughflow in determining the

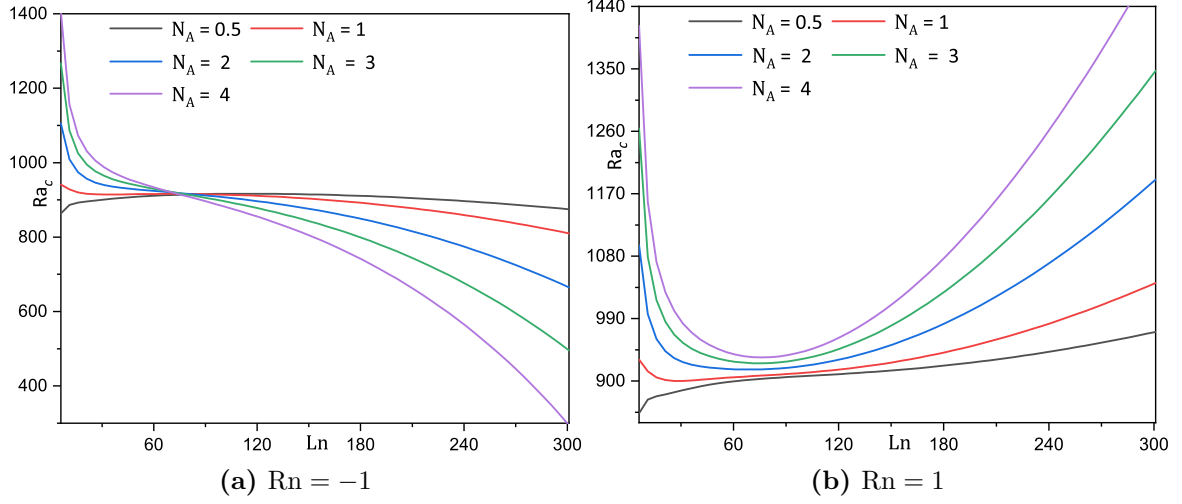


**Fig. 7.** Critical Rayleigh number  $Ra_c$  against  $Ln$  for  $N_A = 0.5, 1, 2, 3, 4$  with  $\tilde{Q} = 0.1, N_B = 2, Rc = 50, Pr = Le = 1$  &  $\lambda_0 = 0.3$ .

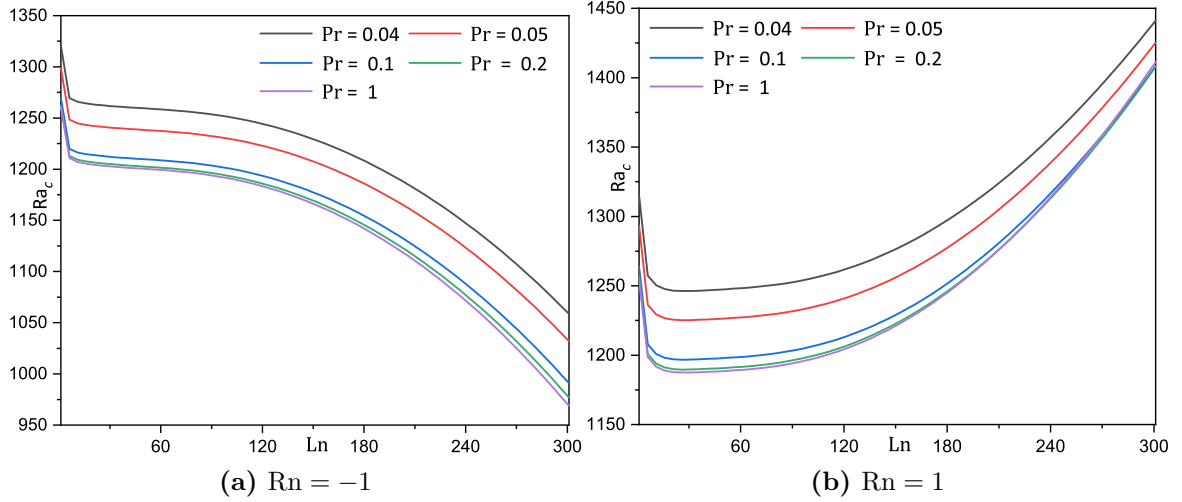


**Fig. 8.** Critical Rayleigh number  $Ra_c$  against  $Ln$  for  $Pr = 0.04, 0.05, 0.1, 0.2, 1$  with  $N_A = 2, \tilde{Q} = 0.1, N_B = 2, Rc = 50, Le = 1$  &  $\lambda_0 = 0.1$ .

critical conditions for convection. The nanofluid model accounts for both Brownian diffusion and thermophoretic transport, while the governing thermal energy equations include standard diffusion as well as cross-diffusion effects. The instability boundaries are determined using the Chebyshev collocation method. The physical implications of the computed stability thresholds are discussed in detail, leading to the following main conclusions:

- The nanoparticle Rayleigh number  $R_n$  is a primary determinant of the linear instability threshold. Negative values ( $R_n < 0$ ) produce a pronounced destabilising effect, especially at larger  $Ln$ , whereas positive values ( $R_n > 0$ ) are strongly stabilising and increase the critical Rayleigh number. Thus, nanoparticle-induced buoyancy can be used as an effective control parameter to either delay or accelerate the onset of convection in nanofluid layers with throughflow and slip.
- The solutal Rayleigh number  $R_c$  acts as an independent destabilising parameter in the linear regime, lowering the critical threshold even when nanoparticle buoyancy is stabilising. This underscores the major role of solutal buoyancy in setting the convection onset and shows that concentration effects can significantly alter the linear transition behaviour.
- It was noted that the vertical throughflow has a dual role as a control mechanism in the linear case, and it can either inhibit or enhance the convection, depending on its strength and the direction of the buoyancy effects of the nanoparticles. This suggests that the throughflow is a flexible means of controlling the onset of linear convection in the nanofluid layer.
- It can be concluded that the boundary slip has a significant effect of increasing the sensitivity of the linear instability threshold to the diffusive effects, which means that the system is very sensitive to the variations of  $Ln$ . This implies that the interfacial properties at the boundaries have a very important role in determining the linear instability threshold.
- From this figure, we can see that the Brownian motion parameter  $N_B$  causes a qualitative change in the neutral stability curves, especially with the appearance and control of the new minimum in  $Ra_c$ . This suggests that the microscale transport processes of the nanoparticles can affect not only the value of the linear instability threshold but also its functional dependence on the Lewis number.
- The findings emphasize again that the role of  $N_A$  in convective instability is not absolute but depends on the sign of  $R_n$ . Contrary to the always destabilizing effect of throughflow or the stabilizing effect of the solute field in the previous figures,  $N_A$  has a dual effect: it is stabilizing for bottom-heavy nanoparticles ( $R_n > 0$ ) but would be destabilizing if top-heavy nanoparticles ( $R_n < 0$ ) are considered. This dual effect emphasizes the crucial need to properly quantify nanoparticle buoyancy in real situations, whether in nanofluid heat transfer or geophysical flows, because the same

thermophoretic mechanism can either stabilize or trigger convection, depending on the density difference between particles and base fluid.

## Author Contributions

All authors contributed to the conceptualization and design of the study. Sattar M. Hassan performed the nondimensional transformation of the system and derived the eigenvalue system for the linear instability analysis. Assma J. Harfash contributed by preparing the programs for the numerical results. Sanaa L. Khalaf provided commentary on the numerical results. Ayad R. Khudair contributed by deriving the steady state solution to the problem analytically, deriving the analytic values of Rayleigh's number, and calculating the errors. Akil J. Harfash proposed the problem, supervised the work, and wrote the abstract, introduction, and conclusion.

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## Conflict of interest

We have no competing interests.

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