

# Supplementary Materials to "Testing Martingale Difference Hypothesis for Functional Time Series"

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The supplementary material is organized as follows. Appendix [A](#) provides some auxiliary lemmas and their proofs. Appendix [B](#) gives all the proofs of propositions and theorems in the main text.

## A Technical lemmas

**Lemma 1.** *Let  $\xi_n = \sum_{j=1}^{n-1} \frac{1}{(j\pi)^2} \xi_{j,n}$  be a sequence of random variables with  $\xi_{j,n} > 0$ . Under the condition (i)  $\xi_{j,n} = o_p(1)$ ,  $\forall 1 \leq j < n$ ; (ii)  $\mathbb{E}\xi_{j,n} \leq C$  uniformly for  $1 \leq j$ , we have  $\xi_n = o_p(1)$ .*

*Proof.* For each fixed integer  $K$  with  $1 \leq K \leq n - 1$ , decompose

$$\xi_n = \sum_{j=1}^K \frac{1}{(j\pi)^2} \xi_{j,n} + \sum_{j=K+1}^{n-1} \frac{1}{(j\pi)^2} \xi_{j,n} := \xi_{1,K} + \xi_{2,K}.$$

Note that by condition (i) we have  $\xi_{1,K} = o_p(1)$ . By condition (ii) we have

$$\mathbb{E} \sum_{j=K+1}^{n-1} \frac{1}{(j\pi)^2} \xi_{j,n} \leq C \sum_{j=K+1}^{\infty} \frac{1}{(j\pi)^2} \rightarrow 0, \text{ as } K \rightarrow \infty.$$

Then Theorem 3.2 of [Billingsley \(1999\)](#) yields that  $\xi_n = o_p(1)$ . □

**Lemma 2.** *Under Assumption 1 and  $H_0$ ,  $n^{-1/2} \sum_{t=1}^n Y_t - \mu$  converges weakly to a Gaussian random element in Hilbert space  $\mathcal{L}^2(\ell)$ .*

*Proof.* Denot  $Y'_t = Y_t - \mu$ . We apply Theorem 2.16 in [Bosq \(2000\)](#) to complete the proof, i.e. verifying the following conditions hold for an orthonormal basis  $\{e_j\}_{j \geq 1}$  of  $\mathcal{L}^2(\ell)$ :

- (i)  $n^{-1/2} \mathbb{E} \left( \max_{1 \leq t \leq n} \|Y'_t\| \right) \rightarrow 0, \quad n \rightarrow \infty,$
- (ii)  $n^{-1} \sum_{1 \leq t \leq n} \langle Y'_t, e_k \rangle \langle Y'_t, e_l \rangle \xrightarrow{\text{a.s.}} \psi_{k,l}, \quad n \rightarrow \infty, l, k \geq 1,$  where  $\{\psi_{k,l}, l, k \geq 1\}$  is a family of real numbers,
- (iii)  $\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \sum_{t=1}^n r_N^2 (n^{-1/2} Y'_t) > \varepsilon \right) = 0, \quad \varepsilon > 0,$  where  $r_N^2(x) = \sum_{i=N}^{\infty} |\langle x, e_i \rangle|^2$  for any  $x \in \mathcal{L}^2(\ell)$ .

To verify the condition (i) we need to show  $\max_{1 \leq t \leq n} \|n^{-1/2} Y'_t\|$  converges to zero in probability and  $\{\max_{1 \leq t \leq n} \|n^{-1/2} Y'_t\|^2, n \geq 1\}$  is uniformly integrable. For any  $\eta > 0$ , note that

$$\mathbb{P} \left( \max_{1 \leq t \leq n} \|Y'_t\| > \sqrt{n}\eta \right) \leq \mathbb{P} \left( \bigcup_{t=1}^n \{\omega : \|Y'_t\| > \sqrt{n}\eta\} \right) \leq \sum_{t=1}^n \mathbb{P} (\|Y'_t\| > \sqrt{n}\eta).$$

Applying Chebyshev inequality and stationarity of  $\{Y'_t\}$  to obtain

$$\begin{aligned}
\sum_{t=1}^n \mathbb{P}(\|Y'_t\| > \sqrt{n\eta}) &= \sum_{t=1}^n \mathbb{P}(\|Y'_t\| 1_{\{\|Y'_t\| \geq \sqrt{n\eta}\}} > \sqrt{n\eta}) \\
&\leq \frac{1}{n\eta^2} \sum_{i=1}^n \mathbb{E}(\|Y'_t\|^2 1_{\{\|Y'_t\| > \sqrt{n\eta}\}}) \\
&= \frac{1}{\eta^2} \mathbb{E}(\|Y'_t\|^2 1_{\{\|Y'_t\| > \sqrt{n\eta}\}}) \rightarrow 0, \quad n \rightarrow \infty.
\end{aligned}$$

On the other hand, we have

$$\mathbb{E}\left(\max_{1 \leq t \leq n} \|n^{-1/2}Y'_t\|^2\right) \leq \sum_{t=1}^n \mathbb{E}\|n^{-1/2}Y'_t\|^2 = \mathbb{E}\|Y'_t\|^2 < \infty,$$

which implies the condition (i) holds.

Now we prove the condition (ii) is satisfied. Let  $(e_j, \lambda_j)$  be the eigenelements of covariance operator  $C_{Y'} = \mathbb{E}(Y'_1 \otimes Y'_1)$ . Since  $\{\langle Y'_t, e_k \rangle \langle Y'_t, e_l \rangle, t \geq 1\}$  is the strictly stationary ergodic sequence, it follows from Birkhoff's Pointwise Ergodic Theorem (see [Petersen and Petersen \(1989\)](#) e.g.) that  $n^{-1} \sum_{1 \leq t \leq n} \langle Y'_t, e_k \rangle \langle Y'_t, e_l \rangle \xrightarrow{\text{a.s.}} \psi_{k,l}$ , where  $\psi_{k,l} = \mathbb{E}(\langle Y'_t, e_k \rangle \langle Y'_t, e_l \rangle) = 1\{k=l\}\lambda_k$  is a real number for  $k, l \geq 1$ .

As for condition (ii), by Markov Theorem we have for any  $\eta > 0$ ,  $\mathbb{P}[\sum_{t=1}^n r_N^2(n^{-1/2}Y'_t) > \eta] \leq \frac{1}{\eta} \mathbb{E}(\sum_{t=1}^n r_N^2(n^{-1/2}Y'_t)) = \frac{1}{\eta} \sum_{t=1}^n \mathbb{E}(\sum_{i=N}^{\infty} |\langle n^{-1/2}Y'_t, e_i \rangle|^2) = \frac{1}{\eta} \sum_{i=N}^{\infty} \langle C_{Y'} e_i, e_i \rangle = \frac{1}{\eta} \sum_{i=N}^{\infty} \lambda_i \rightarrow 0$ , as  $N \rightarrow \infty$ , where the convergence holds by  $\sum_{j=1}^{\infty} \lambda_j = \mathbb{E}\|Y'_t\|^2 < \infty$ . Thus condition (iii) is satisfied and the weak convergence to the Gaussian process holds. □

The next Lemma 3-6 serves as crucial steps for proving Theorem 1.

**Lemma 3.** *Under assumption 1,*

$$\mathbb{E} \int_{\mathbb{R}} \int_{\ell} |\phi_{t-j}(s, \tau_2)|^2 d\tau_2 \omega(ds) = \mathbb{E}\|Y_{t-j} - Y_{t-j}^+\|_1$$

and for any  $t \neq t', t, t' = 1, \dots, n$ ,

$$\mathbb{E} \int_{\mathbb{R}} \int_{\ell} \phi_{t-j}(s, \tau_2) \phi_{t'-j}^c(s, \tau_2) d\tau_2 \omega(ds) = \mathbb{E} \|Y_{t-j} - Y_{t'-j}\|_1 - \mathbb{E} \|Y_{t-j} - Y_{t-j}^+\|_1.$$

*Proof.* It follows directly from Lemma 2 in Wang (2024) and we omit it.  $\square$

**Lemma 4.** Define

$$\hat{Z}_n(\eta) = \sum_{j=1}^{n-1} (n-j)^{1/2} \hat{r}_j(s, \tau_1, \tau_2) \psi_j(\lambda),$$

where

$$\hat{r}_j(s, \tau_1, \tau_2) = \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1)] \phi_{t-j}(s, \tau_2)$$

with  $\phi_{t-j}(s, \tau_2) = \varphi_{t-j}(s, \tau_2) - \mathbb{E}\varphi_{t-j}(s, \tau_2)$ . Then under assumption 1 and  $H_0$ , we have

$$\|\hat{S}_n\|_{\mathcal{A}}^2 = \|\hat{Z}_n\|_{\mathcal{A}}^2 + o_p(1),$$

*Proof.* Note that

$$\begin{aligned} \hat{\gamma}_j(s, \tau_1, \tau_2) &= \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \bar{Y}_{n-j}(\tau_1)] \varphi_{t-j}(s, \tau_2) \\ &= \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1) + \mu(\tau_1) - \bar{Y}_{n-j}(\tau_1)] \varphi_{t-j}(s, \tau_2) \\ &= \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1)] \phi_{t-j}(s, \tau_2) \\ &\quad - \left\{ \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1)] \right\} \left\{ \frac{1}{n-j} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) \right\}, \end{aligned}$$

thus we have  $\hat{S}_n(\eta) = \hat{Z}_n(\eta) - \hat{R}_n(\eta)$ , where

$$\hat{R}_n(\eta) = \sum_{j=1}^{n-1} (n-j)^{1/2} \left\{ \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1)] \right\} \left\{ \frac{1}{n-j} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) \right\} \psi_j(\lambda).$$

Since  $\|\hat{S}_n\|_{\mathcal{A}}^2 = \|\hat{Z}_n\|_{\mathcal{A}}^2 + \|\hat{R}_n\|_{\mathcal{A}}^2 - 2 \operatorname{Re} \left( \left\langle \hat{Z}_n, \hat{R}_n \right\rangle_{\mathcal{A}} \right)$ , to complete the proof we only need to show

$$\|\hat{R}_n\|_{\mathcal{A}}^2 = o_p(1) \quad (\text{A.1})$$

and

$$\left| \left\langle \hat{Z}_n, \hat{R}_n \right\rangle_{\mathcal{A}} \right| = o_p(1). \quad (\text{A.2})$$

For equation (A.1), by the fact that  $\langle \psi_j, \psi_{j'} \rangle = 1_{\{j=j'\}} / (j\pi)^2$  we have

$$\begin{aligned} \|\hat{R}_n\|_{\mathcal{A}}^2 &= \sum_{j=1}^{n-1} \frac{(n-j)}{(j\pi)^2} \left\| \frac{1}{(n-j)} \sum_{t=j+1}^n Y_t - \mu \right\|^2 \\ &\quad \times \left\{ \int_{\mathbb{R}} \int_{\ell} \left| \frac{1}{(n-j)} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) \right|^2 d\tau_2 \omega(ds) \right\} \end{aligned}$$

Apply Lemma 2 and continuous mapping theorem to obtain

$$\left\| (n-j)^{-1/2} \sum_{t=j+1}^n Y_t - \mu \right\|^2 = O_p(1), \quad \forall 1 \leq j < n. \quad (\text{A.3})$$

On the other hand,

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}} \int_{\ell} \left| \frac{1}{(n-j)} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) \right|^2 d\tau_2 \omega(ds) \\ &= \frac{1}{(n-j)^2} \sum_{t=j+1}^n \mathbb{E} \int_{\mathbb{R}} \int_{\ell} |\phi_{t-j}(s, \tau_2)|^2 d\tau_2 \omega(ds) \\ &\quad + \frac{2}{(n-j)^2} \sum_{j+1 \leq t < t' \leq n} \mathbb{E} \int_{\mathbb{R}} \int_{\ell} \phi_{t-j}(s, \tau_2) \phi_{t'-j}^c(s, \tau_2) d\tau_2 \omega(ds) \\ &= \frac{1}{(n-j)} \mathbb{E} \|Y_{t-j} - Y_{t-j}^+\|_1 \\ &\quad + 2 \sum_{m=1}^{n-j-1} \frac{1}{n-j} \left( 1 - \frac{m}{n-j} \right) \left( \mathbb{E} \|Y_t - Y_{t+m}\|_1 - \mathbb{E} \|Y_t - Y_t^+\|_1 \right), \end{aligned}$$

where the last equality holds by Lemma 3. By the ergodic condition and the Toeplitz Lemma, as  $n \rightarrow \infty$ ,

$$\sum_{m=1}^{n-j-1} \frac{1}{n-j} \left(1 - \frac{m}{n-j}\right) (\mathbb{E} \|Y_t - Y_{t+m}\|_1 - \mathbb{E} \|Y_t - Y_t^+\|_1) = o(1)$$

thus

$$\int_{\mathbb{R}} \int_{\ell} \left| \frac{1}{(n-j)} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) \right|^2 d\tau_2 \omega(ds) = o_p(1). \quad (\text{A.4})$$

Then the equation (A.1) holds by (A.3), (A.4) and Lemma 1. Next we consider  $\|\hat{Z}_n\|_{\mathcal{A}}^2$ , denote  $\hat{r}_{j,s}(\tau_1, \tau_2) := \hat{r}_j(s, \tau_1, \tau_2)$ . Then

$$\mathbb{E} \left\| \hat{Z}_n \right\|_{\mathcal{A}}^2 = \sum_{j=1}^{n-1} \frac{n-j}{(j\pi)^2} \mathbb{E} \int_{\mathbb{R}} \|\hat{r}_{j,s}\|^2 \omega(ds) \quad (\text{A.5})$$

where

$$\begin{aligned} & (n-j) \mathbb{E} \int_{\mathbb{R}} \|\hat{r}_{j,s}\|^2 \omega(ds) \\ &= \frac{1}{(n-j)} \sum_{t=j+1}^n \mathbb{E} \left\{ \|Y_t - \mu\|^2 \int_{\mathbb{R}} \int_{\ell} |\phi_{t-j}(s, \tau_2)|^2 d\tau_2 \omega(ds) \right\} \\ &+ \frac{2}{(n-j)} \sum_{j+1 \leq t < t' \leq n} \mathbb{E} \left\{ \langle Y_t - \mu, Y_{t'} - \mu \rangle \int_{\mathbb{R}} \int_{\ell} \phi_{t-j}(s, \tau_2) \phi_{t'-j}^c(s, \tau_2) d\tau_2 \omega(ds) \right\}. \end{aligned}$$

Under assumption 1 and  $H_0$ , for any  $j+1 \leq t < t' \leq n$ ,

$$\mathbb{E} \left\{ \langle Y_t - \mu, Y_{t'} - \mu \rangle \int_{\mathbb{R}} \int_{\ell} \phi_{t-j}(s, \tau_2) \phi_{t'-j}^c(s, \tau_2) d\tau_2 \omega(ds) \right\} = 0$$

where we use the law of iterated expectation. Denote  $\phi_{t-j,s}(\tau_2) := \phi_{t-j}(s, \tau_2)$ , it follows that

$$\begin{aligned}
(n-j)\mathbb{E} \int_{\mathbb{R}} \|\hat{r}_{j,s}\|^2 \omega(ds) &= \mathbb{E} \int_{\mathbb{R}} \|(Y_t - \mu)\phi_{t-j,s}\|^2 \omega(ds) \\
&= \mathbb{E} \left( \|Y_t - \mu\|^2 \int_{\mathbb{R}} \|\phi_{t-j,s}\|^2 \omega(ds) \right) \\
&\leq 2\mathbb{E} [\|Y_t - \mu\|^2 (\|Y_t\|_1 + \mathbb{E}\|Y_t\|_1)] \\
&\leq C < \infty,
\end{aligned} \tag{A.6}$$

where the first inequality holds by Lemma 1 in Székely et al. (2007). Then we have  $\mathbb{E} \|\hat{Z}_n\|_{\mathcal{A}}^2 \leq C \sum_{j=1}^{n-1} \frac{1}{(j\pi)^2} = O(1)$ , which implies

$$\|\hat{Z}_n\|_{\mathcal{A}}^2 = O_p(1). \tag{A.7}$$

Combining (A.1), (A.7) and Cauchy–Schwartz inequality we conclude  $\|\hat{S}_n(\eta)\|_{\mathcal{A}}^2 = \|\hat{Z}_n(\eta)\|_{\mathcal{A}}^2 + o_p(1)$ .  $\square$

Before proceeding with the proof, the following notations are necessary. Let  $\mathcal{A}_\delta := D_\delta \times [0, 1] \times \ell \times \ell$ , where  $D_\delta = \{s \in \mathbb{R} : \delta \leq |s| \leq 1/\delta\}$  with  $\delta \in (0, 1)$ , then  $\mathcal{L}^2(\mathcal{A}_\delta)$  is a separable Hilbert space equipped with the same measure  $\nu$ . Denote by  $\langle f, g \rangle_{\mathcal{A}_\delta}$  and  $\|\cdot\|_{\mathcal{A}_\delta}$  the inner product and norm on  $\mathcal{L}^2(\mathcal{A}_\delta)$ . Let  $\{\alpha_p(s), s \in D_\delta\}_{p=1}^\infty$ ,  $\{\beta_q(\lambda), \lambda \in [0, 1]\}_{q=1}^\infty$  and  $\{u_l(\tau), \tau \in \ell\}_{l=1}^\infty$  be the sequences of orthonormal basis of the Hilbert spaces  $\mathcal{L}^2(D_\delta)$ ,  $\mathcal{L}^2([0, 1])$  and  $\mathcal{L}^2(\ell)$ , respectively. Then  $\{\alpha_p(s)\beta_q(\lambda)u_l(\tau_1)u_r(\tau_2)\}_{p,q,l,r=1}^{+\infty}$  forms a sequence of basis for  $L^2(\mathcal{A}_\delta)$ .

**Lemma 5.** *Under assumption 1 and  $H_0$ ,*

$$\hat{Z}_n^K(\eta) := \sum_{j=1}^K (n-j)^{1/2} \hat{r}_j(s, \tau_1, \tau_2) \psi_j(\lambda) \Rightarrow \mathcal{G}^K(\eta)$$

where " $\Rightarrow$ " denotes the weak convergence in  $\mathcal{L}^2(\mathcal{A}_\delta)$  and  $\mathcal{G}^K$  is a mean zero Gaussian process

whose projection variance is

$$\text{Var}(\langle \mathcal{G}^K, h \rangle_{\mathcal{A}_\delta}) = \sum_{i=1}^{\infty} \sum_{j=1}^K \sum_{j'=1}^K \mathbb{E} \{ \langle (Y_t - \mu) \phi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} \langle (Y_t - \mu) \phi_{t-j'} \psi_{j'}, e_i \rangle_{\mathcal{A}_\delta} \} \langle e_i, h \rangle_{\mathcal{A}_\delta}^2$$

wherer  $\{e_i\}_{i=1}^{\infty}$  is an orthonormal basis of  $\mathcal{L}^2(\mathcal{A}_\delta)$  and  $h \in \mathcal{L}^2(\mathcal{A}_\delta)$ .

*Proof.* By Theorem 17.27 of Henze (2024), we show (i) the projection of  $\hat{Z}_n^K(\eta)$  on arbitrary finite dimensional subspace of  $L^2(\mathcal{A}_\delta)$  are asymptotically normal and (i) the sequence  $\{\hat{Z}_n^K(\eta)\}$  is tight.

(i) Note that for any  $M_1, M_2, M_3, M_4 \geq 1$ , the finite projection can be expressed by

$$P_{\perp}(\hat{Z}_n^K) = \sum_{p=1}^{M_1} \sum_{q=1}^{M_2} \sum_{l=1}^{M_3} \sum_{r=1}^{M_4} \langle \hat{Z}_n^K, \alpha_p \beta_q u_l u_r \rangle_{\mathcal{A}_\delta} \alpha_p \beta_q u_l u_r,$$

where

$$\begin{aligned} & \langle \hat{Z}_n^K, \alpha_p \beta_q u_l u_r \rangle_{\mathcal{A}_\delta} \\ &= \sum_{t=2}^n \langle Y_t - \mu, u_l \rangle \sum_{j=1}^{(t-1) \wedge K} (n-j)^{-1/2} a_{p,q,r}^{t,j} \\ &=: \sum_{t=2}^K S_{p,q,l,r}^{t,K} + \sum_{t=K+1}^n S_{p,q,l,r}^{t,K} \end{aligned} \tag{A.8}$$

with

$$a_{p,q,r}^{t,j} = \int_{D_\delta} \int_0^1 \int_{\ell} \phi_{t-j}(s, \tau_2) \psi_j(\lambda) (\alpha_p(s) \beta_q(\lambda) u_r(\tau_2))^c d\tau_2 d\lambda \omega(ds).$$

Under null hypothesis  $\{S_{p,q,l,r}^{t,K}\}$  is a stationary ergodic difference MDS with respect to filtration

$\mathcal{F}_{t-1}$ . For  $t \leq K$ , the first item

$$\begin{aligned} \sum_{t=2}^K S_{p,q,l,r}^{t,K} &= \sum_{t=2}^K \langle Y_t - \mu, u_l \rangle \sum_{j=1}^{t-1} (n-j)^{1/2} a_{p,q,r}^{t,j} \\ &= n^{-1/2} \sum_{t=2}^K \langle Y_t - \mu, u_l \rangle \sum_{j=1}^{t-1} \left( \frac{n}{n-j} \right)^{-1/2} a_{p,q,r}^{t,j} \\ &= o_p(1), \end{aligned}$$

where the last equality holds by Markov inequality. For  $t \geq K+1$ , by the central limit theorem for stationary ergodic MDS, see [Billingsley \(1961\)](#), the second item

$$\begin{aligned} \sum_{t=K+1}^n S_{p,q,l,r}^{t,K} &= \sum_{t=K+1}^n \langle Y_t - \mu, u_l \rangle \sum_{j=1}^K (n-j)^{-1/2} a_{p,q,r}^{t,j} \\ &= (n-k)^{-1/2} \sum_{t=K+1}^n \langle Y_t - \mu, u_l \rangle \sum_{j=1}^K \left( \frac{n-k}{n-j} \right)^{1/2} a_{p,q,r}^{t,j} \end{aligned}$$

converges to a normal distribution. Thus  $P_{\perp}(\hat{Z}_n^K(\eta))$  converges weakly to a zero mean Gaussian process  $\mathcal{G}^K$  in finite dimensional subspace of  $L^2(\mathcal{A}_{\delta})$ .

(ii) We only need to show that as  $\min(M_1, M_2, M_3, M_4) \rightarrow \infty$ ,

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left\| \hat{Z}_n^K - P_{\perp}(\hat{Z}_n^K) \right\|_{\mathcal{A}_{\delta}}^2 \rightarrow 0,$$

which implies the sequence  $\{\hat{Z}_n^K(\eta)\}$  is tight. Note that

$$\begin{aligned} &\mathbb{E} \left\| \hat{Z}_n^K - P_{\perp}(\hat{Z}_n^K) \right\|_{\mathcal{A}_{\delta}}^2 \\ &= \sum_{p=M_1}^{\infty} \sum_{q=M_2}^{\infty} \sum_{l=M_3}^{\infty} \sum_{r=M_4}^{\infty} \mathbb{E} \left| \langle \hat{Z}_n^K, \alpha_p \beta_q u_l u_r \rangle_{\mathcal{A}_{\delta}} \right|^2 \\ &\rightarrow 0, \end{aligned}$$

as  $\min(M_1, M_2, M_3, M_4) \rightarrow \infty$ , where the equality and convergence are ensured by

$$\begin{aligned}
& \mathbb{E} \sum_{p=M_1}^{\infty} \sum_{q=M_2}^{\infty} \sum_{l=M_3}^{\infty} \sum_{r=M_4}^{\infty} \left| \langle \hat{Z}_n^K, \alpha_p \beta_q u_l(\tau_1) u_r \rangle_{\mathcal{A}_\delta} \right|^2 \\
& \leq \mathbb{E} \sum_{p,q,l,r \geq 1} \left| \langle \hat{Z}_n^K, \alpha_p \beta_q u_l u_r \rangle_{\mathcal{A}_\delta} \right|^2 \\
& = \mathbb{E} \|\hat{Z}_n^K\|_{\mathcal{A}_\delta}^2 < \infty,
\end{aligned}$$

Thus for a fixed  $K$ ,  $\hat{Z}_n^K(\eta) \Rightarrow \mathcal{G}^K(\eta)$  in  $\mathcal{L}^2(\mathcal{A}_\delta)$ . Similar to the Theorem A.1 of [Escanciano and Velasco \(2006\)](#), the asymptotic variance of (A.8) is

$$\sum_{j=1}^K \sum_{j'=1}^K \mathbb{E} \left[ \langle Y_t - \mu, u_l \rangle^2 a_{p,q,r}^{t,j} a_{p,q,r}^{t,j'} \right],$$

then the projected variance of  $\mathcal{G}^K$  is

$$\begin{aligned}
\hat{\sigma}_K^2 & := \text{Var}(\langle \mathcal{G}^K, h \rangle_{\mathcal{A}_\delta}) \\
& = \sum_{p,q,l,r \geq 1} \sum_{j=1}^K \sum_{j'=1}^K \mathbb{E} \left[ \langle Y_t - \mu, u_l \rangle^2 a_{p,q,r}^{t,j} a_{p,q,r}^{t,j'} \right] \langle \alpha_p \beta_q u_l u_r, h \rangle^2 \\
& = \sum_{i=1}^{\infty} \sum_{j=1}^K \sum_{j'=1}^K \mathbb{E} \{ \langle (Y_t - \mu) \phi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} \langle (Y_t - \mu) \phi_{t-j'} \psi_{j'}, e_i \rangle_{\mathcal{A}_\delta} \} \langle e_i, h \rangle_{\mathcal{A}_\delta}^2
\end{aligned}$$

for any  $h \in \mathcal{L}^2(\mathcal{A}_\delta)$ .

□

**Lemma 6.** *Under Assumption 1 and  $H_0$ ,*

$$\hat{Z}_n(\eta) \Rightarrow \mathcal{G}(\eta),$$

where " $\Rightarrow$ " denotes the weak convergence in  $\mathcal{L}^2(\mathcal{A}_\delta)$  and  $\mathcal{G}$  is a mean zero Gaussian process

whose projection variance is

$$\text{Var}(\langle \mathcal{G}, h \rangle_{\mathcal{A}_\delta}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{j'=1}^{\infty} \mathbb{E} \{ \langle (Y_t - \mu) \phi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} \langle (Y_t - \mu) \phi_{t-j'} \psi_{j'}, e_i \rangle_{\mathcal{A}_\delta} \} \langle e_i, h \rangle_{\mathcal{A}_\delta}^2$$

wherer  $\{e_i\}_{i=1}^{\infty}$  is an orthonormal basis of  $\mathcal{L}^2(\mathcal{A}_\delta)$  and  $h \in \mathcal{L}^2(\mathcal{A}_\delta)$ .

*Proof.* Denote

$$\begin{aligned} \hat{Z}_n(\eta) &= \sum_{j=1}^K (n-j)^{1/2} \hat{r}_j(s, \tau_1, \tau_2) \psi_j(\lambda) + \sum_{j=K+1}^{n-1} (n-j)^{1/2} \hat{r}_j(s, \tau_1, \tau_2) \psi_j(\lambda) \\ &:= \hat{Z}_n^K(\eta) + \hat{R}_n^K(\eta) \end{aligned}$$

for some integer  $K$ . Based on the Theorem 4.2 of Billingsley (1961), it suffices to show (i) for any  $K$ ,  $\hat{Z}_n^K(\eta) \Rightarrow \mathcal{G}^K(\eta)$  as  $n \rightarrow \infty$ ; (ii)  $\mathcal{G}^K(\eta) \Rightarrow \mathcal{G}(\eta)$  as  $K \rightarrow \infty$ ; (iii) for any  $\epsilon > 0$ ,  $\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \|\hat{R}_n^K(\eta)\|_{\mathcal{A}_\delta} > \epsilon \right) = 0$ .

(i) it follows from Lemma 5.

(ii) it suffices to show

$$\begin{aligned} \mathbb{E} \|\mathcal{G}^K\|_{\mathcal{A}_\delta}^2 &= \sum_{i=1}^{\infty} \mathbb{E} \left| \langle \mathcal{G}^K, e_i \rangle_{\mathcal{A}_\delta} \right|^2 \langle e_i(\eta), h \rangle^2 \\ &\rightarrow \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{j'=1}^{\infty} \mathbb{E} \{ \langle (Y_t - \mu) \phi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} \langle (Y_t - \mu) \phi_{t-j'} \psi_{j'}, e_i \rangle_{\mathcal{A}_\delta} \} \langle e_i, h \rangle_{\mathcal{A}_\delta}^2 \end{aligned}$$

converges as  $K \rightarrow \infty$ . Denote by

$$\sigma^2 := \sum_{j=1}^{\infty} \sum_{j'=1}^{\infty} \mathbb{E} \{ \langle (Y_t - \mu) \phi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} \langle (Y_t - \mu) \phi_{t-j'} \psi_{j'}, e_i \rangle_{\mathcal{A}_\delta} \}$$

The result follows from

$$\mathbb{E} \|\mathcal{G}^K\|_{\mathcal{A}_\delta}^2 \rightarrow \sum_{i=1}^{\infty} \sigma^2 \langle e_i, h \rangle_{\mathcal{A}_\delta}^2 = \sigma^2 \|h\|_{\mathcal{A}_\delta}^2 < \infty, \quad \text{as } K \rightarrow \infty,$$

where the equality holds by Parseval's identity.

(iii) Note that

$$(n-j)\mathbb{E} \int_{D_\delta} \|\hat{r}_{j,s}\|^2 \omega(ds) \leq (n-j)\mathbb{E} \int_{\mathbb{R}} \|\hat{r}_{j,s}\|^2 \omega(ds) \leq C,$$

thus

$$\begin{aligned} \mathbb{E} \|\hat{R}_n^K\|_{\mathcal{A}_\delta}^2 &= \mathbb{E} \int_{D_\delta} \int_{\ell^2} \sum_{j=K+1}^{n-1} \frac{n-j}{(j\pi)^2} |\hat{r}_j(s, \tau_1, \tau_2)|^2 d\tau_1 d\tau_2 \omega(ds) \\ &= \sum_{j=K+1}^{n-1} \frac{n-j}{(j\pi)^2} \mathbb{E} \int_{D_\delta} \|\hat{r}_{j,s}\|^2 \omega(ds) \\ &\leq C \sum_{j=K+1}^{n-1} \frac{1}{(j\pi)^2} \rightarrow 0, \end{aligned}$$

as  $n, K \rightarrow \infty$ . □

**Lemma 7.** *Under the assumption 1,  $\hat{T}_n^R = \hat{T}_n + o_p(1)$ .*

*Proof.* Note that

$$\begin{aligned} \hat{T}_n^R - \hat{T}_n &= \sum_{j=1}^{n-4} \frac{n-j}{(j\pi)^2} \widetilde{\text{FAMDD}}_n(Y_t | Y_{t-j}) - \sum_{j=1}^{n-1} \frac{n-j}{(j\pi)^2} \text{FAMDD}_n(Y_t | Y_{t-j}) \\ &= \sum_{j=1}^{n-4} \frac{n-j}{(j\pi)^2} \left\{ \widetilde{\text{FAMDD}}_n(Y_t | Y_{t-j}) - \text{FAMDD}_n(Y_t | Y_{t-j}) \right\} - \sum_{j=n-3}^{n-1} \frac{n-j}{(j\pi)^2} \text{FAMDD}_n(Y_t | Y_{t-j}) \\ &=: L_1 + L_2. \end{aligned}$$

First we consider the item  $L_1$ . By simple algebra we have

$$\begin{aligned} &\widetilde{\text{FAMDD}}_n(Y_t | Y_{t-j}) - \text{FAMDD}_n(Y_t | Y_{t-j}) \\ &= \frac{1}{(n-j)(n-j-3)} \sum_{k=j+1}^n \sum_{l=j+1}^n \tilde{A}_{kl}^{(j)} \tilde{B}_{kl} - \frac{1}{(n-j)^2} \sum_{k=j+1}^n \sum_{l=j+1}^n A_{kl}^{(j)} B_{kl} \\ &= \frac{1}{(n-j-3)(n-j)^2} \left\{ (n-j) \sum_{k=j+1}^n \sum_{l=j+1}^n \left( \tilde{A}_{kl}^{(j)} \tilde{B}_{kl} - A_{kl}^{(j)} B_{kl} \right) + 3 \sum_{k=j+1}^n \sum_{l=j+1}^n A_{kl}^{(j)} B_{kl} \right\}. \quad (\text{A.9}) \end{aligned}$$

It can be shown that

$$\begin{aligned}\tilde{a}_{k\cdot}^{(j)} &= a_{k\cdot}^{(j)} + \frac{2}{n-j-2}a_{k\cdot}^{(j)}, & \tilde{a}_{\cdot l}^{(j)} &= a_{\cdot l}^{(j)} + \frac{2}{n-j-2}a_{\cdot l}^{(j)}, \\ \tilde{a}_{\cdot\cdot}^{(j)} &= a_{\cdot\cdot}^{(j)} + \frac{3(n-j)-2}{(n-j-1)(n-j-2)}a_{\cdot\cdot}^{(j)},\end{aligned}$$

and

$$\begin{aligned}\tilde{b}_{k\cdot}^{(j)} &= b_{k\cdot}^{(j)} + \frac{2}{n-j-2}b_{k\cdot}^{(j)}, & \tilde{b}_{\cdot l}^{(j)} &= b_{\cdot l}^{(j)} + \frac{2}{n-j-2}b_{\cdot l}^{(j)}, \\ \tilde{b}_{\cdot\cdot}^{(j)} &= b_{\cdot\cdot}^{(j)} + \frac{3(n-j)-2}{(n-j-1)(n-j-2)}b_{\cdot\cdot}^{(j)},\end{aligned}$$

which implies that

$$\begin{aligned}& \sum_{k=j+1}^n \sum_{l=j+1}^n \tilde{A}_{kl}^{(j)} \tilde{B}_{kl} \\ &= \sum_{k=j+1}^n \sum_{l=j+1}^n \left\{ A_{kl}^{(j)} - \frac{2}{n-j-2}a_{k\cdot}^{(j)} - \frac{2}{n-j-2}a_{\cdot l}^{(j)} + \frac{3(n-j)-2}{(n-j-1)(n-j-2)}a_{\cdot\cdot}^{(j)} \right\} \\ & \quad \times \left\{ B_{kl} - \frac{2}{n-j-2}b_{k\cdot} - \frac{2}{n-j-2}b_{\cdot l} + \frac{3(n-j)-2}{(n-j-1)(n-j-2)}b_{\cdot\cdot} \right\} \\ &= \sum_{k=j+1}^n \sum_{l=j+1}^n A_{kl}^{(j)} B_{kl} - \frac{6(n-j)-4}{(n-j-1)(n-j-2)^2} \sum_{k=j+1}^n \sum_{l=j+1}^n \left\{ a_{k\cdot}^{(j)}b_{\cdot\cdot} + a_{\cdot l}^{(j)}b_{\cdot\cdot} + a_{\cdot\cdot}^{(j)}b_{k\cdot} + a_{\cdot\cdot}^{(j)}b_{\cdot l} \right\} \\ & \quad + \frac{4}{(n-j-2)^2} \sum_{k=j+1}^n \sum_{l=j+1}^n \left\{ a_{k\cdot}^{(j)}b_{k\cdot} + a_{k\cdot}^{(j)}b_{\cdot l} + a_{\cdot l}^{(j)}b_{k\cdot} + a_{\cdot l}^{(j)}b_{\cdot l} \right\} \\ & \quad + \frac{(3(n-j)-2)^2}{(n-j-1)^2(n-j-2)^2} \sum_{k=j+1}^n \sum_{l=j+1}^n a_{\cdot\cdot}^{(j)}b_{\cdot\cdot} \\ &=: \sum_{k=j+1}^n \sum_{l=j+1}^n A_{kl}^{(j)} B_{kl} - L_{1,1} + L_{1,2} + L_{1,3},\end{aligned}$$

where the second equality holds by the fact  $\sum_{l=j+1}^n A_{kl}^{(j)} = \sum_{k=j+1}^n A_{kl}^{(j)} = \sum_{l=j+1}^n B_{kl} = \sum_{k=j+1}^n B_{kl} = 0$ . It follows from a direct calculation that  $\sum_{k,l=j+1}^n a_{k\cdot}^{(j)}b_{\cdot l} = \sum_{k,l=j+1}^n a_{k\cdot}^{(j)}b_{\cdot\cdot} = (n-j)^2 a_{\cdot\cdot}^{(j)}b_{\cdot\cdot}$ . Under the assumption 1, we have  $(n-j)^2 a_{\cdot\cdot}^{(j)}b_{\cdot\cdot} = O_p(1)$  and  $\sum_{k,l=j+1}^n a_{k\cdot}^{(j)}b_{k\cdot} = O_p((n-j)^2)$ , which

implies  $L_{1,1}, L_{1,2}, L_{1,3} = O_p(1)$ . Then combining equation (A.9) to obtain that

$$\widetilde{\text{FAMDD}}_n(Y_t | Y_{t-j}) - \text{FAMDD}_n(Y_t | Y_{t-j}) = O_p((n-j)^{-2}),$$

which implies that  $L_1 = o_p(1)$ . On the other hand,

$$\sum_{j=n-3}^{n-1} \frac{n-j}{(j\pi)^2} \text{FAMDD}_n(Y_t | Y_{t-j}) = \sum_{j=n-3}^{n-1} \frac{n-j}{(j\pi)^2} \int_{\mathbb{R}} \|\hat{\gamma}_{j,s}\|^2 \omega(ds) \leq O_p(1) \sum_{j=n-3}^{n-1} \frac{1}{(j\pi)^2} = o_p(1),$$

which implies that  $L_2 = o_p(1)$ . Thus we complete the proof.  $\square$

## B Proofs of main results

**Proof of Proposition 1.** (1) It follows from Fubini's theorem and the arguments in Section 3 of Wang (2024) that  $\text{FAMDD}(Y_t | Y_{t-j}) = -\mathbb{E} [\langle Y_t - \mu, Y_t^+ - \mu \rangle \|Y_{t-j} - Y_{t-j}^+\|_1]$ . (2) By the fact that  $\|Y_t - Y_t^+\|^2 = \|Y_t\|^2 + \|Y_t^+\|^2 - 2\langle Y_t, Y_t^+ \rangle$  and  $Y_t^+$  is an independent copy of  $Y_t$ , we have

$$\begin{aligned} \mathbb{E}_{Y_t} \left( \|Y_t - Y_t^+\|^2 \right) &= \|Y_t\|^2 + \mathbb{E} \|Y_t^+\|^2 - 2\langle Y_t, \mathbb{E} Y_t^+ \rangle, \\ \mathbb{E}_{Y_t^+} \left( \|Y_t - Y_t^+\|^2 \right) &= \mathbb{E} \|Y_t\|^2 + \|Y_t^+\|^2 - 2\langle \mathbb{E} Y_t, Y_t^+ \rangle, \\ \mathbb{E} \left( \|Y_t - Y_t^+\|^2 \right) &= \mathbb{E} \|Y_t\|^2 + \mathbb{E} \|Y_t^+\|^2 - 2\langle \mathbb{E} Y_t, \mathbb{E} Y_t^+ \rangle. \end{aligned}$$

Then

$$\begin{aligned} B &= \frac{1}{2} \|Y_t - Y_t^+\|^2 - \mathbb{E}_{Y_t} \left( \frac{1}{2} \|Y_t - Y_t^+\|^2 \right) - \mathbb{E}_{Y_t^+} \frac{1}{2} \|Y_t - Y_t^+\|^2 + \mathbb{E} \left( \frac{1}{2} \|Y_t - Y_t^+\|^2 \right) \\ &= \frac{1}{2} (-2\langle Y_t, Y_t^+ \rangle + 2\langle Y_t, \mathbb{E} Y_t^+ \rangle + 2\langle \mathbb{E} Y_t, Y_t^+ \rangle - 2\langle \mathbb{E} Y_t, \mathbb{E} Y_t^+ \rangle) \\ &= -\langle Y_t - \mathbb{E} Y_t, Y_t^+ - \mathbb{E} Y_t^+ \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned}
& \mathbb{E}(-\langle Y_t - \mathbb{E}Y_t, Y_t^+ - \mathbb{E}Y_t^+ \rangle \mathbb{E}_{Y_{t-j}} \|Y_{t-j} - Y_{t-j}^+\|_1) \\
&= \mathbb{E} \mathbb{E} \left\{ -\langle Y_t - \mathbb{E}Y_t, Y_t^+ - \mathbb{E}Y_t^+ \rangle \mathbb{E}_{Y_{t-j}} \|Y_{t-j} - Y_{t-j}^+\|_1 \mid Y_{t-j} \right\} \\
&= \mathbb{E} \left\{ \mathbb{E}_{Y_{t-j}} \|Y_{t-j} - Y_{t-j}^+\|_1 \cdot \mathbb{E}(-\langle Y_t - \mathbb{E}Y_t, Y_t^+ - \mathbb{E}Y_t^+ \rangle \mid Y_{t-j}) \right\} \\
&= 0.
\end{aligned}$$

where the last equality holds by  $\mathbb{E}\langle X, Y \rangle = \langle \mathbb{E}X, \mathbb{E}Y \rangle$  for independent random elements  $X, Y$  in Hilbert space and the fact that  $(Y_t, Y_{t-j})$  and  $(Y_t^+, Y_{t-j}^+)$  are independent. In a similar way we can obtain  $\mathbb{E}(-\langle Y_t - \mathbb{E}Y_t, Y_t^+ - \mathbb{E}Y_t^+ \rangle \mathbb{E}_{Y_{t-j}^+} \|Y_{t-j} - Y_{t-j}^+\|_1) = 0$ . Finally we conclude that

$$\mathbb{E}(A^{(j)}B) = -\mathbb{E}(\langle Y_t - \mathbb{E}Y_t, Y_t^+ - \mathbb{E}Y_t^+ \rangle \|Y_{t-j} - Y_{t-j}^+\|_1),$$

Next we prove the FAMDD( $Y_t \mid Y_{t-j}$ ) is finite. Following [Lyons \(2013\)](#), for  $x, x^+ \in L^2$ , define the distance  $a(x, x^+) = \|x - x^+\|_1$  and let

$$\begin{aligned}
d_a(x, x^+) &= a(x, x^+) - \int a(x, x^+) d\mu(x) - \int a(x, x^+) d\mu(x^+) \\
&\quad \int \int a(x, x^+) d\mu^2(x, x^+),
\end{aligned}$$

where  $\mu$  is the distribution of  $x, x^+$ . Under the assumption that  $\mathbb{E}\|Y_t\| < \infty$ , the  $\mu$  has finite first moment for distance  $a$  and  $b$ , thus  $\int a(x, x^+) d\mu(x, x^+)$  is well defined and finite. It follows from Lemma 2.1 in [Lyons \(2013\)](#) that  $\int d_a(x, x^+) d\mu^2(x, x^+) < \infty$ , i.e.,  $\mathbb{E}(A^{(j)} \cdot A^{(j)}) < \infty$ . On the other hand, under the assumption  $\mathbb{E}\|Y_t^+\|^4 < \infty$ ,

$$\begin{aligned}
\mathbb{E}(B \cdot B) &\leq \mathbb{E}\langle Y_t - \mathbb{E}Y_t, Y_t^+ - \mathbb{E}Y_t^+ \rangle^2 \\
&\leq \mathbb{E}(\|Y_t - \mathbb{E}Y_t\|^2 \cdot \|Y_t^+ - \mathbb{E}Y_t^+\|^2) \\
&\leq \frac{1}{2} \{ \mathbb{E}\|Y_t - \mathbb{E}Y_t\|^4 + \mathbb{E}\|Y_t^+ - \mathbb{E}Y_t^+\|^4 \} < \infty.
\end{aligned}$$

where the last inequity holds by the the fact that  $\mathbb{E}\|Y_t - \mathbb{E}Y_t\|^4 \leq 2^4[\mathbb{E}(\|Y_t\|^4) + (\mathbb{E}\|Y_t\|)^4] < \infty$ .  $\square$

**Proof of Proposition 2.** (1) The proof is similar to the proof of the Lemma 3.1 in Wang (2024) and we omitted. (2) It follows directly from the strong law of large number of  $V$ -statistics of ergodic and stationary sequence (Aaronson et al. (1996)) and the discussion on Proposition 2 in Lee et al. (2020).  $\square$

**Proof of Theorem 1.** We only need to show  $\|\hat{Z}_n\|_{\mathcal{A}}^2 \xrightarrow{d} \|\mathcal{G}\|_{\mathcal{A}}^2$ , then the conclusion follows from Lemma 4. By Theorem 4.2 of Billingsley (1999), it suffices to show (i) for any  $\delta$ ,  $\|\hat{Z}_n\|_{\mathcal{A}_\delta}^2 \xrightarrow{d} \|\mathcal{G}\|_{\mathcal{A}_\delta}^2$ ; (ii)  $\|\mathcal{G}\|_{\mathcal{A}_\delta}^2 \xrightarrow{d} \|\mathcal{G}\|_{\mathcal{A}}^2$  as  $\delta \rightarrow 0$ ; (iii) for any  $\epsilon > 0$ ,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \left| \|\hat{Z}_n\|_{\mathcal{A}}^2 - \|\hat{Z}_n\|_{\mathcal{A}_\delta}^2 \right| > \epsilon \right) = 0.$$

Since condition (i) follows from Lemma 6 and continuous mapping theorem, and condition (ii) is trivial, we only need to prove that the condition (iii) holds. Denote  $D_{1,\delta} = \{s \in \mathbb{R} : |s| < \delta\}$  and  $D_{2,\delta} = \{s \in \mathbb{R} : |s| > 1/\delta\}$ , then

$$\begin{aligned} & \mathbb{E} \left| \|\hat{Z}_n\|_{\mathcal{A}}^2 - \|\hat{Z}_n\|_{\mathcal{A}_\delta}^2 \right| \\ &= \sum_{j=1}^{n-1} \frac{n-j}{(j\pi)^2} \mathbb{E} \int_{D_{1,\delta}} \|\hat{r}_{j,s}\|^2 \omega(ds) + \sum_{j=1}^{n-1} \frac{n-j}{(j\pi)^2} \mathbb{E} \int_{D_{2,\delta}} \|\hat{r}_{j,s}\|^2 \omega(ds) \end{aligned}$$

For the first item, by equation (A.6) we have

$$\sum_{j=1}^{n-1} \frac{n-j}{(j\pi)^2} \mathbb{E} \int_{D_{1,\delta}} \|\hat{r}_{j,s}\|^2 \omega(ds) = \sum_{j=1}^{n-1} \frac{1}{(j\pi)^2} \int_{D_{1,\delta}} \mathbb{E} \|(Y_t - \mu)\phi_{t-j}\|^2 \omega(ds) \rightarrow 0,$$

as first  $n \rightarrow \infty$ , then  $\delta \rightarrow 0$ , where we have used the fact  $\int_{\mathbb{R}} \mathbb{E} \|(Y_t - \mu)\phi_{t-j}\|^2 \omega(ds) < \infty$ . In a

similar way we conclude

$$\sum_{j=1}^{n-1} \frac{n-j}{(j\pi)^2} \mathbb{E} \int_{D_{2,\delta}} \|\hat{r}_{j,s}\|^2 \omega(ds) \rightarrow 0,$$

which yields to condition (iii), thus we can obtain  $\|\hat{Z}_n\|_{\mathcal{A}}^2 \xrightarrow{d} \|\mathcal{G}\|_{\mathcal{A}}^2$ .  $\square$

**Proof of Theorem 3.** Denote

$$\hat{S}_{a,n}(\eta) = \sum_{j=1}^{n-1} (n-j)^{1/2} \hat{\gamma}_{a,j}(s, \tau_1, \tau_2) \psi_j(\lambda),$$

where

$$\hat{\gamma}_{a,j}(s, \tau_1, \tau_2) = \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \bar{Y}_{n-j}(\tau_1) - n^{-1/2} g_t(\tau_1)] \varphi_{t-j}(s, \tau_2).$$

Then  $\hat{S}_n(\eta)$  can be decomposed as

$$\hat{S}_n(\eta) = \hat{S}_{a,n}(\eta) + \hat{\mathcal{V}}_n(\eta), \tag{B.1}$$

where

$$\hat{\mathcal{V}}_n(\eta) = \sum_{j=1}^{n-1} \left( \frac{n-j}{n} \right)^{1/2} \frac{1}{n-j} \sum_{t=j+1}^n g_t(\tau_1) \varphi_{t-j}(s, \tau_2) \psi_j(\lambda).$$

Note that under alternative  $H_{a,n}$ ,  $Y_t - \mu - n^{-1/2} g_t$  is an MDS with respect to filtration  $\mathcal{F}_{t-1}$ , then similar to Lemma 4 and 6,

$$\hat{S}_{a,n}(\eta) \Rightarrow \mathcal{G}(\eta) \tag{B.2}$$

in the Hilbert space  $\mathcal{L}^2(\mathcal{A}_\delta)$ . On the other hand, it follows from mean ergodic theorem that

$$\begin{aligned}\langle \hat{\mathcal{V}}_n, e_i \rangle_{\mathcal{A}_\delta} &= \sum_{j=1}^{n-1} \left( \frac{n-j}{n} \right)^{1/2} \frac{1}{n-j} \sum_{t=j+1}^n \langle g_t \varphi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} \\ &= \sum_{j=1}^{\infty} \mathbb{E} \langle g_t \varphi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} + o_p(1),\end{aligned}\tag{B.3}$$

with  $\{e_i\}_{i=1}^{\infty}$  an orthonormal basis of  $\mathcal{L}^2(\mathcal{A}_\delta)$ . Denote  $\mathcal{V}(\eta) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{E} \langle g_t \varphi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} e_i$ . For any fixed  $K$ , equation B.3 implies

$$\hat{\mathcal{V}}_n^K(\eta) := \sum_{i=1}^K \langle \hat{\mathcal{V}}_n, e_i \rangle_{\mathcal{A}_\delta} e_i \Rightarrow \sum_{i=1}^K \sum_{j=1}^{\infty} \mathbb{E} \langle g_t \varphi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} e_i.$$

On the other hand,

$$\mathbb{E} \|\hat{\mathcal{V}}_n - \hat{\mathcal{V}}_n^K\|^2 = \mathbb{E} \sum_{i=K+1}^{\infty} |\langle \hat{\mathcal{V}}_n, e_i \rangle_{\mathcal{A}_\delta}|^2 \rightarrow 0,$$

as  $n \rightarrow \infty$ , then  $K \rightarrow \infty$ , where the convergence is ensured by  $\mathbb{E} \sum_{i=1}^{\infty} |\langle \hat{\mathcal{V}}_n, e_i \rangle_{\mathcal{A}_\delta}|^2 = \mathbb{E} \|\hat{\mathcal{V}}_n\|_{\mathcal{A}_\delta}^2 < \infty$ . Then by Theorem 17.27 of [Henze \(2024\)](#) we have

$$\hat{\mathcal{V}}_n(\eta) = \mathcal{V}(\eta) + o_p(1),\tag{B.4}$$

where

$$\mathcal{V}(\eta) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{E} \langle g_t \varphi_{t-j} \psi_j, e_i \rangle_{\mathcal{A}_\delta} e_i.$$

Thus combing (B.1), (B.2) and (B.4) we conclude  $\hat{S}_n(\eta) \Rightarrow \mathcal{G}(\eta) + \mathcal{V}(\eta)$  in the Hilbert space  $\mathcal{L}^2(\mathcal{A}_\delta)$ . It follows from the proof of Theorem 1 that  $\|\hat{S}_n\|_{\mathcal{A}} \Rightarrow \|\mathcal{G} + \mathcal{V}\|_{\mathcal{A}}$ .  $\square$

**Proof of Theorem 4.** Denote

$$\hat{Z}_n^*(\eta) = \sum_{j=1}^{n-1} (n-j)^{1/2} \hat{r}_j^*(s, \tau_1, \tau_2) \psi_j(\lambda),$$

with

$$\hat{r}_j^*(s, \tau_1, \tau_2) = \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu_t(\tau_1)] \phi_{t-j}(s, \tau_2) w_t.$$

Note that

$$\begin{aligned} \hat{\gamma}_j^*(s, \tau_1, \tau_2) &= \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1) + \mu(\tau_1) - \bar{Y}_{n-j}(\tau_1)] \phi_{t-j}(s, \tau_2) w_t \\ &= \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu_t(\tau_1)] w_t \phi_{t-j}(s, \tau_2) \\ &\quad - \left\{ \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1)] \right\} \left\{ \frac{1}{n-j} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) w_t \right\} \end{aligned}$$

Then  $\hat{S}_n^*(\lambda, \tau)$  can be decomposed as  $\hat{S}_n^*(\eta) = \hat{Z}_n^*(\eta) - \hat{R}_n^*(\eta)$ , where

$$\begin{aligned} \hat{R}_n^*(\eta) &= \sum_{j=1}^{n-1} (n-j)^{1/2} \left\{ \frac{1}{n-j} \sum_{t=j+1}^n [Y_t(\tau_1) - \mu(\tau_1)] \right\} \\ &\quad \times \left\{ \frac{1}{n-j} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) w_t \right\} \psi_j(\lambda). \end{aligned}$$

Since

$$\left\| \hat{S}_n^* \right\|_{\mathcal{A}}^2 = \left\| \hat{Z}_n^* \right\|_{\mathcal{A}}^2 + \left\| \hat{R}_n^* \right\|_{\mathcal{A}}^2 - 2 \operatorname{Re} \left( \left\langle \hat{Z}_n^*, \hat{R}_n^* \right\rangle_{\mathcal{A}} \right).$$

it suffices to show  $\left\| \hat{R}_n^* \right\|_{\mathcal{A}}^2 = o_p(1)$ , *a.s.* and  $\left| \left\langle \hat{Z}_n^*, \hat{R}_n^* \right\rangle_{\mathcal{A}} \right| = o_p(1)$ , *a.s.* Denote by  $\mathbb{E}^*$  the conditional expectation given  $\{Y_t\}_{t=1}^n$ . Given that the sequence  $\{w_t\}$  is independent with  $\{Y_t\}_{t=1}^n$ , then we have

$$\mathbb{E}^* \int_{\mathbb{R}} \int_{\mathcal{L}} \left| \frac{1}{(n-j)} \sum_{t=j+1}^n \phi_{t-j}(s, \tau_2) w_t \right|^2 d\tau_2 \omega(ds) = o(1), \quad a.s.$$

which implies that  $\left\|\hat{R}_n^*\right\|_{\mathcal{A}}^2 = o_p(1)$ , *a.s.* On the other hand, by equation (A.7) we have  $\left\|\hat{Z}_n^*\right\|_{\mathcal{A}}^2 = O_p(1)$ , *a.s.* thus  $\left\|\hat{S}_n^*\right\|_{\mathcal{A}}^2 = \left\|\hat{Z}_n^*\right\|_{\mathcal{A}}^2 + o_p(1)$ , *a.s.* Following similar arguments in the proof of Lemma 4, 6 and Theorem 3.1, we can prove  $\left\|\hat{S}_n^*\right\|_{\mathcal{A}}^2 \xrightarrow{d} \|\mathcal{G}\|_{\mathcal{A}}^2$ , *a.s.* which completes the proof.  $\square$

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