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Nonlinear Boundary Degeneracy and the Limits of Scalar Diagnostics

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Abstract Scalar order parameters are widely used to describe critical transitions in nonlinear systems, yet their structural validity is rarely stated explicitly. This paper identifies a sharp obstruction to scalar diagnostics that arises when regime boundaries are generated by *interacting degeneracies*. We show that if the reduced boundary ideal is non-principal, then the boundary-induced comparability structure necessarily contains incomparabilities, and *no* scalar index can be faithful (even up to monotone transformation) to the underlying boundary constraints.

Our analysis yields a boundary-first classification. In the codimension-one principal regime, the boundary is a hypersurface with a canonical generator Δ_{red} (unique up to units and admissible reparametrizations), and scalar crisis diagnostics are structurally pinned down to a single channel. Beyond this regime, scalarization fails intrinsically: the failure is ideal-theoretic rather than a calibration artifact.

We illustrate the mechanism by a minimal two-fold construction producing a non-principal boundary, and introduce the positive-degree Betti Profile Index BPI^{*} as a minimal cohomological gatekeeper that becomes informative exactly when the scalar paradigm is no longer defensible.

Keywords boundary ideals · non-principal degeneracy · discriminant channels · crisis orders · scalar diagnostics · cohomological gatekeepers · Betti Profile Index

Keywords nonlinear boundary geometry; singularity-induced orders; discriminant channel; hypersurface vs. multi-constraint boundaries; principal ideal criterion; topological gatekeepers; Betti profiles; failure of scalar reduction

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1 Introduction

From the perspective of nonlinear science, the present work addresses a structural limitation of scalar order parameters in systems with intersecting degeneracies. We show that when crisis boundaries are generated by non-principal ideals, no single scalar diagnostic can faithfully represent the underlying bifurcation geometry. This phenomenon is generic in multi-parameter nonlinear systems with interacting fold-type singularities, and reflects a mismatch between scalar observables and stratified boundary structures.

Catastrophe theory has traditionally explained regime transitions through the classification of critical points of scalar potential functions. This paper develops an algebro-geometric theory of catastrophes in which catastrophic behavior is governed instead by the ideal-theoretic structure of regime boundaries. We show that the principality or non-principality of the reduced boundary ideal constitutes a sharp structural transition separating regimes where faithful scalar diagnostics are possible from those where scalarization is intrinsically obstructed.

Scalar crisis indicators are ubiquitous in economics and the social sciences. They are attractive because they compress complicated regime structures into a single number that can be ranked, compared across scenarios, communicated to non-specialists, and, in principle, optimized. Yet their mathematical status is often left implicit. When is a one-dimensional diagnostic index structurally justified, and when does scalarization conceal genuinely multi-constraint crisis geometry?

The present paper addresses this question from a boundary-based perspective. We treat crisis diagnosis as a representation problem: given a ranking

of scenarios by proximity to a regime-changing boundary, when does this *crisis order* admit a faithful scalar representation? Our analysis shows that the answer depends not on calibration choices or functional forms, but on the algebraic and geometric structure of the regime boundary itself.

Relation to existing work and contribution. The present paper is situated at the intersection of several established research traditions, while addressing a structural question that has remained largely implicit across them.

First, it builds on the axiomatic and representation-theoretic foundations of economics and the social sciences, where weak orders are represented by numerical functions under suitable regularity conditions [Debreu, G. 1954], [Arrow, K.J. 1963],[Krantz, D.H., Luce, R.D., Suppes, P.and Tversky, A. 1971]. In contrast to this classical literature, which focuses on preferences, utilities, or welfare rankings, our object of representation is a *boundary-induced crisis order*: a ranking of scenarios by proximity to a regime-changing boundary.

Second, the paper connects to catastrophe theory and bifurcation analysis, where regime transitions are characterized by degeneracies, multiple roots, and discriminant conditions [Arnold, V. I. 1992],[Thom, R. 1975], [Golubitsky, M. and Schaeffer, D.G. 1985]. While this literature provides canonical normal forms and invariants for boundary phenomena, it does not address a key diagnostic question: when does the underlying boundary geometry justify a *single scalar* diagnostic channel, and when must scalarization fail because multiple constraints bind simultaneously?

Third, our results speak to the extensive applied literature that employs scalar early-warning, fragility, or distance-to-crisis indicators in macroeconomics and finance [Kaminsky, G., Lizondo, S. and Reinhart, C.M. 1998], [Berg, A. and Pattillo, C. 1999]. Rather than proposing another index or calibration strategy, we provide a theoretical principle that applies uniformly to this class of tools: it clarifies when *any* scalar diagnostic can be structurally faithful to the crisis order induced by the regime boundary, and when scalar diagnostics necessarily conceal genuinely multi-constraint crisis geometry.

Fourth, the analysis draws on algebraic and topological methods for describing boundary structure, in particular reduced boundary ideals, principality, and the geometry and topology of their associated varieties [Eisenbud, D. 1995], [Hartshorne, R. 1977], [Iitaka, S. 1982], [Hatcher, A. 2002]. This boundary-first perspective yields a sharp structural classification. We show that scalar crisis diagnostics are essentially unique (up to admissible transformations and monotone normalization) precisely in codimension-one *principal* boundary regimes, while they fail generically once the reduced boundary ideal is non-principal.

The central contribution of the paper is to unify these strands into a single structure theorem linking boundary geometry, diagnostic representability, and the limits of scalarization. Beyond the scalar regime, we introduce the Betti Profile Index BPI^* as a minimal, coordinate-free cohomological gateway indicator that becomes informative exactly when faithful scalar diagnosis is no longer possible. The diagnostic–normative separation emphasized here complements our earlier analysis of welfare losses under regime switching

[Matsuhisa, T. 2025], but the present paper is concerned with the structural limits of diagnosis itself rather than normative evaluation.

We develop the theory in four layers, each isolating a distinct structural question.

Layer (A) formalizes crisis orders and establishes conditions under which such orders admit continuous scalar representations. Layer (B) studies invariance and essential uniqueness in single-channel settings governed by a discriminant-type boundary invariant. Layer (C) introduces a boundary-first algebraic classification via the reduced boundary ideal and identifies codimension-one principality as the precise condition under which scalar diagnostics are structurally pinned down. Layer (D) characterizes the failure of scalar faithfulness in non-principal boundary regimes and introduces a minimal cohomological obstruction indicator.

Each layer is designed to be logically self-contained: it introduces a diagnostic object, states the relevant structural claim, and records the minimal conditions under which the claim is valid. The intended reading mode is therefore modular, while the full paper clarifies how the layers compose into a single boundary-based theory of crisis diagnostics.

Organization. Section 2 introduces scenario spaces, regime boundaries, and crisis orders. Section 3 develops Layer (A). Section 4 develops Layer (B). Section 5 separates diagnosis from welfare and clarifies the diagnostic–normative interface. Section 6 develops Layer (C). Section 7 records operational implications. Section 8 develops Layer (D) and introduces the Betti Profile Index. Section 9 concludes. The geometric constructions underlying Figure 1 are fully reproducible; the exact figure-generation scripts and a minimal run recipe are provided in the *Supplementary Material*.

On proofs. For readability, we place a *Sketch of Proof* immediately after each Lemma/Proposition/Theorem in the main text. Full proofs and auxiliary technical arguments are collected in Appendix C.

2 Scenario space, boundary sets, and crisis orders

2.1 Scenario space and regime boundary

Let S be a set of scenarios (parameter points, calibrated environments, or model instances), equipped with a topology (or metric). A distinguished subset $\Sigma \subset S$ is interpreted as the crisis boundary (or regime boundary), i.e., the locus where equilibrium/stability structure changes via degeneracy (multiple root, rank drop, etc.).

2.2 Crisis order

Definition 1 (Crisis order) A binary relation $\succeq$ on S is called a **crisis order** if $s_1 \succeq s_2$ is interpreted as "scenario s_1 is (weakly) no less crisis-prone than scenario s_2." We write $s_1 \sim_{\succeq} s_2$ for indifference (both $s_1 \succeq s_2$ and $s_2 \succeq s_1$).

Remark 1 (Weak versus strict crisis orders) Throughout the paper we work with *weak* crisis orders (complete and transitive), which are therefore reflexive by definition. When needed, the associated strict crisis order is obtained in the standard way by $s_1 \succ s_2$ if and only if $s_1 \succeq s_2$ and not $s_2 \succeq s_1$.

Example 1 (A crisis order induced by the triple-well boundary (preview of Example 4)) Let $S \subset \mathbb{R}^3$ be a parameter domain with $\theta = (a, b, c)$ and consider the triple-well potential

$$V(x;\theta) = \frac{x^6}{6} - \frac{a}{4}x^4 + \frac{b}{2}x^2 + cx, \qquad R(x;\theta) = \partial_x V = x^5 - ax^3 + bx + c.$$

Let $\Delta(\theta) := \mathrm{Res}_x\big(R, \partial_x R\big)$ denote the discriminant-type resultant whose zero set is the regime boundary

$$\Sigma = \{\theta \in S : \Delta(\theta) = 0\},$$

i.e., the locus where $R(\cdot;\theta)$ has a multiple root (degeneracy / loss of Morse-type structure). Define a crisis order $\succeq$ on S by

$$\theta_1 \succeq \theta_2 \quad :\Longleftrightarrow \quad |\Delta(\theta_1)| \le |\Delta(\theta_2)|.$$

Thus scenarios are ranked by proximity to the degeneracy boundary. This order is complete and transitive, and it is continuous whenever Δ is continuous. Moreover, it satisfies boundary monotonicity with $d(\theta) = |\Delta(\theta)|$. In Layers (B)–(C) we show that, in codimension-one principal settings, any continuous scalar diagnostic representing such an order is essentially unique up to monotone transformation and admissible reparametrizations.

The foundational task is to justify the existence of a continuous scalar diagnostic $D : S \to \mathbb{R}$ representing $\succeq$, and to understand how boundary geometry further restricts admissible diagnostics.

3 Layer (A): crisis orders and representation

3.1 Axioms

Assumption 1 (Weak order) The crisis order $\succeq$ is complete and transitive on S.

Assumption 2 (Topological regularity) S is endowed with a topology (or metric) under which relevant primitives vary continuously, and Σ is closed.

Assumption 3 (Continuity of the order) For every $s \in S$, the upper contour set $\{s' \in S : s' \succeq s\}$ and the lower contour set $\{s' \in S : s \succeq s'\}$ are closed.

Assumption 4 (Boundary monotonicity) There exists a nonnegative proximity functional $d : S \to \mathbb{R}_+$ with $d(s) = 0$ iff $s \in \Sigma$ such that, for any sequence (s_n) with $d(s_n) \downarrow 0$, we have $s_n \succeq s$ for all sufficiently large n and any fixed $s \notin \Sigma$.

Remark 2 Assumption 4 formalizes the minimal commitment: approaching the crisis boundary does not improve the diagnostic ranking.

Remark on assumptions. Assumptions 1–4 (and later Assumptions 6–7 and Assumption 8) are stated for transparency and modularity. Appendix D verifies their *independence* (via counterexamples) and their *soundness* (via a canonical model class satisfying all assumptions), so that no hidden logical redundancy is imposed by the axioms.

3.2 Continuous ordinal representation

Assumption 5 (Regularity for continuous representation) The scenario space S is a connected, separable metric space. Moreover, the weak order $\succeq$ satisfies Assumption 3(closed upper and lower contour sets).

Theorem 1 (Continuous ordinal representation) *Under Assumptions 1–4 and Assumption 5, there exists a continuous function $D : S \to \mathbb{R}$ such that*

$$s_1 \succeq s_2 \iff D(s_1) \geq D(s_2).$$

Moreover, any other continuous representation $\widetilde{D}$ satisfies $\widetilde{D} = \varphi \circ D$ for some strictly increasing φ on $D(S)$.[1]

Proof (Sketch) Under Assumptions 1–3 and Assumption 5, $\succeq$ is a continuous weak order on a connected separable metric space. A Debreu-style ordinal representation theorem (as developed in the measurement literature; see, e.g.,[Debreu, G. 1954] and [Krantz, D.H., Luce, R.D., Suppes, P.and Tversky, A. 1971]) yields a continuous $D : S \to \mathbb{R}$ representing $\succeq$. Uniqueness follows from standard monotone-transform arguments on the image $D(S)$. Boundary monotonicity (Assumption 4) is not needed for existence but supports the intended diagnostic interpretation. A fully quantified proof is given in Appendix C (Theorem 5). □

[1] **What Theorem 1 does not say**: The theorem justifies the existence of a continuous scalar diagnostic under minimal axioms. It does not identify a particular functional form. Functional restriction is provided by invariance and boundary structure (Layers (B)–(D)).

4 Layer (B): catastrophe-invariance and essential uniqueness in single-channel geometry

In many applications the boundary Σ is characterized by a single invariant scalar map (a *discriminant-type object*)

$$\Delta : S \to \mathbb{R}, \qquad \Sigma = \{s \in S : \Delta(s) = 0\}. \tag{1}$$

A *diagnostic measuring proximity* to Σ should then be expressible as a monotone transform of $|\Delta|$ (or of a coordinate-free distance to Σ).

Assumption 6 (Invariance under catastrophe-equivalence) If $s_1 \sim s_2$ (catastrophe-equivalence), then $s_1 \sim_{\succeq} s_2$.

Assumption 7 (Sign irrelevance) Δ is continuous and invariant under $\sim$. Crisis proximity depends on $|\Delta|$ rather than Δ itself.

4.1 Abstract HCFI form

Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be continuous and strictly decreasing, with $\psi(r)$ diverging (or attaining a designated maximal value) as $r \downarrow 0$. Define the **Hazard-Consistent Fragility Index** (HCFI) by

$$\mathrm{HCFI}(s) := \psi(|\Delta(s)|). \tag{2}$$

The choice of ψ is a normalization (reporting) convention.

Lemma 1 (Ordinal uniqueness in the single-channel case) *Assume Theorem 1 and Assumptions 6–7. Suppose, in addition, that crisis comparisons are fully governed by discriminant proximity: for all $s_1, s_2 \in S$,*

$$|\Delta(s_1)| \le |\Delta(s_2)| \Longrightarrow s_1 \succeq s_2. \tag{3}$$

Then every continuous scalar representation D of $\succeq$ satisfies

$$D(s) = \varphi(\mathrm{HCFI}(s)) \quad \textit{for some strictly increasing } \varphi.$$

Proof (Sketch) Under the single-channel premise (3), crisis comparisons are completely determined by the discriminant proximity $|\Delta|$. Hence any continuous representation D must factor through $|\Delta|$, i.e., $D(s) = u(|\Delta(s)|)$ for some u on $\mathrm{im}(|\Delta|)$. Since $\mathrm{HCFI}(s) = \psi(|\Delta(s)|)$ with strictly decreasing ψ, we can reparameterize u as an increasing transform of HCFI and obtain $D = \phi \circ \mathrm{HCFI}$. A complete proof is given in Appendix C (Lemma 4). □

Remark 3 (Single-channel condition) Condition (3) is the "single crisis-geometry channel" premise: crisis comparisons depend only on one invariant proximity channel. When boundary geometry is multi-channel, uniqueness holds only within a specified aggregator class; see Appendix B.

5 Diagnostic–normative interface: linking diagnostics to PPG/PDI

Layers (A)–(C) characterize admissible diagnostics from boundary geometry. By contrast, PPG/PDI-type objects are *normative*: they depend on a choice of evaluative criteria. This section separates diagnosis from welfare and states a transparent sufficient condition under which diagnostic proximity can control normative wedges in a one-way sense.

5.1 Minimal definitions

Fix a scenario space S and two criteria $W_{\mathrm{A}}, W_{\mathrm{B}} : S \to \mathbb{R}$. Define the **Policy Performance Gap** (PPG) by

$$\mathrm{PPG}(s) := W_{\mathrm{B}}(s) - W_{\mathrm{A}}(s), \tag{4}$$

and define the **Policy Discrepancy Index** (PDI) by

$$\mathrm{PDI}(s) := \frac{|\mathrm{PPG}(s)|}{1 + |W_{\mathrm{A}}(s)| + |W_{\mathrm{B}}(s)|}. \tag{5}$$

These are normative objects and are conceptually distinct from proximity indices such as HCFI.

5.2 A minimal sufficient condition for a one-way link

Assumption 8 (Single-channel welfare dominance) There exists a nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with $g(0) = 0$ such that, for all $s \in S$,

$$|\mathrm{PPG}(s)| \le g(\mathrm{HCFI}(s)) \quad \text{and} \quad \mathrm{PDI}(s) \le g(\mathrm{HCFI}(s)).$$

Proposition 1 (One-way control by diagnostic proximity) *Under Assumption 8, if* $\mathrm{HCFI}(s_1) \le \mathrm{HCFI}(s_2)$ *then*

$$|\mathrm{PPG}(s_1)| \le g(\mathrm{HCFI}(s_2)), \qquad \mathrm{PDI}(s_1) \le g(\mathrm{HCFI}(s_2)).$$

Proof (Sketch) This is immediate from Assumption 8. If $\mathrm{HCFI}(s_1) \le \mathrm{HCFI}(s_2)$ and g is nondecreasing, then $g(\mathrm{HCFI}(s_1)) \le g(\mathrm{HCFI}(s_2))$. Combining with the pointwise bounds $|\mathrm{PPG}(s)| \le g(\mathrm{HCFI}(s))$ and $\mathrm{PDI}(s) \le g(\mathrm{HCFI}(s))$ yields the stated one-way control. □

Remark 4 Even when HCFI is structurally pinned down by boundary geometry, no equivalence $\arg\min \mathrm{HCFI} \Longleftrightarrow \arg\min \mathrm{PDI}$ should be asserted without additional structure. Assumption 8 is a strong empirical/theoretical restriction.

Example 2 (Examples and non-examples of Assumption 8)

Example (Assumption 8 holds) Suppose the normative wedge is mediated by the same boundary channel as the diagnostic. Concretely, let $\mathrm{HCFI}(s) = \psi(|\Delta(s)|)$, and define

$$|\mathrm{PPG}(s)| = h(|\Delta(s)|), \qquad \mathrm{PDI}(s) = \widetilde{h}(|\Delta(s)|),$$

where $h, \widetilde{h}$ are nondecreasing with $h(0) = \widetilde{h}(0) = 0$. Then Assumption 8 holds with g defined via the monotone correspondence between $|\Delta|$ and HCFI. This corresponds to a genuinely single-channel environment in which normative distortions are driven by proximity to the same crisis boundary.

Non-example (Assumption 8 fails) Let $S = \mathbb{R}^2$ and define $\Delta_1(x, y) = x$, $\Delta_2(x, y) = y$. Let HCFI depend only on Δ_1, while the normative wedge depends on Δ_2, for instance

$$\mathrm{HCFI}(x, y) = \psi(|x|), \qquad |\mathrm{PPG}(x, y)| = \frac{1}{|y|}.$$

Along sequences with x bounded away from zero and $y \to 0$, the normative wedge diverges while HCFI remains bounded. No nondecreasing function g can satisfy Assumption 8. This illustrates that Assumption 8 excludes genuinely multi-channel normative distortions.

Example 3 (Triple-well: a concrete PPG/PDI–HCFI coupling and its failure mode.) Consider the triple-well representative in Example 2.3/6.9 with $R(x; \theta) = \partial_x V(x; \theta)$ and boundary channel $\Delta(\theta)$. Let $\mathrm{HCFI}(\theta) = \psi(|\Delta(\theta)|)$ with ψ continuous and strictly decreasing.

Case 1 (Assumption 5.1 holds: single-channel normative wedge) Suppose the welfare gap is mediated by boundary proximity, e.g.

$$W_B(\theta) - W_A(\theta) = \mathrm{PPG}(\theta) = h(|\Delta(\theta)|), \qquad \mathrm{PDI}(\theta) = \frac{|h(|\Delta(\theta)|)|}{1 + |W_A(\theta)| + |W_B(\theta)|},$$

where h is nondecreasing with $h(0) = 0$. Then there exists a nondecreasing g (obtained by composing h with the inverse of ψ on $\psi(\mathrm{im}(|\Delta|))$) such that $|\mathrm{PPG}(\theta)| \le g(\mathrm{HCFI}(\theta))$ and $\mathrm{PDI}(\theta) \le g(\mathrm{HCFI}(\theta))$.

Case 2 (Assumption 5.1 fails: a non-boundary welfare channel) Fix (a, b) such that the discriminant channel does not vanish at $c = 0$, i.e. $\Delta(a, b, 0) \neq 0$. By continuity of Δ in c, there exist $\varepsilon > 0$ and $\delta > 0$ such that $|\Delta(a, b, c)| \ge \varepsilon$ for all $|c| \le \delta$. Along $c \to 0$ within this neighborhood, the boundary proximity (hence HCFI) remains bounded away from crisis while a normative wedge may still diverge, e.g.

$$|\mathrm{PPG}(a, b, c)| = \frac{1}{|c|}, \qquad \mathrm{PDI}(a, b, c) = \frac{1}{|c|\,(1 + |W_A| + |W_B|)}.$$

Along $c \to 0$ with $|\Delta(a, b, c)| \ge \varepsilon > 0$, the welfare wedge diverges while HCFI remains bounded. Hence no nondecreasing g can satisfy Assumption 8. This illustrates, within the same triple-well family, how the diagnostic–normative link is a *model restriction* rather than a logical consequence of boundary-based diagnosis.

6 Layer (C): boundary ideals and codimension-one classification

The single-channel premise in Layer (B) is substantive. This section gives a structural selection principle based on the algebraic form of the boundary itself. In the codimension-one principal regime, the reduced boundary ideal admits a canonical generator, providing a boundary-first route to a privileged diagnostic channel and clarifying why potential-based normal forms need not be treated as *a priori* primitives.

Throughout this section, we work over a base field k (typically $k = \mathbb{R}$ in applications) and the parameter ring $k[\theta] = k[\theta_1, \ldots, \theta_m]$. Unless stated otherwise, the parameters are treated as algebraically independent over k, so that $k[\theta]$ is a polynomial ring; this genericity is used only to avoid accidental degeneracies (e.g., repeated factors) and can be relaxed at the cost of heavier notation.

6.1 Boundary ideals

Let $\theta = (\theta_1, \ldots, \theta_m)$ be parameters in a scenario space S and work over $k[\theta]$, where k is a field (typically $k = \mathbb{R}$ in applications). Let $R(x; \theta)$ be an equilibrium predicate (or $R = \partial_x V$). Define the boundary ideal by elimination:

$$I_{\mathrm{bd}}(R) := \langle R, \partial_x R \rangle \cap k[\theta]. \tag{6}$$

Define the reduced boundary ideal by

$$J(R) := \sqrt{I_{\mathrm{bd}}(R)}. \tag{7}$$

The associated set-theoretic boundary is $\Sigma = V(J(R))$.

Remark 5 (Computational aspects of boundary ideals) The construction

$$J(R) = \sqrt{\langle R, \partial_x R \rangle \cap k[\theta]}$$

is constructive in principle: the elimination ideal $\langle R, \partial_x R \rangle \cap k[\theta]$ can be obtained algorithmically via Gröbner bases with an elimination (block) order.

Two caveats matter for the present paper.

(i) Complexity is real. Although elimination, radicalization, and primary decomposition are decidable, their cost can grow rapidly with degree and ambient dimension. Thus "auditing principality" may be computationally demanding in high-dimensional models, even when the underlying boundary geometry is conceptually simple.

(ii) Principality is geometric, not syntactic. Principality of $J(R)$ is not the existence of a singleton generating set, but the hypersurface property of the boundary locus: codimension-one local principality means that *locally* the boundary is cut out by a single equation. In that regime, a reduced generator Δ_{red} (squarefree up to units) provides the canonical single diagnostic channel

(Theorems 2–3). When principality fails, the obstruction is algebraic/structural and cannot be repaired by any scalar normalization.

Section 7(O1) summarizes an Operational verification checklist, while Appendix E records a fully explicit triple-well computation (resultant/Sylvester and elimination viewpoint) illustrating how the check is carried out symbolically.

6.2 Codimension-one principal case

Assumption 9 (Codimension-one principality) $J(R)$ has codimension one and is principal: there exists $\Delta_{\mathrm{red}} \in k[\theta]$ such that $J(R) = \langle \Delta_{\mathrm{red}} \rangle$.

Theorem 2 (Codimension-one boundary classification) *Under Assumption 9, the boundary set is the hypersurface*

$$\Sigma = \{\theta : \Delta_{\mathrm{red}}(\theta) = 0\}.$$

Moreover, irreducible components of Σ correspond to irreducible factors of Δ_{red}.

Proof (Sketch) Under Assumption 9 we have $J(R) = \langle \Delta_{\mathrm{red}} \rangle$ and therefore $\Sigma = V(J(R)) = V(\Delta_{\mathrm{red}}) = \{\theta : \Delta_{\mathrm{red}}(\theta) = 0\}$. Because $J(R)$ is radical, Δ_{red} is squarefree up to units; factoring into distinct irreducibles gives the irreducible decomposition of Σ and the correspondence between components and factors. A complete proof is given in Appendix C (Theorem 6). □

Theorem 3 (Unit-covariance under admissible transformations) *Let R' be obtained from R by an admissible transformation: (i) invertible state-coordinate change, (ii) invertible parameter reparametrization $\theta = \eta(\theta')$, and (iii) multiplication by a unit in the ambient ring. Then $J(R') = \eta_\sharp(J(R))$. In the codimension-one principal case,*

$$\Delta'_{\mathrm{red}}(\theta') = U(\theta') \cdot \Delta_{\mathrm{red}}(\eta(\theta')) \quad \textit{for some unit } U(\theta') \in (k[\theta'])^\times.$$

Proof (Sketch) Admissible state-coordinate changes and parameter reparametrizations induce ring automorphisms/isomorphisms that transport elimination ideals by pullback. Multiplication of R by a unit does not change radicals. Hence $J(R)$ transforms functorially under admissible changes. In the principal codimension-one regime, the transported ideal is again generated by the pullback of Δ_{red}, up to multiplication by a unit in the new ring, yielding $\Delta'_{\mathrm{red}}(\theta') = U(\theta')\Delta_{\mathrm{red}}(\eta(\theta'))$. A complete proof is given in Appendix C (Theorem 7). □

Remark 6 (How this helps the "V is a priori" critique) In codimension one one may proceed boundary-first: (i) identify (or postulate) the reduced boundary class $J(R) = \langle \Delta_{\mathrm{red}} \rangle$ from diagnostic/structural considerations; (ii) choose a normal-form representative whose discriminant realizes Δ_{red} up to admissible transformations; (iii) represent it via a potential V (when convenient) as a representative of the boundary class, not an arbitrary primitive.

6.3 Scalar faithfulness and the obstruction from non-principality

To formalize the limits of scalar diagnosis, we define a boundary-constraint preorder via localization.

Definition 2 (Faithfulness as no-false-comparability) Let $(S, \preceq_\Sigma)$ be the crisis preorder induced by boundary constraints, let $D : S \to \mathbb{R}$ be a scalar diagnostic, and let $\phi : \mathbb{R} \to \mathbb{R}$ be nondecreasing. We say that $\phi \circ D$ is **faithful** (in the sense of *no false comparability*) if it never certifies an order relation that is not present in $\preceq_\Sigma$, i.e.

$$\big(\phi(D(s_1)) \le \phi(D(s_2))\big) \implies s_1 \preceq_\Sigma s_2, \qquad \forall s_1, s_2 \in S.$$

Remark 7 This is intentionally a one-way certificate: it prevents a scalar diagnostic from producing *spurious* comparabilities. A two-way order embedding is generally unattainable in non-principal boundary regimes; see Lemma 2–Theorem 4 and Appendix C.

Lemma 2 (Non-principality yields incomparability) *If $J(R)$ is not principal, then there exist $s_1, s_2 \in S$ such that neither $s_1 \preceq_\Sigma s_2$ nor $s_2 \preceq_\Sigma s_1$ holds.*

Proof (Sketch) Non-principality of $J(R)$ implies the existence of $f, g \in J(R)$ with $f \notin \sqrt{\langle g \rangle}$ and $g \notin \sqrt{\langle f \rangle}$. Choose maximal ideals $m_1 \supset \langle f \rangle$ with $g \notin m_1$ and $m_2 \supset \langle g \rangle$ with $f \notin m_2$. After localizing, g becomes a unit in $k[\theta]_{m_1}$ while f remains a nonunit, and the reverse holds at m_2. Thus the localized ideals $J(R)_{m_1}$ and $J(R)_{m_2}$ are incomparable by inclusion, producing incomparable scenarios under $\preceq_\Sigma$. A complete proof is given in Appendix C (Lemma 5). □

Lemma 3 (Incomparability obstructs faithful scalar diagnostics) *If a preorder $(S, \preceq)$ contains incomparable elements, then no scalar diagnostic $D : S \to \mathbb{R}$ can be faithful to $\preceq$ even up to a monotone transformation.*

Proof (Sketch) If s_1, s_2 are incomparable under $\preceq$, any scalar map $D : S \to \mathbb{R}$ and any monotone transformation ϕ produce a total order on $\{\phi(D(s_1)), \phi(D(s_2))\}$, forcing either $\phi(D(s_1)) \le \phi(D(s_2))$ or the reverse and hence (by faithfulness) forcing comparability of s_1, s_2, a contradiction. A complete proof is given in Appendix C (Lemma 6). □

Theorem 4 (Limits of scalar crisis diagnostics) *Consider a boundary locus $\Sigma = V(J(R))$ induced by equilibrium/feasibility predicates.*

(i) If $J(R)$ is principal in codimension one, then crisis proximity can be represented (ordinally) by a scalar diagnostic based on Δ_{red} up to admissible transformations and monotone normalization.
(ii) If $J(R)$ is non-principal, then no scalar diagnostic index can be faithful in the sense of Definition 2.

Proof (Sketch) Part (i) follows from the codimension-one principal classification (Theorem 2) and unit-covariance (Theorem 3): the boundary channel is pinned down by Δ_{red} up to admissible transformations, and any continuous diagnostic compatible with the induced order is an increasing transform of an Δ_{red}-based proximity functional. Part (ii) follows by combining Lemma 2 (existence of incomparable scenarios under $\preceq_\Sigma$) with Lemma 3 (incomparability obstructs faithful scalar diagnostics). □

6.4 Worked example: a principal triple-well representative

Example 4 (Triple-well representative and principal reduced boundary ideal) Consider

$$V(x;\theta) = \frac{x^6}{6} - \frac{a}{4}x^4 + \frac{b}{2}x^2 + c\,x, \qquad \theta = (a,b,c) \in k^3,$$

so $R(x;\theta) = \partial_x V = x^5 - ax^3 + bx + c$ and $\partial_x R = 5x^4 - 3ax^2 + b$. Eliminating x yields a single generator $\Delta(a,b,c) = \mathrm{Res}_x(R, \partial_x R) \in k[a,b,c]$ and hence $I_{\mathrm{bd}}(R) = \langle \Delta \rangle$ and $J(R) = \langle \Delta_{\mathrm{red}} \rangle$, where Δ_{red} is the squarefree part. Thus $\Sigma = \{\Delta_{\mathrm{red}} = 0\}$ is codimension one and scalar diagnosis is structurally justified.

Remark 8 (Connection to the crisis order) The discriminant Δ_{red} obtained here provides a concrete realization of the crisis order introduced in Definition 1 and illustrated in Example 1: scenarios are ranked by proximity to the degeneracy boundary $\Sigma = \{\Delta_{\mathrm{red}} = 0\}$ via $|\Delta_{\mathrm{red}}|$.

6.5 A natural non-principal boundary from two independent fold constraints

The minimal contrast in Appendix E.3 illustrates that non-principality emerges once two independent constraints bind simultaneously. Here we record a *natural* nonlinear mechanism producing the same phenomenon: two independent saddle-node (fold) boundaries that may intersect.

Example 5 (Two independent folds and a codimension-two boundary intersection) Let (x, y) be two state variables and let $\theta = (a, u, b, v) \in k^4$ be parameters. Consider the decoupled equilibrium predicates

$$F_1(x;\, a, u) = x^3 - ax + u, \qquad F_2(y;\, b, v) = y^3 - by + v.$$

A fold (saddle-node) in the x-subsystem occurs when $F_1 = 0$ and $\partial_x F_1 = 0$ simultaneously; likewise for the y-subsystem.

Define the boundary ideal by elimination in the sense of Section 6.1:

$$I := \langle F_1, \partial_x F_1,\ F_2, \partial_y F_2 \rangle \subset k[x, y, a, u, b, v], \quad I_{\mathrm{bd}} := I \cap k[a, u, b, v],$$
$$J := \sqrt{I_{\mathrm{bd}}}.$$

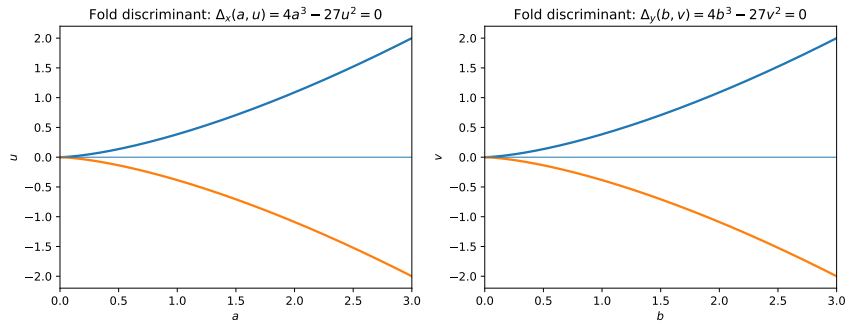


Fig. 1 Two-condition degeneracy as a natural source of non-principal boundary geometry (Example 5).

A direct elimination shows that the boundary constraints factor into the two classical cubic discriminants:

$$\Delta_x(a,u) := \mathrm{Res}_x(F_1, \partial_x F_1) = 4a^3 - 27u^2,$$
$$\Delta_y(b,v) := \mathrm{Res}_y(F_2, \partial_y F_2) = 4b^3 - 27v^2,$$

and hence

$$J = \langle \Delta_x(a,u),\ \Delta_y(b,v) \rangle \subset k[a,u,b,v].$$

Therefore the boundary locus is the *intersection* of two independent hypersurfaces:

$$\Sigma = V(J) = \{\Delta_x = 0\} \cap \{\Delta_y = 0\}.$$

Unless one discriminant divides the other (non-generic), J is *not* principal. Geometrically, the regime boundary is intrinsically multi-constraint (codimension two): the system crosses a single fold surface generically, but the *simultaneous* fold event lies on the intersection, and the resulting boundary-constraint preorder admits incomparabilities (Lemma 2), obstructing faithful scalar diagnostics (Theorem 4).

Remark 9 (Nonlinear-science interpretation) Example 5 is deliberately standard: it represents a two-mode system in which each mode has a fold bifurcation surface. The single-channel regime corresponds to tracking proximity to one fold surface at a time. The intrinsically multi-channel regime arises when the boundary is governed by multiple independent degeneracy constraints, i.e. when "which fold matters" is itself scenario-dependent. This is precisely the ideal-theoretic non-principal transition emphasized in Layers (C)–(D).

Figure 1 should be interpreted as a minimal geometric model of intersecting fold discriminants, illustrating how boundary intersections generate non-principal crisis ideals in multi-parameter systems. Additional computational details and figure-generation scripts are provided in the Supplementary Material.

7 Operational implications and implementation protocols

This section is an *operational interlude*: it outlines how the structural results can be used to examine scalar diagnostics in practice. The formal definition and theory of the cohomological gateway indicator BPI^* are given in Section 8 (Layer (D)).

Layers (A)–(C) establish admissibility and (in the single-channel regime) essential uniqueness. This section records operational validation checks relevant once a diagnostic is deployed in empirical pipelines. These are not extra axioms, but rather recommended verification protocols.

Operational implication of BPI^*. When $\mathrm{BPI}^* = 0$ (boundary-trivial in positive degrees), scalar threshold rules are structurally defensible under codimension-one principality. When $\mathrm{BPI}^* \geq 1$, single-index optimization or single-threshold early-warning rules should be treated as structurally fragile.

(O1) **Algebraic principality test (boundary-first gate).** Given a response predicate $R(x;\theta)$, compute the elimination ideal

$$I_{\mathrm{bd}}(R) := \langle R, \partial_x R \rangle \cap k[\theta]$$

(e.g. via a Gröbner basis with an elimination/block order, or by an explicit resultant when available).

- (a) **Codimension check:** verify $\operatorname{codim} I_{\mathrm{bd}}(R) = 1$ (hypersurface boundary candidate).
- (b) **Principality check:** test whether $J(R) = \sqrt{I_{\mathrm{bd}}(R)}$ is (locally) principal. If yes, extract a reduced generator Δ_{red} (squarefree up to units) so that $J(R) = \langle \Delta_{\mathrm{red}} \rangle$ and use $|\Delta_{\mathrm{red}}|$ as the canonical single channel.
- (c) **Exit rule:** if principality fails (or multiple independent generators persist), treat the system as intrinsically multi-channel and bypass scalar diagnostics, switching to the cohomological gatekeeper BPI^* (Section 8).

Remark 5 clarifies computational caveats, and Appendix E gives a fully explicit triple-well example as a symbolic template.

(O2) **Robustness:** small perturbations of calibrated inputs should not induce large swings in HCFI.

(O3) **Ranking stability:** rankings remain stable under reasonable preprocessing variants.

(O4) **Early-warning alignment:** along realized paths, HCFI builds up before observed regime shifts (up to tolerance).

(O5) **Comparability:** cross-context comparisons require an explicit convention (choice of ψ).

Algorithm 1 Boundary-first verification pipeline for the scalar paradigm

Require: Polynomial predicate $R(x;\theta) \in k[x,\theta]$ (or a finite family of predicates), scenario domain $S \subseteq k^m$
Ensure: Decision: ScalarOK (principal hypersurface channel) or MultiChannel (non-principal / multi-constraint)
1: Form the ideal $I \leftarrow \langle R, \partial_x R \rangle \subset k[x,\theta]$ (or the appropriate Jacobian/rank-drop ideal in multivariate state)
2: Eliminate state variables: $I_{\mathrm{bd}} \leftarrow I \cap k[\theta]$ (Gröbner elimination / resultants when available)
3: Radicalize: $J \leftarrow \sqrt{I_{\mathrm{bd}}}$ and set $\Sigma \leftarrow V(J)$
4: Compute/estimate $\mathrm{codim}(J)$ in $k[\theta]$
5: **if** $\mathrm{codim}(J) \neq 1$ **then**
6: **return** MultiChannel {Boundary is not a hypersurface (intrinsically multi-constraint)}
7: **end if**
8: Test (local) principality of J on Σ (e.g. local generator test / primary decomposition)
9: **if** J is (locally) principal **then**
10: Extract a reduced generator Δ_{red} (squarefree up to units) such that $J = \langle \Delta_{\mathrm{red}} \rangle$
11: **return** ScalarOK with canonical channel $|\Delta_{\mathrm{red}}|$
12: **else**
13: **return** MultiChannel {Invoke Layer (D): compute cohomological gatekeepers (e.g. BPI*)}
14: **end if**

8 Layer (D): a minimal cohomological gateway beyond principality: BPI*

When $J(R)$ is not principal, the boundary is genuinely multi-constraint. A minimal next-step invariant should (i) depend only on the boundary locus, (ii) be coordinate invariant, and (iii) detect the first topological obstruction degree. We introduce BPI* for this role.

Assumption 10 (Topology for cohomological diagnostics) The scenario space S is a subspace of $\mathbb{R}^m$ (with the subspace topology), or more generally a paracompact Hausdorff space. We restrict attention to boundary loci $Z \subset S$ that are triangulable (e.g., semi-algebraic), and we use singular cohomology $H^k(Z;k)$ over a fixed coefficient field k. Let $Z := \Sigma = V(J(R)) \subset S$ and denote the *Betti number*:

$$b_k(Z) := \dim_k H^k(Z;k).$$

Scope of "minimal gateway". The term *minimal* in BPI* is used in a restricted, structural sense: among invariants that (i) are functorial under restriction to boundary neighborhoods, (ii) detect the onset of non-principality as an obstruction to scalar faithfulness, and (iii) remain stable under monotone reparameterizations of scalar diagnostics, BPI* provides the lowest-complexity cohomological witness. We do not claim minimality with respect to *all* conceivable diagnostics or all invariants.[2]

Definition 3 (Betti Profile Index) Define the **Betti Profile Index** (BPI) by

$$\mathrm{BPI}(Z) := \min\{k \geq 0 : b_k(Z) \neq 0\}$$

and define the **Positive-degree Betti Profile Index** (BPI^*) by

$$\mathrm{BPI}^*(Z) := \min\{k \geq 1 : b_k(Z) \neq 0\},$$

with the convention $\mathrm{BPI}^*(Z) = +\infty$ if $b_k(Z) = 0$ for all $k \geq 1$.

Example 6 (A non-principal boundary with $\mathrm{BPI}^* = 1$*)* Let $S = \mathbb{R}^3$ with $\theta = (u, v, w)$ and boundary constraints $\Delta_1 = u^2+v^2-1 = 0$ and $\Delta_2 = w = 0$. Then $J = \langle \Delta_1, \Delta_2 \rangle$ is not principal and $Z \cong S^1$, hence $b_1(Z) = 1$ and $\mathrm{BPI}^*(Z) = 1$.

Example 7 (A non-principal boundary with $\mathrm{BPI}^* = 2$*)* Let $S = \mathbb{R}^4$ with $\theta = (u, v, w, t)$ and impose two independent boundary constraints

$$\Delta_1(u, v, w, t) := u^2 + v^2 + w^2 - 1 = 0, \qquad \Delta_2(u, v, w, t) := t = 0.$$

Then $J = \langle \Delta_1, \Delta_2 \rangle$ is not principal (generically), and the boundary locus is

$$Z = V(J) \cong S^2.$$

Hence $b_2(Z) = 1$ and $b_k(Z) = 0$ for $k \geq 1$ with $k \neq 2$, so $\mathrm{BPI}^*(Z) = 2$.

Remark 10 (What the positive degree detects) Examples 6 and 7 illustrate that BPI^* is not merely a "non-principal flag". It also conveys a coarse but coordinate-free signature of the boundary locus: $\mathrm{BPI}^* = 1$ witnesses a non-trivial one-dimensional loop structure (e.g. S^1-type), while $\mathrm{BPI}^* = 2$ witnesses a two-dimensional spherical obstruction (e.g. S^2-type). In operational terms, distinct positive degrees suggest different failure modes of scalar thresholding: loop-type boundaries invite channel-switching ambiguity, while sphere-type boundaries suggest enclosure/compartment effects in boundary neighborhoods.

Proposition 2 (Silence in the principal hypersurface regime) *In the codimension-one principal setting (Assumption 9),* Δ_{red} *already provides a complete scalar boundary channel up to admissible transformations. In this regime* BPI^* *is optional, while it becomes informative once principality fails.*

Proof (Sketch) In the codimension-one principal regime, Δ_{red} already determines the hypersurface class and hence the complete single-channel boundary description up to admissible transformations (Theorems 2– 3). Therefore no further invariant is needed for boundary-channel classification. When principality fails, no single generator exists and scalar faithfulness fails (Theorem 4), so topological invariants of $Z = V(J(R))$, including BPI^*, provide additional coordinate-free information not reducible to a single discriminant scalar. A complete proof is given in Appendix C (Proposition C.8). □

[2] **Scope of the "minimal" claim**: By "minimal" we mean minimal *within the class of functorial, reparametrization-invariant topological diagnostics* that survive after scalar reductions fail: BPI^* is the first positive-degree cohomological obstruction to boundary-triviality that remains invariant under the admissible transformations of Layers (A)–(C).

9 Discussion and conclusion

Outlook. This paper separates three notions often conflated in applied crisis modeling: (i) existence of a scalar diagnostic, (ii) essential uniqueness of its ordinal content (in single-channel geometry), and (iii) normative status.

Layer (A) provides an axiomatic justification for existence of a continuous scalar representation of a crisis order. Layer (B) clarifies that functional restriction beyond normalization requires a substantive single-channel premise. Layer (C) identifies when such a single channel is structurally warranted: codimension-one principality of the reduced boundary ideal yields a canonical generator Δ_{red}. When principality fails, scalar faithfulness fails generically; this is a structural obstruction, not a calibration artifact. Layer (D) proposes BPI^* as a minimal gateway diagnostic to signal when the scalar paradigm should be abandoned. In particular, BPI^* does not compete with scalar indices inside the scalar regime; it diagnoses when the system has exited that regime.

Relation to catastrophe theory and algebraic geometry. Classical catastrophe theory (Thom–Arnold) explains qualitative regime transitions through equivalence classes of critical points of *scalar* generating potentials, and thereby implicitly presumes that a faithful scalar observable is available. The present paper shifts the focus from potentials to *regime boundaries* and their algebraic structure. Rather than asking which singularities arise from a scalar potential, we ask when the boundary separating admissible regimes can be faithfully probed by a single scalar observable at all. This change of viewpoint reveals that the scalar paradigm rests on a strong structural assumption: *codimension-one principality* of the reduced boundary ideal, yielding a canonical generator Δ_{red} (up to units and admissible reparametrizations). When principality fails, the boundary is intrinsically multi-constraint and induces order incomparabilities that obstruct faithful scalar representation. In this sense, the principal/non-principal transition is a boundary-theoretic degeneracy transition that separates single-channel from genuinely multi-channel catastrophe regimes. The Betti Profile Index BPI^* is not a competing scalar invariant; it is a minimal, coordinate-free gatekeeper that becomes informative precisely when the scalar paradigm ceases to be structurally defensible.

Future research. Several directions follow naturally.

(i) **Multi-channel diagnostics.** When principality fails, a systematic theory of admissible *multi-channel* diagnostics is needed. A first step is to formalize minimal axioms under which ordinal uniqueness can be recovered within an aggregator class $\Psi(\psi_1(|\Delta_1|), \ldots, \psi_K(|\Delta_K|))$, and to relate the choice of channels to generators and localizations of the boundary ideal.

(ii) **Computational audit protocols.** An important extension concerns explicit computational verification pipelines for testing the scalar paradigm.

The present framework suggests a three-stage protocol: (i) compute the elimination ideal $I_{\mathrm{bd}}(R) = \langle R, \partial_x R\rangle \cap k[\theta]$ using Gröbner bases or related symbolic methods; (ii) test whether the resulting boundary is of codimension one and admits a (locally) principal description, yielding a reduced generator Δ_{red}; (iii) when principality fails or multiple boundary components persist, abandon scalar diagnostics and deploy cohomological indicators such as BPI*. Notably, this pipeline highlights a dual limitation. Even when scalar diagnostics fail for structural reasons (non-principal boundary ideals), the computational task of verifying principality itself may be demanding in high-dimensional systems. This reinforces the central message of the paper: both normative interpretation and computational feasibility place intrinsic limits on scalar crisis diagnostics.

(iii) **Dynamic and pathwise extensions.** Extend the framework to path spaces where boundary proximity and comparability evolve over time, clarifying when early-warning scalarization remains structurally warranted and when regime shifts are necessarily multi-channel.

Conclusion. The results can be interpreted as a structural degeneracy transition within catastrophe theory itself. Crossing from a principal to a non-principal boundary changes not merely the local normal form of a singularity, but the qualitative nature of observable structure: from regimes where a single scalar channel suffices to regimes where no scalar observable can capture the boundary order without loss. In this sense, non-principal boundary geometry represents a higher-order catastrophe—not reducible to classical scalar potential models—because the obstruction is encoded in the boundary ideal itself rather than in a chosen scalar generating function.

Declarations

AI support. The author used AI tools (ChatGPT, Google Gemini) for language polishing and proofreading only. The author takes full responsibility for the content and any remaining errors.

Competing interests. The author declares that he has no competing interests.

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A Proof roadmap and location of full proofs

This appendix provides a compact roadmap for the proof structure and indicates where complete proofs are located.

- **Main text (all sections):** each Lemma/Proposition/Theorem is followed by a *Sketch of Proof* intended to make the logical dependencies transparent.
- **Appendix C:** full proofs of all main results, including technical reductions and auxiliary lemmas omitted from the sketches.
- **Reading guide:** Layer (A) results in Section 3 support Layer (B) in Section 4; Sections 6–7 (boundary ideals and cohomological obstructions) rely on the algebraic setup introduced in Section 2 and the uniqueness analysis in Section 4.

Remark The sketches are designed to be self-contained at the level of ideas; Appendix C provides the full argumentation needed for verification and reuse in related models.

B Multi-channel extensions (optional)

If crisis geometry depends on multiple invariant channels $\Delta_1, \dots, \Delta_K$ with $\Sigma = \{s : \Delta_1(s) = \cdots = \Delta_K(s) = 0\}$, an admissible diagnostic may take the form

$$\mathrm{HCFI}^{(K)}(s) = \Psi\Big(\psi_1(|\Delta_1(s)|), \dots, \psi_K(|\Delta_K(s)|)\Big),$$

where each ψ_k is decreasing and Ψ is increasing in each argument. Ordinal uniqueness can be recovered within the chosen aggregator class under corresponding axioms.

C Complete definitions and proofs

This appendix provides complete (fully quantified) definitions and full proofs for the main-text results. Each item is cross-referenced to its main-text counterpart to avoid divergence between the "readable" main text and the technical archive.

C.1 Complete definitions

Definition 4 (Complete local form of scalar diagnostic faithfulness) Let $\Sigma \subset S$ denote the regime boundary and let $J(R) \subset k[\theta]$ be the reduced boundary ideal as in (6)–(7). For each scenario $s \in S$, write $\theta(s) \in \mathrm{Spec}(k[\theta])$[3]for its parameter point and let $\mathfrak{m}_s \subset k[\theta]$ be the corresponding maximal ideal (i.e. $\mathfrak{m}_s = \{f \in k[\theta] : f(\theta(s)) = 0\}$). Define the **local boundary-constraint ideal** at s by

$$J_s \;:=\; J(R)_{\mathfrak{m}_s} \;\subset\; k[\theta]_{\mathfrak{m}_s}. \tag{8}$$

Definition 5 We equip S with the *boundary-constraint preorder* $\preceq_\Sigma$ defined by

$$s_1 \preceq_\Sigma s_2 \quad :\Longleftrightarrow \quad J_{s_2} \;\subseteq\; J_{s_1}. \tag{9}$$

A *scalar diagnostic index* $D : S \to \mathbb{R}$ is **faithful to the boundary structure** if there exists a nondecreasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that, for all $s_1, s_2 \in S$,

$$\varphi\big(D(s_1)\big) \;\le\; \varphi\big(D(s_2)\big) \quad\Longrightarrow\quad s_1 \preceq_\Sigma s_2. \tag{10}$$

Equivalently, faithfulness is understood here as a *one-way* certification property: whenever the scalar ranking asserts $\phi(D(s_1)) \le \phi(D(s_2))$ (for nondecreasing ϕ), the underlying preorder must satisfy $s_1 \preceq_\Sigma s_2$.[4]

C.2 Complete proof of the representation theorem (Layer A)

Theorem 5 (Complete version of Theorem 1) *Assume Assumptions 1–4 and Assumption 5. Then there exists a continuous function $D : S \to \mathbb{R}$ such that*

$$s_1 \succeq s_2 \iff D(s_1) \ge D(s_2).$$

Moreover, if $\widetilde{D} : S \to \mathbb{R}$ is another continuous representation, then $\widetilde{D} = \varphi \circ D$ on S for some strictly increasing φ on $D(S)$.

Proof We separate the argument into: (i) existence of an ordinal representation on a topological scenario space, (ii) continuity, and (iii) uniqueness up to increasing transforms.

Step 1 (weak order and quotient) By Assumption 1, $\sim_\succeq$ is an equivalence relation. Let $S/\!\sim_\succeq$ denote the quotient set of indifference classes and write $[s]$ for the class of s.

Step 2 (order-continuity) By Assumption 3, upper and lower contour sets are closed. This is a standard premise in Debreu-type representation results.

Step 3 (Debreu-style representation) Invoke an appropriate representation theorem for a continuous weak order on a separable metric space under the relevant auxiliary assumptions (e.g. connectedness when required by the chosen reference). Conclude existence of a continuous ordinal representation $D : S \to \mathbb{R}$.

Step 4 (uniqueness) If D and $\widetilde{D}$ are continuous ordinal representations of the same weak order, define φ on $D(S)$ by $\varphi(D(s)) := \widetilde{D}(s)$. Well-definedness follows from indifference preservation, and strict monotonicity follows from strict preference preservation. □

[3] We denote by $\mathrm{Spec}(k[\theta])$ the set of all the emphnon-trivial prime ideals in $k[\theta]$.

[4] We do *not* require a two-way order embedding, which is precisely what fails in non-principal boundary regimes

C.3 Complete proof of discriminant-based ordinal uniqueness (Layer B)

Lemma 4 (Complete version of Lemma 1) *Assume Theorem 1, Assumptions 6–7, and* (3)*. Then every continuous scalar representation D of $\succeq$ satisfies $D = \varphi \circ \mathrm{HCFI}$ for some strictly increasing φ on* $\mathrm{HCFI}(S)$*.*

Proof By (3) and completeness/transitivity, $s_1 \sim_{\succeq} s_2$ whenever $|\Delta(s_1)| = |\Delta(s_2)|$. Hence any representation D factors through $|\Delta|$, i.e. $D(s) = u(|\Delta(s)|)$ for some u on $\mathrm{im}(|\Delta|)$. Since $\mathrm{HCFI}(s) = \psi(|\Delta(s)|)$ with strictly decreasing ψ, define φ on $\mathrm{HCFI}(S)$ by $\varphi(\mathrm{HCFI}(s)) := D(s)$. Then φ is strictly increasing because D and HCFI represent the same order. □

C.4 Complete proofs for the boundary-ideal obstruction and codimension-one classification (Layer C)

Lemma 5 (Complete version of Lemma 2) *Assume the setup of Section 6. If the reduced boundary ideal $J(R) \subset k[\theta]$ is not principal, then there exist scenarios $s_1, s_2 \in S$ such that neither $s_1 \preceq_\Sigma s_2$ nor $s_2 \preceq_\Sigma s_1$ holds.*

Proof Because $J(R)$ is radical and not principal, choose $f, g \in J(R)$ such that $f \notin \sqrt{\langle g \rangle}$ and $g \notin \sqrt{\langle f \rangle}$. Then there exist maximal ideals $\mathfrak{m}_{s_1} \supset \langle f \rangle$ with $g \notin \mathfrak{m}_{s_1}$ and $\mathfrak{m}_{s_2} \supset \langle g \rangle$ with $f \notin \mathfrak{m}_{s_2}$. Localizing at $\mathfrak{m}_{s_1}$ makes g a unit while f remains a nonunit; the reverse holds at $\mathfrak{m}_{s_2}$. Thus $J(R)_{\mathfrak{m}_{s_1}}$ and $J(R)_{\mathfrak{m}_{s_2}}$ are incomparable by inclusion, hence s_1, s_2 are incomparable under $\preceq_\Sigma$. □

Lemma 6 (Complete version of Lemma 3) *Let $(S, \preceq)$ be a preorder. If there exist incomparable $s_1, s_2 \in S$, then there is no scalar diagnostic $D : S \to \mathbb{R}$ that is faithful to $\preceq$ even up to monotone transformation.*

Proof Assume D and a nondecreasing ϕ satisfy $\phi(D(s_1)) \leq \phi(D(s_2)) \Rightarrow s_1 \preceq s_2$ for all pairs. Because $\mathbb{R}$ is totally ordered, either $\phi(D(s_1)) \leq \phi(D(s_2))$ or the reverse holds, forcing comparability of s_1, s_2, contradiction. □

Theorem 6 (Complete version of Theorem 2) *Assume Assumption 9. Then $\Sigma = V(J(R)) = \{\Delta_{\mathrm{red}} = 0\}$, and irreducible components of Σ correspond bijectively to irreducible factors of Δ_{red}.*

Proof Because $J(R) = \langle \Delta_{\mathrm{red}} \rangle$, we have $\Sigma = V(J(R)) = V(\Delta_{\mathrm{red}})$. Since $J(R)$ is radical, Δ_{red} is squarefree up to units; writing $\Delta_{\mathrm{red}} = \prod_j p_j$ as a product of distinct irreducibles yields $V(\Delta_{\mathrm{red}}) = \bigcup_j V(p_j)$, the irreducible decomposition. □

Theorem 7 (Complete version of Theorem 3) *Under the admissible transformations in Theorem 3, the reduced boundary ideal transforms by pullback. In the principal case, Δ_{red} transforms by composition with the parameter reparametrization and multiplication by a unit.*

Proof Invertible state changes induce automorphisms of the ambient polynomial ring and preserve elimination ideals up to isomorphism. Invertible parameter reparametrizations induce a ring isomorphism on $k[\theta]$ and transport ideals by pullback. Multiplication by a unit leaves radicals unchanged. Therefore $J(R')$ is the transported ideal. If $J(R) = \langle \Delta_{\mathrm{red}} \rangle$, then $J(R')$ is generated by the pulled-back Δ_{red} up to a unit. □

C.5 Complete proof for the "minimal gateway" claim (Layer D)

Proposition 3 (Complete version of Proposition 2) *In the codimension-one principal regime (Assumption 9),* Δ_{red} *already provides a complete single-channel boundary description up to admissible transformations. In this regime,* BPI* *is not needed for classification, but it becomes informative when* $J(R)$ *is not principal.*

Proof By Theorem 2 and Theorem 3, the boundary hypersurface class is determined (up to admissible changes) by Δ_{red}; thus no additional invariant is required to fix the scalar boundary channel. When $J(R)$ is not principal, no single generator exists; cohomological invariants of $Z = V(J(R))$, including BPI*, supply coordinate-free information not reducible to one discriminant scalar. □

D Independence and soundness of Assumptions 1–4, Assumptions 6–7, and Assumption 8

This appendix illustrates (i) independence of selected assumptions via counterexamples and (ii) consistency via a canonical model class.

D.1 Independence of selected assumptions

Independence of Assumption 4 Let $S = \mathbb{R}$ with boundary $\Sigma = \{0\}$. Define a weak order by $s_1 \succeq s_2$ iff $|s_1| \geq |s_2|$ except that values near the boundary are reversed, e.g. declare $0.01 \prec 1$. Completeness, transitivity, and order-continuity can be arranged, but approaching the boundary does not weakly increase crisis ranking.

Independence of Assumption 7 Let $\Delta(s) = s$ on $S = \mathbb{R}$ and define a crisis order by $s_1 \succeq s_2$ iff $\Delta(s_1) \geq \Delta(s_2)$. The order is complete and continuous with boundary $\Sigma = \{0\}$, but the sign of Δ matters.

Independence of Assumption 8 Let HCFI be governed by an invariant channel Δ_1, while welfare wedges depend on an orthogonal channel Δ_2. Then there need not exist any nondecreasing g such that $|\mathrm{PPG}| \leq g(\mathrm{HCFI})$.

D.2 Soundness: a model satisfying all assumptions

Let $S \subset \mathbb{R}^m$ be open, let $\Delta : S \to \mathbb{R}$ be continuous, and define $\Sigma = \{\Delta = 0\}$. Define

$$s_1 \succeq s_2 \iff |\Delta(s_1)| \leq |\Delta(s_2)|.$$

Completeness/transitivity are immediate; order-continuity follows from continuity of Δ; Assumption 4 holds with $d(s) = |\Delta(s)|$.

Define $\mathrm{HCFI}(s) = \psi(|\Delta(s)|)$ with ψ continuous and strictly decreasing, ensuring Assumptions 6–7. Finally, define normative wedges by $|\mathrm{PPG}(s)| = h(|\Delta(s)|)$ and $\mathrm{PDI}(s) = \widetilde{h}(|\Delta(s)|)$ for nondecreasing $h, \widetilde{h}$ with $h(0) = \widetilde{h}(0) = 0$; then Assumption 8 holds after composing with ψ.

E Computational verification of codimension-one principality: an explicit triple-well template

This appendix complements Remark 5 and the operational protocol Section 7(O1) by recording a fully explicit symbolic audit pipeline for codimension-one principality, using the triple-well representative (Example 4) as a transparent template.

E.1 Elimination via Gröbner bases: a minimal verification recipe

Let k be a field of characteristic 0 (typically $k = \mathbb{R}$ in applications), let $\theta = (\theta_1, \dots, \theta_m)$ and x denote the state variable. Given a polynomial predicate $R(x;\theta) \in k[x,\theta]$, define

$$I := \langle R, \partial_x R\rangle \subset k[x,\theta], \qquad I_{\mathrm{bd}}(R) := I \cap k[\theta], \qquad J(R) := \sqrt{I_{\mathrm{bd}}(R)}.$$

Elimination theorem (computational form). Fix a block elimination monomial order with $x \succ \theta$ (e.g. lex with x highest). Let G be a Gröbner basis of I for this order. Then

$$I_{\mathrm{bd}}(R) = \langle G \cap k[\theta]\rangle.$$

Thus $I_{\mathrm{bd}}(R)$ is obtained by discarding all Gröbner basis elements involving x.

Checklist. A direct symbolic verification of the scalar paradigm proceeds as:

(i) compute $I_{\mathrm{bd}}(R)$ by elimination (Gröbner basis with block order);
(ii) compute $\operatorname{codim} I_{\mathrm{bd}}(R)$ (hypersurface test);
(iii) compute $J(R) = \sqrt{I_{\mathrm{bd}}(R)}$ (radical);
(iv) test local principality / codimension-one principality of $J(R)$;
(v) if principal, extract Δ_{red} (squarefree generator up to units) so that $J(R) = \langle \Delta_{\mathrm{red}}\rangle$ and $\Sigma = \{\Delta_{\mathrm{red}} = 0\}$.

Squarefree reduction (characteristic 0). If a principal generator Δ is known for $I_{\mathrm{bd}}(R) = \langle \Delta\rangle$, then a canonical choice of reduced generator is the squarefree part

$$\Delta_{\mathrm{red}} := \frac{\Delta}{\gcd\big(\Delta, \partial_{\theta_1}\Delta, \dots, \partial_{\theta_m}\Delta\big)} \quad \text{(up to a unit in } k[\theta]).$$

In codimension one, this identifies the hypersurface boundary class used in Layer (C).

E.2 Triple-well: explicit resultant/Sylvester eliminant

Consider the triple-well potential in Example 4:

$$V(x;a,b,c) = \frac{x^6}{6} - \frac{a}{4}x^4 + \frac{b}{2}x^2 + cx, \qquad R(x;a,b,c) = \partial_x V = x^5 - ax^3 + bx + c.$$

Then

$$\partial_x R = 5x^4 - 3ax^2 + b.$$

Let $I = \langle R, \partial_x R\rangle \subset k[x,a,b,c]$.

Resultant eliminant. Define the eliminant by the resultant

$$\Delta(a,b,c) := \mathrm{Res}_x\big(R, \partial_x R\big) \in k[a,b,c].$$

Equivalently, Δ is the determinant of the $(5+4)\times(5+4) = 9\times 9$ Sylvester matrix. Writing

$$R = x^5 + 0\cdot x^4 + (-a)x^3 + 0\cdot x^2 + bx + c, \qquad \partial_x R = 5x^4 + 0\cdot x^3 + (-3a)x^2 + 0\cdot x + b,$$

the Sylvester matrix $\mathrm{Syl}(R, \partial_x R)$ is

$$\mathrm{Syl}(R,\partial_x R) = \begin{pmatrix}
1 & 0 & -a & 0 & b & c & 0 & 0 & 0\\
0 & 1 & 0 & -a & 0 & b & c & 0 & 0\\
0 & 0 & 1 & 0 & -a & 0 & b & c & 0\\
0 & 0 & 0 & 1 & 0 & -a & 0 & b & c\\
0 & 0 & 0 & 0 & 1 & 0 & -a & 0 & b\\
5 & 0 & -3a & 0 & b & 0 & 0 & 0 & 0\\
0 & 5 & 0 & -3a & 0 & b & 0 & 0 & 0\\
0 & 0 & 5 & 0 & -3a & 0 & b & 0 & 0\\
0 & 0 & 0 & 5 & 0 & -3a & 0 & b & 0
\end{pmatrix}, \qquad \Delta = \det(\mathrm{Syl}(R,\partial_x R)).$$

Elimination ideal and principality. In this case the elimination ideal is principal:

$$I \cap k[a, b, c] = \langle \Delta(a, b, c) \rangle.$$

Hence

$$I_{\mathrm{bd}}(R) = \langle \Delta \rangle, \qquad J(R) = \sqrt{\langle \Delta \rangle} = \langle \Delta_{\mathrm{red}} \rangle,$$

where Δ_{red} is the squarefree part (up to units). Therefore the triple-well boundary is codimension one and principal, and the canonical single-channel boundary equation is $\Delta_{\mathrm{red}} = 0$ as used in Layer (C).

E.3 How non-principality arises: a minimal contrast inside the same parameter space

The triple-well family is intentionally "single-channel" once the boundary is defined purely by degeneracy. Non-principality arises as soon as two independent boundary constraints are simultaneously operative. For instance, impose the additional structural constraint $c = 0$ (symmetry / zero-bias) together with degeneracy:

$$J_{\mathrm{two}} := \langle \Delta_{\mathrm{red}},\, c \rangle \subset k[a, b, c].$$

Unless c divides Δ_{red} (non-generic), J_{two} is not principal. Geometrically, the boundary is no longer a hypersurface but an intersection of two constraints, producing multi-constraint boundary geometry and, generically, incomparabilities in the boundary-constraint preorder (Lemma 2), thereby obstructing faithful scalar diagnostics (Theorem 4).

Takeaway. This explicit contrast supports the main message of Layers (C)–(D): "failure of scalarization" is not a calibration artifact but an ideal-theoretic boundary transition.