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Eco-Epidemiological Model and analysis of Iris Yellow Spot Disease (IYSD) Dynamics in Onion Plants with Optimal Control Strategies

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Abstracts

Iris Yellow Spot Disease (IYSD), caused by Iris Yellow Spot Virus (IYSV) and transmitted by Thrips tabaci, poses a serious threat to onion production worldwide. In this study, an eco-epidemiological model is formulated to investigate the transmission dynamics of IYSD in onion plants. The model incorporates logistic growth of the host population and a nonlinear saturated incidence rate described by a Holling type II functional response. The basic reproduction number is derived using the next-generation matrix approach, and both local and global stability analyses of the equilibria are conducted. The analysis reveals the existence of a forward bifurcation at $R_0 = 1$. Furthermore, global stability of the endemic equilibrium is established using Lyapunov functions and LaSalle's invariance principle. Numerical simulations are performed to validate the analytical results and to assess the effectiveness of optimal control strategies. The findings provide valuable insights into the mechanisms driving IYSD spread and offer guidance for developing efficient disease management strategies in onion production systems.

Key words*:* Modeling, IYSV, basic reproduction number, stability analysis, sensitivity analysis, Optimal Control Theory, Pontryagin's maximum principle.

1. INTRODUCTION

Allium cepa L., often known as onions, is an essential vegetable crop that plays a major role in both economic stability and food security worldwide. The onion is rich in minerals, vitamins,

carbs, and proteins. It has also been reported to include ascorbic acid, niacin, riboflavin, and thiamine. Onion is therefore recognized for both its therapeutic qualities and nutritional worth in the human diet (Gelaye *et al*., 2024). The Onion thrips is originated in the Mediterranean area. Originally discovered in Hawaii in 1915, it may now be found anywhere onions are planted worldwide (Zereabruk, 2017). It has mostly expanded as a food item, adding value to many cuisines and serving as cousins (Megersa, 2017). Grown throughout Africa, particularly in tropical areas, it provides farmers with a cash crop (Patiene *et al.*, 2022). The crop is highly valued in Ethiopia for its usage as a spice, in daily stews, and in a variety of vegetable-based food preparation applications. Nevertheless, in spite of all the prospects, its production status is not well-developed and structured (Megersa, 2017). The low yield and poor quality of onions produced in Ethiopia are caused by a variety of issues, including diseases, insect pests, and inadequate pre- and post-harvest management techniques (Ermias, 2018). Furthermore, the primary issue with onion harvests in Ethiopia is onion thrips, which can completely damage onion fields, particularly during dry seasons (Tadele, 2014). According to Negash *et al* (2018) onion thrips is a major vector of the Iris yellow spot virus (IYSV), which can result in 100% crop losses, and is considered a worldwide pest of onions. An important factor affecting onion yield and profitability is thrips control. Management strategies include cultural, host plant resistance, biological and chemical protection. Nonetheless, in many nations, like Ethiopia, the primary method of controlling thrips in commercial onion production is still the use of chemical pesticides.

Many studies in epidemiological modeling have been conducted using mathematical modeling to describe the dynamics of IYSV disease in onion plants. For example, Modeling and stability study of an eco-epidemiological model of IYSV disease dynamics in onion plants with nonlinear saturation incidence rate and logistic growth have been emphasized by Kawe *et al.* (2022). They computed the threshold quantity R_0 using the next-generation matrix method and carried out the stability analysis of the model equilibria. Additionally, they used the partial rank correlation coefficient and Latin Hypercube sampling methodologies to carry out the model's global sensitivity analysis. Sensitivity study revealed that R_0 is most susceptible to the thrips vectors' mortality rate. Lastly, in order to confirm his analytical conclusions, they ran numerical simulations and visually presented the results.

The other study done by, Patiene *et al.* (2022) suggested stability and Hopf bifurcation analysis of a delayed eco-epidemiological model of IYSV disease dynamics in onion plants with nonlinear saturated incidence and logistic growth. In this work, they use a set of delay differential equations to develop and evaluate a non-linear mathematical model that explores the dynamics of IYSV illness in onion plants. They use latent periods and incubation as time delays components. The local stability of the disease-free equilibrium in the presence of delays was examined using Descartes' rule of signs. Lastly, numerical simulations were run and visually shown to validate the analytical findings.

To the best of the researchers' knowledge, no epidemiological model study has been carried out so far regarding Iris Yellow Spot Disease (IYSD) of onion plants. Based on the description above, every study was used ordinary differential equation methods to split the population of onion plants into various compartments, such as Susceptible $S(t)$, Infected $I(t)$, and others, and to split the population of thrips vectors into two subclasses: Susceptible $X(t)$ and Infected $W(t)$. However, none of them took optimal control strategies into account, and no research using optimal control has been done. In order to close this gap, this inspired us to conduct this study. Consequently, motivated by Kawe *et al.* (2022) references and considering the above discussion, in this work, we were propose eco-epidemiological model and analysis of Iris Yellow Spot Disease (IYSD) dynamics in onion plants with a nonlinear saturated incidence of the Holling type II form and logistic growth by applying optimal control problem which takes into account the permissible controls in the form of using Prevention strategies, Quarantine strategy, and Chemical control on the control of Iris Yellow Spot Disease (IYSD) of onion plants.

This paper is organized as follows: In Section 2, the Mycobacterium tuberculosis model was described and formulated using fractional-order differential equations approach. In Section 3, qualitative and sensitivity analysis of the model were discussed and interpreted. In Section 4, the MATLAB software was used to investigate numerical simulation results. Finally, numerical results and discussion, conclusions and recommendations are presented in Section 5, 6 and 7 respectively.

2. Model Formulation and Description

2.1. Assumptions of the model

We make the following assumptions in formulating the model.

1. In the presence of the disease, the onion plant population is divided into two classes namely, the susceptible onion plant $S(t)$, infected onion plant $I(t)$, while the thrips vector is divided into two classes namely, susceptible thrips vector $X(t)$ and infected thrips vector $W(t)$. We don't consider the recovered individuals since there is no cure for infected plants. Hence, the total onion plant and vector population at time t are given by: $N_o(t) = S(t) + I(t)$, and $N_v(t) = X(t) + W(t)$.
2. In the absence of the disease, the susceptible onion plant population grows logistically as $rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right)$, where, $r > 0$ is the intrinsic growth rate and K the carrying capacity of onions farm.
3. A susceptible onion plant become infected from contact with an infected thrips and a susceptible thrips becomes infected from contact with an infected onion plant.
4. Both susceptible and infected onion plants are harvested at a rate g, and infected onion plants die from IYSD at a rate ρ.
5. Let δ_1 be the transmission rate from susceptible onion plants to infected thrips vectors, and δ_2 be the transmission rate from infected onion plants to susceptible thrips vectors. These transmission process is modelled using a nonlinear saturated incidence rate of the Holling type II form given by $\frac{\delta_1 SW}{1+m_1 W}, \frac{\delta_2 SW}{1+m_2 W}$ where m_1, m_2 are the saturation constants.

2.2. The model derivation

Our proposed model divides the total onion plants population $N_o(t)$ into two subclasses of susceptible $S(t)$ and infected $I(t)$. Then the total onion plants population at any time t is given by $N_o(t) = S(t) + I(t)$. The total thrips vectors population $N_v(t)$ is divided into two subclasses of susceptible $X(t)$ and infected $W(t)$. Then the total thrips vectors population at any time t is given by $N_o(t) = X(t) + W(t)$. The susceptible onion plants population are generated by the logistic growth function at a rate $rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right)$ with r being the intrinsic growth rate and K the carrying capacity of the farm. It is decreased following effective contact

with infected thrips vectors at a rate $\frac{\delta_1 S(t)W(t)}{1+m_1 W}$ where $\delta_1$represents the disease transmission rate from infected thrips vectors to susceptible onion plants and the parameter m_1 denotes saturation level of the force of infection. The growth of susceptible onion plants is further reduced by harvesting at a rate g. The population of infected onion plants is generated by the infection of susceptible onion plants at a rate $\frac{\delta_1 S(t)W(t)}{1+m_1 W}$. The population of infected onion plants is further decreased by harvesting at a rate g, and upon infection and display of symptoms, the plants die from IYSD infection at a rate ρ. The population of susceptible thrips vectors is generated by the recruitment of individuals at a rate π and is decreased after the effective contact with infected onion plants at a rate $\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)}$ where $\delta_2$denotes the disease transmission rate from infected onion plants to susceptible thrips vectors and the parameter m_2 denotes saturation level of the force of infection. This population is further decreased by natural death at a rate σ. The population of infected thrips vectors is generated by the infection of susceptible thrips vectors at a rate $\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)}$. This population is further decreased by natural death at a rate σ. Based on the above description, we have the following compartmental diagram for the transmission dynamics of IYSD.

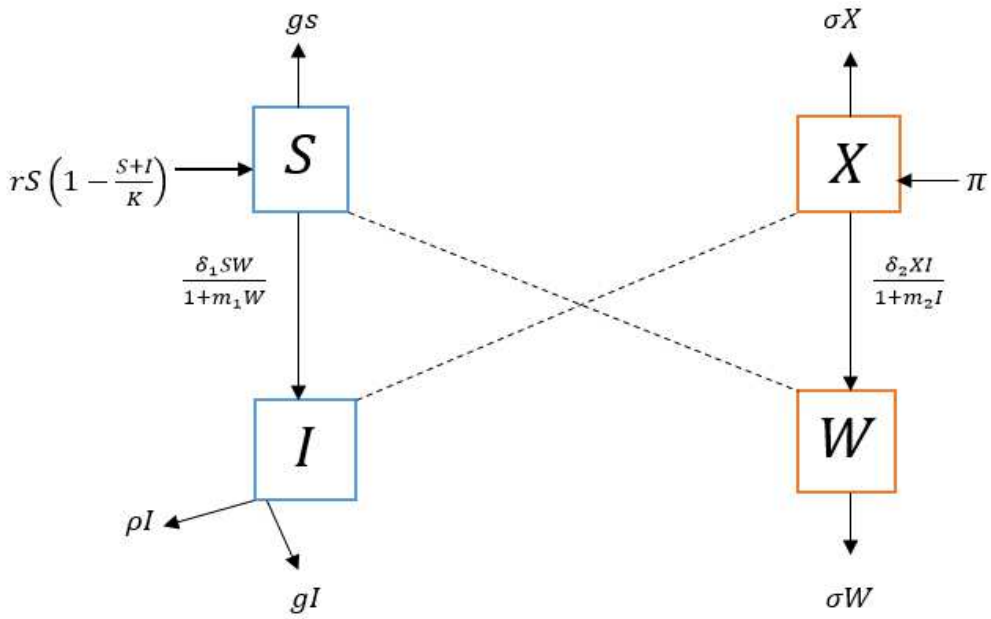


Figure 1: Compartmental flow diagram for IYSD infection. The dotted lines without arrows indicate transmission or contact. The solid lines represent transfer from one compartment to another.

Thus, based on the above assumptions, description and schematic diagram, the proposed model describing the transmission dynamics of IYSV disease can be formulated mathematically by the following deterministic system of nonlinear differential equations:

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - gS(t) \\ \frac{dI(t)}{dt} = \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - \rho I(t) - gI(t) \\ \frac{dX(t)}{dt} = \pi - \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma X(t) \\ \frac{dW(t)}{dt} = \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma W(t) \end{cases} \quad (1)$$

Where given initial conditions are $S(0) = S_0 \geq 0,\ I(0) = I_0 \geq 0, X(0) = X_0 \geq 0,\ W(0) = W_0 \geq 0$.

3. Qualitative analysis of the model

In this section, we discuss some basic qualitative behaviors of the model, such as the positivity of solution, invariant region, disease -free equilibrium point, endemic equilibrium point, basic reproduction number, global stability of disease -free equilibrium point and global stability of endemic equilibrium point as well as sensitivity analysis of the model were discussed as follows.

3.1. Positivity solution

In this subsection, we proved all solutions of the models (4.1) remain positive for future time.

Theorem 1: For all $t > 0$, the solution of the system equation of model (1) are non-negative, with their respective initial values are positive.

Proof: To prove this theorem let us take first equation of system equation of model (1).

$$\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - gS(t)$$

Let $f(S(t), I(t), W(t)) = r\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\delta_1 W(t)}{1+m_1 W}$, then the above equation becomes

$$\frac{dS(t)}{dt} = S(t)[f(S(t), I(t), W(t)) - g]. \quad (2)$$

Let us we apply separation of variables method in equation (2) we get

$$\frac{dS(t)}{S} = [f(S(t), I(t), W(t)) - g]dt. \quad (3)$$

Then take integration both sides of equation (3) we get

$$\int_0^t \frac{dS(t)}{S} = \int_0^t [f(S(t), I(t), W(t)) - g]\mathrm{dt}$$

$$\ln \frac{S(t)}{S(0)} = \int_0^t [f(S(t), I(t), W(t)) - g]\mathrm{dt}$$

$$\frac{S(t)}{S(0)} = exp\left(\int_0^t [f(S(t), I(t), W(t)) - g]\mathrm{dt}\right)$$

$$S(t) = S(0)exp\left(\int_0^t \left[f\left(S(t), I(t), W(t)\right) - g\right]\mathrm{dt}\right) > 0$$

Since $S(0) = S_0 \geq 0$, then $S(t) > 0$ for all $t \geq 0$.

In the same process, the positivity of other compartments is demonstrated, leading to the following solutions: $I(t) \geq I(0)e^{-(\rho+g)\mathrm{t}}, X(t) \geq X(0)e^{-\sigma\mathrm{t}}$, and $W(t) \geq W(0)e^{-\mu_T\mathrm{t}}$. Since the value of the exponential function is positive, then $S(t) > 0, I(t) > 0, X(t) > 0$, and $W(t) > 0$. Therefore, all the solution sets are positive for $t > 0$. So the model of the system (1) has appropriate biological meaning.

3.2. Invariant Region

In this subsection, we discussed the invariant region, in which the model solution is bounded.

Theorem 2: The solution of the system equations of model (1) are enters and bounded in the feasible set $\Omega = \{(S, I, X, W) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2\}$.

Proof: For this model, the overall onion plants population size at time t is given by

$$N_o = S(t) + I(t). \quad (4)$$

Then, differentiating both side of equation (4) with respect to t gives

$$\frac{dN_o}{dt} = \frac{dS(t)}{dt} + \frac{dI(t)}{dt}. \qquad (5)$$

After substituting system equation of (1) to (5) we obtain

$$\frac{dN_o}{dt} = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - gS(t) - \rho I(t) - gI(t)$$

Since $N_o = S(t) + I(t)$ and $S(t) \leq N_o(t)$, we get

$$\frac{dN_o}{dt} \leq rN_o(t)\left(1 - \frac{N_o(t)}{K}\right) - gN_o(t) \qquad (6)$$

Solving equation (6) by separation of variable principle and using the standard comparison principle, we obtain

$$\lim_{t\to\infty} sup\, N_o(t) \leq \frac{K(r-g)}{r}, \text{ when } r > g.$$

Therefore $0 \leq N_o(t) \leq \frac{K(r-g)}{r}$, when $r > g$. Define

$$\Omega_o = \left\{S(t), I(t) \in \mathbb{R}^2_+ : 0 \leq S(t) + I(t) \leq \frac{K(r-g)}{r}\right\}.$$

Similarly, let $N_v = X(t) + W(t)$ be the total population of thrips vectors. Then the rate at which the total population of thrips vectors changes is given by

$$\frac{dN_v}{dt} = \frac{dS(t)}{dt} + \frac{dI(t)}{dt} \leq \pi - \sigma N_v(t).$$

By separating the variable and integrating, we get $N_v(t) \leq \frac{\pi}{\sigma} + \left(N_v(0) - \frac{\pi}{\sigma}\right)e^{-\sigma t}$, with $\sigma \neq 0$. Using the standard comparison principle, we have

$$\lim_{t\to\infty} sup\, N_v(t) \leq \frac{\pi}{\sigma}.$$

Therefore $0 \leq N_v(t) \leq \frac{\pi}{\sigma}$. Define

$$\Omega_v = \left\{X(t), W(t) \in \mathbb{R}^2_+ : 0 \leq X(t) + W(t) \leq \frac{\pi}{\sigma}\right\}.$$

Therefore, the feasible solution set of the IYSD model (1) remain in the following region

$$\Omega = \Omega_v \times \Omega_v = \left\{(S, I, X, W) \in \mathbb{R}_+^4 : 0 \leq S(t) + I(t) \leq \frac{K(r-g)}{r}, 0 \leq X(t) + W(t) \leq \frac{\pi}{\sigma}\right\}.$$

Thus, all solutions of equation of model (5.1) that initially in Ω remains in Ω for all $t \geq 0$.

3.3. Disease-Free Equilibrium points

The disease-free equilibrium point (DFE) is a steady state solution where there is no IYSD in onion plants population and no IYSV in the thrips vectors population. Hence it is determined by setting right hand side equation of the model (1) equal to zero.

$$\begin{aligned} &rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - gS(t) = 0 \\ &\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - \rho I(t) - gI(t) = 0 \\ &\pi - \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma X(t) = 0 \\ &\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma W(t) = 0 \end{aligned} \tag{7}$$

Assume that there is no IYSD in onion plants population and no IYSV in the thrips vectors population that is $I(t) = W(t) = 0$. Then, from equation (7) we obtain

$$E_0 = \left(\frac{K(r-g)}{r}, 0, \frac{\pi}{\sigma}, 0\right). \tag{8}$$

3.4. Basic reproduction number

In this section, we discuss basic reproduction number (R_0) which is defined as the average number of secondary infections arising from one newly infected individual introduced into a healthy population. The next generation matrix method was used to compute basic reproduction number by considering the generation of infected compartments (Martcheva, 2015).

The infected compartments are:

$$\begin{cases} \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - \rho I(t) - gI(t), \\ \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma W(t). \end{cases} \tag{9}$$

Then by the principle of next-generation matrix, the Jacobian matrices of the rate of new infections f and the rate of transfer in and out of the infected classes v are obtained as follows.

So, from the equation of (9), we have

$$f = \begin{bmatrix} \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} \\ \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} \end{bmatrix}, \tag{10}$$

$$v = \begin{bmatrix} (\rho + g)I(t) \\ \sigma W(t) \end{bmatrix}. \tag{11}$$

Next we determine the Jacobian matrix of f and v with respect to $I(t)$ and $W(t)$ at disease-free equilibrium point (DFEP) $E_0 = \left(\frac{K(r-g)}{r}, 0, \frac{\beta}{\sigma}, 0\right)$ as follows.

The Jacobian matrices of f and v at disease-free equilibrium point (E_0)become F and V respectively, then

$$F = \begin{bmatrix} \frac{\partial F_1}{\partial I} & \frac{\partial F_1}{\partial W} \\ \frac{\partial F_2}{\partial I} & \frac{\partial F_2}{\partial W} \end{bmatrix} = \begin{bmatrix} \frac{\partial\left(\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)}\right)}{\partial I} & \frac{\partial\left(\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)}\right)}{\partial W} \\ \frac{\partial\left(\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)}\right)}{\partial I} & \frac{\partial\left(\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)}\right)}{\partial W} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\delta_1 K(r-g)}{r} \\ \frac{\delta_2 \pi}{\sigma} & 0 \end{bmatrix}. \tag{12}$$

And

$$V = \begin{bmatrix} \frac{\partial V_1}{\partial I} & \frac{\partial V_1}{\partial W} \\ \frac{\partial V_2}{\partial I} & \frac{\partial V_2}{\partial W} \end{bmatrix} = \begin{bmatrix} \frac{\partial((\rho+g)I(t))}{\partial I} & \frac{\partial((\rho+g)I(t))}{\partial W} \\ \frac{\partial(\sigma W(t))}{\partial I} & \frac{\partial(\sigma W(t))}{\partial W} \end{bmatrix}.$$

Then evaluate Jacobian matrices of V at disease -free equilibrium point gives

$$V = \begin{bmatrix} \rho + g & 0 \\ 0 & \sigma \end{bmatrix}.$$

Next we find inverse of V as follows:

$$V^{-1} = \frac{adjV}{\det(V)} = \frac{\begin{bmatrix} \sigma & 0 \\ 0 & \rho+g \end{bmatrix}}{\sigma(\rho+g)} = \begin{bmatrix} \frac{1}{\rho+g} & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix}. \tag{13}$$

Then product of F and V^{-1} become

$$FV^{-1} = \begin{bmatrix} 0 & \frac{\delta_1 K(r-g)}{r} \\ \frac{\delta_2 \pi}{\sigma} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\rho+g} & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\delta_1 K(r-g)}{\sigma r} \\ \frac{\delta_2 \pi}{\sigma(\rho+g)} & 0 \end{bmatrix}. \quad (14)$$

Then from (14) we obtain eigenvalue using $\det(FV^{-1} - \lambda I_n)$, where I_n is an identity matrix.

That is $\begin{bmatrix} -\lambda & \frac{\delta_1 K(r-g)}{\sigma r} \\ \frac{\delta_2 \pi}{\sigma(\rho+g)} & -\lambda \end{bmatrix} = 0.$ (15)

Then after multiply and rearrange from (15), we obtain the following characteristics equation.

$$\lambda^2 - \left(\frac{\delta_1 K(r-g)}{\sigma r}\right)\left(\frac{\delta_2 \pi}{\sigma(\rho+g)}\right) = 0. \quad (16)$$

Hence, from characteristics equation of (16) we obtain $\lambda_1 = -\sqrt{\frac{\delta_1 \delta_1 \pi K(r-g)}{r\sigma^2(\rho+g)}}$, and $\lambda_2 = \sqrt{\frac{\delta_1 \delta_1 \pi K(r-g)}{r\sigma^2(\rho+g)}}$.

Now using the principle of next-generation matrix the largest eigenvalue is the basic reproduction number (R_0). Therefore, our basic reproduction numbers for model equations (1) is $R_0 = \sqrt{\frac{\delta_1 \delta_2 \pi K(r-g)}{r\sigma^2(\rho+g)}}$. (17)

The square roots found in R_0 means that two generations are required for transmission of IYSD in onion farms. That is from infected onion plants to healthy thrips vectors and then from infected thrips vectors to healthy onion plants. Choosing $R_{0h} = \sqrt{\frac{\delta_1 K(r-g)}{r(\rho+g)}}$ and $R_{0v} = \sqrt{\frac{\delta_2 \pi}{\sigma^2}}$, equation (17) can be written as $R_0 = \sqrt{R_{0h} \times R_{0v}}$. Here, R_{0h} describes the number of onion plants infected by one infectious thrips vector over its infection period, and R_{0v} is the number of thrips vectors infected by one infectious onion plant during the period of infectiousness.

3.5. Local stability of Disease -Free Equilibrium Point

In this subsection, we discuss stability analysis of the disease -free equilibrium point of the model. Thus, stability analysis of the disease-free equilibrium point of the system equation of model is determined using linearizing the non-linear system of equations (Martcheva, 2015).

Theorem 3: When $r > g$, the disease-free equilibrium point of system (1) is locally asymptotically stable (LAS) if and only if $R_0 < 1$ and unstable (US) if $R_0 > 1$.

Proof: To prove stability analysis of the disease -free equilibrium point of the model, we computed the Jacobian matrix of the system equation of model (1) at the disease-free equilibrium (IFE) as follows.

The Jacobian matrix of the system (1) is

$$J = \begin{bmatrix} \frac{\partial L_1}{\partial S} & \frac{\partial L_1}{\partial I} & \frac{\partial L_1}{\partial X} & \frac{\partial L_1}{\partial W} \\ \frac{\partial L_2}{\partial S} & \frac{\partial L_2}{\partial I} & \frac{\partial L_2}{\partial X} & \frac{\partial L_2}{\partial W} \\ \frac{\partial L_3}{\partial S} & \frac{\partial L_3}{\partial I} & \frac{\partial L_3 S}{\partial X} & \frac{\partial L_3}{\partial W} \\ \frac{\partial L_4}{\partial S} & \frac{\partial L_4}{\partial I} & \frac{\partial L_4}{\partial X} & \frac{\partial L_4}{\partial W} \end{bmatrix} \qquad (18)$$

Where

$$L_1 = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - gS(t),\ L_2 = \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - \rho I(t) - gI(t),$$

$$L_3 = \pi - \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma X(t),\ L_4 = \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma W(t).$$

Then

$$J = \begin{bmatrix} r - g - \left(\frac{2rS+rI}{K}\right) - \frac{\delta_1 W}{1+m_1 W} & -\frac{rS}{K} & 0 & -\frac{\delta_1 S}{(1+m_1 W)^2} \\ \frac{\delta_1 W}{1+m_1 W} & -(\rho+g) & 0 & \frac{\delta_1 S}{(1+m_1 W)^2} \\ 0 & -\frac{\delta_2 X}{(1+m_2 I)^2} & -\frac{\delta_2 I}{1+m_2 I} - \sigma & 0 \\ 0 & \frac{\delta_2 X}{(1+m_2 I)^2} & \frac{\delta_2 I}{1+m_2 I} & \sigma \end{bmatrix}.$$

By evaluate Jacobian matrix at the disease-free equilibrium point IFE we obtain

$$J_{DFE} = \begin{bmatrix} -(r-g) & -(r-g) & 0 & -\delta_1 k\left(\frac{r-g}{r}\right) \\ 0 & -(\rho+g) & 0 & \delta_1 k\left(\frac{r-g}{r}\right) \\ 0 & -\frac{\delta_2 \pi}{\sigma} & -\sigma & 0 \\ 0 & \frac{\delta_2 \pi}{\sigma} & 0 & -\sigma \end{bmatrix} \tag{19}$$

Then, we obtain characteristic equation from (19) using $\det(J_{IFE} - \lambda I_n) = 0$, where I_n is an identity matrix.

$$\begin{bmatrix} -(r-g)-\lambda & -(r-g) & 0 & -\delta_1 k\left(\frac{r-g}{r}\right) \\ 0 & -(\rho+g)-\lambda & 0 & \delta_1 k\left(\frac{r-g}{r}\right) \\ 0 & -\frac{\delta_2 \pi}{\sigma} & -\sigma-\lambda & 0 \\ 0 & \frac{\delta_2 \pi}{\sigma} & 0 & -\sigma-\lambda \end{bmatrix} = 0. \tag{20}$$

$$\Rightarrow \big((r-g)+\lambda\big)(\sigma+\lambda)\begin{vmatrix} -(\rho+g)-\lambda & \delta_1 k\left(\frac{r-g}{r}\right) \\ \frac{\delta_2 \pi}{\sigma} & -\sigma-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow \big((r-g)+\lambda\big)(\sigma+\lambda)\left[\big((\rho+g)+\lambda\big)(\sigma+\lambda)+\sigma(\rho+g)-\frac{\delta_1\delta_2 k\pi}{\sigma}\left(\frac{r-g}{r}\right)\right] = 0.$$

Then after simplify and rearranging we obtain

$$\big((r-g)+\lambda\big)(\sigma+\lambda)\left[\lambda^2+\lambda(\rho+g+\sigma)+\sigma(\rho+g)-\frac{\delta_1\delta_2 k\pi}{\sigma}\left(\frac{r-g}{r}\right)\right] = 0.$$

$$\Rightarrow \begin{cases} (r-g)+\lambda = 0, \\ \sigma+\lambda = 0, \\ \lambda^2+\lambda(\rho+g+\sigma)+\sigma(\rho+g)-\frac{\delta_1\delta_2 k\pi}{\sigma}\left(\frac{r-g}{r}\right) \end{cases} \tag{21}$$

From the first two equations in (21), we have

$(r-g)+\lambda = 0 \Rightarrow \lambda = -(r-g) \leq 0,$

$\sigma+\lambda = 0 \Rightarrow \lambda = -\sigma \leq 0$, or

From the last equations in (21), we have

$$\lambda^2 + b_1\lambda + b_2 = 0. \tag{22}$$

Where, $b_1 = \rho + g + \sigma$ and $b_2 = \sigma(\rho + g) - \frac{\delta_1 \delta_2 k\pi}{\sigma}\left(\frac{r-g}{r}\right)$.

Since roots of $\lambda_1 = -(r - g), \lambda_3 = -\sigma$ are negative real parts but roots of λ_2 and λ_4 are not known. So that to determine the root of the rest Eigen value from characteristics equation (22) by applying Rough–Hurwitz stability criterion. Therefore according to Rough–Hurwitz stability criterion of the polynomial function degree two of characteristic equations of (22) have strictly negative real roots if and only if both coefficients are non-negative (Getachew et *al*, 2022).

Then

$b_1 \geq 0 \Longrightarrow \rho + g + \sigma \geq 0$, since it is addition of positive parameter.

$b_2 \geq 0 \Longrightarrow \sigma(\rho + g) - \frac{\delta_1 \delta_2 k\pi}{\sigma}\left(\frac{r-g}{r}\right) \geq 0$

$\Longrightarrow \sigma(\rho + g) \geq \frac{\delta_1 \delta_2 k\pi}{\sigma}\left(\frac{r-g}{r}\right)$

$\Longrightarrow \frac{\delta_1 \delta_2 k\pi}{\sigma}\left(\frac{r-g}{r}\right) < \sigma(\rho + g)$

$\Longrightarrow \frac{\delta_1 \delta_2 \pi K(r-g)}{r\sigma^2(\rho+g)} < 1$

$\Longrightarrow \sqrt{\frac{\delta_1 \delta_2 \pi K(r-g)}{r\sigma^2(\rho+g)}} < 1$

$\Longrightarrow R_0 < 1.$

Therefore, infection-free equilibrium point E_0 is locally asymptotically stable if $R_0 < 0$. Consequently, when $r > g$ the disease-free equilibrium point E_0 is locally asymptotically stable whenever $R_0 < 1$.

3.6. Global Stability of the Disease-Free Equilibrium Point

In this section, we investigate the global stability of the disease -free equilibrium R_0 using the theorem by Carlos and Baojun (2004) as done in the paper (Tonny et al., 2015). To do so, we write system equation (1) as

$$\begin{cases} \frac{dU}{dt} = F(U,V) \\ \frac{dU}{dt} = G(U,V), G(U,0) = 0 \end{cases} \tag{23}$$

Where $U = (S(t), X(t)) \in R^2$ denotes the healthy population and $V = (I(t), W(t)) \in R^2$ represents the infected population. Let $E_0 = (U^*, 0)$ denote the disease-free equilibrium of the system. Furthermore, we list two conditions and if met will guarantee the global asymptotic stability of E_0.

i. For $\frac{dU}{dt} = F(U,0)$, U^* is globally asymptotically stable;

ii. $\frac{dV}{dt} = D_V G(U^*, 0)V - \hat{G}(U,V)$, $\hat{G}(U,V) \geq 0$ for all $U, V \in \Omega$

Where $D_V G(U^*, 0)$ a Metzler and the Jacobian matrix of is $G(U,V)$ take in $(I(t), W(t))$ and evaluated at $E_0 = (U^*, 0) = \left(\frac{K(r-g)}{r}, 0, \frac{\pi}{\sigma}, 0\right)$.

Theorem 4 When $r > g$, the fixed point $E_0 = (U^*, 0)$ is a globally asymptotic stable -free equilibrium of model equation (1) provided that $R_0 < 1$ and that the condition (i) and (ii) are satisfied.

Proof. From the model system (4.1) we obtain $F(U,V)$ and $G(U,V)$ as:

$$F(U,V) = \begin{bmatrix} rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\delta_1 S(t) W(t)}{1+m_1 W(t)} - gS(t) \\ \pi - \frac{\delta_2 X(t) I(t)}{1+m_2 I(t)} - \sigma X(t) \end{bmatrix} \tag{24}$$

$$G(U,V) = \begin{bmatrix} \frac{\delta_1 S(t) W(t)}{1+m_1 W(t)} - \rho I(t) - gI(t) \\ \frac{\delta_2 X(t) I(t)}{1+m_2 I(t)} - \sigma W(t) \end{bmatrix} \tag{25}$$

From condition (i), we consider the reduced system $\frac{dU}{dt} = F(U,0)$ and we get:

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - gS(t) \\ \frac{dS(t)}{dt} = \pi - \sigma X(t) \end{cases} \tag{26}$$

$U^* = \left(\frac{K(r-g)}{r}, \frac{\pi}{\sigma}\right)$ is globally asymptotically stable equilibrium point for the reduced System $\frac{dU}{dt} = F(U, 0)$. We proved this by finding the solution of the equations in the system (4.12). From the first equation, we have

$$\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - gS(t)$$

By separation of variable and integrating factor technique, we get

$$S(t) = \frac{K(r-g)e^{r(t+C)}}{re^{r(t+C)} - e^{g(t+C)}}, \text{ (where C is a constant of integration)}$$

Which approaches $\frac{K(r-g)}{r}$ as $t \to \infty$. Similarly, from the second equation we obtain $X(t) = \frac{\pi}{\sigma} + \left(X(0) - \frac{\pi}{\sigma}\right)e^{-\sigma t}$, which approaches $\frac{\pi}{\sigma}$as $t \to \infty$. This asymptotic dynamics is independent of the initial condition in the domain Γ; hence the convergence of the solution of the reduced system (4.12) is global in the domain Γ. Moreover, the Jacobian matrix of $G(U, V)$ taken in $(I(t), W(t))$ and evaluate at $E_0(U^*, 0)$ is found to be

$$D_V G(U^*, 0) = \begin{bmatrix} -r - g & \delta_1 K\left(\frac{r-g}{r}\right) \\ \frac{\delta_2 \pi}{\sigma} & -\sigma \end{bmatrix}$$

Therefore, from the formula in condition (ii), we get the following expression

$$\hat{G}(U, V) = D_V G(U^*, 0)V - G(U, V),$$

and we want to show that $\hat{G}(U, V) \geq 0$, which is obtained as:

$$\hat{G}(U, V) = \begin{bmatrix} \delta_1 W(t)\left(K\left(\frac{r-g}{r}\right) - \frac{S(t)}{1+m_1 W(t)}\right) \\ \delta_2 I(t)\left(\frac{\pi}{\sigma} - \frac{X(t)}{1+m_1 I(t)}\right) \end{bmatrix}$$

From the invariant region $\Omega = \Omega_o \times \Omega_v$, since $r > g$, we have that $K\left(\frac{r-g}{r}\right) \geq S(t)$ and $\frac{\pi}{\sigma} \geq X(t)$. Therefore, $\hat{G}(U, V) \geq 0$ for all $(U, V) \in \Omega$, and we conclude that the DFE is globally asymptotically stable whenever $R_0 < 1$.

3.7. Existence of Endemic Equilibrium point

In this section, we explore the existence of the endemic equilibrium point E^*. It is a positive steady solution where the IYSD persist in the population of onion plants. The endemic equilibrium $E^* = (S^*, I^*, X^*, W^*)$ of the system (4.1) is the solution of the algebraic equation

$$\begin{aligned} &rS(t)\left(1-\frac{S(t)+I(t)}{K}\right)-\frac{\delta_1 S(t)W(t)}{1+m_1W(t)}-gS(t)=0\\ &\frac{\delta_1 S(t)W(t)}{1+m_1W(t)}-\rho I(t)-gI(t)=0\\ &\pi-\frac{\delta_2 X(t)I(t)}{1+m_2I(t)}-\sigma X(t)=0\\ &\frac{\delta_2 X(t)I(t)}{1+m_2I(t)}-\sigma W(t)=0 \end{aligned} \tag{27}$$

From the third equation of system (27), we isolate X^* as;

$$X^* = \frac{\pi(1+m_2I^*)}{\sigma+I^*(\delta_2+\sigma m_2)} \tag{28}$$

From the fourth equation of system (28), we obtain W^*as;

$$W^* = \frac{\pi\delta_2 I^*}{\sigma^2+I^*(\sigma\delta_2+\sigma^2 m_2)} \tag{29}$$

By substituting (29) in to the second equation of system (27), we obtained;

$$S^* = \frac{(\rho+g)\sigma^2+(\rho+g)(\sigma\delta_2+\sigma^2m_2+\pi\delta_2m_1)I^*}{\pi\delta_1\delta_2}. \tag{30}$$

Hence, we summarize $X^*,\ W^*,\ S^*$ as

$$\begin{cases} X^* = \frac{\pi(1+m_2I^*)}{\sigma+I^*(\delta_2+\sigma m_2)},\\ W^* = \frac{\pi\delta_2 I^*}{\sigma^2+I^*(\sigma\delta_2+\sigma^2 m_2)},\\ S^* = \frac{(\rho+g)\sigma^2+(\rho+g)(\sigma\delta_2+\sigma^2m_2+\pi\delta_2m_1)I^*}{\pi\delta_1\delta_2} \end{cases} \tag{31}$$

And I^* is the positive root of the following equation

$$H(I^*) = b_2{I^*}^2 + b_1I^* + b_0 = 0 \tag{32}$$

With

$b_2 = -r(\sigma\delta_2 + \sigma^2 m_2 + \pi\delta_2 m_1)[(\rho + g)(\sigma\delta_2 + \sigma^2 m_2 + \pi\delta_2 m_1) + \pi\delta_1\delta_2]$,

$b_1 = -r\sigma^2[2(\rho + g)(\sigma\delta_2 + \sigma^2 m_2 + \pi\delta_2 m_1) + \pi\delta_1\delta_2] + \pi\delta_1\delta_2 K[(r - g)(\sigma\delta_2 + \sigma^2 m_2 + \pi\delta_2 m_1) - \pi\delta_1\delta_2]$,

$b_0 = -r(\rho + g)\sigma^4\left[1 - {R_0}^2\right]$.

3.8. Local Stability of the Endemic Equilibrium Point

This section explores the local stability of the endemic equilibrium point E^*. We can see from (4.39) that the endemic equilibrium point has long expressions and the standard linearization method which consist of finding the eigenvalues of the Jacobian matrix around the endemic equilibrium point can be mathematically complicated. Hence, in order to investigate the local asymptotic stability of the endemic equilibrium E^*, we use the result based on the center manifold theory described in Carlos and Baojun (2004) to investigate if the model system (4.1) exhibits a forward or backward bifurcation when $R_0 = 1$. When the bifurcation is forward, it implies that the endemic equilibrium point is locally asymptotically stable for $R_0 > 1$. This result is reproduced here for convenience.

Theorem 5 Center manifold theory Carlos and Baojun (2004).

Proof: Consider the following general system of ordinary differential equations with a parameter ϕ.

$$\frac{dx}{dt} = f(x, \phi) f: R^n \times R \to R \; and \; f \in C^n(R\hat{}n \times R). \tag{33}$$

Without loss of generality, it is assumed that 0 is an equilibrium for system (33) for all values of the parameter ϕ, (that is $f\ (0;\ \phi)\ \equiv\ 0$). Assume

1. $A\ =\ D_x\, f(0,0) = \left(\frac{\partial f_i}{\partial x_i}(0,0)\right)$ is the linearized matrix of system (33) around the equilibrium 0 with ϕ evaluated at 0. Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real parts;
2. Matrix A has a non-negative right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue. Let f_k be the k^{th} component of f and

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0) \tag{34}$$

$$b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0) \tag{35}$$

The local dynamics of the system (33) around the equilibrium point 0 are totally determined by the signs of a and b.

i. $a > 0, b > 0$ when $\phi < 0$ with $|\phi| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium, when $0 < \phi \ll 1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium.

ii. $a < 0, b < 0$, when $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable; when $0 < \phi \ll 1$, asymptotically stable, and there exits a positive unstable equilibrium.

iii. $a > 0, b < 0$, when $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable; and there exists a locally asymptotically stable negative equilibrium, when $0 < \phi \ll 1$, 0 is stable and a positive equilibrium appears.

iv. $a < 0, b > 0$, when $\phi < 0$ changes from negative to positive, 0 changes its stability from stable to unstable. Corresponding to a negative unstable equilibrium becomes positive and locally asymptotically stable.

Particularly, if $a < 0$ and $b > 0$, then the bifurcation is forward; if $a > 0$ and $b > 0$, then the bifurcation is backward.

To apply the above theorem, we make the following simplification and change of variables in the system (4.1). Let $S = y_1, I = y_2, X = y_3, W = y_4$. Further by introducing the vector notation $\boldsymbol{y} = (y_1, y_2, y_3, y_4)^T$, system (4.1) has the form $\frac{dy}{dt} = F(\boldsymbol{y})$, where $F = (f_1, f_1, f_1, f_1)^T$, as follows

$$\begin{cases} \frac{dy_1}{dt} = f_1 = r y_1 \left(1 - \frac{y_1 + y_2}{K}\right) - \frac{\delta_1 y_1 y_4}{1 + m_1 y_4} - g y_1 \\ \frac{dy_2}{dt} = f_2 = \frac{\delta_1 y_1 y_4}{1 + m_1 y_4} - \rho y_2 - g y_2 \\ \frac{dy_3}{dt} = f_3 = \pi - \frac{\delta_2 y_2 y_3}{1 + m_2 y_2} - \sigma y_3 \\ \frac{dy_4}{dt} = f_4 = \frac{\delta_2 y_2 y_3}{1 + m_2 y_2} - \sigma y_4 \end{cases} \tag{36}$$

The above theorem is used to determine whether the model (2.1) exhibits a forward or backward bifurcation at $R_0 = 1$. We set the transmission rate δ1 as the bifurcation parameter. Solving for δ1 the equation $R_0 = 1$ gives $\delta_1 = \delta_1^*$ as follows

$$\delta_1 = \delta_1^* = \frac{r\sigma^2(\rho+g)}{\delta_2\pi K(r-g)} \tag{37}$$

Linearization of the system (37) at the disease free equilibrium point $E_0 = \left(\frac{K(r-g)}{r}, 0, \frac{\beta}{\sigma}, 0\right)$ with $\delta_1 = \delta_1^*$ is

$$J(E_0) = \begin{bmatrix} -(r-g) & -(r-g) & 0 & -\delta_1 k\left(\frac{r-g}{r}\right) \\ 0 & -(\rho+g) & 0 & \delta_1 k\left(\frac{r-g}{r}\right) \\ 0 & -\frac{\delta_2\pi}{\sigma} & -\sigma & 0 \\ 0 & \frac{\delta_2\pi}{\sigma} & 0 & -\sigma \end{bmatrix} \tag{38}$$

The above matrix $J(E_0)$ has a simple zero eigenvalue. Moreover, let $\boldsymbol{v} = (v_1, v_2, v_3, v_4)^T$ be the right eigenvector of (38) associated with the simple zero eigenvalue. Then **v** is obtained by solving $J(E_0)\boldsymbol{v} = 0$. That is

$$\begin{cases} -v_1(r-g) - v_1(r-g) - v_4\delta_1 k\left(\frac{r-g}{r}\right) = 0 \\ -v_2(\rho+g) - v_4\delta_1 k\left(\frac{r-g}{r}\right) = 0 \\ -v_2\frac{\delta_2\pi}{\sigma} - v_3\sigma = 0 \\ v_2\frac{\delta_2\pi}{\sigma} - v_4\sigma = 0 \end{cases} \tag{39}$$

Solving (39) by direct calculation, we get:

$$\boldsymbol{v} = \left(-\frac{(r+\rho)}{r-g}v_2, v_2, -\frac{\delta_2\pi}{\sigma^2}v_2, \frac{\delta_2\pi}{\sigma^2}v_2\right)^T \tag{40}$$

where $v_2 = v_2 > 0$. Next, let $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$ be the left eigenvector of (40) associated with the simple zero eigenvalue. It satisfies $\boldsymbol{z}.\boldsymbol{v} = 1$ and the matrix $J(E0)$ should be transposed so that $J^T(E0)\,\boldsymbol{z} = 0$. That is

$$\begin{cases} -z_1(r-g)=0 \\ -z_1(r-g)-z_2(\rho+g)-\ z_3\frac{\delta_2\pi}{\sigma}+z_4\frac{\delta_2\pi}{\sigma}=0 \\ -z_3\sigma=0 \\ -z_1\left(\frac{\sigma^2(\rho+g)}{\delta_2\pi}\right)+z_2\left(\frac{\sigma^2(\rho+g)}{\delta_2\pi}\right)=0 \end{cases} \quad (41)$$

Solving (41), the left eigenvector that satisfies $\boldsymbol{z}.\boldsymbol{v}\ =\ 1$ is given by

$$\boldsymbol{z}\ =\ (0, z_2, 0, \frac{\sigma(\rho+g)}{\delta_2\pi}z_2)^T \quad (42)$$

Equation $\boldsymbol{z}.\boldsymbol{v}\ =\ 1$ results in $z_2v_2 + z_4v_4 = 1$. That is

$$z_2v_2 + \frac{(\rho+g)}{\sigma}\ z_2v_2 = 1 \quad (43)$$

Assume $v2\ =\ \sigma$, we get $z2\ = \frac{1}{\sigma+\rho+g}$ and then, the right and left eigenvectors turn out to be:

$$\boldsymbol{v}\ = \left(\frac{-\sigma(r+\rho)}{r-g}, \sigma, -\frac{\delta_2\pi}{\sigma}, \frac{\delta_2\pi}{\sigma}\right)^T, \boldsymbol{z}\ = \left(0, \frac{1}{\sigma+\rho+g}, 0, \frac{\sigma(\rho+g)}{\delta_2\pi(\sigma+\rho+g)}\right)^T \quad (44)$$

Based on the above theorem, we have to compute **a** and **b** where

$a = \sum_{k,i,j=1}^{4} z_k v_i v_j \frac{\partial^2 f_k}{\partial y_i \partial y_j}(E_0, \delta_1^*)$ and $b = \sum_{k,i=1}^{n} z_k v_i \frac{\partial^2 f_k}{\partial y_i \partial \delta_1^*}(E_0, \delta_1^*)$

Computation of a: we will only consider when $k\ =\ 2$ and $k\ =\ 4$ since $z_1 = z_3 = 0$ for $k = 1, .., 4$. That is the functions

$$f_2 = \frac{\delta_1 y_1 y_4}{1+m_1 y_4} - \rho y_2 - g y_2,\ f_4 = \frac{\delta_2 y_2 y_3}{1+m_2 y_2} - \sigma y_4 \quad (45)$$

will be used to determine **a**. The associated non-zero partial derivatives of f_2 and f_4 evaluated at (E_0, δ_1^*) which we need to compute **a** are given by

$$\begin{aligned}
&\frac{\partial^2 f_2}{\partial y_1 \partial y_4}(E_0, \delta_1^*) = \frac{\partial^2 f_2}{\partial y_4 \partial y_1}(E_0, \delta_1^*) = \frac{r\sigma^2(\rho+g)}{\delta_2 \pi K(r+g)},\\
&\frac{\partial^2 f_2}{\partial y_4^2}(E_0, \delta_1^*) = \frac{-2m_1\sigma^2(\rho+g)}{\delta_2 \pi K(r+g)},\\
&\frac{\partial^2 f_4}{\partial y_2^2}(E_0, \delta_1^*) = \frac{-2m_2\delta_2\pi}{\sigma},\\
&\frac{\partial^2 f_4}{\partial y_2 \partial y_3}(E_0, \delta_1^*) = \frac{\partial^2 f_4}{\partial y_3 \partial y_2}(E_0, \delta_1^*) = \delta_2.
\end{aligned} \tag{46}$$

Furthermore,

$$\frac{\partial^2 f_2}{\partial y_4 \partial \delta_1^*}(E_0, \delta_1^*) = \frac{K(r-g)}{r} \tag{47}$$

and all the other second order partial derivatives are zero. Substituting the eigenvectors (43) and the above partial derivatives into **a** and **b**, we then have

$$a = 2z_2 v_1 v_4 \frac{\partial^2 f_2}{\partial y_1 \partial y_4}(E_0, \delta_1^*) + z_2 v_4^2 \frac{\partial^2 f_2}{\partial y_4^2}(E_0, \delta_1^*) + z_4 v_2^2 \frac{\partial^2 f_4}{\partial y_2^2}(E_0, \delta_1^*) +$$

$$2z_4 v_2 v_3 \frac{\partial^2 f_4}{\partial y_2 \partial y_3}(E_0, \delta_1^*),$$

$$a = 2\left(\frac{\sigma(\rho + g)}{\delta_2 \pi(\sigma + \rho + g)}\right)\left[\frac{r\delta_2\pi(r+\rho)}{K(r-g)^2} + \frac{m_1\delta_2^2\pi^2}{\sigma} + m_2\delta_2\pi\sigma + \delta_2^2\pi\right] < 0.$$

(4.55)

And

$$b = z_2 v_4 \frac{\partial^2 f_2}{\partial y_4 \partial \delta_1^*}(E_0, \delta_1^*) = \frac{\delta_2 \pi K(r-g)}{r\sigma(\sigma + \rho + g)} \tag{48}$$

Therefore, **a** < 0 and **b** > 0 at bifurcation parameter $\delta_1 = \delta_1^*$. This scenario indicates that the IYSD model exhibits a forward bifurcation at $R_0 = 1$. Its biological meaning is that as long as $R_0 < 1$, the IYSD can be eliminated from the onion plants population. Hence the unique endemic equilibrium point $E^* = (S^*, I^*, X^*, W^*)$ is locally asymptotically stable whenever $R_0 > 1$.

3.9. Global Stability Analysis of Endemic Equilibrium point

In this subsection we investigated the global stability of the five equilibria points. It is analyzed by constructing proper lyapunov function as described by somewhere else (Martcheva, 2015).

Theorem 6 The endemic equilibrium E^*, of the model (1) is globally stable if $R_1 > 1$, and condition (60) and (61) is hold.

Proof: To prove this theorem, we construct the following Lyapunov function for the equilibrium point E^* .

$$L(S^*, I^*, X^*, W^*) = \delta_2 X^* f_2(I^*)\left(S - S^* - S^* \ln\left(\frac{S}{S^*}\right)\right) + \delta_2 X^* f_2(I^*)\left(I - I^* - I^* \ln\left(\frac{I}{I^*}\right)\right)$$

$$+\delta_1 S^* f_1(W^*)\left(X - X^* - X^* \ln\left(\frac{X}{X^*}\right)\right) + \delta_1 S^* f_1(W^*)\left(W - W^* - W^* \ln\left(\frac{W}{W^*}\right)\right) \quad (49)$$

Where $f_1(W^*) = \frac{W^*}{1+m_1 W^*}$ and $f_2(I^*) = \frac{I^*}{1+m_2 I^*}$ for convenience. Hence L is C_1 on the interior of Ω, E^* is the global maximum of L on Ω, and we then have $L(S^*, I^*, X^*, W^*) = 0$. Then by apply derivative to both side of equation (4.57)with respect to t gives

$$\frac{dL(t)}{dt} = \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{S^*}{S}\right)\frac{dI}{dt} + \left(1 - \frac{S^*}{S}\right)\frac{dX}{dt} + \left(1 - \frac{S^*}{S}\right)\frac{dW}{dt}. \quad (50)$$

By substituting system equation of (1) to equation (50) we obtain

$$\frac{dL(t)}{dt} = \left(1 - \frac{S^*}{S}\right)\left[rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - \frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - gS(t)\right] + \left(1 - \frac{I^*}{I}\right)\left[\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - \rho I(t) - gI(t)\right] + \left(1 - \frac{X^*}{X}\right)\left[\pi - \frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma X(t)\right] + \left(1 - \frac{W^*}{W}\right)\left[\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - \sigma W(t)\right] \quad (51)$$

Let $rS\left(1 - \frac{N_0)}{K}\right) = \delta_1 S f_1(W^*) + gS,\ \delta_1 S^* f_1(W^*) - (\rho + g)I^* = 0,\ \pi = \delta_2 X^* f_2(I^*) + \sigma X^*$ and $\delta_2 X^* f_2(I^*) - \sigma W^* = 0$ then equation (51) becomes

$$\frac{dL(t)}{dt} = \left(1 - \frac{S^*}{S}\right)[\delta_1 S f_1(W^*) - \delta_1 S f_1(W)] + \left(1 - \frac{I^*}{I}\right)[\delta_1 S f_1(W) - \delta_1 S^* f_1(W^*)] + \left(1 - \frac{I^*}{I}\right)[(\rho + g)I - (\rho + g)I^*] + \left(1 - \frac{X^*}{X}\right)[\delta_2 X^* f_2(I^*) - \delta_2 X f_2(I)] + \left(1 - \frac{X^*}{X}\right)[\sigma X^* - \sigma X] + \left(1 - \frac{W^*}{W}\right)[\delta_2 X f_2(I) - \delta_2 X^* f_2(I^*)] + \left(1 - \frac{W^*}{W}\right)[\sigma W^* - \sigma W] \quad (52)$$

By direct calculation from equation(52), we obtain

$$\frac{dL(t)}{dt} = \delta_1 S^* f_1(W^*)\left(1-\frac{S^*}{S}\right)\left[\frac{S}{S^*}-\frac{Sf_1(W)}{S^*f_1(W^*)}\right] + \delta_1 S^* f_1(W^*)\left(1-\frac{I^*}{I}\right)\left[\frac{Sf_1(W)}{S^*f_1(W^*)}-1\right] + (\rho +$$
$$g)I^*\left(1-\frac{I^*}{I}\right)\left[\frac{I}{I^*}-1\right] + \delta_2 X^* f_2(I^*)\left(1-\frac{X^*}{X}\right)\left[1-\frac{Xf_2(I)}{X^*f_2(I^*)}\right] + \sigma X^*\left(1-\frac{X^*}{X}\right)\left[1-\frac{X}{X^*}\right] +$$
$$\delta_2 X^* f_2(I^*)\left(1-\frac{W^*}{W}\right)\left[\frac{Xf_2(I)}{X^*f_2(I^*)}-1\right] + \sigma W^*\left(1-\frac{W^*}{W}\right)\left[1-\frac{W}{W^*}\right] \quad (53)$$

This implies that

$$\frac{dL(t)}{dt} = \delta_1 S^* f_1(W^*)\left[\frac{S}{S^*}-\frac{Sf_1(W)}{S^*f_1(W^*)}-1+\frac{Sf_1(W)}{S^*f_1(W^*)}\right] + \delta_1 S^* f_1(W^*)\left[\frac{Sf_1(W)}{S^*f_1(W^*)}-1-\frac{SI^*f_1(W)}{S^*If_1(W^*)}+\right.$$
$$\left.\frac{I^*}{I}\right] + (\rho+g)I^*\left[2-\frac{I}{I^*}-\frac{I^*}{I}\right] + \delta_2 X^* f_2(I^*)\left[1-\frac{Xf_2(I)}{X^*f_2(I^*)}-\frac{X^*}{X}+\frac{f_2(I)}{f_2(I^*)}\right] + \sigma X^*\left[2-\frac{X^*}{X}-\frac{X}{X^*}\right] +$$
$$\delta_2 X^* f_2(I^*)\left[\frac{Xf_2(I)}{X^*f_2(I^*)}-1-\frac{XW^*f_2(I)}{X^*Wf_2(I^*)}+\frac{W^*}{W}\right] + \sigma W^*\left[2-\frac{W^*}{W}-\frac{W}{W^*}\right] \quad (53)$$

After collecting like terms and rearranging from (53), we obtain

$$\frac{dL(t)}{dt} = \delta_1 S^* f_1(W^*)\left[\frac{S}{S^*}-2+\frac{Sf_1(W)}{S^*f_1(W^*)}-\frac{SI^*f_1(W)}{S^*If_1(W^*)}+\frac{I^*}{I}+\frac{S^*If_1(W^*)}{SI^*f_1(W)}-\frac{S^*If_1(W^*)}{SI^*f_1(W)}\right] +$$
$$\delta_2 X^* f_2(I^*)\left[\frac{f_2(I)}{f_2(I^*)}-\frac{X^*}{X}-\frac{XW^*f_2(I)}{X^*Wf_2(I^*)}+\frac{W^*}{W}+\frac{Wf_2(I^*)}{W^*f_2(I)}-\frac{Wf_2(I^*)}{W^*f_2(I)}\right] + (\rho+g)I^*\left[2-\frac{I}{I^*}-\frac{I^*}{I}\right] +$$
$$\sigma X^*\left[2-\frac{X^*}{X}-\frac{X}{X^*}\right] + \sigma W^*.\left[2-\frac{W^*}{W}-\frac{W}{W^*}\right] \quad (54)$$

From equations (54) after making some simplification and arrangement, we obtain

$$\frac{dL(t)}{dt} = \delta_1 S^* f_1(W^*)\left[\frac{S}{S^*}+\frac{Sf_1(W)}{S^*f_1(W^*)}+\frac{I^*}{I}+\frac{S^*If_1(W^*)}{SI^*f_1(W)}-4\right] + \delta_1 S^* f_1(W^*)\left[2-\frac{SI^*f_1(W)}{S^*If_1(W^*)}-\right.$$
$$\left.\frac{S^*If_1(W^*)}{SI^*f_1(W)}\right] + \delta_2 X^* f_2(I^*)\left[\frac{f_2(I)}{f_2(I^*)}+\frac{W^*}{W}+\frac{Wf_2(I^*)}{W^*f_2(I)}-3\right] + \delta_2 X^* f_2(I^*)\left[3-\frac{X^*}{X}-\frac{XW^*f_2(I)}{X^*Wf_2(I^*)}-\right.$$
$$\left.\frac{Wf_2(I^*)}{W^*f_2(I)}\right] + (\rho+g)I^*\left[2-\frac{I}{I^*}-\frac{I^*}{I}\right] + \sigma X^*\left[2-\frac{X^*}{X}-\frac{X}{X^*}\right] + \sigma W^*\left[2-\frac{W^*}{W}-\frac{W}{W^*}\right] \quad (55)$$

From equation (55) by arithmetic geometric inequality we have.

$$\left[2-\frac{SI^*f_1(W)}{S^*If_1(W^*)}-\frac{S^*If_1(W^*)}{SI^*f_1(W)}\right] \leq 0, \quad (56)$$

$$\left[3-\frac{X^*}{X}-\frac{XW^*f_2(I)}{X^*Wf_2(I^*)}-\frac{Wf_2(I^*)}{W^*f_2(I)}\right] \leq 0, \quad (57)$$

$$\left[2-\frac{I}{I^*}-\frac{I^*}{I}\right] \leq 0, \quad (58)$$

$$\left[2-\frac{W^*}{W}-\frac{W}{W^*}\right]\leq 0. \tag{59}$$

And if

$$\left[\frac{S}{S^*}+\frac{Sf_1(W)}{S^*f_1(W^*)}+\frac{I^*}{I}+\frac{S^*If_1(W^*)}{SI^*f_1(W)}-4\right]\leq 0, \tag{60}$$

$$\left[\frac{f_2(I)}{f_2(I^*)}+\frac{W^*}{W}+\frac{Wf_2(I^*)}{W^*f_2(I)}-3\right]\leq 0. \tag{61}$$

Hence, by $(56)-(61)$ we have that $\frac{dL(t)}{dt}\leq 0$ and $\frac{dL(t)}{dt}=0$ if and only if $S=S^*, I=I^*, X=X^*, and\ W=W^*$. Thus, the largest compact invariant set in $\left\{(S^*,I^*,X^*,W^*)\in \Omega: \frac{dL(t)}{dt}=0\right\}$ is the singleton E^*. By LaSalle's invariant principle, it follows that the endemic equilibrium point E^* is globally asymptotically stable (GAS) in the domain Ω.

3.10. Sensitivity Analysis and it's interpretation

In this sub section, we preformed the sensitivity analysis of basic reproduction number in order to determine the effect parameter on the disease transmission and controls. Thus to obtain sensitivity analysis of the basic reproduction number of model R_0, we used the formula given below:

$$P_{N_i}^{R_0}=\frac{N_i}{R_0}\times\frac{\partial R_0}{\partial N_i}, \tag{62}$$

with N_i is the parameter in R_0, where, $R_0=\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}$. $R_{0h}=\sqrt{\frac{\delta_1 K(r-g)}{r(\rho+g)}}$ and $R_{0v}=\sqrt{\frac{\delta_2\pi}{\sigma^2}}$

For $N_i=\delta_1$, then $P_\beta=\frac{\delta_1}{R_0}\times\frac{\partial}{\partial\delta_1}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right)=\frac{\delta_1}{2R_0^2}\left(\frac{\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}\right)$

$$=\frac{1}{2}\times\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}\times\frac{r\sigma^2(\rho+g)}{\delta_1\delta_2\pi K(r-g)}=\frac{1}{2}>0.$$

For $N_i=\delta_2$, then $P_\beta=\frac{\delta_2}{R_0}\times\frac{\partial}{\partial\delta_2}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right)=\frac{\delta_2}{2R_0^2}\left(\frac{\delta_1\pi K(r-g)}{r\sigma^2(\rho+g)}\right)$

$$=\frac{1}{2}\times\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}\times\frac{r\sigma^2(\rho+g)}{\delta_1\delta_2\pi K(r-g)}=\frac{1}{2}>0.$$

For $N_i = \pi$, then $P_\beta = \frac{\pi}{R_0} \times \frac{\partial}{\partial \pi}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right) = \frac{\pi}{2R_0^2}\left(\frac{\delta_1\delta_2 K(r-g)}{r\sigma^2(\rho+g)}\right)$

$$= \frac{1}{2} \times \frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)} \times \frac{r\sigma^2(\rho+g)}{\delta_1\delta_2\pi K(r-g)} = \frac{1}{2} > 0.$$

For $N_i = K$, then $P_\beta = \frac{K}{R_0} \times \frac{\partial}{\partial K}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right) = \frac{K}{2R_0^2}\left(\frac{\delta_1\delta_2\pi(r-g)}{r\sigma^2(\rho+g)}\right)$

$$= \frac{1}{2} \times \frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)} \times \frac{r\sigma^2(\rho+g)}{\delta_1\delta_2\pi K(r-g)} = \frac{1}{2} > 0.$$

For $N_i = r$, then $P_\beta = \frac{r}{R_0} \times \frac{\partial}{\partial r}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right) = \frac{r}{2R_0^2} \times \frac{\partial}{\partial r}\left(\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}\right)$

$$= \frac{r}{2R_0^2} \times \left(\frac{\delta_1\delta_2\pi K g\sigma^2(p+g)}{r^2\sigma^4(\rho+g)^2}\right) = \frac{g}{2(r-g)}.$$

For $N_i = g$, then $P_\beta = \frac{g}{R_0} \times \frac{\partial}{\partial g}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right) = \frac{g}{2R_0^2} \times \frac{\partial}{\partial g}\left(\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}\right)$

$$= \frac{-g}{2R_0^2} \times \left(\frac{\delta_1\delta_2\pi K(r+p)}{r\sigma^2(\rho+g)^2}\right) = \frac{-g(r+\rho)}{2(\rho+g)} < 0.$$

For $N_i = \sigma$, then $P_\beta = \frac{\sigma}{R_0} \times \frac{\partial}{\partial \sigma}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right) = \frac{\sigma}{2R_0^2} \times \frac{\partial}{\partial \sigma}\left(\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}\right)$

$$= \frac{-1}{2R_0^2} \times \frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)} = -1 < 0.$$

For $N_i = \rho$, then $P_\beta = \frac{\rho}{R_0} \times \frac{\partial}{\partial \rho}\left(\sqrt{\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}}\right) = \frac{\rho}{2R_0^2} \times \frac{\partial}{\partial \rho}\left(\frac{\delta_1\delta_2\pi K(r-g)}{r\sigma^2(\rho+g)}\right)$

$$= \frac{-\rho}{2R_0^2} \times \frac{\delta_1\delta_2\pi K(r-g)r\sigma^2}{r^2\sigma^4(\rho+g)^2} = -\frac{\rho}{2(\rho+g)} < 0.$$

Hence, we summarized parameters, symbols and sensitivity indices, and it is value in the R_0 (**Table 1)**.

Table 1. Sensitivity indices of integer order model.

Parameter	Description	Indices sensitivity	Value
$\boldsymbol{\delta_1}$	Transmission rate from infected thrips to healthy onion plants	$+ve$	0.5
$\boldsymbol{\delta_2}$	Transmission rate from infected onion plants to healthy thrips	$+ve$	0.5
$\boldsymbol{\pi}$	Birth or immigration rate of thrips vectors	$+ve$	0.5
$\boldsymbol{K}$	Carrying capacity onion farm	$+ve$	0.5
$\boldsymbol{r}$	Intrinsic growth rate of onion plants	$+ve$	0.0102
$\boldsymbol{g}$	Harvest rate of onion plants	$-ve$	-0.02239
$\boldsymbol{\sigma}$	Natural death rate for thrips vectors	$-ve$	-1
$\boldsymbol{\rho}$	Death rate of onion plants due to IYSD disease	$-ve$	-0.4878

In general, the above sensitivity index is interpreted as follows: Parameters with positive indices have a greater impact on the leading to an increase of the spread of IYSD in the farm because their values increase when other parameters are kept constant. A parameter with a negative index has a greater impact on the reduction of IYSD in the farm when its value increases while another parameter is kept constant.

4. OPTIMAL CONTROL

In Section 4 we presented a mathematical model for malaria transmission dynamics and studied both mathematically and epidemiological well posedness without giving intention mechanism. Detail intervention strategies are done in this sections where, we introduced controls that

minimize IYSV disease transmission. After we present the optimal control formulation, we study the transmission dynamics through numerical.

4.1. Extension of the model into an optimal control

Now we introduced the time-dependent controls in the model (1) for the aim of controlling IYSV disease and study the strategies that control IYSV disease transmission. We applied optimal control strategies on the model (1) to identify the best intervention strategies that help to eradicate the disease in the specified time.

The optimal control model is an extension IYSV disease model by including the following three controls defined as:

i) The control u_1 represents Prevention strategies, which protect susceptible from contacting the disease.

ii) The control u_2 represents Quarantine strategy, which minimize the expansion of the Iris Yellow Spot Disease (IYSD) disease in the field by uprooting infectious onion and burning it.

iii) The control u_3 represents Chemical control, which protect and minimizes the invasion of Iris Yellow Spot Disease (IYSD) of onion plants.

After incorporating, $u_1, u_2,$ and u_3 in the model (1), we obtain the following optimal control model:

$$g(x,u,t) = \begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - (1-u_1)\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - gS(t) \\ \frac{dI(t)}{dt} = (1-u_1)\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - (u_2+\rho+g)I(t) \\ \frac{dX(t)}{dt} = \pi - (1-u_2)\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - (u_3+\sigma)X(t) \\ \frac{dW(t)}{dt} = (1-u_2)\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - (u_3+\sigma)W(t) \end{cases} \quad (63)$$

The purpose of introducing controls in the model is to find the optimal level of the intervention strategy preferred to minimize the spread of infection and cost of implementation of the control. The control variables u_1, u_2 and u_3 are minimized subject to the differential equation (63) and formulate the objective functional as

$$J(u_1, u_2, u_3) = \min_{u_1,u_2,u_3} \int_0^T \left(A_1 I + A_2 W + \frac{1}{2}(B_1 u_1^2 + B_2 u_2^2 + B_3 u_3^2) \right) dt \qquad (64)$$

Subjected to $g(x, u, t)$.

Where T represents the final time of control implementation, and quantities A_1 and A_2 are weights constants or balance factors of the infected onion plants population and infected thrips vectors population respectively, while B_1, B_2, and B_3 are weight constants used for Prevention strategies, Quarantine strategy, and Chemical control (u_i for $i = 1,2,3$). We further assume that, due to technical reasons, the cost of controls u_i for $i = 1,2,3$ is non linear and quadratic as seen in the cost function (64) and $B_i u_i^2$ for $i = 1,2,3$ represents the cost of controls u_i for $i = 1,2,3$.

The main goal is to achieve the best control $u_1^*,\ u_2^*,$ and u_3^* so that

$$J(u_1, u_2, u_3) = \min_U J(u_1^*, u_2^*, u_3^*), \qquad (65)$$

where,

$$U = \{(u_1, u_2, u_3) | u_i \text{ is measureable on } [0, T], 0 \leq u_i \leq 1, i = 1,2,3\}.$$

4.2. Hamiltonian and Uniqueness of the Optimality System

In this subsection, we examine conditions that can assure the Hamiltonian and Uniqueness of a solution to the optimal control problem (63).

4.2.1. Hamiltonian and optimality system

According to the Pontryagin's maximum principle, if $u^*(\cdot) \in U$ is optimal for problem (65) with fixed final time T, then there exists a non-trivial absolutely continuous mapping $\lambda: [0,\ T] \to \mathbb{R}^4,\ \lambda = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$ called the adjoint vector, such that:

1. the Hamiltonian function is defined as

$$\mathcal{H}(S, I, X, W, t) = R(I, W, u_1, u_2, u_3, t) + \lambda_1 \frac{dS}{dt} + \lambda_2 \frac{dI}{dt} + \lambda_3 \frac{dX}{dt} + \lambda_4 \frac{dW}{dt} \qquad (66)$$

Where $R(I,W,u_1,u_2,u_3,t) = A_1I + A_2W + \frac{1}{2}(B_1u_1^2 + B_2u_2^2 + B_3u_3^2), \lambda_i, i = 1,2,3,4$ are the adjoint variable functions to be obtained properly by applying Pontryagin's maximal principle (Pontryagin *et al*., 1992) and also using (Pontryagin *et al*., 1992) for the existence of the optimal control pairs.

2. the control system

$$S' = \frac{\partial \mathcal{H}}{\partial \lambda_1}, I' = \frac{\partial \mathcal{H}}{\partial \lambda_2}, X' = \frac{\partial \mathcal{H}}{\partial \lambda_3} \quad W' = \frac{\partial \mathcal{H}}{\partial \lambda_4}$$

3. The adjoint system

$$\lambda_1' = \frac{-\partial \mathcal{H}}{\partial S}, \lambda_2' = \frac{-\partial \mathcal{H}}{\partial I}, \lambda_3' = \frac{-\partial \mathcal{H}}{\partial X}, \lambda_4' = \frac{-\partial \mathcal{H}}{\partial W}$$

4. and the optimality condition

$$\mathcal{H}(x^*, u^*, \lambda^*) = \min_{u \in \mathrm{U}} \mathcal{H}(x, u, \lambda) \quad \text{holds for almost all } t \in [0,T].$$

5. Moreover, the transversality condition

$$\lambda_i(T) = 0, \quad i = 1, \dots, 4 \quad \text{also holds true.}$$

Theorem 4.1: The optimal control problem (5.3) with fixed final time T admits a unique optimal solution $x^* = (S^*, I^*, X^*, W')$ associated with an optimal control $u^* = (u_1^*, u_2^*, u_3^*)$ for all $t \in [0,T]$. Moreover, there exist adjoint function $\lambda_i^*, \ i = 1, 2, \dots, 4$ such that

$$\begin{cases} \frac{d\lambda_1}{dt} = -\frac{dH}{dS} = -\lambda_1 r\left(1 - \frac{2S+I}{K}\right) + (1-u_1)\frac{\delta_1 W}{1+m_1 W}(\lambda_1 - \lambda_2) + \lambda_1 g \\ \frac{d\lambda_2}{dt} = -\frac{dH}{dI} = -A_1 + \frac{\lambda_1 Sr}{K} + \lambda_2(u_2 + \rho + g) + (1-u_2)\frac{\delta_2 X}{(1+m_2 I)^2}(\lambda_3 - \lambda_4) \\ \frac{d\lambda_3}{dt} = -\frac{dH}{dX} = (1-u_2)\frac{\delta_2 I}{1+m_2 I}(\lambda_3 - \lambda_4) - (u_3 + \sigma)\lambda_3 \\ \frac{d\lambda_4}{dt} = -\frac{dH}{dW} = -A_2 + (1-u_1)\frac{\delta_2 S}{(1+m_1 W)^2}(\lambda_1 - \lambda_2) + (u_3 + \sigma)\lambda_4 \end{cases} \tag{67}$$

with transversality conditions

$$\lambda_i^*(T) = 0, \quad i = 1, \dots, 4. \tag{68}$$

Furthermore, we obtain the control set $u^* = (u_1^*, u_2^*, u_3^*)$ characterized by

$u_1^*(t) = \max\{0, \min(1, \Phi_1)\}$,

$u_2^*(t) = \max\{0, \min(1, \Phi_2)\}$,

$u_3^*(t) = \max\{0, \min(1, \Phi_3)\}$

Where,

$\Phi_1 = \frac{\delta_1 SW}{B_1(1+m_1W)}(\lambda_2 - \lambda_1)$,

$\Phi_2 = \frac{\delta_2 XI(\lambda_4 - \lambda_3) + \lambda_2 I(1 + m_2 I)}{B_2(1 + m_2 I)}$

$\Phi_3 = \frac{\lambda_3 X + \lambda_4 W}{B_3}$

Proof: The adjoint equation and transversality conditions are standard results obtained from Pontryagin's maximum principle (Pontryagin *et al*., 1992). We differentiate Hamiltonian in Equation (66) with respect to states S, I, X and W respectively, and then the adjoint system can be written as:

$$\frac{d\lambda_1}{dt} = -\frac{dH}{dS} = -\lambda_1 r\left(1 - \frac{2S+I}{K}\right) + (1-u_1)\frac{\delta_1 W}{1+m_1 W}(\lambda_1 - \lambda_2) + \lambda_1 g$$
$$\frac{d\lambda_2}{dt} = -\frac{dH}{dI} = -A_1 + \frac{\lambda_1 Sr}{K} + \lambda_2(u_2 + \rho + g) + (1-u_2)\frac{\delta_2 X}{(1+m_2 I)^2}(\lambda_3 - \lambda_4)$$
$$\frac{d\lambda_3}{dt} = -\frac{dH}{dX} = (1-u_2)\frac{\delta_2 I}{1+m_2 I}(\lambda_3 - \lambda_4) - (u_3 + \sigma)\lambda_3$$
$$\frac{d\lambda_4}{dt} = -\frac{dH}{dW} = -A_2 + (1-u_1)\frac{\delta_2 S}{(1+m_1 W)^2}(\lambda_1 - \lambda_2) + (u_3 + \sigma)\lambda_4$$

And also for characterization of the optimal control we used the following partial differential equation:

$\frac{\partial H}{\partial u_i} = 0$ at $u_i = u_i^*$where $i = 1; 2; 3.$

For $i = 1\ \frac{\partial H}{\partial u_1} = 0$ at $u_1 = u_1^*$

$\Longrightarrow u_1^* = \frac{\delta_1 SW}{B_1(1+m_1 W)}(\lambda_2 - \lambda_1)$

For $i = 2\frac{\partial H}{\partial u_2} = 0$ at $u_2 = u_2^*$

$\Longrightarrow u_2^* = \frac{\delta_2 XI}{B_2(1+m_2 I)}(\lambda_4 - \lambda_3) + \frac{\lambda_2 I}{B_2}$

For $i = 3\frac{\partial H}{\partial u_3} = 0$ at $u_3 = u_3^*$

$\Longrightarrow u_3^* = \frac{\lambda_3 X + \lambda_4 W}{B_3}$

When we write by using standard control arguments involving the bounds on the controls, we conclude:

$$u_1^* = \begin{cases} \Phi_1, if\ 0 < \Phi_1 < 1 \\ 0, if\ \Phi_1 \leq 0 \\ 1, if\ \Phi_1 \geq 1 \end{cases}$$

$$u_2^* = \begin{cases} \Phi_2, if\ 0 < \Phi_2 < 1 \\ 0, if\ \Phi_2 \leq 0 \\ 1, if\ \Phi_2 \geq 1 \end{cases}$$

$$u_3^* = \begin{cases} \Phi_3, if\ 0 < \Phi_3 < 1 \\ 0, if\ \Phi_3 \leq 0 \\ 1, if\ \Phi_3 \geq 1 \end{cases}$$

Since $0 < u_i^* < 1$, we can wrote in a compact notation:

$u_1^* = \max\{0, \min(1, \Phi_1)\}$,

$u_2^* = \max\{0, \min(1, \Phi_2)\}$,

$u_3^* = \max\{0, \min(1, \Phi_3)\}$

Where,

$\Phi_1 = \frac{\delta_1 SW}{B_1(1+m_1 W)}(\lambda_2 - \lambda_1)$,

$$\Phi_2 = \frac{\delta_2 XI}{B_2(1 + m_2 I)}(\lambda_4 - \lambda_3) + \frac{\lambda_2 I}{B_2}$$

$$\Phi_3 = \frac{\lambda_3 X + \lambda_4 W}{B_3}$$

The optimality system is formed from the optimal control system (the state system) and the adjoint variable system by incorporating the characterized control set and initial and transversal condition

$$\begin{cases} \frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)+I(t)}{K}\right) - (1-u_1)\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - gS(t) \\ \frac{dI(t)}{dt} = (1-u_1)\frac{\delta_1 S(t)W(t)}{1+m_1 W(t)} - (u_2+\rho+g)I(t) \\ \frac{dX(t)}{dt} = \pi - (1-u_2)\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - (u_3+\sigma)X(t) \\ \frac{dW(t)}{dt} = (1-u_2)\frac{\delta_2 X(t)I(t)}{1+m_2 I(t)} - (u_3+\sigma)W(t) \\ \frac{d\lambda_1}{dt} = -\lambda_1 r\left(1 - \frac{2S+I}{K}\right) + (1-u_1)\frac{\delta_1 W}{1+m_1 W}(\lambda_1 - \lambda_2) + \lambda_1 g \\ \frac{d\lambda_2}{dt} = -A_1 + \frac{\lambda_1 Sr}{K} + \lambda_2(u_2+\rho+g) + (1-u_2)\frac{\delta_2 X}{(1+m_2 I)^2}(\lambda_3 - \lambda_4) \\ \frac{d\lambda_3}{dt} = (1-u_2)\frac{\delta_2 I}{1+m_2 I}(\lambda_3 - \lambda_4) - (u_3+\sigma)\lambda_3 \\ \frac{d\lambda_4}{dt} = -A_2 + (1-u_1)\frac{\delta_2 S}{(1+m_1 W)^2}(\lambda_1 - \lambda_2) + (u_3+\sigma)\lambda_4 \end{cases} \tag{69}$$

$\lambda_i(T) = 0, \quad i = 1, \dots, 4, S(0) = S_0,\ I(0) = I_0, X(0) = X_0,\ W(0) = W_0.$

4.2.2. Uniqueness of the Optimality System

Since the state and adjoint variables are bounded and also the obtained ordinary differential equations have Lipschitz in their structure, it is possible to show the uniqueness, hence the following theorem.

Theorem 4.2 For $t \in [0, T]$; the bounded solutions to the optimality system are unique.

Proof: See (Fister *et al*., 1998) for the proof of this theorem.

5. NUMERICAL ANALYSIS

This section present and discusses numerical results for the system (1) and (63) for different values of parameters given in the model. The simulation process is carried out using *MATLAB* software. We performed numerical simulations of our proposed model (1) and the optimal control problem (63) to support some of the analytical results. The parameter values used in the

numerical simulation were obtained from the different literatures (Table 2), but we estimated the few from data found in (Birithia et al., 2011) and (Munk., 2009) to draw realistic scenarios. Particularly, since the average life span of thrips vectors is 14–30 days (Birithia et al., 2011), it follows that it natural mortality rate is $\sigma = \frac{1}{\frac{(14+130)}{2}} = 0.045\ day^{-1}$. Moreover, mature onion plants could be ready for harvest at about 80 to 120 days (Munk., 2009), meaning that the harvest rate can be estimated as $g = \frac{1}{\frac{(80+120)}{2}} = 0.01\ day^{-1}$. Additionally, we used the initial values of the model are set as: $S(0) = 600, I(0) = 150, X(0) = 200, W(0) = 15$. We choose the following set as weight factors: $A_1 = 80, A_2 = 60, A_3 = 60, A_4 = 100, B_1 = 60, B_2 = 100$ and $B_3 = 80$ to perform numerical stimulations.

Table 2: Parameter values for the model (1).

Parameter	Description	Value	Reference
$\boldsymbol{\delta_1}$	Transmission rate from infected thrips to healthy onion plants	$0.05\ vector^{-1}day^{-1}$	MJ Jeger *et al.* (2015)
$\boldsymbol{\delta_2}$	Transmission rate from infected onion plants to healthy thrips	$0.05\ plant^{-1}day^{-1}$	MJ Jeger *et al.* (2015)
$\boldsymbol{\pi}$	Birth or immigration rate of thrips vectors	$0.003\ day^{-1}$	Patiene *et al.* (2022)
$\boldsymbol{K}$	Carrying capacity onion farm	4000	Assumed
$\boldsymbol{r}$	Intrinsic growth rate of onion plants	$0.5\ day^{-1}$	Patiene *et al.* (2022)
$\boldsymbol{g}$	Harvest rate of onion plants	$0.01\ day^{-1}$	Estimated
$\boldsymbol{\sigma}$	Natural death rate for thrips vectors	$0.045\ day^{-1}$	Estimated
$\boldsymbol{\rho}$	Death rate of onion plants due to IYSD disease	$0.4\ day^{-1}$	Assumed
$\boldsymbol{m_1}$	Saturation rate of healthy onion plants due to infected thrips	0.01	Patiene *et al.* (2022)

m_2	Saturation rate of healthy thrips due to infected onion plants	0.02	Patiene *et al.* (2022)

To determine the impact of each controls on the reduction of IYSD of onion plant, we used the following three strategies with different combination of two controls at a time and three controls at a time.

Strategy A (Prevention and quarantine)

Strategy B (Prevention and chemical control)

Strategy C (Quarantine and chemical control)

Strategy D (All the three control strategies)

Strategy A: Applying the treatment of Prevention u_1 and quarantine u_2.

Under this strategy, we used control prevention and quarantine to minimize the objective function without chemical control and **Figure 4**, shows that due to the control strategies there was a significant decrease in the number of infected onion plants and increase infected thrips vectors when compared to with no control.

Strategy B: Applying the treatment of Prevention u_1 and chemical control u_3.

In utilizing the strategies, prevention and chemical while setting quarantine strategy zero, we observed from the **Figure 5**, that there was a significant drop in the number infected thrips vectors when there was control compared to that without control, but there was no change in the infected onion plants.

Strategy C: Applying the treatment of quarantine u_2 and chemical control u_3.

Under this strategy, we simulated the model using quarantine control and chemical control optimize the objective function while prevention control was set to zero. In **Figure 6**, it is observed that there is a significant drop in the number of infected onion plants and infected thrips vectors when there is control compared to the situation when there is no control.

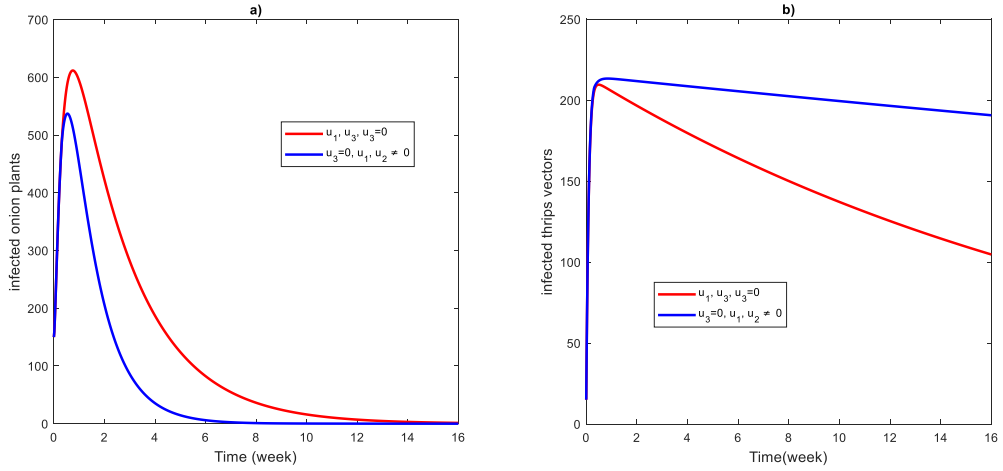


Figure 1: Simulations of the IYSD disease model with prevention and quarantine controls.

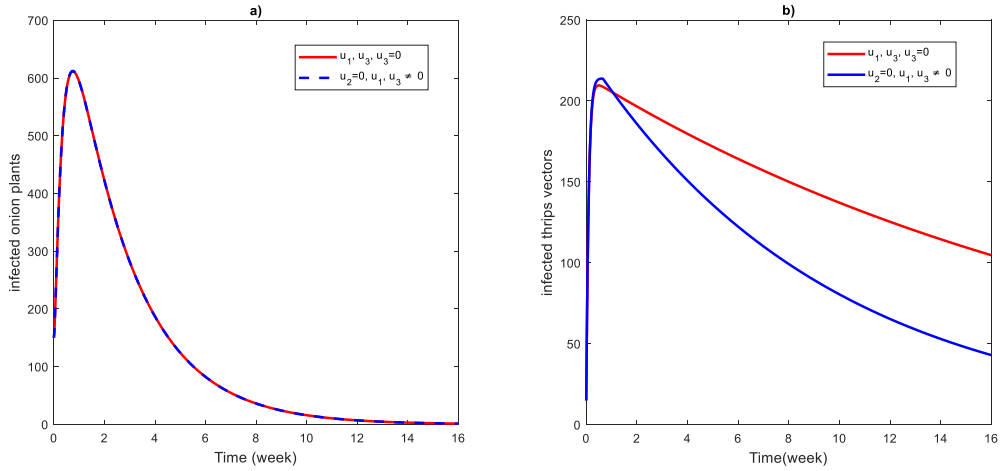


Figure 2: Simulations of the IYSD disease model with Prevention and chemical controls.

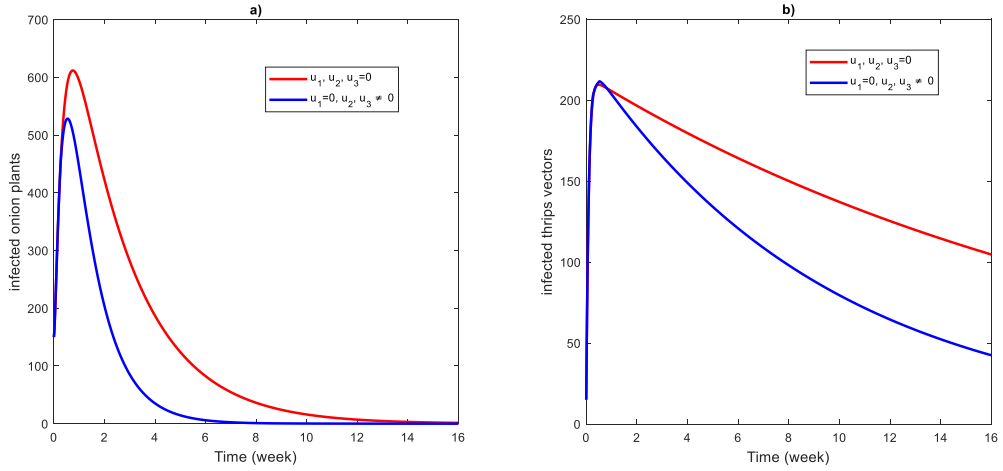


Figure 3: Simulations of the IYSD disease model with quarantine and chemical controls.

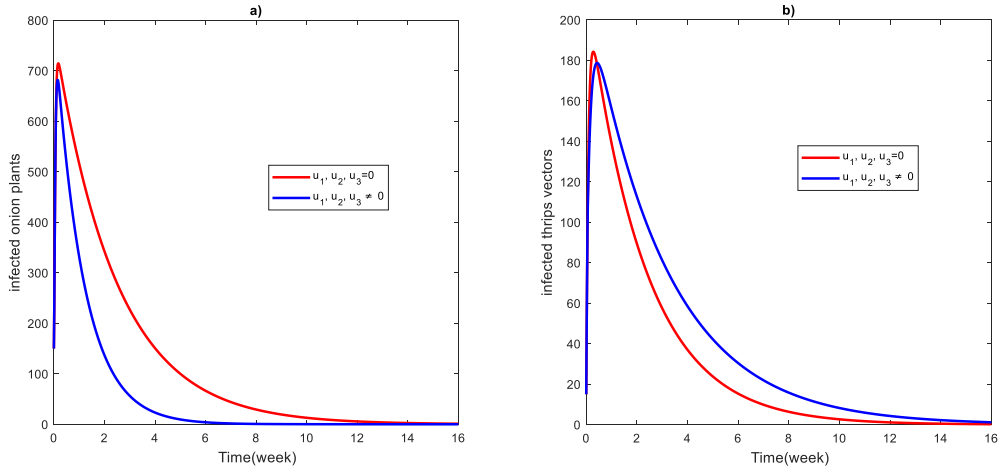


Figure 4: Simulations of the IYSD disease model with Prevention, quarantine and chemical controls.

Strategy D: Applying all the three control strategies of Prevention u_1, quarantine u_2 and chemical control u_3.

We simulated the model using of all strategies, namely, prevention control $u1$, the quarantine control and chemical control was to optimize the objective function. The simulation results in the **Figure 7**, showed that there was a significant drop in the number of infected onion plants and increase infected thrips vectors when there is control compared to the situation without control.

From all the above the strategies A to D, we conclude that the **strategy C** (combination of quarantine u_2 and chemical control u_3) is the best strategy to minimize the number of infected onion plants and infected thrips vectors in the IYSD.

6. CONCLUSION

In this study, we have formulated and analyzed a mathematical eco-epidemiological model describing the transmission dynamics of Iris Yellow Spot Virus (IYSV) disease in onion plants with nonlinear saturated incidence of Holling type II form and logistic growth to add some realism. We computed the threshold quantity R_0 which led to the following findings: the disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$, as confirmed by the Routh-Hurwitz criterion, and the endemic equilibrium E^* is globally asymptotically stable if $R_0 > 1$, as confirmed by Lyapunov's method and LaSalle's invariance principle. Further, we applied the center manifold theory to establish that the model undergoes the forward bifurcation at $R_0 = 1$. From the basic model, an optimal control problem is formulated by incorporating three control variables; Prevention, treatment and screening. The Hamiltonian, adjoint variables, characterization of the controls and the optimality system are derived from the optimal control problem and also numerically simulated by considering combination of two control at a time and by applying all the three control variables. The numerical results indicate that the **strategy C** controls (combination of quarantine u_2 and chemical control u_3) is the best strategy to minimize the number of infected onion plants and infected thrips vectors in the IYSD population.

7. Authorship contribution statement

Abdella Kereme participated in the project design, method development, validation, analyzing and interpreting the data, drafting and writing the manuscript. **Mohammed Obsiye** design, conceptualization, methodologically supervised, supervision, reviewing and editing the manuscript. **Tasew Ayele** designed, supervised, interpreted data, and reviewed and edited the manuscript. All authors have read and approved the final manuscript.

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10. Availability of the date

All data generated or analysed during this study are included in this published article and it is cited properly in this manuscript

11. Competing interest

The authors declare that they have no competing interest

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